Saddle point states in two-dimensional superconducting films biased near the depairing current

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The structure and energy of saddle point (SP) states in a two-dimensional (2D) superconducting film of finite width w with transport current I are found in the framework of the Ginzburg-Landau model. We show that very near the depairing current $I_{\rm dep}$, a SP state with a vortex does not exist; it transforms to a 2D nucleus state, which is a finite region with partially suppressed order parameter. It is also shown that for slightly lower currents the contribution of the vortex core energy is important for a SP state with a vortex and it cannot be neglected for $I \gtrsim 0.6 I_{\rm dep}$. It is demonstrated that in a film with local current concentration near a bend, the energy of the SP state may be much smaller than that in the straight film, and this favors the effect of fluctuations in such samples.

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I. INTRODUCTION

Many physical systems have several metastable states at fixed external parameters (temperature, magnetic field, etc.) which correspond to different local minima of their free energy. Examples are the different configurations of DNA molecules,[1] vortex "molecules" in mesoscopic superconductors,[2,3] and magnetic states of a lattice of magnetic nanocaps.[4] The metastable states are usually separated from each other by an energy barrier ΔF which can be overcome due to thermal activation. If the height of the energy barrier is much larger than the thermal energy $k_B T$, then such transitions are relatively rare events, and one may estimate their rate as $\sim\exp(-\Delta F/k_B T)$.[5] The standard procedure to calculate ΔF is to find the saddle point states which correspond to the local maxima of the free energy and find the energy difference between them and the metastable states.[1–3,5]

In superconductors with transport current less than the critical one, this idea can also be applied for calculation of the finite resistance appearing due to fluctuations. For one-dimensional (1D) superconducting wires, Langer and Ambegaokar[6] (LA) found in the framework of the Ginzburg-Landau (GL) model that in the saddle point state the superconducting order parameter $\psi = |\psi| e^{i\varphi}$ is partially suppressed in a finite region along the wire and the amplitude of the suppression depends on the current. If one starts from such a 1D nucleus state the time evolution of the order parameter will inevitably lead to phase slip[6] and to a finite resistance $R_{\rm wire}$. LA found a dependence of ΔF on both the current and temperature which with good accuracy[7] could be written as

$$\frac{\Delta F_{LA}}{F_0} = \frac{2\sqrt{2}}{3\pi}\frac{w}{\xi}\left(1 - \frac{I}{I_{\rm dep}}\right)^{5/4}, \tag{1}$$

where $F_0 = \Phi_0^2 d/16\pi^2\lambda^2$, Φ_0 is the magnetic flux quantum, w is the width and d is the thickness of the wire, respectively, λ is the London penetration depth, ξ is the coherence length, and $I_{\rm dep} = c\Phi_0 w d/12\sqrt{3}\pi^2\xi\lambda^2$ is the depairing current in the Ginzburg-Landau model. According to the general concept, $R_{\rm wire} = \nu \exp(-\Delta F_{LA}/k_B T)$, where the preexponential factor ν is calculated in Refs. 8 and 9.

In contrast to 1D wires, in thin ($d \ll \lambda$) 2D films several saddle point states exist at a given value of the current. Further, we discuss the case of a relatively narrow film with $\xi \ll w < \lambda^2/d$ (this condition ensures the uniformity of the current distribution over the width of a superconductor with transport current in the ground state). At the present time three types of saddle point states in such samples are distinguished: (i) the state with a single vortex,[10–15] (ii) the vortex-antivortex (VA) state,[14–17] and (iii) the LA state extended to the 2D case.[13–15] In Ref. 15 it was argued that the VA state has at least twice larger energy than the single-vortex state and the LA-like state is the most energetically unfavorable among the considered states at all currents $I \leqslant I_{\rm dep}$. However, in Ref. 15, the current dependence of ΔF_{LA} [see Eq. (1)] was not taken into account. If one takes the result for the energy of the single-vortex (V) state near the depairing current $1 - I/I_{\rm dep} \ll 1$ (found in the London model[13–15]),

$$\frac{\Delta F_V}{F_0} \simeq 1 - \frac{I}{I_{\rm dep}}, \tag{2}$$

and compares it with Eq. (1), then it is easy to see that even for wide films $w \gg \xi$ there is a finite region of currents $1 - (3\pi\xi/2\sqrt{2}w)^4 < I/I_{\rm dep} \leqslant 1$ where $\Delta F_{LA} < \Delta F_V$. But such a quantitative comparison is not correct because Eq. (2) does not contain the energy of the vortex core ($E_{\rm core} \sim 0.38 F_0$ for a vortex located far from the edges[18]) and, as we show below, it seriously underestimates ΔF_V at $I \gtrsim 0.6 I_{\rm dep}$.

