Dispersion relations and low relaxation of spin waves in thin magnetic films

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We study spin excitations in thin magnetic films in the Heisenberg model with magnetic dipole and exchange interactions by the spin operator diagram technique and make comparison of their parameters with characteristics of spin waves in thick films. Dispersion relations of spin waves in thin magnetic films (in two-dimensional magnetic monolayer and bilayer lattices) and the spin-wave resonance spectrum in N-layer structures are found. For thick magnetic films, spin excitations are determined by simultaneous solution of the generalized Landau-Lifshitz equations and the equation for the magnetostatic potential. Generalized Landau-Lifshitz equations are derived from first principles and have the integral (pseudodifferential) form. It is found that dispersion relations of spin waves in monolayers and in bilayers differ from dispersion relations of spin waves in continuous thick magnetic films. For normal magnetized ferromagnetic films, the spin-wave damping is calculated in the one-loop approximation for a diagram expansion of the Green functions at low temperature. In thick magnetic films, the magnetic dipole interaction makes a major contribution to the relaxation of long-wavelength spin waves. Thin films have a region of the low relaxation of long-wavelength spin waves. In thin magnetic films, four-spin-wave processes take place and the exchange interaction makes a major contribution to the damping. It is found that the damping of spin waves propagating in a magnetic monolayer is proportional to the quadratic dependence on the temperature and is very low for spin waves with small wave vectors.

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I. INTRODUCTION

Nanosized magnetic films are of great interest due to their perspective applications in spin-wave devices. At present, the most important spin-wave devices—microwave filters, delay lines, signal-to-noise enhancers, and optical signal processors—have been realized on the base of magnetic films of microsized thickness.[1–3] Nanosized films give us an opportunity to construct spin-wave devices of small sizes and to design devices with new functional properties. Recently, new applications of spin waves have been proposed such as spin-wave computing,[4,5] spin-wave filtering using width-modulated nanostrip waveguides,[6] and transmission of electrical signals by spin-wave interconversion in an insulator garnet $Y_3Fe_5O_{12}$ (YIG) film based on the spin-Hall effect.[7] Spin-wave logic elements have been done on the basis of a Mach-Zehnder-type interferometer[6,8,9] and can be realized on magnonic crystals.[5] Using nanosized magnetic films, we have a probability to construct an array of logic elements of small sizes. Ferromagnetic monolayers, bilayers, and trilayers are of great interest for magnetic sensors and spin-wave devices. Spin excitations in these thin magnetic film structures are theoretically investigated and are studied by the Brillouin light-scattering method.[10–14]

In order to design new spin-wave devices based on nanosized magnetic films, it is necessary to determine dispersion relations and damping of spin excitations in thin films. In the phenomenological model with the magnetic dipole interaction (MDI) and the exchange interaction,[15–18] the magnetization dynamics in thick magnetic films is described by the Landau-Lifshitz equations, which are differential with respect to spatial variables. The differential form of equations is postulated. In this connection, the following question arises: is this form of Landau-Lifshitz equations correct for thin nanosized films? Determination of the dispersion relations depends on the answer of this question. In phenomenological models, the spin-wave damping is described by relaxation terms in Gilbert, Landau-Lifshitz, or Bloch forms.[18] Properties of intrinsic relaxation processes are not taken into account in these terms and, therefore, the calculated spin-wave damping may be incorrect. The above-mentioned leads us to the main question of the paper: what are the dispersion relations and damping of spin waves in thin films and can they be derived from first principles? In order to answer this question, we consider generalized Landau-Lifshitz equations, spin excitations, and relaxation of spin waves in thin films in the framework of the Heisenberg model with the MDI and the exchange interaction. In the paper, we suppose that films are thin in two cases. (1) For the case, when we calculate dispersion relations of spin waves, we say that an N-layer structure is thin, if N is a low number (for example, monolayer, bilayer, trilayer). (2) For the case of relaxation processes a film is thin, if the spin-wave energy is less than energy gaps between spin-wave modes and, therefore, three-spin-wave processes are forbidden.

The above-mentioned problems have not yet been investigated comprehensively. One of the cause of these problems is the long-range action of the MDI. The spin-wave relaxation and the spin-wave dynamics become dependent on the dimensions and shapes of ferromagnetic samples. In order to analyze the Heisenberg model with the MDI and the exchange interaction, we use the spin operator diagram technique.[19–23] Advantages of the spin operator diagram technique give us the opportunity to calculate the spin-wave damping at high temperatures and obtain more exact relationships describing spin-wave scattering and excitations in comparison with methods based on diagram techniques for creation and annihilation magnon Bose operators.[24–32] In Refs. 23 and 33, the spin operator diagram technique is generalized for models with arbitrary internal Lie-group dynamics.

In Sec. II, we consider spin operator diagram technique for the Heisenberg model with the MDI and the exchange interaction. Spin-wave excitations are determined by poles of the $\mathcal{P}$ matrix: the matrix of the effective Green functions and interaction lines. On the basis of this diagram technique, dispersion relations of spin waves in a magnetic monolayer and in a bilayer and the spectrum of spin-wave resonances in an N-layer structure are found (see Sec. III). It is found that dispersion relations of spin waves in monolayer and bilayer lattices differ from dispersion relations of spin waves in continuous thick magnetic films. This difference is due to the discreetness of the lattice. For the case when the MDI is equal or greater than the exchange interaction, for example, for monolayer consisted of magnetic nanoparticles on the lattice, this difference becomes essential and is taken into account. For thick magnetic films, it is more convenient to present the $\mathcal{P}$-matrix-pole equation describing spin-wave excitations in the form of the Landau-Lifshitz equations and the equation for the magnetostatic potential (see Sec. IV). Spin excitations are determined by simultaneous solution of these equations. Landau-Lifshitz equations are integral (pseudodifferential) equations, but not differential ones with respect to spatial variables. In the common case, the reduction of Landau-Lifshitz equations to differential equations with exchange boundary conditions is incorrect and their solutions give dispersion relations different from dispersion relations calculated on the basis of integral (pseudodifferential) Landau-Lifshitz equations. The contradiction is removed, if the pinning parameter is equal to the spin-wave wave vector. In Sec. V, we consider spin-wave relaxation in thick and thin magnetic films. In thick films, three-spin-wave processes take place and the MDI makes a major contribution to the relaxation of long-wavelength spin waves. Thin films have a region of low relaxation of long-wavelength spin waves. In this case, three-spin-wave processes are forbidden and the exchange interaction makes a major contribution to the relaxation process.

II. HEISENBERG MODEL WITH MAGNETIC DIPOLE AND EXCHANGE INTERACTIONS

A. Spin operator diagram technique

Let us consider the Heisenberg model with the exchange interaction and the MDI on a lattice.[22,23] The exchange interaction is short ranged and the MDI is long ranged. Operators $S^{\pm} = S^x \pm iS^y$ and S^z satisfy the commutation relations:

$$[S^z(\vec{1}),S^+(\vec{1}')] = S^+(\vec{1})\delta_{\vec{1}\vec{1}'},$$
$$[S^z(\vec{1}),S^-(\vec{1}')] = -S^-(\vec{1})\delta_{\vec{1}\vec{1}'},$$
$$[S^+(\vec{1}),S^-(\vec{1}')] = 2S^z(\vec{1})\delta_{\vec{1}\vec{1}'},$$

where $\vec{1} \equiv \vec{r}_1, \vec{1}' \equiv \vec{r}_1{}'$ is the abridged notation of lattice sites.

The Hamiltonian of the Heisenberg model is

$$\mathcal{H} = -g\mu_B \sum_{\vec{1}} H(\vec{1})S^z(\vec{1}) - g\mu_B \sum_{\vec{1}} h_\mu(\vec{1})S^\mu(\vec{1}) - \frac{1}{2}\sum_{\vec{1},\vec{1}'} J_{\mu\nu}(\vec{1}-\vec{1}')S^\mu(\vec{1})S^\nu(\vec{1}'), \tag{1}$$

where H ($\vec{H} \parallel Oz$) is the external magnetic field, h_μ is the auxiliary infinitesimal magnetic field, and $\mu = -, +, z$. It is supposed that the summation in Eq. (1) and in all following relations is performed over all repeating indices μ,ν. The summation is carried out over the lattice sites $\vec{1},\vec{1}'$ in the volume V of the ferromagnetic sample. g and μ_B are the Landé factor and the Bohr magneton, respectively. $J_{\mu\nu}(\vec{1}-\vec{1}') = J_{\nu\mu}(\vec{1}'-\vec{1})$ is the interaction between spins, which is the sum of the exchange interaction $I_{\mu\nu}$ and the MDI:

$$J_{\mu\nu}(\vec{1}-\vec{1}') = I_{\mu\nu}(\vec{1}-\vec{1}') - 4\pi(g\mu_B)^2\nabla_\mu\Phi(\vec{r}-\vec{r}\,')\nabla'_\nu|_{\vec{r}=\vec{1},\vec{r}\,'=\vec{1}'}, \tag{2}$$

where the function $\Phi(\vec{r}-\vec{r}\,')$ in the MDI term is determined by the equation

$$\Delta\Phi(\vec{r}-\vec{r}\,') = \delta(\vec{r}-\vec{r}\,'),$$
$$\nabla_\mu = \{\nabla_-,\nabla_+,\nabla_z\} = \left\{\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\frac{\partial}{\partial z}\right\}. \tag{3}$$

In three-dimensional space, $\Phi(\vec{r}-\vec{r}\,') = -1/4\pi|\vec{r}-\vec{r}\,'|$ and the MDI term in Hamiltonian (1) can be written as

$$\mathcal{H}^{(\mathrm{dip})} = \frac{(g\mu_B)^2}{2}\sum_{\vec{1},\vec{1}'}\left[\frac{(\vec{S}(\vec{1}),\vec{S}(\vec{1}'))}{|\vec{1}-\vec{1}'|^3} - \frac{3(\vec{S}(\vec{1}),\vec{1}-\vec{1}')(\vec{S}(\vec{1}'),\vec{1}-\vec{1}')}{|\vec{1}-\vec{1}'|^5}\right].$$

For the following calculations of spin-wave dispersion relations in magnetic films, we use a more convenient form of the MDI determined by relations (2) and (3).

Spin excitations, interaction of spin waves, spin-wave relaxation, and other parameters of excitations in the canonical spin ensemble are determined by the generating functional[19,23,33,34]

$$Z[h] = \mathrm{Sp}\exp[-\beta\mathcal{H}(h)] = \sum_{n=0}^{\infty}\sum_{\substack{\vec{1},\ldots,\vec{n}\\ \mu_1,\ldots,\mu_n}}\int_0^\beta\cdots\int_0^\beta Q^{\mu_1\ldots\mu_n}(\vec{1},\ldots,\vec{n},\tau_1,\ldots,\tau_n) \times h_{\mu_1}(\vec{1},\tau_1)\ldots h_{\mu_n}(\vec{n},\tau_n)\,d\tau_1\ldots d\tau_n, \tag{4}$$

where $\beta = 1/kT$, k is the Boltzmann constant, and T is the temperature, $h = \{h_{\mu_i}\}$. Coefficients $Q^{\mu_1\ldots\mu_n}$ are proportional to the temperature Green function without vacuum loops:

$$G^{\mu_1\ldots\mu_n}(\vec{1},\ldots,\vec{n},\tau_1,\ldots,\tau_n) \equiv \langle\langle\mathbf{T}\hat{S}^{\mu_1}(\vec{1},\tau_1)\ldots\hat{S}^{\mu_n}(\vec{n},\tau_n)\rangle\rangle = (\beta g\mu_B)^{-n}Z^{-1}\left.\frac{\delta^n Z[h]}{\delta h_{\mu_1}(\vec{1},\tau_1)\ldots\delta h_{\mu_n}(\vec{n},\tau_n)}\right|_{h\to 0}, \tag{5}$$

where $\hat{S}^\alpha(\vec{j},\tau) = \exp(\tau\mathcal{H})S^\alpha(\vec{j})\exp(-\tau\mathcal{H})$ are the spin operators in the Euclidean Heisenberg representation, $\tau \in [0,\beta]$. $\mathbf{T}$ is the τ-time ordering operator. Variable τ is added in the auxiliary field h_μ in order to take into account $\mathbf{T}$ ordering. $\langle\langle\ldots\rangle\rangle$ denotes averaging of spin operators calculated with $\exp(-\beta\mathcal{H})/\mathrm{Sp}\exp(-\beta\mathcal{H})$. The symbol Sp denotes the trace.

The frequency representation of the expansion (4) is more convenient for calculations. The Fourier transforms of $Q^{\mu_1\cdots\mu_n}$ are defined in terms of the Matsubara frequencies $\omega^{(1)}_{m_1} = 2\pi m_1/\hbar\beta, \ldots, \omega^{(n)}_{m_n} = 2\pi m_n/\hbar\beta$[35] ($m_1, \ldots, m_n$ are integers):

$$Q^{\mu_1\cdots\mu_n}(\vec{1},\ldots,\vec{n},\omega^{(1)}_{m_1},\ldots,\omega^{(n)}_{m_n}) = \int_0^\beta \cdots \int_0^\beta Q^{\mu_1\cdots\mu_n}(\vec{1},\ldots,\vec{n},\tau_1,\ldots,\tau_n) \times \exp\left[-i\hbar\left(\omega^{(1)}_{m_1}\tau_1 + \cdots + \omega^{(n)}_{m_n}\tau_n\right)\right] d\tau_1 \ldots d\tau_n. \quad (6)$$

The coefficients $Q^{\mu_1\cdots\mu_n}$ can be expanded with respect to the interaction $J_{\mu\nu}(\vec{1}-\vec{1}')$ [see Eq. (2)].[19–23,33] Each term of this expansion is represented by a diagram constructed of propagators, vertices, blocks and interaction lines.

