

Telegraph noise and the Fabry-Perot quantum Hall interferometer

B. Rosenow¹ and Steven H. Simon²

¹*Institut für Theoretische Physik, Universität Leipzig, D-04103 Leipzig, Germany*

²*Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxfordshire OX1 3NP, United Kingdom*

(Received 22 December 2011; published 14 May 2012)

We consider signatures of Abelian and non-Abelian quasiparticle statistics in quantum Hall Fabry-Perot interferometers. When quasiparticles enter and exit the interference cell, for instance due to glassy motion in the dopant layer, the anyonic phase can be observed in phase jumps. In the case of the non-Abelian $\nu = 5/2$ state, if the interferometer is small, we argue that free Majoranas in the interference cell are either strongly coupled to one another or are strongly coupled to the edge. We analyze the expected phase jumps and in particular suggest that changes in the fermionic parity of the ground state should give rise to characteristic jumps of π in the interference phase.

DOI: [10.1103/PhysRevB.85.201302](https://doi.org/10.1103/PhysRevB.85.201302)

PACS number(s): 73.43.Fj, 73.43.Cd

The search for low-energy Majorana fermions has become a focus of both condensed-matter physics and quantum information.^{1,2} The experimental system that appears closest to finding evidence for the existence of Majoranas is the $\nu = 5/2$ quantum Hall state, where recent experiments have found evidence for excitations with one quarter of the electron charge,^{3–5} which are expected if the $\nu = 5/2$ state is due to pairing of electrons. From a theoretical point of view, the most likely description is the Moore-Read Pfaffian⁶ state (or its particle-hole conjugate,⁷ which for the purpose of this Rapid Communication is identical) with charge $e/4$ excitations that host Majorana fermions and have non-Abelian statistics in the Ising universality class,^{8–12} and thus have the potential for topological quantum computation.²

The non-Abelian statistics of $\nu = 5/2$ quasiparticles (QPs) was theoretically predicted to be observable in Fabry-Perot (FP) interference experiments.^{13–16} Under ideal conditions where all localized QPs are far apart from each other and far away from the edge of the interferometer, the non-Abelian statistics manifests itself in the even-odd effect:^{15,16} The fundamental harmonic of the interference signal disappears when an odd number of QPs is inside the interferometer cell. For an even number of QPs in the interior of the interferometer, there are degenerate states due to the Majorana zero modes in the interference cell, and the interference phase may switch by π depending on the parity of the fermion number within the cell. For experimental realizations of FP interferometers in micron-scale structures, such as the one by Willett *et al.*¹⁷ and by Kang *et al.*,¹⁸ the idealized picture as described above may not apply. The goal of this Rapid Communication is to consider a possible set of experimental conditions which could potentially be in agreement with the behavior of the experiments. One of the key features we focus on is the effect of telegraph noise:¹⁹ to what extent it can influence the measurement and how it can be used as a tool for extracting physical information from the measurements. In this Rapid Communication we propose a qualitative understanding of the experiments of Ref. 17 at $\nu = 5/2$ and Ref. 18 at both $\nu = 5/2$ and $7/3$. Some key results are that via telegraph noise measurements the statistical phase of QPs may be measured and, under certain circumstances, may be unperturbed by Coulomb charging effects. We also focus on the effects of averaging over time scales larger than the noise, which may be

relevant in the case of Ref. 17. For the $5/2$ case we describe the importance of Majorana coupling and how that is likely to affect the measured phases.

At an integer filling fraction and in the absence of interaction effects, the addition of a single QP (an electron or hole) to the cavity does not change the interference phase (since an electron on the edge encircling an electron or hole in the interior of the cavity accumulates a total phase of $\pm 2\pi$). However, at fractional filling ν where QPs have anyonic statistics (let us say, Abelian statistics for now) a QP on the edge encircling a QP in the interior accumulates a phase which is a fractional multiple of 2π . Thus, if a QP enters the cavity, it changes the phase of the interference. Typically, the conductance will be of the form (again assuming Abelian statistics)

$$G = G_0 + G_1 \cos(\theta), \quad (1)$$

where

$$\theta = 2\pi e^*(\phi + \beta V_G) + N_L \theta_a + \delta\theta_C(\phi, V_G, N_L). \quad (2)$$