The effect of the vortex core was taken into account in recent work[13] using the GL model. But the authors focused on the region of small currents $I \ll I_{\rm dep}$ (where the contribution of $E_{\rm core}$ to ΔF_V is relatively small) and they concluded that for films with $w > w_c \simeq 4.4\xi$ the energy of the single-vortex state is lower than the energy of the LA state. Below we show that a vortex saddle point (SP) state does not exist at $I \sim I_{\rm dep}$; it transforms to a *vortex-free* 2D nucleus (which is a finite region with partially suppressed order parameter) located near the edge of the film. We confirm our estimation above that very near $I_{\rm dep}$ even for wide films $w \gg w_c$ the saddle point state of Langer and Ambegaokar has the lowest energy among the SP states. We also study the practically important case of a film with one 180° bend, which models the part of the superconducting meander used in superconducting single-photon detectors.[19–21] We show that due to current concentration near the bend the jump to the saddle point state needs much less energy at $I \to I_c$ in comparison with that in a 2D film with uniform current distribution, which promotes the effects of fluctuations in such samples. This result is rather general and could be applied to any superconducting system with a local current concentration.

II. METHOD

To find the state which is a solution of the stationary Ginzburg-Landau equations but is unstable (a saddle point state), we use the following numerical procedure. When we search for a SP state with a vortex we first put (as an initial condition) the phase distribution corresponding to the vortex located at the point (n,m) of the discrete grid and additionally fix the phase difference between adjacent points $\varphi(n,m+1)-\varphi(n,m-1)=\pi$ [which, in some respects, pins the vortex at the point (n,m)] at any time step. We find the solution of the stationary GL equations by using the relaxation method (by adding the time derivative $\partial\psi/\partial t$ in the GL equation for the order parameter and waiting until it goes to zero). By variation of the current (as an external parameter) it is possible to find a stationary state when the vortex does not move from the point (n,m). At low currents there are several points where the vortex position could be fixed by this method for given I. By our definition the one with the highest energy corresponds to the SP state (it corresponds to the local maximum of the Gibbs energy as a function of the vortex position in the London approach[10–12,14,15]). After reaching this state the energy difference can be found using the expression

$$\Delta F = F_{\text{saddle}} - F_{\text{ground}} - \frac{\hbar}{2e} I \Delta\varphi, \tag{3}$$

where $\Delta\varphi$ is an additional phase difference between the ends of the film which appears in the SP state in comparison with the ground state, and F_{saddle} and F_{ground} are the Ginzburg-Landau free energies of the saddle point and ground states, respectively.

To find a vortex-free saddle point state, we fix the magnitude of the order parameter $|\psi|>0$ at one point at the edge and allow ψ to vary at all other points. Than we increase the current up to the moment when the state becomes nonstationary. By our definition we find the vortex-free SP state corresponding to the given value of the current. We checked that if we start from this state and let ψ vary everywhere in the film, the vortex is nucleated at the point where we initially fix $|\psi|$ and passes across the film. This finding is an extension to the 2D case of the main idea of LA that if one starts from a SP state with finite $|\psi|$ everywhere in the sample the time evolution of the order parameter will inevitably lead to phase slip and a voltage pulse.

The proposed method is much simpler (from the point of view of the numerical procedure) than the methods used in Refs. 22 and 23 for finding SP states in a mesoscopic superconducting disk in a magnetic field or 2D film with a transport current.[13,17] A similar procedure to fix the vortex position was used, for example, in Ref. 24. Moreover, our method could be easily applied to 2D samples of arbitrary geometry (triangles, disks, stars, etc.) and to the 3D case to find the vortex-free saddle point states. We checked that the method gives the same results for the 1D saddle point state (for both the spatial dependence of the order parameter and the excess energy ΔF) as analytically found in Ref. 6. The validity of the proposed method is also supported by our results for 2D films because they coincide with results found in the London model at low currents.

In the numerical calculations the step of the rectangular grid was $\delta x=\delta y=\xi/4$ and the width of the film varied from 4.5ξ up to 30ξ. The length of the film was chosen as $L=4w$ (which is long enough to allow neglect of the effect of finite length). The boundary conditions, written here in dimensionless units (for units see, for example, Ref. 23), $\psi^*\nabla\psi|_{y=\pm L/2}=iI/wd$ and $\nabla\psi|_{x=\pm w/2}=0$.