1. Propagators. Spin propagators

$$D_\pm(\vec{1},\vec{1}',\omega_m) = \frac{\delta_{\vec{1}\vec{1}'}}{p_0 \pm i\beta\hbar\omega_m}, \quad (7)$$

where $p_0 = \beta g\mu_B H$, are determined for the spin ensemble without any interaction between spins. The propagators $D_\pm(\vec{1},\vec{1}',\omega_m)$ are represented by directed lines in diagrams [see Fig. 1(a)]. The directions of arrows show the direction of growth of the frequency variable ω_m.

2. Vertices. There are five types of vertices [see Fig. 1(b)]. Vertices a and b are the start and end points of propagators, respectively. In analytical expressions of diagrams the vertex a corresponds with the factor 2 and the vertex b with the factor 1. The vertex c ties three propagators and corresponds with the factor (-1) in analytical expressions. The vertex d with

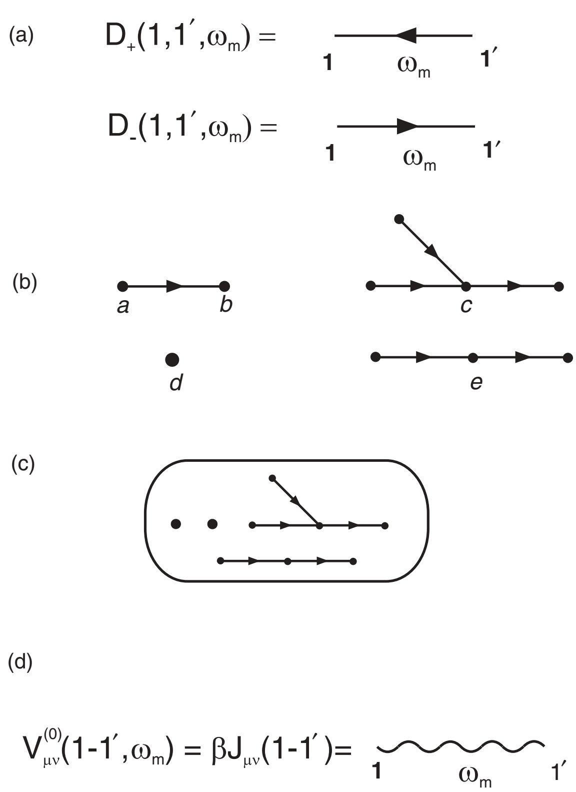

FIG. 1. (a) Propagators $D_\pm$, (b) vertices, (c) block with isolated parts and (d) interaction lines $V^{(0)}_{\mu\nu}$.

the factor 1 is defined as a single vertex. The vertex e ties two propagators. The factor of the e vertex is equal to (-1).

3. Blocks. Blocks contain propagators and isolated vertices d [see Fig. 1(c)]. Propagators can be connected through vertices c and e. In analytical expressions of the diagram expansion, each block corresponds with the block factor $B^{[\kappa-1]}(p_0)$, where κ is the number of isolated parts in the block. The factor $B^{[\kappa-1]}(p_0)$ is expressed by partial derivatives of the Brillouin function B_S for the spin S with respect to p_0:

$$B(p_0) = \langle\langle S^z\rangle\rangle_0 = SB_S(Sp_0), \quad B^{[n]}(p_0) = S\frac{\partial^n B_S(Sp_0)}{\partial p_0^n}, \quad (8)$$

where $\langle\langle\ldots\rangle\rangle_0$ denotes the statistical averaging performed over the states described by the Hamiltonian $\mathcal{H}$ (1) without the interaction $J_{\mu\nu}$ between spins. $B_S(x) = (1 + 1/2S)\coth[(1 + 1/2S)x] - (1/2S)\coth(x/2S)$.

4. Interaction lines. The interaction line $V^{(0)}_{\mu\nu}(\vec{1}-\vec{1}',\omega_m) = \beta J_{\mu\nu}(\vec{1}-\vec{1}')$ connects two vertices in a diagram [see Fig. 1(d)]. The correspondence between the first index μ of the interaction line $V^{(0)}_{\mu\nu}$ and the vertex type is the following. (1) If $\mu = -$, then the left end point of $V^{(0)}_{-\nu}$ is bound to the vertex a; (2) if $\mu = +$, then this end point is bound to the vertices b or c; and (3) if $\mu = z$, then the end is bound to the vertices d or e. The analogous correspondence is satisfied for the right end ν of $V^{(0)}_{\mu\nu}$.

Coefficients $Q^{\mu_1\cdots\mu_n}$ in the expansion (4) in the frequency representation (6) are the sum of N topologically nontrivial diagrams $\sum_t^N Q_t^{\mu_1\cdots\mu_n}$ $(t = 1,\ldots,N)$. The general form of the analytical expression of the diagram in the frequency representation is written as[19–23]

$$Q_t^{\mu_1\cdots\mu_n}(\vec{1},\ldots,\vec{n},\omega^{(1)}_{m_1},\ldots,\omega^{(n)}_{m_n}) = (-1)^L 2^{m_a}\frac{P_k}{2^k k!}\prod_l B^{[\kappa_l-1]}(p_0)\prod_{\vec{i},\vec{j}\in l}^{\kappa_l}\delta_{\vec{i}\vec{j}} \times \sum_{\substack{\vec{1}',\ldots\vec{k}'\\ \vec{1}''\ldots\vec{k}''}}\sum_{m_i} V^{(0)}_{\alpha\gamma}(\vec{1}'-\vec{1}'',\omega_{m_1}) \times\cdots\times V^{(0)}_{\rho\sigma}(\vec{k}'-\vec{k}'',\omega_{m_k}) \times \prod_{\vec{s},\vec{s}'}^{I_D} D_-(\vec{s},\vec{s}',\omega_{m_s})\prod_v^{I_v}\delta\left(\sum_{r\in v}\beta\hbar\omega_{m_r}\right), \quad (9)$$

where $\vec{1},\ldots,\vec{n},\omega^{(1)}_{m_1},\ldots,\omega^{(n)}_{m_n}$ are the external lattice and frequency variables corresponded to the auxiliary fields h_{μ_i} in the expansion (4). m_a is the number of a vertices in a diagram. L is the number of c and e vertices. P_k is the number of topological equivalent diagrams. $2k$ is the number of vertices connected with k interaction lines $V^{(0)}_{\alpha\gamma}\ldots V^{(0)}_{\rho\sigma}$. The product $\prod_l$ is performed over all blocks of a diagram. κ_l is the number of isolated parts in block l. The term $\prod_{\vec{i},\vec{j}\in l}^{\kappa_l}\delta_{\vec{i}\vec{j}}$ denotes that all isolated parts in block l are determined on a single lattice site. I_D is the number of propagators in a diagram. I_v is the number of vertices in a diagram. $\sum_{m_i}$ denotes the summation performed over all inner frequency variables. The term $\prod_v^{I_v}\delta(\sum_{r\in v}\beta\hbar\omega_{m_r})$ gives the frequency conservation in each vertex v, i.e., the sum of frequencies of propagators and interaction lines, which come in and go out from the vertex v, is equal to 0. The vertex d can be connected with the single interaction line. In the analytical expression, this corresponds to the factor $\delta(\beta\hbar\omega_m)$. The lattice variables $\vec{s}$ and $\vec{s}\,'$ of propagators D_- can be inner or external. In the first case, end points of propagators are connected with the end points $\{\vec{1}',\vec{1}'',\ldots,\vec{k}',\vec{k}''\}$ of interaction lines $V^{(0)}_{\alpha\gamma}\ldots V^{(0)}_{\rho\sigma}$ and the summation $\sum_{\substack{\vec{1}',\ldots\vec{k}'\\ \vec{1}'',\ldots\vec{k}''}}\sum_{m_i}$ is performed. In the second case, end points of propagators are not connected with interaction lines.

The first approximation of the diagram expansion (4) is the self-consistent field approximation in which the effective field acting on spins is derived and the self-consistent field $H^{(c)}_\mu$ induced by the neighboring spins is taken into account.[19,22,23] This leads to the substitution $p_0 \to p = \beta g\mu_B|\vec{H} + \vec{H}^{(c)}|$ in the propagator D_- (7) and in the block factor $B^{[\kappa_l-1]}$ (8) in the analytical expression (9). The self-consistent field is the sum of exchange and magnetic dipole self-consistent fields, $H^{(c)}_\mu = H^{(\mathrm{exch})}_\mu + H^{(m)}_\mu$, where

$$H^{(\mathrm{exch})}_\mu(\vec{1}) = (g\mu_B)^{-1}\sum_{\vec{1}'} I_{\mu\nu}(\vec{1}-\vec{1}')\langle\langle S^\nu(\vec{1}')\rangle\rangle$$

$$H^{(m)}_\mu(\vec{1}) = -4\pi g\mu_B\nabla_\mu\sum_{\vec{1}'}\Phi(\vec{r}-\vec{r}\,')\nabla'_\nu\langle\langle S^\nu(\vec{r}\,')\rangle\rangle\Big|_{\substack{\vec{r}=\vec{1}\\ \vec{r}\,'=\vec{1}'}}. \tag{10}$$

The second approximation of the expansion (4) is the approximation of the effective Green functions and interactions. In this approximation, the poles of the matrix of the effective Green functions and interactions are determined and the dispersion curves are obtained. The next terms in the diagram expansion determine the imaginary and real corrections to the poles of the matrix of the effective Green functions and interactions. The imaginary parts of the poles give the relaxation parameters of spin excitations and the real parts determine the corrections to the dispersion curves. In the next section, we consider the approximation of the effective Green functions and interactions.

B. Effective Green functions and interaction lines

In the framework of this approximation, the matrix of the effective Green functions and effective interactions $\mathcal{P} = \|P_{AB}(\vec{1},\vec{1}',\omega_m)\|$ is introduced.[22,23] We compose the $\mathcal{P}$ matrix from analytical expressions of connected diagrams with two external sites. These sites are end points of propagators, single vertices d, or end points of interaction lines. Accordingly, multi-indices $A = (a\mu)$, $B = (b\nu)$ are the double indices, where $\mu,\nu = \{-,+,z\}$ and indices a, b point out that A, B belong to a propagator or to a d vertex ($a,b = 1$), or belong to an interaction line ($a,b = 2$). The zero-order approximation $\mathcal{P}^{(0)}$ of the $\mathcal{P}$ matrix is determined by the matrix of the bare interaction $\mathcal{V}^{(0)} = \|V^{(0)}_{\mu\nu}(\vec{1}-\vec{1}',\omega_m)\|$ and by the two-site Green functions (5) in the self-consistent-field approximation $\mathcal{G}^{(0)} = \|G^{(0)}_{\mu\nu}\|$, given on a lattice site:

$$\mathcal{P}^{(0)} = \begin{pmatrix} \|P^{(0)}_{(1\mu)(1\nu)}\| & \vdots & \|P^{(0)}_{(1\mu)(2\nu)}\| \\ \cdots & \cdots & \cdots \\ \|P^{(0)}_{(2\mu)(1\nu)}\| & \vdots & \|P^{(0)}_{(2\mu)(2\nu)}\| \end{pmatrix} = \begin{pmatrix} \|G^{(0)}_{\mu\nu}\| & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & \|V^{(0)}_{\mu\nu}\| \end{pmatrix},$$

where

$$\|G^{(0)}_{\mu\nu}\| = \begin{pmatrix} 0 & G^{(0)}_{-+} & 0 \\ G^{(0)}_{+-} & 0 & 0 \\ 0 & 0 & G^{(0)}_{zz} \end{pmatrix} = 2B(p)\begin{pmatrix} 0 & D_-(\vec{1},\vec{1}',\omega_m) & 0 \\ D_+(\vec{1},\vec{1}',\omega_m) & 0 & 0 \\ 0 & 0 & \frac{B^{[1]}(p)}{2B(p)}\delta_{\vec{1}\vec{1}'}\delta_{m0} \end{pmatrix} \tag{11}$$

with the propagator (7) in which the substitution $p_0 \to p = \beta g\mu_B|\vec{H} + \vec{H}^{(c)}|$ is performed.