Here, ϕ is the (dimensionless) flux through a reference area A_0 for the interferometer, e^* is the charge of QPs in units of the electron charge, N_L is the number of QPs inside the interference loop, and θ_a is the anyonic phase, which is $2\pi/3$ for $\nu = 1/3$ (or $4/3$ or $7/3$ etc.). V_G denotes a change in side-gate voltage measured relative to a reference value, and the coupling of voltage to the interferometer area is described by the parameter $\beta = \frac{B}{\phi_0} \frac{\partial A_0}{\partial V_G}$. Here $\delta\theta_C(\phi, V_G, N_L)$ is the Coulomb correction to the interference phase. Ideally, one would like to observe the anyonic phase θ_a directly.

The early theoretical discussions of the quantum Hall FP interferometer^{13–16,20} neglected the strong Coulomb interaction (and hence the correction $\delta\theta_C$) that can occur in a pinched-off FP cavity and focused on the physics deep in the so-called Aharonov-Bohm (AB) regime, where $\delta\theta_C$ is small. However, more recent theoretical work^{21,22} supported by several experiments^{23–25} showed that a different regime where the strong Coulomb interaction dominates the physics [the so-called “Coulomb dominated (CD) regime”] more typically occurs. Thus, it may be necessary to disentangle the anyonic phase θ_a from the Coulomb correction.

Fluctuations in the number of localized QPs, N_L , can have different origins. One possible source is voltage fluctuations

caused by the glassy dynamics of charges in the donor layer. We consider two models of how this may result in noise measured in the conductance.

In our first model (model A), we consider the noise to be equivalent to a random change in the gate voltage, which may attract discrete QPs into the interferometer. Far in the AB regime, this should result in phase slips in the interference pattern given by the anyonic phase θ_a . To remain in the AB regime, the Coulomb effects must not be too large, and one can bound the magnitude of the Coulomb correction $\delta\theta_C$. The derivation is straightforward and is given in Supplemental Material A,²⁶ with the result that (at $\nu = 1/3, 4/3, 7/3$) the Coulomb contribution $\delta\theta_C$ to the phase slip is always negative, with its magnitude at most half as big as the anyonic contribution θ_a . In the CD regime, however, $\delta\theta_C$ can be as big as θ_a in magnitude.

We now consider a second model (model B), where the fluctuations in the donor layer are strongly coupled to the Coulomb charge of the interferometer. For simplicity we consider two different configurations of donor impurities and, correspondingly, two possible values of N_L . In this model fluctuations between the two states occur only when there is a near degeneracy of the energy of the two states. If the fluctuations in the donor layer occur physically close to the 2DEG in the interferometer, we can have a situation where $\delta\theta_C$ is very small. A detailed calculation is provided in Supplemental Material B.²⁶ In essence the charging effect of the fluctuation of charge in the donor layer is roughly canceled by the addition of the QP charge, and this cancellation is enforced by the requirement that the two possible states of the system are energetically degenerate. Thus, one may measure the ideal phase jump value even deep into the CD regime.

Experimentally, by examining the phase jumps that occur in the telegraph noise as the side gate voltage is changed smoothly, one can attempt to measure the statistical parameter θ_a . Recent experiments at $\nu = 7/3$ by Kang¹⁸ have made precisely this type of measurement and have observed a phase jump in agreement with the ideal anyonic phase θ_a . This result can be explained if either the system is deep into the AB regime (which is unlikely for a small device) or the above-described screening cancellation of model B is being realized.