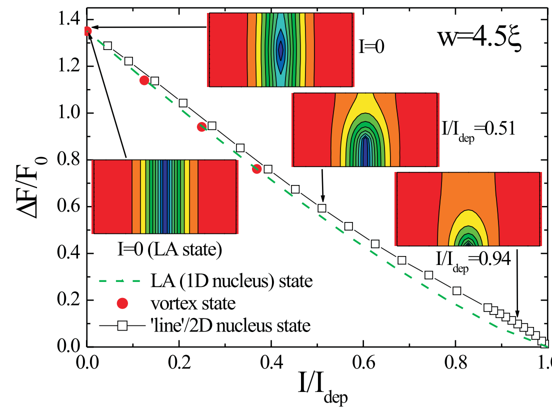

FIG. 1. (Color online) Energy of saddle point states of three kinds: LA (green dashed curve), vortex ("line") [red circles (widely spaced empty squares)], and 2D nucleus (closely spaced empty squares) in the film with $w=4.5\xi$. In the insets we present contour plots of $|\psi|$ at different currents and different kinds of SP states.

III. RESULTS

In Fig. 1 the dependence of ΔF on the current for different SP states is presented for the film with $w=4.5\xi$. We should note that a vortex in such a narrow film has a strongly deformed core (see the upper inset in Fig. 1) and it resembles more a Josephson than an Abrikosov vortex.[25] In wider films (see Figs. 2 and 3), deformation of a vortex core occurs when the vortex is near the edge at the distance $\Delta x \lesssim 2\xi$ (see the insets in Figs. 2 and 3). A similar result was found for a vortex placed near an artificial defect (see Fig. 2 in Ref. 24) and near the edge of a superconducting disk (see Fig. 3 in Ref. 22) or film (see Fig. 2 in Ref. 13).

Unfortunately our numerical method does not allow us to find a SP state with a vortex when $\Delta x<1.5\xi$ (the last red circle at the largest current in Figs. 1–3 corresponds to $\Delta x=1.5\xi$), because we were not able to find a stationary solution of the Ginzburg-Landau equation with the additional condition $\varphi(n,m+1)-\varphi(n,m-1)=\pi$ (this effective pinning "force" becomes insufficiently strong to pin the vortex very near the edge). But we notice that if we fix $|\psi|=0$ along a line of finite length near the edge (see the inset in Fig. 1 where the contour plot of $|\psi|$ at $I/I_{\text{dep}}=0.51$ is shown) and find the stationary solution of the GL equations with this additional condition, then the excess energy ΔF of such a "line" state (see the widely spaced empty squares in Figs. 1–3) is close to the energy of the vortex state (when the vortex is relatively close to the edge—see Figs. 2 and 3). Because of that one may approximate the energy of the vortex state when $\Delta x<1.5\xi$ by the energy of the "line" state with length $l<2\xi$.

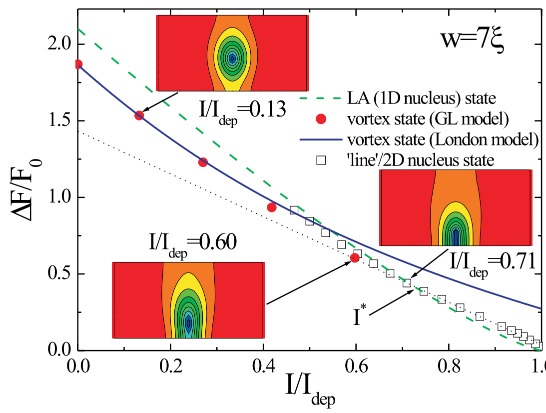

FIG. 2. (Color online) Energy of saddle point states of three kinds: LA (green dashed curve), vortex ("line") [red circles (widely spaced empty squares)], and 2D nucleus (closely spaced empty squares) in the film with $w = 7\xi$. The blue curve corresponds to Eq. (4). The black dotted line is the dependence $\Delta F/F_0 = 1.43(1 - I/1.026 I_{\rm dep})$ which fits well (deviation less than 2%) our numerical results at $0.6 \leqslant I/I_{\rm dep} \leqslant 0.97$. At $I > I^* \simeq 0.73 I_{\rm dep}$ the LA state has the lowest energy.