The $\mathcal{P}$ matrix is obtained by means of the summation of the $\mathcal{P}^{(0)}$ matrix: the summation of all diagram chains consisted of the bare Green functions $G^{(0)}_{\mu\nu}$ and the bare interaction lines $V^{(0)}_{\mu\nu}$ (Fig. 2). These chains of propagators and interaction lines

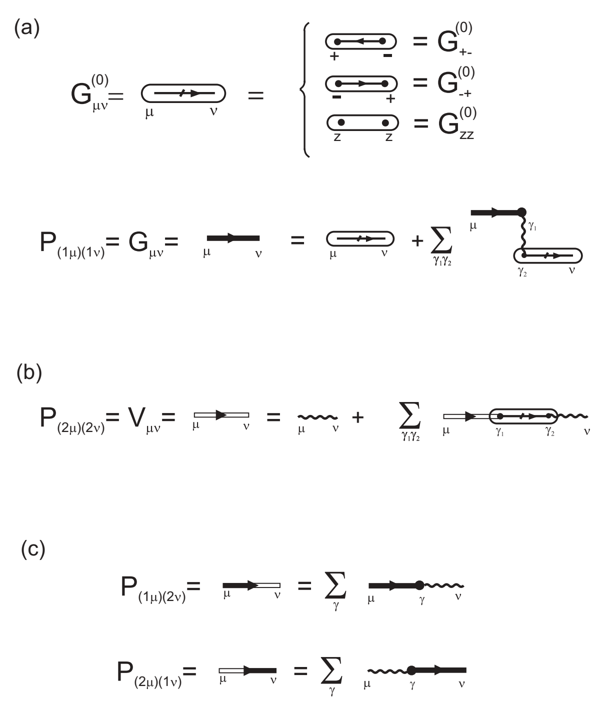

FIG. 2. (a) Definition of the effective Green functions $P_{(1\mu)(1\nu)} = G_{\mu\nu}$ via the bare two-site Green functions $G^{(0)}_{\mu\nu}$. (b) Definition of effective interaction lines $P_{(2\mu)(2\nu)} = V_{\mu\nu}$. (c) Definition of intersecting terms $P_{(1\mu)(2\nu)}$, $P_{(2\mu)(1\nu)}$.

do not have any loop insertion. Analytical expressions of the considered diagrams can be written in accordance with relation (9). The summation gives equation of the Dyson type, which forms the relationship between $\mathcal{P}^{(0)}$- and $\mathcal{P}$-matrices,

$$\mathcal{P} = \mathcal{P}^{(0)} + \mathcal{P}\sigma\mathcal{P}^{(0)}, \tag{12}$$

where

$$\sigma = \begin{pmatrix} 0 & \vdots & \mathcal{E} \\ \cdots & \cdots & \cdots \\ \mathcal{E} & \vdots & 0 \end{pmatrix},$$

$\mathcal{E} = \|\delta_{\mu\nu}\|$ is the diagonal matrix.

The $\mathcal{P}$-matrix consists of the two-site effective Green functions $\mathcal{G} = \|G_{\mu\nu}\| = \mathcal{G}^{(0)}(\mathcal{E} - \mathcal{V}^{(0)}\mathcal{G}^{(0)})^{-1}$, where $G_{\mu\nu} = P_{(1\mu)(1\nu)}$, effective interactions $\mathcal{V} = \|V_{\mu\nu}\| = \mathcal{V}^{(0)}(\mathcal{E} - \mathcal{G}^{(0)}\mathcal{V}^{(0)})^{-1}$, where $V_{\mu\nu} = P_{(2\mu)(2\nu)}$, and intersecting terms $P_{(1\mu)(2\nu)}$, $P_{(2\mu)(1\nu)}$ (see Fig. 2). The effective Green functions, effective interactions, and intersecting terms are denoted in diagrams by directed thick lines, empty lines, and compositions of the thick and empty lines, respectively. The $\mathcal{P}$ matrix determines the spectrum of quasiparticle excitations in the spin ensemble. Spectrum relations for spin excitations are given by the $\mathcal{P}$ matrix poles by zero eigenvalues of the operator $1 - \sigma\mathcal{P}^{(0)}$ or, equivalently, by $\mathcal{E} - \mathcal{V}^{(0)}\mathcal{G}^{(0)}$ under the analytical continuation

$$\begin{aligned} i\omega_m &\to \omega + i\varepsilon \,\mathrm{sign}\,\omega, \\ \delta(\beta\hbar\omega_m) &= \delta_{m0} \to \frac{1}{\beta\hbar(\omega + i\varepsilon\,\mathrm{sign}\,\omega)} \quad (\varepsilon \to +0). \end{aligned} \tag{13}$$

Since zero eigenvalues of the operator $\mathcal{E} - \mathcal{V}^{(0)}\mathcal{G}^{(0)}$ may correspond to different eigenfunctions and can determine different excitation modes, we introduce the spectral parameter λ for the eigenfunctions $h_\mu^{(\lambda)}(\vec{1},\omega_m)$ of the operator $\mathcal{E} - \mathcal{V}^{(0)}\mathcal{G}^{(0)}$. The spectral parameter λ can be discrete or continuous. Taking into account the above mentioned, we get the equation describing spin-wave excitations:

$$h_\mu^{(\lambda)}(\vec{1},\omega_m) - \sum_{\vec{1}',\vec{1}'',\nu,\rho} V_{\mu\nu}^{(0)}(\vec{1} - \vec{1}',\omega_m) \times G_{\nu\rho}^{(0)}(\vec{1}',\vec{1}'',\omega_m)h_\rho^{(\lambda)}(\vec{1}'',\omega_m)\bigg|_{i\omega_m \to \omega + i\varepsilon\mathrm{sign}\omega} = 0. \tag{14}$$

III. SPIN WAVES IN MAGNETIC FILMS

A. Spin-wave equations for magnetic films

Let us consider spin waves with the wave vector $\vec{q}$ in normal and in-plane magnetized films consisted of N monolayers at low temperature. Monolayers can be regarded as layers consisting of ions with strong exchange interaction or layers consisting of magnetic nanoparticles. In the second case, the exchange interaction between nanoparticles can reach low values in comparison with the MDI. The external magnetic field H is parallel to the z axis. At low temperature, derivatives of the Brillouin function in $B^{[n]}(p)$ in relation (8) tend to zero exponentially with decreasing temperature. Thus it follows that diagrams containing blocks with isolated parts can be dropped, the Green function $G_{zz}^{(0)}$ in relation (11) is negligible and only the Green functions $G_{-+}^{(0)}$, $G_{+-}^{(0)}$ are taken into account in Eq. (14). Indices μ, ν of interactions $V_{\mu\nu}^{(0)}$ in Eq. (14) are $\{-,+\}$. We suppose that on monolayers, spins are placed on quadratic lattice sites with the lattice constant a and the spin orientation is parallel to the z axis. The exchange interaction acts between neighboring spins and is isotropic between spins in monolayers, $2I_{-+}^{(\mathrm{mon})} = 2I_{+-}^{(\mathrm{mon})} = I_{zz}^{(\mathrm{mon})} = I_0$, and between neighboring layers, $2I_{-+}^{(\mathrm{lay})} = 2I_{+-}^{(\mathrm{lay})} = I_{zz}^{(\mathrm{lay})} = I_d$.

As we consider spin waves in two-dimensional layers and films, it is necessary to discuss restrictions imposed by the Mermin-Wagner theorem.[36] The Mermin-Wagner theorem states that continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions in dimensions $\leqslant 2$. In accordance with the theorem, the isotropic spin Heisenberg model can be neither ferromagnetic nor antiferromagnetic. The theorem extends to N-layer films: for any finite temperature and for any finite number of layers, a phase transition is ruled out.[37,38] In the case of the Heisenberg model with the Hamiltonian $\mathcal{H}$ (1), the Mermin-Wagner theorem is not applied: the $O(3)$ rotational symmetry of the Hamiltonian $\mathcal{H}$ is broken by the MDI and by the external magnetic field H. Therefore the two-dimensional layers and films considered below have nonzero finite magnetization. We suppose that the magnitude of the magnetic field is sufficient to achieve magnetic saturation and to eliminate a domain structure.

1. *Normal magnetized films*

In normal magnetized films, x and y axes are in the monolayer plane and the z axis is normal to monolayers. The magnetic field H is normal to monolayers. The Fourier transform of the exchange interaction with respect to the longitudinal lattice variables $\vec{1}_{xy}$ is

$$\begin{aligned} \bar{I}(\vec{q},1_z - 1'_z) &= \sum_{\vec{1}_{xy}-\vec{1}'_{xy}} I(\vec{1}_{xy} - \vec{1}'_{xy},1_z - 1'_z) \\ &\quad \times \exp[-i\vec{q}(\vec{1}_{xy} - \vec{1}'_{xy})] \\ &= \bar{I}(0,1_z - 1'_z) + 2I_0[\cos(q_x a) \\ &\quad + \cos(q_y a)]\delta_{1_z 1'_z}, \end{aligned} \tag{15}$$

where $\vec{1}_{xy}$ and $\vec{1}'_{xy}$ are lattice sites in monolayers, 1_z and $1'_z$ are z positions of layers, $\vec{q} = (q_x,q_y)$ is the longitudinal wave vector in monolayers, and $\bar{I}(0,1_z - 1'_z)$ is the exchange interaction at $\vec{q} = 0$, which is equal to I_d between spins of neighboring layers. The corresponding exchange part of the interaction line $V_{\mu\nu}^{(0)} = V_{\mu\nu}^{(\mathrm{exch})} + V_{\mu\nu}^{(\mathrm{dip})}$ [see Fig. 1(d)] is

$$V_{\mu\nu}^{(\mathrm{exch})}(\vec{q},1_z - 1'_z) = \beta\bar{I}(\vec{q},1_z - 1'_z)/2, \tag{16}$$

where $\mu\nu = (-+),(+-)$. For indices $\mu\nu = (--)$ and $(++)$, $V_{\mu\nu}^{(\mathrm{exch})} = 0$. The MDI part $V_{\mu\nu}^{(\mathrm{dip})}$ is determined by the Fourier transform of Eq. (3):

$$\left(-q^2 + \frac{\partial^2}{\partial z^2}\right)\Phi(\vec{q},z-z') = S_a^{-1}\delta(z-z')$$

with the solution

$$\Phi(\vec{q},1_z - 1'_z) = \Phi(\vec{q},z-z')|_{z=1_z,z'=1'_z} = \frac{-1}{2qS_a}\exp(-q|1_z - 1'_z|), \qquad (17)$$

where $S_a = a^2$, $q = |\vec{q}|$. According to the solution (17), the corresponding MDI part of the interaction line is

$$V^{(\mathrm{dip})}_{\mu\nu}(\vec{q},1_z - 1'_z) = \frac{-2\pi\beta(g\mu_B)^2 q_\mu q_\nu}{qS_a}\exp(-q|1_z - 1'_z|), \qquad (18)$$

where

$$\mu,\nu = \{-,+\}\; q_- = \tfrac{1}{2}(q_x + iq_y), \quad q_+ = \tfrac{1}{2}(q_x - iq_y).$$

Taking into account relations (16) and (18), from Eq. (14), we obtain equations for spin-wave modes with the wave vector $\vec{q}$ in N-layer magnetic films:

$$h^{(\lambda)}_\mu(\vec{q},1_z,\omega_m) - \sum_{\vec{1}'_z}\big[V^{(0)}_{\mu-}(\vec{q},1_z - 1'_z,\omega_m) \times G^{(0)}_{-+}(1'_z,1'_z,\omega_m)h^{(\lambda)}_+(\vec{q},1'_z,\omega_m) + V^{(0)}_{\mu+}(\vec{q},1_z - 1'_z,\omega_m) \times G^{(0)}_{+-}(1'_z,1'_z,\omega_m)h^{(\lambda)}_-(\vec{q},1'_z,\omega_m)\big]\big|_{i\omega_m\to\omega+i\varepsilon\,\mathrm{sign}\omega} = 0, \qquad (19)$$

where

$$G^{(0)}_{\substack{-+\\(+-)}}(1_z,1'_z,\omega_m) = \frac{2B(p)\delta_{1_z1'_z}}{p \pm i\beta\hbar\omega_m},$$

$\lambda = 1,\ldots,N$ is the mode number, $V^{(0)}_{\mu\nu}(\vec{q},1_z - 1'_z,\omega_m) = V^{(\mathrm{exch})}_{\mu\nu}(\vec{q},1_z - 1'_z) + V^{(\mathrm{dip})}_{\mu\nu}(\vec{q},1_z - 1'_z)$, $\mu,\nu = \{-,+\}$. Eigenvalues of equations (19) give dispersion relations of spin waves in normal magnetized films.

2. In-plane magnetized films

In in-plane magnetized films, x and z axes are in the monolayer plane and the y axis is normal to monolayers. The Fourier transform of the exchange interaction with respect to the longitudinal lattice variables $\vec{1}_{xz}$ is given by relation (15), where substitutions $\vec{1}_{xy} \to \vec{1}_{xz}$, $\vec{1}_z \to \vec{1}_y$, and $q_y \to q_z$ should be done. The MDI part of the interaction is

$$V^{(\mathrm{dip})}_{\substack{--\\(++)}}(\vec{q},1_y - 1'_y) = \pi\beta(g\mu_B)^2\left(q_x^2 \pm 2q_x\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}\right)\Phi(\vec{q},y-y')\Big|_{y=1_y,y'=1'_y}$$
$$V^{(\mathrm{dip})}_{+-}(\vec{q},1_y - 1'_y) = V^{(\mathrm{dip})}_{-+}(\vec{q},1_y - 1'_y) = \pi\beta(g\mu_B)^2\left(q_x^2 - \frac{\partial^2}{\partial y^2}\right)\Phi(\vec{q},y-y')\Big|_{y=1_y,y'=1'_y}, \qquad (20)$$

where

$$\Phi(\vec{q},y-y') = \frac{-1}{2qS_a}\exp(-q|y-y'|),$$

$q = (q_x^2 + q_z^2)^{1/2}$ is the longitudinal wave vector. Taking into account relation (15) with the above mentioned substitutions and relation (20), from Eq. (14), we obtain equations for spin-wave modes in in-plane magnetized films analogous to Eq. (19), where the substitution $\vec{1}_z \to \vec{1}_y$ should be done. In next sections, we find spin-wave dispersion relations for the cases of monolayer and two-layer films and spin-wave resonance relations for the case of N-layer structures.