In Fig. 1 (top) a simulated data set is displayed for $\nu = 7/3$. (In this simulation, and in further simulations in this Rapid Communication, we assume that either we are deep in the AB regime or that we have model B. Either condition would give the same result.) The plotted conductance is given by Eq. (1) where the variable $t = e^* \beta V_G$ is varied smoothly and N_L is the integer part of a constant times t plus a random component with a correlation “time” of τ_c . This simulated data look qualitatively similar to the experiments of Ref. 18. If one were to “time average” over this telegraph noise, a very different signal could result, as shown in Fig. 1 (middle and bottom). Here, the time averaging is achieved by Fourier transforming t to ω , multiplying the transform by $e^{-\omega\tau}$, and inverting the transform. This procedure simulates a lock-in amplifier with time constant τ . In the limit where $\tau \gg \tau_c$ the observed G in Eq. (1) is replaced by its average over N_L for each value of V_G . In this case the observed period in t will shift from 1 to $1/(1 + \frac{\theta_a}{2\pi} \langle d\langle N_L \rangle / dt \rangle)$. In Fig. 1 we show the drastic effect of a long measurement time constant if the

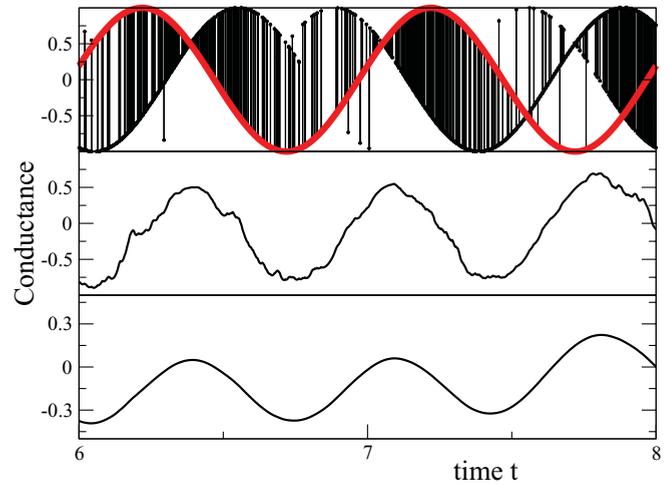


FIG. 1. (Color online) A simulation of random telegraph noise during a scan of a $\nu = 1/3$ (or $7/3$) FP interferometer. The vertical axis is $\cos(2\pi t + 2\pi N_L/3)$, where $t = e^* \beta V_G$ is the horizontal axis, which we can think of as time, assuming a constant voltage sweep rate. Here $N_L = \lfloor \eta t + \xi r \rfloor$, with $\lfloor x \rfloor$ denoting the greatest integer smaller or equal to x and r being a Gaussian random variable with unit variance. The analytic results are then passed through a simulated lock-in amplifier with an averaging time τ . Here $\eta = 1.25$ means roughly (Ref. 27) that 1.25 QPs is added to the dot per cycle and $\xi = 0.3$ is the amplitude of the noise in units of the QP number. The correlation “time” of the noise constant of the Gaussian noise is $\tau_c = 0.001$ in the above units. (Top) Observed conductance with an averaging time shorter than the noise correlation time. The thick (red) curve is a pure sine wave for reference. (Middle) Observed conductance using an averaging time $\tau = 0.0066$ larger than τ_c . Note that the detailed structure is hidden by the averaging. (Bottom) Observed conductance using a very long averaging time $\tau = 0.066$. Note that the period of oscillation is distorted toward smaller values as discussed in the text.

telegraph noise is fast. It should be understood that the model for random QP motion in the dot (explained in the caption) is quite crude. Nonetheless, it gives a feel for the physics.

We now turn to study the situation at $\nu = 5/2$. For a moment, let us assume all of the QPs are stationary (no telegraph noise) and are far from each other and far from the edge. We also focus here on interference of the non-Abelian $e/4$ QPs traveling around the cavity and ignore the Abelian $e/2$ QPs for the moment. If there are an odd number of QPs in the cavity, no interference should be observed.^{15,16} For an even number of QPs in the cavity, there are degenerate zero modes (or qubits) due to the non-Abelian degrees of freedom associated with the QPs. Interference should be seen, but the interference phase may be switched by π depending on the quantum number of the non-Abelian degrees of freedom (i.e., the setting of the non-Abelian qubits in the cavity). However, if the Majoranas in the bulk of the cavity are coupled to the edge modes, then the qubits in the cavity can flip and, if the measurement time scale is sufficiently long, the two possible (opposite) phases of signals are both seen equally, resulting in cancellation of the interference signal.²⁸ The rate at which the qubit is expected to flip is determined by the bulk-edge coupling (set by the distance from the bulk Majorana to the edge). While this coupling is expected to decay exponentially