At currents $I \sim I_{\rm dep}$ the vortex ("line") state transforms to the vortex-free SP state (closely spaced empty squares in Figs. 1–3) when the phase circulation along any closed contour in the film is equal to zero and $|\psi| > 0$ everywhere in the film. To find it we fix the amplitude of the order parameter $|\psi|$ at one point at the edge. Because of the proximity effect and $I \sim I_{\rm dep}$, this leads to suppression of $|\psi|$ in a relatively large region around this point (see the inset in Fig. 1 at $I/I_{\rm dep} = 0.94$). Further, we call it a 2D nucleus state to distinguish it from the 1D nucleus state of LA (compare the insets in Fig. 1 at $I = 0$ and at $I/I_{\rm dep} = 0.94$).

In Figs. 2 and 3 we also plot the dependence of ΔF on the energy of the vortex SP state found in the London limit[13–15]

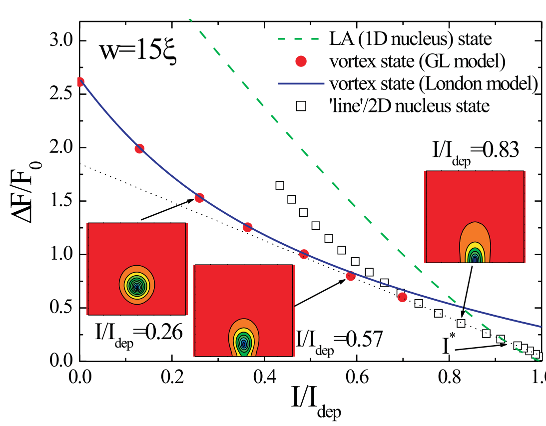

FIG. 3. (Color online) Energy of saddle point states of three kinds: LA (green dashed curve), vortex ("line") [red circles (widely spaced empty squares)], and 2D nucleus (closely spaced empty squares near depairing current) in the film with $w = 15\xi$. The blue curve corresponds to Eq. (4). The black dotted line is the dependence $\Delta F/F_0 = 1.85(1 - I/1.028 I_{\rm dep})$ which fits well (deviation less than 2%) our numerical results at $0.6 \leqslant I/I_{\rm dep} \leqslant 0.97$. At $I > I^* \simeq 0.92 I_{\rm dep}$ the LA state has the lowest energy.

TABLE I. Values of coefficients in the fitting expressions for the energy of vortex and 2D nucleus saddle point states [Eq. (5)] and vortex core energy when the vortex is in the center of the film [the coefficient ϵ in Eq. (4)].

w/ξ	A	B	C	n	ϵ
7	1.43	1.026	0.89	0.7	0.37
10	1.67	1.026	1.02	0.7	0.38
15	1.85	1.028	0.88	0.6	0.38
30	1.88	1.034	0.68	0.5	0.38

(solid blue curve)

$$\frac{\Delta F_V}{F_0} = -\frac{1}{2}\ln\left(1+\frac{I^2}{\alpha^2 I_{\rm dep}^2}\right) - \frac{I}{\alpha I_{\rm dep}}\tan^{-1}\left[\frac{\alpha I_{\rm dep}}{I}\right] + \epsilon + \ln\left(\frac{2w}{\pi\xi}\right), \qquad (4)$$

where $\alpha = 3\sqrt{3}\pi\xi/4w$ and we add the energy of the vortex core $E_{\rm core} = \epsilon F_0$. The numerical coefficient ϵ is found from comparison of Eq. (4) with the numerical result at $I = 0$ and it is presented in Table I for different widths. Notice the good agreement between the GL and London models at $I \lesssim 0.6 I_{\rm dep}$. At larger currents the vortex is located at the distance $\Delta x \lesssim 2\xi$ from the edge, and one has to take into account deformation of the core, which provides the dependence $\epsilon(I)$. This leads to a large discrepancy between the London (with $\epsilon = $ const) and GL models at $I \gtrsim 0.6 I_{\rm dep}$ (see Figs. 2 and 3). Moreover, at $I \sim I_{\rm dep}$ the vortex state transforms to the 2D nucleus state, which cannot be found in the London limit.

Our numerical results at $I/I_{\rm dep} \gtrsim 0.6$ could be fitted (examples of the fitting are presented in Figs. 2 and 3 and the inset in Fig. 5) by the following functions:

$$\frac{\Delta F}{F_0} \simeq \begin{cases} A(1 - I/B I_{\rm dep}), & 0.6 \leqslant I/I_{\rm dep} \leqslant 0.97, \\ C(1 - I/I_{\rm dep})^n, & 0.95 \leqslant I/I_{\rm dep} \leqslant 1. \end{cases} \qquad (5)$$

The coefficients A, B, and C and power n for different widths are listed in Table I. Note that the coefficient A is almost twice larger than the result which follows from the London model [see Eq. (2)]. It reflects the contribution of $E_{\rm core}(I)$ to ΔF which for $I \gtrsim 0.6 I_{\rm dep}$ cannot be neglected, and $E_{\rm core}$ gradually decreases with increasing I.