B. Spin waves in magnetic monolayer

1. Normal magnetized monolayer films

Dispersion relations of spin waves in normal magnetized monolayer lattice are determined by the determinant of Eq. (19) for variables $h^{(1)}_-$ and $h^{(1)}_+$. Taking into account relations (16) and (18), we find

$$\omega^2(\vec{q}) = \Omega(\vec{q})[\Omega(\vec{q}) + 2\pi\gamma\sigma_m q], \qquad (21)$$

where

$$\Omega(\vec{q}) = \gamma(H + H^{(m)}) + \frac{2B(p)I_0}{\hbar}[2 - \cos(q_x a) - \cos(q_y a)],$$

$\gamma = g\mu_B/\hbar$ is the gyromagnetic ratio, $H^{(m)} = |\vec{H}^{(m)}|$ is the depolarizing magnetic field (10), $\sigma_m = g\mu_B B(p)/S_a$ is the surface magnetic moment density, and $q = (q_x^2 + q_y^2)^{1/2}$. As one can see from relation (21), in the monolayer lattice, spin waves have the one-mode character.

In the next sections, we compare dispersion relations (21) with dispersion relations in thick magnetic films. Therefore we calculate the dispersion curve for a monolayer film with parameters analogous to YIG films. YIG films have the magnetization $4\pi M = 4\pi g\mu_B B(p)/a^3 = 1750$ Oe and the exchange interaction constant $\alpha = B(p)I_0 a^2/\hbar\gamma 4\pi M = 3.2 \times 10^{-12}$ cm^2 at room temperature.[18] Magnetic parameters of monolayer with $\langle\langle S^z\rangle\rangle_0 = B(p) = 1/2$ are analogous to YIG, if the lattice constant $a = 0.4$ nm and the exchange interaction between neighboring spins $I_0 = 0.085$ eV. Figure 3 presents the dispersion curve (21) of spin waves propagating in the monolayer film. The spin-wave wave vector $\vec{q}$ is parallel to the x axis ($q_x = q$, $q_y = 0$) and is in the range $[0,\pi/a]$. Calculations have been done at the sum of

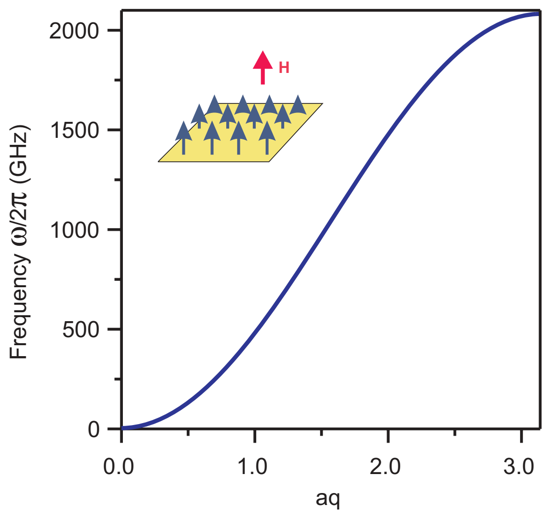

FIG. 3. (Color online) Dispersion curve of spin waves propagating in the normal magnetized monolayer film with quadratic lattice ($a = 0.4$ nm) at the sum of magnetic fields $H + H^{(m)} = 3$ kOe. Exchange interaction I_0 is 0.085 eV.

magnetic fields $H + H^{(m)} = 3$ kOe. For the given monolayer film, the exchange interaction makes a major contribution to the dispersion. The relatively weak MDI is significant for the dispersion at small values of the wave vector $q < q_0$, where

$$q_0 = \frac{\hbar\pi\gamma\sigma_m}{B(p)I_0a^2} = \frac{a}{4\alpha}.$$

At $q \to 0$, the group velocity of spin waves is positive $v = \pi\gamma\sigma_m$. These spin waves are analogous to forward volume magnetostatic spin waves propagating in magnetic films.[1–3,18]

2. *In-plane magnetized monolayer films*

Dispersion relations of spin waves in in-plane magnetized monolayers are determined by the determinant of Eq. (19) for the variables $h_-^{(1)}$ and $h_+^{(1)}$ with the substitution $\vec{1}_z \to \vec{1}_y$. Taking into account relations (20), we obtain

$$\omega^2(\vec{q}) = [\Omega(\vec{q}) + \Omega_M - 2\pi\gamma\sigma_m q]\cdot\left[\Omega(\vec{q}) + 2\pi\gamma\sigma_m\frac{q_x^2}{q}\right], \tag{22}$$

where

$$\Omega(\vec{q}) = \gamma H + \frac{2B(p)I_0}{\hbar}[2 - \cos(q_x a) - \cos(q_z a)],$$

$q = (q_x^2 + q_z^2)^{1/2}$, and $\Omega_M = 4\pi\gamma\sigma_m/a$. Spin waves propagating along the x axis ($\vec{q} \perp \vec{H}, q = q_x, q_z = 0$) at $q \to 0$ have the positive group velocity

$$v = \frac{\pi\gamma\sigma_m\Omega_M}{[\Omega(0)(\Omega(0) + \Omega_M)]^{1/2}}$$

and, in this sense, are analogous to surface magnetostatic spin waves propagating in magnetic films.[1–3,18] In contrast with this, spin waves propagating along the z axis ($\vec{q} \parallel \vec{H}, q = q_z, q_x = 0$) at $q \to 0$ have the negative group velocity

$$v = -\frac{\pi\gamma\sigma_m\Omega(0)^{1/2}}{(\Omega(0) + \Omega_M)^{1/2}}$$

and have features of backward volume magnetostatic spin waves. These backward spin waves propagate in the sector $[-\theta,\theta]$, where $\sin\theta = \Omega(0)/[\Omega(0) + \Omega_M]$.

C. Spin waves in magnetic bilayer

Let us consider spin waves in magnetized structures consisted of two monolayers of the quadratic lattice with the lattice constant a. The distance between layers is equal to d and the exchange interaction between spins of layers is I_d.

1. *Normal magnetized films*

Dispersion relations for two spin-wave modes in normal magnetized bilayer are determined by eigenvalues of Eq. (19) for variables $h_-^{(1)}$, $h_+^{(1)}$, $h_-^{(2)}$, and $h_+^{(2)}$ and can be written as

$$\omega^{(1)2}(\vec{q}) = \Omega(\vec{q})\{\Omega(\vec{q}) + 2\pi\gamma\sigma_m q[1 + \exp(-qd)]\},$$
$$\omega^{(2)2}(\vec{q}) = \left[\Omega(\vec{q}) + \frac{2B(p)I_d}{\hbar}\right]\left\{\Omega(\vec{q}) + \frac{2B(p)I_d}{\hbar} + 2\pi\gamma\sigma_m q[1 - \exp(-qd)]\right\}, \tag{23}$$

where $q = (q_x^2 + q_y^2)^{1/2}$ and $\Omega(\vec{q})$ is defined in relation (21). For the first mode, spins in different layers change their orientations in phase. In this case, spin waves of the first mode correspond to spin waves in monolayer (21). At $q \to 0$, the group velocity of spin waves $v = 2\pi\gamma\sigma_m$ is two times higher than the group velocity in monolayer. For the second mode, spins in different layers change orientations in antiphase and the energy of the spin wave with the given longitudinal wave vector q is higher than the energy of the spin wave of the first mode. For $q \to 0$, the spin-wave group velocity v tends to zero. Dispersion curves of spin waves determined by relations (23) are shown in Fig. 4. Spin waves propagate along the x axis. Calculations have been done for the exchange interactions $I_0 = I_d = 0.085$ eV and for the distance between layers $d = a = 0.4$ nm at the sum of magnetic fields $H + H^{(m)} = 3$ kOe.

2. *In-plane magnetized films*

Dispersion relations of spin waves in in-plane magnetized bilayers are determined by eigenvalues of Eq. (19) for variables $h_-^{(1)}$, $h_+^{(1)}$, $h_-^{(2)}$, and $h_+^{(2)}$ with the substitution $\vec{1}_z \to \vec{1}_y$. Taking into account relations (20), for spin waves propagating along

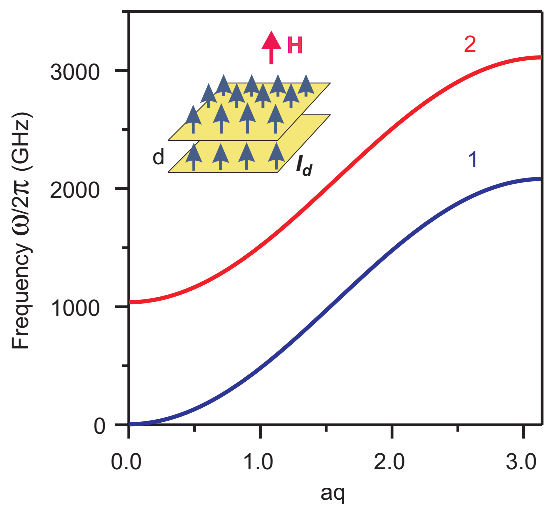

FIG. 4. (Color online) Dispersion curve of spin waves propagating in the normal magnetized bilayer with quadratic lattice ($a = 0.4$ nm) at the sum of magnetic fields $H + H^{(m)} = 3$ kOe. Exchange interactions, $I_0 = I_d = 0.085$ eV. The distance between monolayers d is equal to the lattice constant a. 1 and 2 are the first and the second modes of spin waves, respectively.

the x axis ($\vec{q} \perp \vec{H}, q = q_x, q_z = 0$), we obtain

$$\omega^{(n)2}(q) = \left[\Omega(q) + \frac{B(p)I_d}{\hbar}\right]\left[\Omega(q) + \Omega_M + \frac{B(p)I_d}{\hbar}\right] + \left[\frac{B(p)I_d}{\hbar}\right]^2 + Q\{\Omega_M - Q[1 + 2\exp(-2qd)]\} \pm \left(\left\{\left[2\Omega(q) + \frac{2B(p)I_d}{\hbar} + \Omega_M\right]\frac{B(p)I_d}{\hbar} + Q\exp(-qd)(2Q - \Omega_M)\right\}^2 + 4Q^2\exp(-2qd) \times \left\{Q^2\exp(-2qd) - \left[\frac{B(p)I_d}{\hbar}\right]^2\right\}\right)^{1/2}, \tag{24}$$

where $Q = 2\pi\gamma\sigma_m q$, $n = 1,2$ is the mode number, $\Omega(q)$ and Ω_M are defined in relation (22). At $q \to 0$, the group velocity of the first mode

$$v = \frac{2\pi\gamma\sigma_m\Omega_M}{\{\Omega(0)[\Omega(0) + \Omega_M]\}^{1/2}}$$

is two times higher than the group velocity of spin waves in monolayer. For the second mode, v tends to zero.

For spin waves propagating along the z axis ($\vec{q} \parallel \vec{H}, q = q_z, q_x = 0$) dispersion relations of two modes are

$$\omega^{(1)2}(q) = \Omega(q)\{\Omega(q) + \Omega_M - 2\pi\gamma\sigma_m q[1 + \exp(-qd)]\},$$
$$\omega^{(2)2}(q) = \left[\Omega(q) + \frac{2B(p)I_d}{\hbar}\right]\left\{\Omega(q) + \Omega_M + \frac{2B(p)I_d}{\hbar} - 2\pi\gamma\sigma_m q[1 - \exp(-qd)]\right\}. \tag{25}$$

For small wave vectors q, the group velocity of the first mode is negative and at $q \to 0$ is equal to

$$v = -\frac{2\pi\gamma\sigma_m\Omega(0)^{1/2}}{(\Omega(0) + \Omega_M)^{1/2}}.$$

The group velocity of the second mode tends to zero with the wave vector decrease.

D. Spin-wave resonance in N-layer structure

In this section, we consider a spin-wave resonance in a normal magnetized structure consisted of N uniform monolayer lattices with the exchange interaction I_d between spins of layers. The distance between layers is equal to d. The spin-wave resonance is the limit case of a spin wave when the longitudinal wave vector $q \to 0$. Therefore the MDI terms $V^{(\text{dip})}_{\mu\nu}(\vec{q}, 1_z - 1'_z)$ in Eq. (19) can be dropped and the equations with variables $h_+^{(\lambda)}$ and $h_-^{(\lambda)}$ are separated and eigenvalues are determined by the zero values of the determinant (we write the determinant $\mathcal{D}^{(+)}$ for equations with the $h_+^{(\lambda)}$):

$$\mathcal{D}^{(+)} = G^{(0)}(1)\ldots G^{(0)}(N)\det\begin{pmatrix} [G^{(0)-1}(1) - V^{(0)}(11)] & -V^{(0)}(12) & 0 & \vdots \\ -V^{(0)}(21) & [G^{(0)-1}(2) - V^{(0)}(22)] & -V^{(0)}(23) & \vdots \\ 0 & -V^{(0)}(32) & [G^{(0)-1}(3) - V^{(0)}(33)] & \vdots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

where $V^{(0)}(kj)$ and $G^{(0)}(k)$ are the abridged notation of $V^{(\text{exch})}_{+-}(\vec{q}, k_z - j_z, \omega_m)|_{\vec{q}=0}$ and $G^{(0)}_{-+}(k,k,\omega_m)$ at $i\omega_m \to \omega + i\varepsilon\,\text{sign}\,\omega$, respectively. (k, j) are indices of layers. Taking into account that spins of outer layers ($k = 1, N$) interact with spins of one inner layer and spins of inner layers interact with spins of two layers and introducing the variable for inner layers in the determinant $\mathcal{D}^{(+)}$,

$$x = \frac{G^{(0)-1}(k) - V^{(0)}(kk)}{-V^{(0)}(jk)} = \frac{\hbar}{B(p)I_d}[\omega - \gamma(H + H^{(m)})] - 2 \quad (k \neq 1, N,\ j = k \pm 1),$$

we obtain that the spin-wave resonance spectrum is determined by roots of the polynomial

$$R_N(x) = \begin{vmatrix} (x+1) & 1 & 0 & 0 & \vdots & 0 & 0 \\ 1 & x & 1 & 0 & \vdots & 0 & 0 \\ 0 & 1 & x & 1 & \vdots & 0 & 0 \\ 0 & 0 & 1 & x & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & x & 1 \\ 0 & 0 & 0 & 0 & \vdots & 1 & (x+1) \end{vmatrix} = (x+1)^2 P_{N-2}(x) - 2(x+1)P_{N-3}(x) + P_{N-4}(x) = 0 \quad (N \geqslant 2),$$

where $P_{-2}(x) = -1$, $P_{-1}(x) = 0$, $P_0(x) = 1$, $P_N(x) = x P_{N-1}(x) - P_{N-2}(x)$. Polynomial $R_N(x)$ has N roots:

$$x^{(n)} = -2\cos\left(\frac{\pi n}{N}\right),$$

where $n = 0,1,\ldots,N-1$. Taking into account the form of the roots $x^{(n)}$, we can introduce the transverse wave vector $q_z^{(n)} = \pi n/Nd$. Then the spin-wave resonance spectrum can be written as

$$\omega^{(n)} = \gamma(H + H^{(m)}) + \frac{2B(p)I_d}{\hbar}\left[1 - \cos\left(q_z^{(n)}d\right)\right]. \tag{26}$$

For the first mode ($n = 0$), spins in different layers change their orientations in phase. For the highest mode ($n = N - 1$), spins in different layers change orientations in antiphase and the energy of spin-wave resonance is highest. Figure 5 presents the spin-wave resonance spectrum (26) for the structure with $N = 40$ layers. One can see that at low values of the transverse wave vector, the resonance spectrum is proportional to the quadratic dependence on $q_z^{(n)}$.