with distance, given that the cavities in the experiments of Refs. 17 and 18 are extremely small, an estimate of the decay length given by Ref. 11 suggests that for a measurement time scale on the order of seconds, the coupling of bulk to edge should always be sufficiently strong to destroy the interference signal.²⁸ This raises the question of why interference should be seen at all.

So far we have been assuming that the QPs in the cavity are sufficiently far from each other that the non-Abelian degrees of freedom are all zero energy. However, again, given that the cavity is small, the Majoranas couple to each other and the zero energy modes split. If this splitting is larger than the temperature, then the qubits freeze into their lowest energy state and interference would again be seen for the case of an even number of QPs in the cavity, but not for odd. We thus need to consider the spectrum of the coupled Majoranas, which depends on their detailed positions. We estimate²⁹ that there are roughly 20 QPs in the dot and that they may be spaced on the order of $0.1 \mu\text{m}$. The spectrum should be roughly given by energy levels equally spaced by $\Delta E = t/N$, with t the neighbor hopping possibly as large¹¹ as 200 mK and N the number of QPs. This spacing of $\Delta E \approx 10$ mK is potentially large enough to allow interference to be seen at accessibly low temperatures.³⁰ Although these numerical estimates are optimistic, they are not out of the question.

While this seems like a good explanation for why interference is seen at $\nu = 5/2$, there is a complication again associated with the bulk-edge coupling. The situation of having a bulk Majorana coupled to the edge has been discussed in detail in Refs. 28, 31, and 32 (see also Ref. 33) An important limit is when the Majorana is strongly coupled to the edge compared to the measurement voltage e^*V (which experimentally is roughly on the scale of the temperature). In this case the Majorana is absorbed into the edge, and the situation becomes as if that particular Majorana were no longer in the cavity. Considering again that the dot is very small and the bulk-edge coupling is likely to be substantial, this effect is one which we must address.

Unfortunately, determining the size of the bulk-edge coupling is even more uncertain, requiring detailed knowledge of the structure of the dot and the edge. Attempts at electrostatic simulation²⁹ to determine positions of QPs and edges suggests that it is not easily possible to have an excitation gap higher than the temperature and yet always weak enough coupling to the edge that a lone Majorana will not be absorbed into the edge. Instead, we assume the opposite inequality that the bulk edge coupling is larger (or on order of) e^*V . In this case, a lone Majorana is always absorbed into the edge and interference is observed when there are an odd number of QPs in the dot as well as when there are an even number of QPs in the dot (so long as the excitation gap is larger than or on order of T). Note that if e^*V is not much less than the bulk-edge coupling, then the Majorana is not very strongly coupled to the edge, and the amplitude of interference can be reduced and the phase slightly shifted.^{28,32}

We next consider the phase of the interference pattern for both the $e/2$ or $e/4$ QPs and how it may change if QPs are hopping in and out of the dot, analogous to what we considered for $\nu = 1/3$ above. For interference of $e/2$ QPs traveling around the interferometer, the interference is given by Eq. (1),

with N_L being the number of $e/4$ QPs inside the interferometer (with an $e/2$ counting as two $e/4$'s) and $\theta_a = \pi/2$. We do not study this case further since it is not very different from the above-discussed $\nu = 7/3$, and it is likely that the tunneling of $e/2$ QPs is less than that of $e/4$ at any rate.

