PHYSICAL REVIEW B **85**, 165144 (2012)

Anyonic Bloch oscillations

Stefano Longhi and Giuseppe Della Valle
Dipartimento di Fisica, Politecnico di Milano, and Istituto di Fotonica e Nanotecnologie del Consiglio Nazionale delle Ricerche, Piazza L. da Vinci 32, I-20133 Milano, Italy

(Received 28 February 2012; revised manuscript received 9 April 2012; published 26 April 2012)

The onset of Bloch oscillations (BOs) for two correlated anyons hopping on a one-dimensional lattice is theoretically investigated in the framework of the anyon-Hubbard model. It is shown that, even in the absence of on-site particle interaction, BOs are degraded for a nonvanishing statistical phase exchange owing to the nonlocal quasiparticle nature of anyons. A remarkable exception is found for pseudofermions, i.e., particles that, although they are bosons on site, behave as fermions off site. In this case, if the ratio of forcing to the hopping rate is smaller than ∼0.5, in the absence of on-site interaction long-lived BOs are observed at a frequency which is half the BO frequency of single particles. This is notably distinct from results for previously investigated BOs of two strongly correlated bosons or fermions, in which particle correlation leads to a doubling of the BO frequency.

DOI: 10.1103/PhysRevB.85.165144 PACS number(s): 71.10.Pm, 71.10.Fd, 05.30.Pr, 67.85.−d

I. INTRODUCTION

Bloch oscillations (BOs) represent one of the most striking predictions of the semiclassical theory of electronic transport in crystals; they were predicted by Bloch and Zener in two seminal papers more than 80 years ago.[1] Owing to dephasing effects, BOs have never been observed in natural crystals, and solely with the advent of semiconductor superlattices has their existence been finally demonstrated as terahertz radiation emitted from coherently oscillating electrons.[2] More recently, the analogs of electronic BOs have been predicted and experimentally observed for ultracold atoms[3] and Bose-Einstein condensates[4] in tilted optical lattices,[5] for optical waves in arrays of evanescently coupled optical waveguides[6] and photonic superlattices,[7] and for sound waves in acoustic superlattices.[8] The inclusion of lattice disorder, nonlinearities, inhomogeneities, or defects has generally a detrimental effect on BOs.[9–11] In particular, it is known that BOs may be affected rather dramatically by particle interactions, for both bosons and fermions. The Hubbard and Bose-Hubbard models have been recently introduced to highlight the role of on-site particle interaction in BOs beyond mean-field models.[11–16] As correlation is generally responsible for decoherence of BOs,[11,12] interesting novel phenomena have been predicted for BOs of *few* interacting particles in the strong-interaction regime,[13–15] such as the frequency doubling of BOs of two correlated particles[13,14] and, more generally, fractional BOs for N-bound-particle states.[15]

In addition to bosons and fermions, quasiparticles with intermediate statistics interpolating from Bose statistics to Fermi statistics, so-called anyons, were suggested more than 30 years ago.[17,18] Anyons can be regarded as topological states of matter whose quasiparticle excitations obey fractional statistics. For two anyons, the wave function acquires a fractional phase $\exp(i\theta)$ under particle exchange, where $0 < \theta < \pi$ is the statistical exchange phase; bosons and fermions are retrieved for $\theta = 0$ and $\theta = \pi$, respectively. Although a clean experimental demonstration of particles with fractional statistics is missing, their existence has been generally associated with the fractional quantum Hall effect.[19] Recently, there have been a few atom-optics proposals of Abelian anyons hopping on a one-dimensional (1D) lattice.[20,21] In particular, in Ref. 21 it was shown that anyons moving on a 1D lattice can be realized by ordinary bosons on a lattice with conditional hopping amplitudes. We are interested here in BOs of anyonic particles, as well as the impact of the statistical phase θ on the BO motion.

In this work we study BOs for two anyons hopping on a 1D lattice in the framework of the anyon-Hubbard model.[21] The main result of our analysis is that a nonvanishing statistical phase θ degrades BOs even in the absence of on-site particle interaction. This behavior occurs because anyons in 1D lattices can be effectively regarded as nonlocal quasiparticles in which many-body effects occur even without on-site particle interactions.[21] A remarkable exception is found for pseudofermions, i.e., particles that, although they are bosons on site, are fermions off site. In this case, if the ratio of forcing to the hopping rate is smaller than ∼0.5 and in the absence of on-site interaction, long-lived BOs do appear, but at a frequency which is half the BO frequency of single (noninteracting) particles. This is notably distinct from results for previously investigated BOs of strongly correlated bosons, in which the frequency of BOs is an integer multiple of the single-particle BO frequency.[13–15]