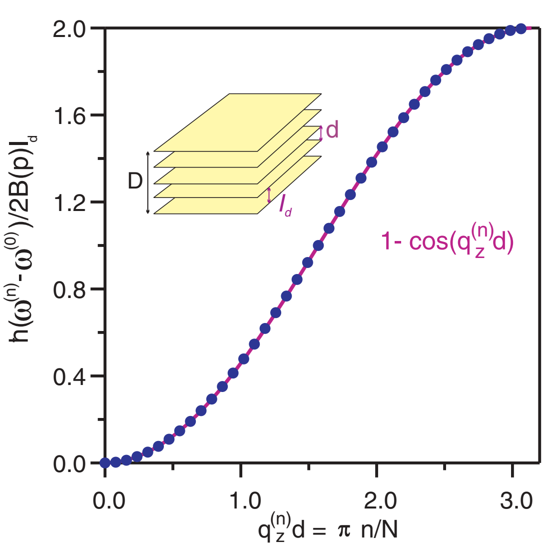

FIG. 5. (Color online) Spin-wave resonance spectrum $\omega^{(n)}$ ($n = 0,1,\ldots,N-1$) for the structure with $N = 40$ layers. $q_z^{(n)}$ is the transverse wave vector, d is the distance between layers, and I_d is the exchange interaction between spins of layers.

IV. LANDAU-LIFSHITZ EQUATIONS AND SPIN-WAVE EXCITATIONS IN THICK MAGNETIC FILMS

A. Linearized Landau-Lifshitz equations

Equations (14) and (19) describe spin-wave excitations. Solutions of these equations for magnetic samples of great volumes and for thick N-layer magnetic films with $N \gg 1$ become difficult, because determinants of Eqs. (14) and (19) have high orders. In order to overcome the difficulty and to find the spin-wave spectrum for these samples, we derive Landau-Lifshitz equations.[22,23] Dispersion relations for spin excitations are determined by the $\mathcal{P}$-matrix poles (12) that coincides with poles of the matrix $\mathcal{G}$ of effective propagators. Accordingly, the dispersion relations can be derived from the eigenvalues of equation

$$\mathcal{G} = \mathcal{G}^{(0)} + \mathcal{G}(\mathcal{V}^{(\mathrm{exch})} + \mathcal{V}^{(\mathrm{dip})})\mathcal{G}^{(0)}, \tag{27}$$

where $\mathcal{G}^{(0)} = \|G^{(0)}_{\mu\nu}\|$ is the matrix of bare propagators (11). Since the considered interaction is the sum of exchange and magnetic dipole interactions, we can obtain the eigenvalues and eigenfunctions of equation (27) by a two-step procedure. In the first stage, we perform the summation of diagrams, take into account the exchange interaction, and find the propagator matrix $\mathcal{G}^{(1)} = \|G^{(1)}_{\mu\nu}\|$

$$\mathcal{G}^{(1)} = \mathcal{G}^{(0)} + \mathcal{G}^{(0)}\mathcal{V}^{(\mathrm{exch})}\mathcal{G}^{(1)}. \tag{28}$$

In the second stage, the summation of diagrams with dipole interaction lines is performed. This gives the equation for the matrix $\mathcal{G}$ of effective propagators expressed in terms of the matrix $\mathcal{G}^{(1)}$:

$$\mathcal{G} = \mathcal{G}^{(1)} + \mathcal{G}\mathcal{V}^{(\mathrm{dip})}\mathcal{G}^{(1)}. \tag{29}$$

Thus the solution of Eq. (27), which determines the matrix $\mathcal{G}$, is equivalent to the solution of Eqs. (28) and (29). After the performed two-step summation, Eq. (14) for eigenfunctions $h^{(\lambda)}_{\mu}$ is written in the more convenient form

$$h^{(\lambda)}_{\mu}(\vec{1},\omega_m) - \sum_{\substack{\rho,\sigma \\ \vec{1}'\vec{1}''}} V^{(\mathrm{dip})}_{\mu\rho}(\vec{1} - \vec{1}',\omega_m) \times G^{(1)}_{\rho\sigma}(\vec{1}',\vec{1}'',\omega_m)h^{(\lambda)}_{\sigma}(\vec{1}'',\omega_m)\Big|_{i\omega_m \to \omega + i\varepsilon\,\mathrm{sign}\omega} = 0. \tag{30}$$

The solution of simultaneous equations (28) and (30) gives the dispersion relations for spin excitations. These equations can be reduced to linearized Landau-Lifshitz equations in the generalized form and the equation for the magnetostatic potential. In order to perform this transformation, one needs to make a transition to the retarded Green functions. We transform the matrix equation (28) to equations describing small variations of the magnetic moment density (or the variable magnetization), m_ν. The variable magnetization m_ν under the action of the magnetic field $h_\nu = \bar{h}_\nu$, which is generated by the MDI $\mathcal{V}^{(\mathrm{dip})}$, is given by the retarded Green functions, which are determined by the analytical continued values of the propagator matrix $\mathcal{G}^{(1)}$:[39]

$$m_\nu(\vec{1},\omega) = \frac{\beta(g\mu_B)^2}{V_a} \sum_{\rho,\vec{1}'} G^{(1)}_{\nu\rho}(\vec{1},\vec{1}',\omega_m)\Big|_{i\omega_m\to\omega-i\varepsilon} \bar{h}_\rho(\vec{1}',\omega), \tag{31}$$

where V_a is the atomic volume. The analytical continuation $i\omega_m \to \omega - i\varepsilon$ defines the retarded Green functions. $\bar{h}_\rho(\vec{1},\omega)$ is the field of the magnetic dipole-dipole interaction acting on spins. By multiplying matrix equation (28) by $\mathcal{G}^{(0)-1}$ from the left and by $\bar{h}_\rho$ from the right, performing the analytical continuation $i\omega_m \to \omega - i\varepsilon$, $\delta(\beta\hbar\omega_m) \to [\beta\hbar(\omega - i\varepsilon)]^{-1}$ and taking into account relation (31), we get the matrix equation (28) in the form of simultaneous equations:

$$\sum_{\nu,\vec{1}'} \left[G^{(0)-1}_{\rho\nu}(\vec{1},\vec{1}',\omega) - \beta I_{\rho\nu}(\vec{1}-\vec{1}')\right] m_\nu(\vec{1}',\omega) = \frac{\beta(g\mu_B)^2}{V_a}\bar{h}_\rho(\vec{1},\omega). \tag{32}$$

For isotropic exchange interaction, $2I_{-+} = 2I_{+-} = I_{zz} = I$, equations (32) have the form

$$\hat{E}_\pm m_\pm(\vec{1},\omega) = 2\gamma M(\vec{1})\bar{h}_\mp(\vec{1},\omega), \tag{33}$$

$$\hat{E}_z m_z(\vec{1},\omega) = \frac{B^{[1]}(p)}{B(p)}\gamma M(\vec{1})\bar{h}_z(\vec{1},\omega), \tag{34}$$

where $M(\vec{1}) = g\mu_B B(p)/V_a$ is the magnetic moment density at the low-temperature approximation. We say that the operators $\hat{E}_\pm$ and $\hat{E}_z$,

$$\hat{E}_\pm m_\pm(\vec{1},\omega) = [\gamma(H(\vec{1}) + H^{(m)}(\vec{1})) \pm \omega] m_\pm(\vec{1},\omega) + \frac{B(p)}{\hbar V_b}\sum_{\vec{1}'}\int_{V_b}[\bar{I}(0) - \bar{I}(\vec{q})] \times \exp[i\vec{q}(\vec{1}-\vec{1}')] m_\pm(\vec{1}',\omega)\, d^3q$$

and

$$\hat{E}_z m_z(\vec{1},\omega) = \omega m_z(\vec{1},\omega) - \frac{B^{[1]}(p)}{\hbar V_b}\sum_{\vec{1}'}\int_{V_b}\bar{I}(\vec{q})\exp[i\vec{q}(\vec{1}-\vec{1}')] \times m_z(\vec{1}',\omega)\, d^3q,$$

are Landau-Lifshitz operators. For a cubic crystal lattice, the Fourier transform of the exchange interaction with respect to the lattice variables is $\bar{I}(\vec{q}) = \sum_{\vec{1}} I(\vec{1})\exp(-i\vec{q}\vec{1}) = 2I_0[\cos(q_x a) + \cos(q_y a) + \cos(q_z a)]$, where I_0 is the interaction between neighboring spins. The field $H^{(m)}(\vec{1})$ is defined by relation (10) and depends on the magnetic moment density $M(\vec{1})$; $V_b = (2\pi)^3/V_a$ is the volume of the first Brillouin zone. Equations (33) and (34) have the generalized form of the Landau-Lifshitz equations.[18] Solutions $m_\pm$ of Eq. (33) depend on temperature, because $\beta = 1/kT$ is contained in the variable p of the function $B(p)$ (8), through which the magnetic moment density $M(\vec{1})$ is expressed. Equation (34) describes longitudinal variations of the variable magnetization under the influence of the field $\bar{h}_z$. At low temperature, the derivative of the function $B^{[1]}(p)$ tends to zero and the longitudinal variable magnetization m_z is negligible.

From the form of the magnetic dipole interaction in relations (2) and (3), it follows that the field $h_\nu = \bar{h}_\nu$, which is generated by the MDI $\mathcal{V}^{(\mathrm{dip})}$, in relation (31) is magnetostatic, i.e., it is expressed in terms of the magnetostatic potential φ: $\bar{h}_\nu = -\nabla_\nu\varphi$. We transform equation (30) to the equation for the magnetostatic potential $\varphi(\vec{r},\omega)$. Taking into account relation (31) and the explicit form of the magnetic dipole interaction in relations (2) and (3), performing the derivation ∇_μ, the analytical continuation $i\omega_m \to \omega - i\varepsilon$ and the summation of equation (30) over the index μ, we obtain the equation expressed in terms of φ, m_ν:

$$-\Delta\varphi(\vec{r},\omega) + 4\pi\nabla_\nu m_\nu(\vec{1},\omega)|_{\vec{1}\to\vec{r}} = 0. \tag{35}$$

Equation (35) gives the boundary conditions for the normal component of the field $\vec{b} = -\nabla\varphi + 4\pi\vec{m}$,

$$(\vec{b},\vec{n})|_{+\partial V} = (\vec{b},\vec{n})|_{-\partial V}, \tag{36}$$

where $\vec{n}$ is the normal to the boundary, ∂V, $+\partial V$, and $-\partial V$ denote different sides of the boundary. Thus, in consideration of the Landau-Lifshitz equations (33) and (34), the dispersion relations of spin excitations are given by the eigenvalues of Eq. (35).

If the scale of the spatial distribution of the variable magnetization $m_\nu(\vec{1},\omega)$ and the sample size are much greater than the lattice constant a, then the sum over the lattice variables $\sum_{\vec{1}}$ in Landau-Lifshitz operators $\hat{E}_\pm$ and $\hat{E}_z$ can be converted into an integral over the sample volume $V_a^{-1}\int d^3r$. In this approximation, we suppose that the film is continuous over the thickness and, therefore, one can use methods of differential and integral calculus. Let us consider the case of normal and in-plane magnetized homogeneous films when the temperature is low. Then we obtain that $m_z \to 0$ and Eq. (34) is dropped.