For interference of $e/4$ traveling around the interferometer, again assuming that any lone Majoranas are strongly coupled to the edge²⁸ and the remaining non-Abelian modes are thermally frozen into a particular state, we find an interference pattern *also* of the form of Eq. (1), where $\theta_a = \pi/4$ (again with any $e/2$ QP in the interferometer counting as two $e/4$'s). Thus, deep in the AB regime we would expect to observe phase slips with an ideal value of $\pi/4$ due to QP addition. For model A, we can derive that within the AB regime the maximum Coulomb correction to the ideal $\pi/4$ phase slip is negative like in the $\nu = 7/3$ case, but with $|\delta\theta_c| = 3\pi/8$ quite large in magnitude compared to θ_a . In the CD regime, $|\delta\theta_c|$ can be up to $\frac{3\pi}{4}$, three times larger than the statistical phase itself. Within model B again we predict that $|\delta\theta_c|$ can be quite small, even within the CD regime.

In addition to these phase slips due to QP addition, the interference pattern may be flipped (shifted by an additional phase of π) depending on the state of the frozen non-Abelian degrees of freedom within the cavity. Indeed, each time the QPs in the dot have their positions rearranged, this degree

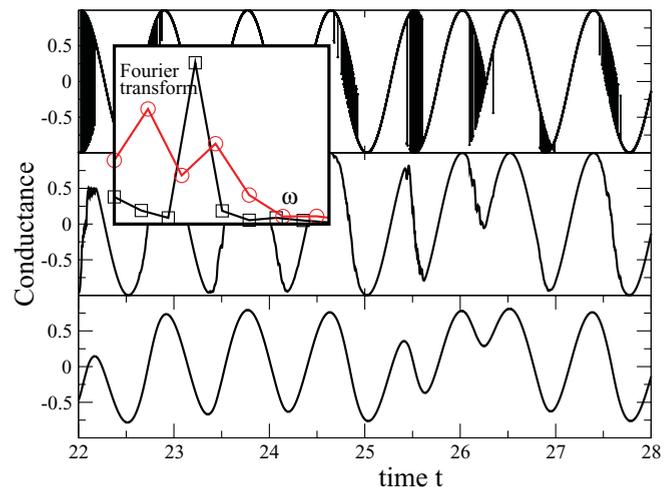


FIG. 2. (Color online) Simulation of telegraph noise during a scan of a $\nu = 5/2$ FP interferometer. The vertical axis is $\cos(2\pi t + \pi f(N_L)/4)$, where $t = e^*\beta V_G$ is the horizontal axis and $N_L = \lfloor \eta t + \xi r \rfloor$, with r again being a Gaussian random variable of unit magnitude. Here $f(n)$ is an integer valued function where $f(n+1) - f(n)$ takes the value 4 with probability 0.3 (corresponding to a π phase slip), and the value 1 with probability 0.5 (corresponding to $\pi/4$) and value 5 with probability 0.2 (corresponding to $5\pi/4$ phase slips). For this plot the noise correlation time is $\tau_c = 0.001$, $\eta = 1.45$, and $\gamma = 0.08$. (Top) Results for an averaging time shorter than τ_c . The phase shifts of π , $\pi/4$, and $5\pi/4$ are easily observed. (Middle) Observed with an averaging time $\tau = 0.0033$ these phase shifts are obscured. (Bottom) Observed with an averaging time $\tau = 0.033$ the smooth behavior appears as if there are fast and slow periods. (Inset) Windowed Fourier transforms of the bottom panel. (Black squares) transform of region $t = 22.5-25$ (red circles) transform of region between $t = 25$ and $t = 27$. Qualitatively similar pictures are obtained with many different parameters used in the simulation.

of freedom may be changed since the lower energy state of the qubit depends on the detailed configuration of QPs.¹¹ We should thus expect to see (ideally) phase slips of both $\pi/4$ and π , where the slips of π may occur concurrent with the slips of $\pi/4$ (resulting in $5\pi/4$) or may occur separately.²⁷ A rough simulation of this type of physics is shown in Fig. 2 (top) and, as above, a low-pass filter (middle and bottom) is shown of the same data. We would like to mention that similarly to the case without bulk-edge and bulk-bulk coupling,³⁴ the 331 state can give rise to the same types of phase jumps. Slips of π are expected due to QP spin flips in the bulk. In the special case where only one of the two spin edge channels is interfering, then slips of $\pm\pi/4$ and $\pm5\pi/4$ are expected when the different spin QPs enter or exit the dot. However, two recent experiments^{35,36} have indicated that the $5/2$ state is spin polarized, which would be inconsistent with the conventional picture of an unpolarized 331 state.³⁷

Relation to experiment. The experimental results¹⁸ are in very good agreement with a model assuming strong coupling between both bulk Majorana degrees of freedom and strong bulk-edge coupling. The fact that the experimentally measured values of phase slips are described by the statistical

phases without Coulomb correction can be explained by our model B.