At $I/I_{\rm dep} \sim 1$ the power $n < 1$ (see Table I) in Eq. (5) tells one that there is a finite (but rather narrow for wide films with $w \gg \xi$) interval of currents $I^*(w) < I < I_{\rm dep}$ where the 1D nucleus (LA) state has the lowest energy (see Figs. 2 and 3). This result, counterintuitive at first sight, is explained by the presence of the last term in the right-hand side of Eq. (3). Although in the LA state the order parameter is suppressed over the whole width (see the inset at $I = 0$ in Fig. 1) and it costs more condensation energy than in the vortex or 2D nucleus state, the phase difference $\Delta\varphi$ is much larger in the LA state than in other SP states at $I \sim I_{\rm dep}$ and this causes the above result.

Previously, we considered only the single-vortex state and a 2D nucleus which is located near the edge of the film. In Fig. 4 we demonstrate that the energy of a 2D nucleus located in the center of the film is larger than the energy of the edge 2D nucleus. The vortex-antivortex state into which the 2D nucleus

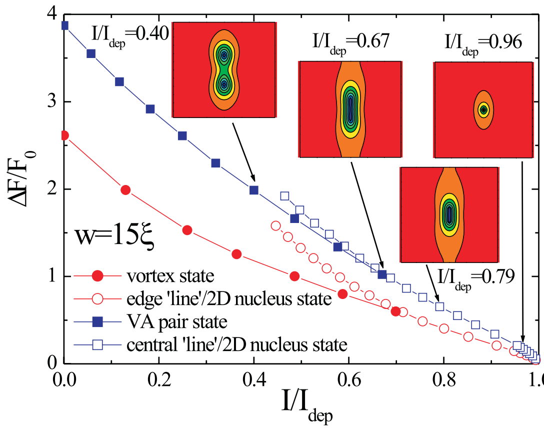

FIG. 4. (Color online) Energy of saddle point states of different kinds: single vortex (red circles), vortex-antivortex pair (blue squares), and "line" (2D) nucleus located at the edge [empty widely (closely) spaced circles] and in the center [empty widely (closely) spaced squares] of the film.

state is transformed at lower currents (see the solid and widely spaced empty squares in Fig. 4) has energy larger than the single-vortex state. A similar result was found in the London limit where the difference reaches a factor of 2 between ΔF_V and ΔF_{VA}.[15,17]

We also find the saddle point states in a 2D superconducting film with a 180° bend—see the left inset in Fig. 5. In our simulations we choose the width of the film $w = 10\xi$, the length of the sample in the bend region $L = 2w$, and two widths of the slit: $w_{\rm slit} = 2.5\xi$ (shown in the left inset in Fig. 5) and $w_{\rm slit} = 10\xi$. In Fig. 5 we present our results for the energy of single-vortex and edge 2D nucleus states. Note that the current in Fig. 5 is normalized to the critical current of the sample and not to $I_{\rm dep}$ as in Figs. 1–4 ($I_c = 0.85 I_{\rm dep}$ for the film with $w_{\rm slit} = 2.5\xi$, $I_c = 0.91 I_{\rm dep}$ for the film with $w_{\rm slit} = 10\xi$, and $I_c = I_{\rm dep}$ for the straight film).

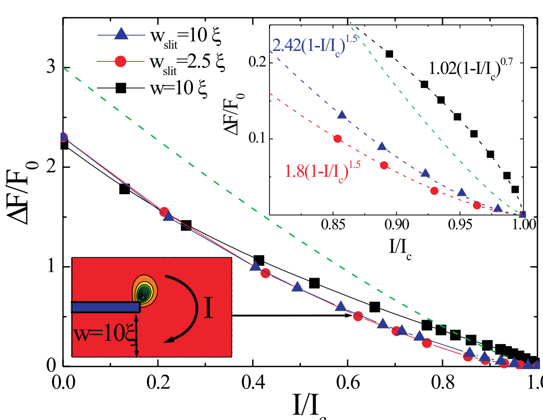

FIG. 5. (Color online) Energy of saddle point states (vortex and 2D nucleus) for the film with a bend and slits of different widths $w_{\rm slit} = 2.5\xi$ (red circles) and $w_{\rm slit} = 10\xi$ (blue triangles) and the film without a bend with $w = 10\xi$ (black squares). In the right inset we present a zoom at $I \sim I_c$ which shows the energy of the 2D nucleus SP states with the dashed curves as a fitting of the numerical results. The green dashed curve shows the energy of the LA state for a straight film with $w = 10\xi$.