1. Normal magnetized films

The dispersion relations of spin waves in a normal magnetized film with thickness D are determined by Eqs. (33) and (35). Taking into account that the magnetic field $H^{(m)}$ in normal magnetized films is equal to $-4\pi M$,[18] we find the dispersion relations of spin waves:

$$\omega^{(n)2}(\vec{q}) = \Omega^{(n)}(\vec{q})\left[\Omega^{(n)}(\vec{q}) + \Omega_M q^2/q_0^{(n)2}\right], \tag{37}$$

where $n = 1,2,3,\ldots$ is the mode number, $\vec{q} = (q_x,q_y)$ is the two-dimensional longitudinal wave vector of spin waves,

$q = |\vec{q}|$,

$$\Omega^{(n)}(\vec{q}) = \gamma H - \Omega_M + \frac{2B(p)I_0}{\hbar} \times \left[3 - \cos(q_x a) - \cos(q_y a) - \cos\left(q_z^{(n)} a\right)\right],$$

$\Omega_M = 4\pi\gamma M$, $q_0^{(n)} = (q^2 + q_z^{(n)2})^{1/2}$, $q_z^{(n)}$ is the transverse vector. The magnetostatic potential over the thickness $z \in [-D/2, D/2]$ of the magnetic film is

$$\varphi^{(n\vec{q})}(x,y,z) = (2\pi)^{-1} f^{(n)-1/2} \exp(iq_x x + iq_y y) \times \cos\left[q_z^{(n)} z + \pi(n-1)/2\right], \tag{38}$$

where $f^{(n)} = D/2 + q/q_0^{(n)2}$. The boundary conditions (36) gives the relationship between the transverse $q_z^{(n)}$ and the longitudinal q wave vectors:

$$2 \cot q_z^{(n)} D = \frac{q_z^{(n)}}{q} - \frac{q}{q_z^{(n)}}. \tag{39}$$

For low values of q and for small mode numbers n, we can neglect the exchange term in the $\Omega^{(n)}(\vec{q})$. Then, in this case, dispersion relations (37) correspond to dispersion relations of forward volume magnetostatic spin waves.[1–3,18]

2. In-plane magnetized films

Let us consider the case, when x and z axes are in the film plane and the y-axis is normal to the plane. The magnetic field $\vec{H}$ is parallel to the z axis. Spin waves propagate along the x axis. Dispersion relations of spin waves in an in-plane magnetized film with the thickness D are determined by Eqs. (33) and (35) with boundary conditions (36). Taking into account that the magnetic field $H^{(m)}$ in in-plane magnetized films is equal to zero, we find the dispersion relations of surface spin waves:

$$\omega^{(s)2}(q) = \Omega^2(q) + \Omega(q)\Omega_M + \frac{\Omega_M^2}{4}[1 - \exp(-2qD)], \tag{40}$$

where

$$\Omega(q) = \gamma H + \frac{4B(p)I_0}{\hbar}[1 - \cos(qa)],$$

and dispersion relations of high spin-wave modes,

$$\omega^{(n)2}(q) = \Omega^{(n)}(q)[\Omega^{(n)}(q) + \Omega_M], \tag{41}$$

where

$$\Omega^{(n)}(q) = \gamma H + \frac{2B(p)I_0}{\hbar}\left[2 - \cos(qa) - \cos\left(q_y^{(n)} a\right)\right],$$

$q_y^{(n)} = \pi n/D$, $n = 1,2,3,\ldots$. For low values of q, we can neglect the exchange term in the $\Omega(q)$ in relation (40). In this case, the dispersion relations correspond to dispersion relations of Damon-Eshbach surface magnetostatic spin waves.[40]

Let us consider the case when spin waves propagate along the z axis ($q = q_z$). Then the solution of Eqs. (33) and (35) gives the spin-wave dispersion relations

$$\omega^{(n)2}(q) = \Omega^{(n)}(q)\left[\Omega^{(n)}(q) + \Omega_M - \frac{\Omega_M q^2}{q_y^{(n)2} + q^2}\right], \tag{42}$$

where $\Omega^{(n)}(q)$ is defined in relation (41). The transverse wave vector $q_y^{(n)}$ is determined by relation (39), where the substitution $q_z^{(n)} \to q_y^{(n)}$ should be done. For low values of q and for small mode numbers n, dispersion relations (42) correspond to dispersion relations of backward volume magnetostatic spin waves.[1–3,18]

B. Difference between dispersion relations of spin waves in monolayers, bilayers and in thick magnetic films

We can single out the MDI part in the dispersion relations of spin waves in monolayers, bilayers, and in thick magnetic films. Taking into account that for monolayers and for bilayers $\Omega_M = 4\pi\gamma\sigma_m/a$, we can write the dispersion relations (21), (23), and (37) of spin waves propagating in normal magnetized films in the form

$$\omega^{(n)2}(\vec{q}) = \Omega^{(n)}(\vec{q})[\Omega^{(n)}(\vec{q}) + \Omega_M \eta^{(n)}(qD)], \tag{43}$$

where n is the mode number. $\eta^{(n)}(qD)$ is the function of qD, where for monolayers $D = a$, for bilayers $D = 2d = 2a$ (we consider the case $d = a$), and in the case of thick films, D is the thickness. $\Omega^{(n)}(\vec{q})$ is defined in relation (37). The η function determines the action of the MDI.

For spin waves propagating in in-plane magnetized films along the x axis ($\vec{q} \perp \vec{H}, q = q_x, q_z = 0$), dispersion relations (22) and (24) and surface spin-wave relation (40) can be written in the form

$$\omega^2(q) = \Omega^2(q) + \Omega(q)\Omega_M + \Omega_M^2 \eta,$$

where for the case of monolayers and of thick films, η is a function of qD. For bilayers, η is a function of qD, $\Omega(q)/\Omega_M$, and $B(p)I_0/\Omega_M$. $\Omega(q)$ is defined in relation (22).

For spin waves propagating in in-plane magnetized films along the z axis ($\vec{q} \parallel \vec{H}, q = q_z, q_x = 0$), dispersion relations (22), (25), and (42) have the form

$$\omega^{(n)2}(q) = \Omega^{(n)}(q)[\Omega^{(n)}(q) + \Omega_M - \Omega_M \eta^{(n)}(qD)]$$

with $\Omega^{(n)}(q)$ defined in relation (41). The η function for backward volume spin waves coincides with the η function for forward waves in relation (43).

For the first forward and backward spin-wave modes and for spin waves propagating in in-plane magnetized films along the x axis, η functions are presented in Fig. 6. For in-plane magnetized bilayers, Fig. 6(b) shows the η function of $I_0 = 0$ and of $B(p)I_0/\Omega_M \to \infty$. For these cases, the η function is independent on the variable $\Omega(q)/\Omega_M$. One can see that η functions of spin waves in monolayers, bilayers, and in thick magnetic films are close for $qD < 1$. Thus, in order to calculate dispersion relations of spin waves in N-layer films ($N = 1,2,\ldots$) consisted of monolayers for $qD < 1$, we can consider the N-layer film as continuous. For example, for a quadratic lattice monolayer with the lattice constant a, the parameters of this continuous film are the following: the thickness D is equal to a and the volume magnetic moment density M is determined by the surface magnetic moment density σ_m, $M = \sigma_m/a$. Spin excitations in thin magnetic films, bilayers, and trilayers are calculated in Refs. 10–14 in the continuous-film approximation for low wave vectors, $qD \ll 1$. In accordance with the above mentioned, in this case, the usage of the continuous-film approximation is correct.

For $qD > 1$, the difference between the η function of monolayer and the η function of continuous thick magnetic

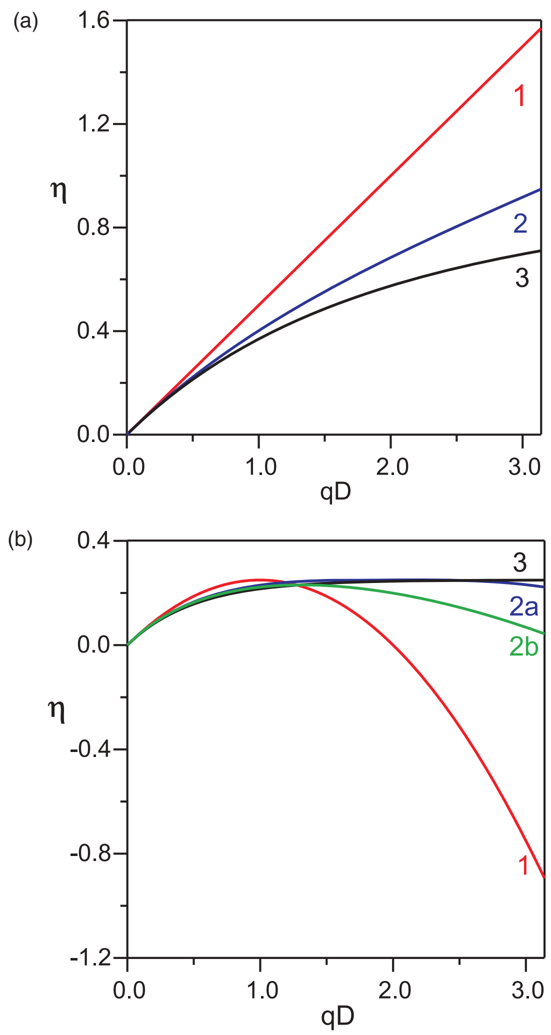

FIG. 6. (Color online) (a) η function characterizing the action of the MDI on dispersion curves of forward spin waves in normal magnetized films and backward spin waves in in-plane magnetized films vs the normalized wave vector qD. 1 is a monolayer, 2 is a bilayer, and 3 is a thick film case. (b) η function for surface spin waves in in-plane magnetized films. 1 is a monolayer, 2 is a bilayer [2a for $I_0 = 0$, and 2b for $B(p)I_0/\Omega_M \to \infty$], 3 is a thick film case.

films is considerable. The difference is due to the discreetness of the lattice. If the exchange interaction is much greater than the MDI, the difference between dispersion relations of spin waves in monolayers and in continuous thick magnetic films determined by η functions is insignificant in comparison with the exchange interaction and can be dropped. But, when the MDI is equal or greater than the exchange interaction, this difference becomes essential and should be taken into account. It is important for the case of monolayer lattice with magnetic nanoparticles on lattice sites.

C. Exchange boundary conditions

Let us consider the case when the size of a homogeneous film is much greater than the lattice constant a and the sum $\sum_{\vec{1}}$ in operators $\hat{E}_\pm$ (33) can be converted into an integral over the sample volume $V_a^{-1}\int d^3r$. The magnetic fields H and $H^{(m)}$ are homogeneous. If we restrict ourself to the second term in the Fourier transform of the exchange interaction $\bar{I}(\vec{q}) - \bar{I}(0) = -I_0a^2q^2$, then the operators $\hat{E}_\pm$ can be written in the pseudodifferential form of order 2:[41]

$$\hat{E}_\pm m_\pm(\vec{r},\omega) = [\gamma(H + H^{(m)}) \pm \omega]m_\pm(\vec{r},\omega) + \frac{4\pi\gamma\alpha M}{(2\pi)^3}\int_V\int_{V_b} q^2\exp[i\vec{q}(\vec{r}-\vec{r}')] \times m_\pm(\vec{r}',\omega)\, d^3q\, d^3r', \tag{44}$$

where $\alpha = B(p)I_0a^2/\hbar\gamma 4\pi M$ is the exchange interaction constant, V is the volume of the ferromagnetic sample.

In Refs. 15–18,42–46, the pseudodifferential Landau-Lifshitz operators are reduced to the differential operators with respect to spatial variables:

$$\hat{E}_\pm(\vec{r},\omega) = \gamma[H + H^{(m)} - 4\pi\alpha M\Delta] \pm \omega. \tag{45}$$

For solvability of Eq. (33) with differential Landau-Lifshitz operators (45), the exchange boundary conditions are imposed:

$$\left.\frac{\partial m_\nu}{\partial \vec{n}} + \xi m_\nu\right|_{\partial V} = 0, \tag{46}$$

where $\vec{n}$ is the inward normal to the boundary ∂V and ξ is the pinning parameter. As it is found in Appendix for the case of forward volume spin waves propagating in a normal magnetized film, simultaneous equations (33) with operators (45) and with boundary conditions (46) and Eq. (35) with boundary conditions (36) have no solutions due to incompatibility of conditions (36) and (46). In order to evaluate the influence of the exchange boundary conditions on the dispersion relations, we formally drop out the boundary conditions (36). Then the exchange boundary conditions (46) give the relation for the transverse wave vector $q_z^{(n)}$ (see Appendix):

$$2\cot q_z^{(n)}D = \frac{q_z^{(n)}}{\xi} - \frac{\xi}{q_z^{(n)}}, \tag{47}$$

where n is the mode number. Dispersion relations (37) of the first spin-wave mode propagating in the YIG film of the thickness $D = 0.5$ μm with $4\pi M = 1750$ Oe, and $\alpha = 3.2\times10^{-12}$ cm^2 at the applied magnetic field $H = 3000$ Oe are shown in Fig. 7 for the transverse wave vector $q_z^{(1)}$ (47) with different pinning parameters ξ. In contrast with these curves, we show dispersion relations based on pseudodifferential operators (44) with the boundary conditions (36) (the curve A). One can see that there does not exist any constant pinning parameter ξ at which the curve A calculated on the basis of relation (39) coincides with the curves calculated on the basis of the exchange boundary conditions.

In order to overcome the contradiction based on simultaneous solvability of relations (39) and (47), we should require that the pinning parameter $\xi = q$. Only in this case, the curve A calculated on the basis of equations with pseudodifferential Landau-Lifshitz operators (44) coincides with the curves calculated on the basis of differential equations with the exchange boundary conditions (46).