With respect to the experiment of Ref. 17, it is possible that at least part of the nonsinusoidal behavior observed there is a result of time averaging over phase jumps due to telegraph noise. Unfortunately, it is extremely hard to estimate what the fundamental time scale of the telegraph noise should be (and it may differ from sample to sample) since it almost certainly is related to glassy behavior. Furthermore, in Ref. 17, nonsinusoidal behavior is also observed at $\nu = 2$ which, as mentioned above, does not have fractional phase shifts. In this case, the nonsinusoidal behavior is most likely to be caused by Coulomb charging effects.²¹ In the $\nu = 5/2$ regime, it is possible that time averaging over π phase slips mimics a reduction of the period of resistance oscillations (see Fig. 2).

We would like to thank W. Kang for helpful discussions and for making his data available to us before publication and acknowledge helpful discussions with A. Stern and B. I. Halperin. This work was supported in part by EPSRC Grants No. EP/I031014/1 and No. EP/I032487/1, by the Aspen Center for Physics, and by BMBF.

-
- ¹B. G. Levi, *Phys. Today* **64**, 20 (2011).
²C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, *Rev. Mod. Phys.* **80**, 1083 (2008).
³I. P. Radu, J. B. Miller, C. M. Marcus, M. A. Kastner, L. N. Pfeiffer, and K. W. West, *Science* **320**, 899 (2008).
⁴M. Dolev, M. Heiblum, V. Umansky, A. Stern, and D. Mahalu, *Nature (London)* **452**, 829 (2008).
⁵V. Venkatachalam, A. Yacoby, L. Pfeiffer, and K. West, *Nature (London)* **469**, 185 (2011).
⁶G. Moore and N. Read, *Nucl. Phys. B* **360**, 362 (1991).
⁷S. S. Lee, S. Ryu, C. Nayak, and M. P. A. Fisher, *Phys. Rev. Lett.* **99**, 236807 (2007); M. Levin, B. I. Halperin, and B. Rosenow, *ibid.* **99**, 236806 (2007).
⁸Strictly speaking, the braiding statistics of QPs the $5/2$ state are expected to be Ising up to an Abelian phase.
⁹Parsa Bonderson, Victor Gurarie, and Chetan Nayak, *Phys. Rev. B* **83**, 075303 (2011).
¹⁰Y. Tserkovnyak and S. H. Simon, *Phys. Rev. Lett.* **90**, 016802 (2003).
¹¹M. Baraban, G. Zikos, N. Bonesteel, and S. H. Simon, *Phys. Rev. Lett.* **103**, 076801 (2009).
¹²M. Storni and R. H. Morf, *Phys. Rev. B* **83**, 195306 (2011).
¹³E. Fradkin, C. Nayak, A. Tsvetlik, and F. Wilczek, *Nucl. Phys. B* **516**, 704 (1998).
¹⁴S. Das Sarma, M. Freedman, and C. Nayak, *Phys. Rev. Lett.* **94**, 166802 (2005).
¹⁵A. Stern and B. I. Halperin, *Phys. Rev. Lett.* **96**, 016802 (2006).
¹⁶P. Bonderson, A. Kitaev, and K. Shtengel, *Phys. Rev. Lett.* **96**, 016803 (2006).
¹⁷R. L. Willett, L. N. Pfeiffer, and K. W. West, *Phys. Rev. B* **82**, 205301 (2010); *Proc. Natl. Acad. Sci. USA* **106**, 8853 (2009).
¹⁸Sanghun An, P. Jiang, H. Choi, W. Kang, S. H. Simon, L. N. Pfeiffer, K. W. West, and K. W. Baldwin, e-print arXiv:1112.3400; W. Kang, 2011 Trieste Workshop and School on Topological Aspects of Condensed Matter Physics (smr2248), [<http://www.ictp.tv/eya/topologicalCMP.php?day=05&month=07>].