We want to stress here that at $I \sim I_c$ the energy of SP states is considerably lower in the film with a bend than in one without it (compare the results presented in the right inset in Fig. 5) taken at *the same ratio* I/I_c. Also, the stronger the current concentration near the bend (which is manifested in a lower value of I_c for smaller $w_{\rm slit}$), the smaller ΔF. We expect that the smallest ΔF could be reached in the case of an infinitesimally narrow and long crack near the edge of the film,[26,27] which provides the maximal current concentration and maximal suppression of I_c. We explain this effect by the partial suppression of the order parameter (on the scale of about several ξ) in the region with the strongest current concentration even in the ground state. As a result it takes less energy to jump to the SP state from the ground state with already locally suppressed $|\psi|$. The proof of this idea also comes from our results at low currents where ΔF differs a little for straight films and films with a bend (see Fig. 5). At low currents the suppression of $|\psi|$ near the bend is weak and it only slightly influences ΔF.

A decrease of ΔF in a 1D wire with constriction was also found in Ref. 28. But the authors of Ref. 28 did not present ΔF as a function of I/I_c and it is not clear whether their result is connected with reduction of I_c (which gives trivially $\Delta F \to 0$ at current $I = I_c < I_{\rm dep}$ and $\Delta F < \Delta F_{LA}$ at $I < I_c$) or with a change in the power n in the asymptotic $\Delta F \sim (1 - I/I_c)^n$ as in our case (see the right inset in Fig. 5).

IV. CONCLUSION

It is found that in a 2D superconducting film with spatially uniform current distribution in the ground state the lowest-energy saddle point state at $I \sim I_{\rm dep}$ corresponds to the state of Langer and Ambegaokar found for a 1D wire. Only at $I < I^*$ [where $I^*(w) < I_{\rm dep}$ even for wide films $w \gg \xi$] does the lowest-energy SP state correspond either to the edge 2D nucleus state or to a state with a vortex located next to the edge and having a strongly modified core. At currents $I \lesssim 0.6 I_{\rm dep}$ (for films with $w \gtrsim 7\xi$) the lowest-energy saddle point state corresponds to a single vortex with an ordinary core, and the results of the London model are recovered. We demonstrate that in a film with current concentration, which leads to local spatial variation of the order parameter in the ground state, the energy of the 2D nucleus SP state may be much lower than the energy of any saddle point states in a film with uniform current distribution taken at the same ratio $I/I_c \sim 1$.

The last result has an important practical consequence. It shows that if in the sample there are places with strong current concentration (bends, geometrical defects of the edge, and so on) this favors the effect of fluctuations near I_c because of the much lower energy barrier, in comparison with a straight film without defects. For example, if $k_B T = 0.1 F_0$ the switching of the straight film with $w = 10\xi$ to the resistive state roughly occurs at $I \simeq 0.96 I_c = 0.96 I_{\rm dep}$ (at this current $\Delta F = 0.1 F_0$—see the right inset in Fig. 5). But for the film with a bend and $w_{\rm slit} = 2.5\xi$, the energy barrier $\Delta F = 0.1 F_0$ is reached at $I \simeq 0.85 I_c \simeq 0.72 I_{\rm dep}$ and transition to a resistive state occurs at a much lower current than one might expect from the results for a straight film.

We believe that the results found are valid not only near T_c (where the Ginzburg-Landau model is quantitatively valid) but at lower temperatures too. Indeed, in recent work[29] the asymptotic $\Delta F \sim (1 - I/I_{\mathrm{dep}})^{5/4}$ for $I \to I_{\mathrm{dep}}$ was confirmed for a LA-like state in a "dirty" 1D superconducting wire at any temperature. In a "pure" 1D wire only 15% difference from the LA result at $T \to 0$ was noticed in Ref. 30 in the limit $I \to 0$.[31]

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[31] In Ref. 30, the authors did not take into account the additional term in the free energy, which describes the work done by the external source of current $\sim I\Delta\varphi$ (see the discussion in Ref. 29), and they find finite ΔF even at $I \to I_{\mathrm{dep}}$ (see the inset in Fig. 1 in Ref. 30).