V. SPIN-WAVE RELAXATION

In this section we answer the question: what is the value of spin-wave relaxation in the model with magnetic dipole

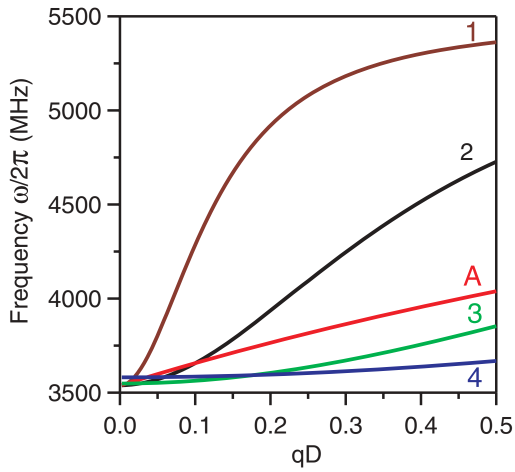

FIG. 7. (Color online) Dispersion curves of the first spin-wave mode propagating in the YIG film of the thickness $D = 0.5\ \mu$m with $4\pi M = 1750$ Oe, $\alpha = 3.2 \times 10^{-12}$ cm^2 at the applied magnetic field $H = 3000$ Oe. The curve A is calculated on the base of relation (39) for the case of pseudodifferential Landau-Lifshitz operators (44). Curves 1–4 are calculated for the case of differential Landau-Lifshitz operators (45) on the base of relation (47) with different pinning parameters ξ. (1) $\xi D = 0.01$, (2) 0.1, (3) 1, and (4) 10.

and exchange interactions derived from first principles? The answer depends on the ratio of the spin-wave energy to intervals between modes of the spin-wave spectrum and is different for thick and for thin magnetic films. In thick films, the spin-wave energy is greater than energy gaps between modes and a three-spin-wave process takes place. If the exchange interaction is isotropic, it cannot induce three-magnon processes and, therefore, the MDI makes a major contribution to the relaxation. We consider the spin-wave damping in thick films in the one-loop approximation. In thin magnetic films (for example, in nanosized films), the energy of long-wavelength spin waves is less than energy gaps between modes and three-spin-wave processes are forbidden. In this case, four-spin-wave processes take place, the exchange interaction makes a major contribution to the relaxation, and the spin-wave damping has lower values in comparison with the damping in thick films. We calculate the spin-wave relaxation for four-spin-wave processes in thin films for long-wavelength spin waves in the two-loop approximation.

A. Spin-wave relaxation in thick films

The spin-wave relaxation induced by a three-spin-wave process in normal magnetized homogeneous ferromagnetic films is considered in Refs. 22 and 23 in the one-loop approximation for spin waves with small longitudinal wave vectors at low temperature. The relaxation is determined by self-energy diagram insertions $\Sigma_{(1+)(1-)}$ to the $\mathcal{P}$ matrix given by relation (12) (see Fig. 8). Damping of the j-mode excitation is

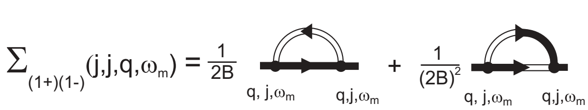

FIG. 8. Self-energy diagrams in the one-loop approximation at low temperature. B is determined by relation (8).

defined by the imaginary part of the pole of the effective Green functions $G_{-+} = P_{(1-)(1+)}$ with insertions $\Sigma_{(1+)(1-)}$ under the analytical continuation (13):

$$\Delta^{(j)}(\vec{q}) = \frac{\delta\omega^{(j)}(\vec{q})}{\omega^{(j)}(\vec{q})} = \frac{2B(p)V_a}{\beta\hbar\omega^{(j)}(\vec{q})}\,\mathrm{Im}\,\Sigma_{(1+)(1-)} \times (j,j,\vec{q},\omega_m)\Big|_{i\omega_m\to\omega+i\varepsilon\,\mathrm{sign}\,\omega}$$
$$= \frac{V_a}{2\beta\hbar\omega^{(j)}}\,\mathrm{Im}\sum_{n,i,k}\int F^{(i)}F^{(k)}[\bar{P}_{(1-)(1+)}(i,-\vec{q}_1,-\omega_n) \times \bar{P}_{(2z)(2z)}(k,\vec{q}-\vec{q}_1,\omega_m-\omega_n) + \frac{1}{8B(p)}\bar{P}_{(1-)(2z)}(i,\vec{q}_1,\omega_n)\bar{P}_{(2z)(1+)} \times (k,\vec{q}-\vec{q}_1,\omega_m-\omega_n)]N^2(j,\vec{q};i,\vec{q}_1;k,\vec{q}-\vec{q}_1) \times d^2q_1\Big|_{i\omega_m\to\omega+i\varepsilon\,\mathrm{sign}\,\omega}, \tag{48}$$

where

$$\bar{P}_{(1-)(1+)}(j,\vec{q},\omega_m) = 2\rho V_a^2(\Omega^{(j)} + 2\eta_{-+}^{(j)} + i\omega_m),$$
$$\bar{P}_{(1-)(2z)}(j,\vec{q},\omega_m) = -2\eta_{+z}^{(j)}(\Omega^{(j)} + i\omega_m),$$
$$\bar{P}_{(2z)(1+)}(j,\vec{q},\omega_m) = -2\eta_{z-}^{(j)}(\Omega^{(j)} + i\omega_m),$$
$$\bar{P}_{(2z)(2z)}(j,\vec{q},\omega_m) = F^{(j)-1}\beta V_a\bar{I}(\vec{q}) - \rho^{-1}\eta_{zz}^{(j)}(\Omega^{(j)2} + i\omega_m^2),$$
$$F^{(j)} = \left(\omega^{(j)2} + \omega_m^2\right)^{-1}, \quad \rho = \frac{B(p)}{\beta\hbar V_a},$$
$$\eta_{\mu\nu}^{(j)} = \frac{\Omega_M q_\mu q_\nu}{q_0^{(j)2}} \quad (\mu,\nu = -,+,z),$$
$$q_\pm = \frac{1}{2}(q_x \mp iq_y),$$
$$\bar{I}(\vec{q}) = 2I_0\left[\cos(q_xa) + \cos(q_ya) + \cos\left(q_z^{(j)}a\right)\right]$$

is the Fourier transform of the exchange interaction,

$$N(j_1,\vec{q}_1;\,j_2,\vec{q}_2;\,j_3,\vec{q}_3) = \frac{1}{8\pi V_a}\prod_{k=1}^{3}\frac{1}{f^{(j_k)1/2}}\sum_{\sigma_1,\sigma_2,\sigma_3}\frac{\sin\left[\left(\sum_{k=1}^{3}\sigma_kq_z^{(j_k)}\right)D/2\right]}{\sum_{k=1}^{3}\sigma_kq_z^{(j_k)}} \times \exp\left[i\sum_{k=1}^{3}\sigma_k\pi(j_k-1)/2\right]$$

is the block factor in the representation of the functions (38), $f^{(j)} = D/2 + q/q_0^{(j)2}$, $\sigma_k = \pm 1$; $\sum_{\sigma_1,\sigma_2,\sigma_3}$ denotes the summation over all sets $\{\sigma_1,\sigma_2,\sigma_3\}$. The spin-wave frequency $\omega^{(j)}$ and the transverse wave vector $q_z^{(j)}$ are determined by relations (37) and (39), respectively. The damping $\Delta^{(j)}$ increases directly proportionally to the temperature.

Relation (48) describes relaxation of the spin-wave j mode caused by inelastic scattering on thermal excited spin-wave modes. Relaxation occurs through the confluence of the j mode with the k mode to form the i mode. From the explicit form of the block factor N in relation (48), it follows that the confluence processes take place when the sum of mode numbers $j + i + k$ is equal to an odd number. The confluence processes are induced by the MDI and are accompanied by

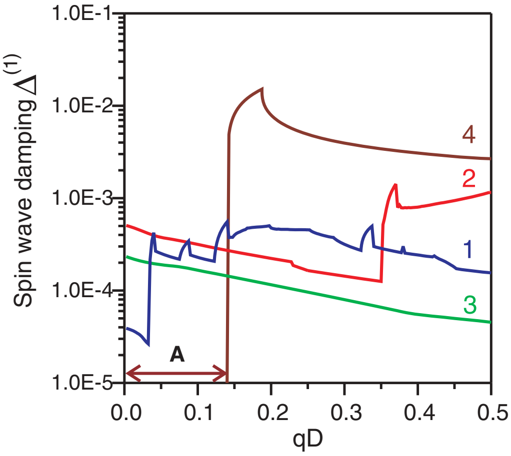

FIG. 9. (Color online) Spin-wave damping $\Delta^{(1)} = \delta\omega^{(1)}/\omega^{(1)}$ of the first mode in normal magnetized YIG film with the magnetization $4\pi M = 1750$ Oe and the exchange interaction constant $\alpha = 3.2 \times 10^{-12}$ cm^2 at $H = 3000$ Oe, $T = 300$ K at different film thickness D. (1) $D = 500$, (2) 300, (3) 200, (4) 120 nm. A is the low-relaxation region.

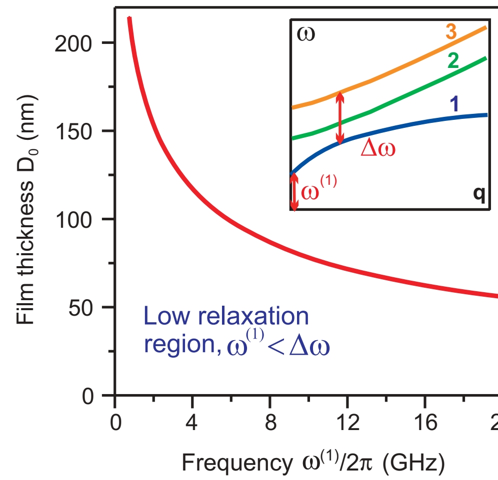

FIG. 10. (Color online) Film thickness D_0 of YIG film versus the excitation frequency $\omega^{(1)}(\vec{q})/2\pi$ of the first mode at the wave vector $\vec{q} \to 0$. Low-relaxation region of the first spin-wave mode exists for YIG films with the thickness $D < D_0$.

transitions between thermal excited i and k modes. Transitions take place when the equation

$$\omega^{(j)}(\vec{q}) = \omega^{(i)}(\vec{q}^{\,(s)}) - \omega^{(k)}(\vec{q} - \vec{q}^{\,(s)}) \tag{49}$$

has at least one solution $\vec{q}^{\,(s)}$ for the given $\vec{q}$, i, j, k. The existence of solutions $\vec{q}^{\,(s)}$ of Eq. (49) depends on the thickness of the magnetic film. With decreasing film thickness D, the density of dispersion curves of modes on the plane (ω,q) decreases and the frequency of the spacings between curves increase. The least frequency spacing occurs between the first $(i = 1)$ and the third $(k = 3)$ modes. Figure 9 shows the damping $\Delta^{(1)}$ of the first spin-wave mode versus the longitudinal wave vector q normalized by the film thickness D at different film thicknesses. Calculations have been done for a YIG film with the magnetization $4\pi M = 1750$ Oe and the exchange interaction constant $\alpha = 3.2 \times 10^{-12}$ cm^2 at $H =$ 3000 Oe and $T = 300$ K. One can see that for the YIG film with the thickness $D = 120$ nm in the region $qD < 0.14$ the damping $\Delta^{(1)}$ is equal to zero due to the absence of transitions between modes. Thus, in thin magnetic films, a low spin-wave relaxation region takes place. We define the low-relaxation region as a region in the (ω,q) space, where spin wave has no damping induced by three-spin-wave processes. For the given j mode, this region appears when the excitation frequency $\omega^{(j)}(\vec{q})$ is less than the difference $\omega^{(3)}(\vec{q}^{\,(s)}) - \omega^{(1)}(\vec{q} - \vec{q}^{\,(s)})$ at any values of the wave vector $\vec{q}^{\,(s)}$. For the first mode $\omega^{(1)}$ in the YIG film, the film thickness, when the low spin-wave relaxation region appears, is shown in Fig. 10 at $q \to 0$. If

$$\omega^{(1)}(0) < \omega^{(3)}(\vec{q}^{\,(s)}) - \omega^{(1)}(\vec{q}^{\,(s)}), \tag{50}$$

the first mode has low values of the spin-wave damping $\Delta^{(1)}$. Taking into account dispersion relations (37), from inequality (50), we can obtain an estimation of the characteristic thickness for the given frequency ω:

$$D_0 = \frac{2\pi(\alpha\Omega_M)^{1/2}}{[\omega(\omega + \Omega_M)]^{1/4}}.$$

We say that a film is thin with respect to the relaxation process, if the film thickness $D < D_0$. In the next section, we consider the relaxation of spin waves in thin films.