¹⁹Eytan Grosfeld, Steven H. Simon, and Ady Stern, *Phys. Rev. Lett.* **96**, 226803 (2006).
²⁰C. de C. Chamon, D. E. Freed, S. A. Kivelson, S. L. Sondhi, and X. G. Wen, *Phys. Rev. B* **55**, 2331 (1997).
²¹B. Rosenow and B. I. Halperin, *Phys. Rev. Lett.* **98**, 106801 (2007).
²²Bertrand I. Halperin, Ady Stern, Izhar Neder, and Bernd Rosenow, *Phys. Rev. B* **83**, 155440 (2011).
²³Yiming Zhang, D. T. McClure, E. M. Levenson-Falk, C. M. Marcus, L. N. Pfeiffer, and K. W. West, *Phys. Rev. B* **79**, 241304(R) (2009).
²⁴N. Ofek, Aveek Bid, M. Heiblum, Ady Stern, V. Umansky, and D. Mahalu, *Proc. Natl. Acad. Sci. USA* **107**, 5276 (2010).
²⁵H. Choi, P. Jiang, M. D. Godfrey, W. Kang, S. H. Simon, L. N. Pfeiffer, K. W. West, and K. W. Baldwin, *New J. Phys.* **13**, 055007 (2011); e-print arXiv:0708.2448.
²⁶See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevB.85.201302> for information on the two models used in this Rapid Communication.
²⁷The sign of the phase slip may be positive or negative depending on whether QPs or holes are added to the dot as the gate voltage is swept. This sign may not be directly related to the sign of the gate voltage depending on details such as whether the gate is an area gate or a density gate.
²⁸Bernd Rosenow, Bertrand I. Halperin, Steven H. Simon, and Ady Stern, *Phys. Rev. B* **80**, 155305 (2009); *Phys. Rev. Lett.* **100**, 226803 (2008).
²⁹S. Simon *et al.*, 2011 Trieste Stig Lundquist Conference (smr2240), [<http://www.ictp.tv/eya/Lundqvist2011.php>]; S. H. Simon and C. von Keyserlingk (unpublished).

- ³⁰For a set of excitation energies E_n the visibility of interference scales as $\prod_n \tanh(E_n/2T)$. If the energy levels are equally spaced, the visibility (in the case of even number of QPs in the cavity) is reduced to below half at $\Delta E = T$, is 1/10 at twice this temperature, and drops drastically at higher temperatures.
- ³¹B. J. Overbosch and X.-G. Wen, e-print [arXiv:0706.4339](https://arxiv.org/abs/0706.4339); B. J. Overbosch, Ph.D. thesis, MIT, 2008.
- ³²Waheb Bishara and Chetan Nayak, *Phys. Rev. B* **80**, 155304 (2009).
- ³³David J. Clarke and Kirill Shtengel, *New J. Phys.* **13**, 055005 (2011).
- ³⁴A. Stern, B. Rosenow, R. Ilan, and B. I. Halperin, *Phys. Rev. B* **82**, 085321 (2010).
- ³⁵M. Stern, B. A. Piot, Y. Vardi, V. Umansky, P. Plochocka, D. K. Maude, and I. Bar-Joseph, *Phys. Rev. Lett.* **108**, 066810 (2012).
- ³⁶L. Tiemann, G. Gamez, N. Kumada, and K. Muraki, *Science* **335**, 828 (2012).
- ³⁷An unpolarized 331 state could also be proposed (see, for example, B. J. Overbosch and X.-G. Wen, e-print [arXiv:0804.2087](https://arxiv.org/abs/0804.2087)). However, there is no indication from any numerics or analytics that such a state is realized for any Hamiltonian.