B. Relaxation in thin magnetic films

What is the value of spin-wave damping in the low relaxation region in thin magnetic films? We consider four-spin-wave processes in the normal magnetized monolayer of the quadratic lattice with the lattice constant a at small longitudinal wave vector values $\vec{q} = (q_x,q_y)$ at low temperature. As isotropy of the exchange interaction cannot forbid four-spin-wave processes and the value of the exchange interaction is much greater than the MDI, only the exchange interaction will be taken into account in diagrams. We suppose that the exchange interaction acts between neighboring spins and is equal to I_0. In order to calculate self-energy diagram insertions to the effective Green functions in the two-loop approximation, we use the ladder expansion (see Fig. 11). At small values of wave vectors the bare Γ_0 vertex [see Fig. 11(a)] is

$$\begin{aligned}\Gamma_0(1,2;3,4) &\equiv \Gamma_0(\vec{k},\vec{s}+\vec{q}-\vec{k};\vec{q},\vec{s})\\ &= \beta[\bar{I}(\vec{k}-\vec{q}) + \bar{I}(\vec{k}-\vec{s}) - \bar{I}(\vec{s}) - \bar{I}(\vec{q})]\\ &= 2\beta I_0 a^2(\vec{q},\vec{s}),\end{aligned}$$

where 1,2; 3,4 is the abridged notation of two-dimensional wave vectors, which are variables of Γ_0 vertex; $|\vec{k}|,|\vec{q}|,|\vec{s}| \ll a^{-1}$:

$$\begin{aligned}\bar{I}(\vec{q}) &= \sum_{\vec{1}_{xy}-\vec{1}'_{xy}} I(\vec{1}_{xy} - \vec{1}'_{xy}) \exp[-i\vec{q}(\vec{1}_{xy} - \vec{1}'_{xy})]\\ &= 2I_0[\cos(q_x a) + \cos(q_y a)].\end{aligned}$$

The Γ vertex in the ladder approximation [see Fig. 11(b)] is determined by the relationship

$$\begin{aligned}&\Gamma(1,2;3,4)\\ &\quad\equiv \Gamma(\vec{k},\omega_1,\vec{s}+\vec{q}-\vec{k},\omega_3+\omega_4-\omega_1;\vec{q},\omega_3,\vec{s},\omega_4)\\ &\quad= \Gamma_0(\vec{k},\vec{s}+\vec{q}-\vec{k};\vec{q},\vec{s})\end{aligned}$$

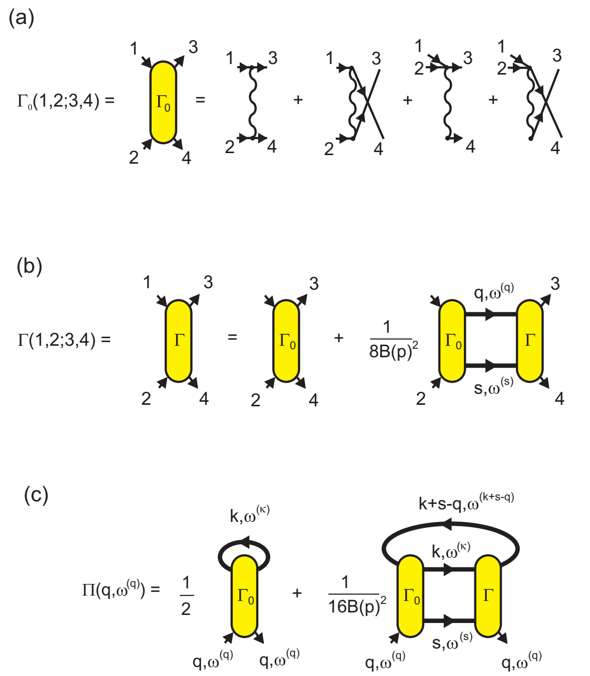

FIG. 11. (Color online) (a) Bare Γ_0 vertex. (b) Ladder approximation. (c) Self-energy diagram insertion.

$$+\frac{1}{8B^2(p)S_b}\sum_{\omega_m^{(q)}}\int \Gamma_0(\vec{k},\vec{s}+\vec{q}-\vec{k};\vec{q}',\vec{s}+\vec{q}-\vec{q}')$$
$$\times G_{-+}\big(\vec{q}',\omega_m^{(q)}\big)G_{-+}\big(\vec{s}+\vec{q}-\vec{q}',\omega_3+\omega_4-\omega_m^{(q)}\big)$$
$$\times\Gamma\big(\vec{q}',\omega_m^{(q)},\vec{s}+\vec{q}-\vec{q}',\omega_3+\omega_4-\omega_m^{(q)};\vec{q},\omega_3,\vec{s},\omega_4\big)\,d^2q',$$

where

$$G_{-+}(\vec{q},\omega_m)=\frac{2B(p)}{\beta\hbar(\omega(\vec{q})-i\omega_m)}$$

is the effective Green function determined by the $\mathcal{P}$ matrix (12), $\omega(\vec{q})$ is the frequency of spin excitations in monolayer (21), and S_b is the volume of the two-dimensional first Brillouin zone. The coefficient $1/8B^2(p)$ is due to the fact that the substitution of the bare Green function to effective ones in diagrams are performed inside blocks. The self-energy diagram insertion [see Fig. 11(c)] is given by

$$\Pi(\vec{q},\omega_m^{(q)})=\frac{1}{2S_b}\sum_{\omega_n^{(k)}}\int \Gamma_0(\vec{q},\vec{k};\vec{q},\vec{k})G_{-+}\big(\vec{k},\omega_n^{(k)}\big)\,d^2k$$
$$+\frac{1}{16B^2(p)S_b^2}\sum_{\omega_n^{(k)},\omega_l^{(s)}}\int\int \Gamma_0(\vec{q},\vec{s}+\vec{k}-\vec{q};\vec{s},\vec{k})$$
$$\times G_{-+}\big(-\vec{s}-\vec{k}+\vec{q},\,-\omega_n^{(k)}-\omega_l^{(s)}+\omega_m^{(q)}\big)$$
$$\times G_{-+}\big(\vec{k},\omega_n^{(k)}\big)G_{-+}\big(\vec{s},\omega_l^{(s)}\big)$$
$$\times\Gamma\big(\vec{s},\omega_l^{(s)},\vec{k},\omega_n^{(k)};\vec{q},\omega_m^{(q)},\vec{s}+\vec{k}-\vec{q},$$
$$\omega_n^{(k)}+\omega_l^{(s)}-\omega_m^{(q)}\big)\,d^2k\,d^2s. \qquad (51)$$

The damping of spin-wave excitations at $\omega=\omega(\vec{q})$ is expressed by the imaginary part of the self-energy $\Pi(\vec{q},\omega_m^{(q)})$:

$$\Delta(\vec{q})=\frac{\delta\omega(\vec{q})}{\omega(\vec{q})}=\frac{\mathrm{Im}\,\Pi\big(\vec{q},\omega_m^{(q)}\big)}{\beta\omega}\bigg|_{i\omega_m^{(q)}\to\omega+i\varepsilon\,\mathrm{sign}\,\omega}. \qquad (52)$$

Taking into account the self-energy $\Pi(\vec{q},\omega_m^{(q)})$ in the Born approximation, namely, substituting $\Gamma\to\Gamma_0$ in relation (51), integrating over $\vec{k}$ and $\vec{s}$, and summing over the frequency variables $\omega_n^{(k)}$ and $\omega_l^{(s)}$, from equation (52) at $\hbar\omega(\vec{q})<kT$, we obtain

$$\Delta(\vec{q})=\frac{C(qa)^2(kT)^2}{16\pi B^2(p)I_0\varepsilon^{(0)}},$$

where $C=1.12$, k is the Boltzmann constant and $\varepsilon^{(0)}=\hbar\gamma(H+H^{(m)})$ is the Zeeman energy. In order to evaluate the damping of spin waves, we calculate $\Delta(\vec{q})$ for spin waves with the wavelength $\lambda=5\ \mu$m propagating in the monolayer film with the lattice constant $a=0.4$ nm and with the exchange interaction between neighboring spins $I_0=0.085$ eV, $B(p)=1/2$ at $T=300$ K. Then, taking into account that $q=2\pi/\lambda$, for $\varepsilon^{(0)}/h=10$ GHz, we obtain $\Delta(\vec{q})=4.28\times10^{-6}$. Thus one can see that the damping of spin waves of small wave vectors is low.

VI. CONCLUSIONS

The results of the investigations can be summarized as follows. (1) Spin excitations in thin magnetic films in the Heisenberg model with magnetic dipole and exchange interactions are studied by the spin operator diagram technique. Dispersion relations of spin waves in two-dimensional magnetic monolayer and bilayer lattices and the spin-wave resonance spectrum in N-layer structures are obtained. It is found that dispersion relations of spin waves in monolayer and bilayer lattices differ from dispersion relations of spin waves in continuous thick magnetic films. This difference is due to the discreetness of the lattice. For the case when the magnetic dipole interaction is equal or greater than the exchange interaction, for example, for monolayer consisted of magnetic nanoparticles on the lattice, this difference becomes essential and is taken into account.

(2) Generalized Landau-Lifshitz equations for thick magnetic films, which are derived from first principles, have the integral (pseudodifferential) form, but not differential one with respect to spatial variables. Spin excitations are determined by simultaneous solution of the Landau-Lifshitz equations and the equation for the magnetostatic potential. It is found that the model based on differential Landau-Lifshitz equations with exchange boundary conditions is contradictory. The contradiction is removed, if the pinning parameter ξ is equal to the spin-wave wave vector q.

(3) The magnetic dipole interaction makes a major contribution to the relaxation of long-wavelength spin waves in thick magnetic films. The spin-wave damping is determined by diagrams in the one-loop approximation, which correspond to three-spin-wave processes. The three-spin-wave processes are accompanied by transitions between thermal excited spin-wave modes. The damping increases directly proportionally to the temperature.

(4) Thin films have a region of low relaxation of long-wavelength spin waves. In thin magnetic films, the energy of these waves is less than energy gaps between spin-wave modes, therefore, three-spin-wave processes are forbidden, four-spin-wave processes take place and, as a result of this, the exchange interaction makes a major contribution to the relaxation. It is found that the damping of spin waves propagating in a magnetic monolayer has the form of the quadratic dependence on the temperature and is very low for spin waves with small wave vectors.

Low-damping spin waves can be observed in YIG films of nanometer thickness. Thin (nanosized) magnetic films can be used in spin-wave devices. The low damping of long-wavelength spin waves gives us an opportunity to construct tunable narrow-band spin-wave filters with high quality at the microwave frequency range.

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APPENDIX: SPIN-WAVE MODEL WITH EXCHANGE BOUNDARY CONDITIONS

Let us consider forward volume spin waves propagating in a normal magnetized film homogeneous through the thickness D. The applied magnetic field $\vec{H}$ is parallel to the z axis. In order to understand the role of the exchange boundary conditions in Refs. 15 and 16, we consider Landau-Lifshitz equations (33) with differential operators (45) and with boundary conditions (46). The dispersion relations of spin waves are given by the eigenvalues of equation for the magnetostatic potential φ (35). Taking into account that the magnetic field of spin waves $\vec{h}_\nu = -\nabla_\nu \varphi$, after the Fourier transform with respect to the longitudinal variables x and y, we can write Landau-Lifshitz equations (33) in the form

$$\left[\Omega + \alpha\Omega_M\left(q^2 - \frac{\partial^2}{\partial z^2}\right) \pm \omega\right] m_\pm = \frac{-i\Omega_M q}{4\pi}\varphi, \quad \text{(A1)}$$

where $\Omega = \gamma(H + H^{(m)})$ and $\Omega_M = 4\pi\gamma M$, q is the longitudinal wave vector. Without loss of a generality, we suppose that $q = q_x$ and $q_y = 0$. The equation for the magnetostatic potential φ (35) is written as

$$\left(q^2 - \frac{\partial^2}{\partial z^2}\right)\varphi + 2\pi i q(m_+ + m_-) = 0. \quad \text{(A2)}$$

The boundary conditions (36) for the normal component of the field $\vec{b} = -\nabla\varphi + 4\pi\vec{m}$ at boundaries $z = -D/2$ and $D/2$ are reduced to the form

$$\left.\frac{\partial}{\partial z}\varphi\right|_{+\partial V} = \left.\frac{\partial}{\partial z}\varphi\right|_{-\partial V}, \quad \text{(A3)}$$

where $+\partial V$ and $-\partial V$ denotes different sides of the boundary. The magnetostatic potential φ is continuous in the boundary region

$$\varphi|_{+\partial V} = \varphi|_{-\partial V}. \quad \text{(A4)}$$

The exchange boundary conditions (46) can be written as

$$\left.\frac{\partial m_\pm}{\partial z} + \xi m_\pm\right|_{\partial V} = 0 \quad (z = -D/2) \quad \text{(A5)}$$

and

$$\left.-\frac{\partial m_\pm}{\partial z} + \xi m_\pm\right|_{\partial V} = 0 \quad (z = D/2), \quad \text{(A6)}$$

where ξ is the arbitrary constant pinning parameter.

In accordance with the form of Eqs. (A1) and (A2), we find the magnetic moment density $m_\pm$ and the potential φ over the film thickness $z \in [-D/2, D/2]$ in the form

$$m_\pm(z) = A_\pm \exp(iq_z z) + B_\pm \exp(-iq_z z), \quad \text{(A7)}$$

$$\varphi(z) = C\exp(iq_z z) + D\exp(-iq_z z). \quad \text{(A8)}$$

Taking into account Eq. (A2), the potential φ in the external region of the film is given as

$$\varphi(z) = E\exp(qz) \quad (z < -D/2), \quad \text{(A9)}$$

$$\varphi(z) = F\exp(-qz) \quad (z > D/2). \quad \text{(A10)}$$

Relations (A7), (A8), (A9), and (A10) contain eight unknown variables $A_\pm$, $B_\pm$, C, D, E, and F. In order to find these variables, we have eight equations: two equations (A3) at boundaries $z = D/2$ and $-D/2$, two equations (A4) at $z = D/2$ and $-D/2$, and four equations (A5) and (A6). The substitution of φ (A8), (A9), and (A10) in Eqs. (A3) and (A4) gives the relation for the transverse wave vector q_z:

$$2\cot q_z D = \frac{q_z}{q} - \frac{q}{q_z}. \quad \text{(A11)}$$

From Eqs. (A5) and (A6) and the form (A7), we obtain the relation

$$2\cot q_z D = \frac{q_z}{\xi} - \frac{\xi}{q_z}. \quad \text{(A12)}$$

Thus we have different relations (A11) and (A12) for determination q_z. In the common case, for given q, D, and the constant ξ, simultaneous solvability of Eqs. (A11) and (A12) is impossible and there is no solution of q_z. This leads us to the conclusion that the phenomenological model[15,16] based on the exchange boundary conditions (46) is internally contradictory.

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