II. THE ANYON-HUBBARD MODEL

The starting point of our analysis is provided by the anyon-Hubbard model,[21] which describes the hopping dynamics of correlated anyons on a 1D lattice, extended to include the external forcing. The Hamiltonian of the model reads explicitly

$$\hat{H} = -J\sum_l(\hat{a}_l^\dagger\hat{a}_{l+1} + \hat{a}_{l+1}^\dagger\hat{a}_l) + \frac{U}{2}\sum_l \hat{n}_l(\hat{n}_l - 1) + F\sum_l l\hat{n}_l, \tag{1}$$

where J is the tunneling rate between adjacent sites, U is the on-site interaction energy, the operators $\hat{a}_l^\dagger$,$\hat{a}_l$ create or annihilate an anyon at site l, and $\hat{n}_l = \hat{a}_l^\dagger\hat{a}_l$ is the particle number operator at site l. As in Ref. 21, we assume that the operators $\hat{a}_l^\dagger$,$\hat{a}_l$ satisfy the generalized commutation relations

$$\hat{a}_l\hat{a}_k^\dagger = \delta_{l,k} + \exp[-i\theta\epsilon(l-k)]\hat{a}_k^\dagger\hat{a}_l, \tag{2}$$

$$\hat{a}_l \hat{a}_k = \exp[i\theta\epsilon(l-k)]\hat{a}_k \hat{a}_l, \tag{3}$$

where θ is the statistical exchange phase and the function $\epsilon(l-k)$ is defined as

$$\epsilon(l-k) = \begin{cases} 1, & l > k, \\ 0, & l = k, \\ -1, & l < k. \end{cases} \tag{4}$$

Note that, since $\epsilon(l-k) = 0$ for $l = k$, two particles on the same site behave as ordinary bosons. Moreover, anyons with statistics $\theta = \pi$ are pseudofermions, i.e., they behave as ordinary fermions when they occupy different lattice sites.[21]

As in Refs. 13 and 14, here we focus our analysis to study the motion of two correlated particles, and we wish to highlight the impact of the statistical phase exchange θ on the BO dynamics. With this aim, let us indicate by $c_{n,m}(t)$ the amplitude probability of finding one anyon at site n and the other one at site m of the lattice, i.e., let us expand the state vector $|\psi(t)\rangle$ of the system in Fock space as

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \sum_{n,m} c_{n,m}(t)\hat{a}_n^\dagger \hat{a}_m^\dagger |0\rangle. \tag{5}$$

The evolution equations for the amplitudes $c_{n,m}$, as obtained from the Schrödinger equation $i\partial_t|\psi\rangle = \hat{H}|\psi\rangle$ with $\hbar = 1$, read explicitly

$$i\frac{dc_{n,m}}{dt} = -J[c_{n+1,m} + c_{n-1,m} + c_{n,m-1}\exp(-i\varphi_{n,m-1}) + c_{n,m+1}\exp(i\varphi_{n,m})] + [U\delta_{n,m} + F(n+m)]c_{n,m}, \tag{6}$$

where $\varphi_{n,m}$ is given by

$$\varphi_{n,m} = \begin{cases} -\theta, & n = m, m+1, \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

Equation (6) shows that the dynamics of two anyons on a one-dimensional lattice is equivalent to the dynamics of a single particle hopping on a two-dimensional lattice with *defects* on the diagonal $n = m$. The defects arise not only from the on-site interaction energy U, but also from a nonvanishing value of the statistical phase θ, which makes the hopping rates phase sensitive at $m = n \pm 1$. Hence two noninteracting anyons with $\theta \neq 0$ are expected to show correlated dynamics, in spite of the absence of particle interaction. This remarkable property of anyons can be explained after observing that they can be viewed as nonlocal quasiparticles, which thus show many-body effects even in the absence of on-site particle interaction. In fact, by a fractional version of the Jordan-Wigner transformation the anyon-Hubbard model (1) can be exactly mapped into a Bose-Hubbard model with occupation-dependent hopping amplitudes.[21] This feature indicates that anyonic dynamics can be realized using bosons in optical lattices. An experimental setup to create anyons in one-dimensional lattices with fully tunable exchange statistics has been proposed in Ref. 21. As discussed in Ref. 21, anyons are created by bosons in a driven optical lattice with effective occupation-dependent hopping amplitudes. The latter are realized by assisted Raman tunneling following a scheme originally proposed by Jaksch and Zoller to create Peierls phases in optical lattices[22] (see also Ref. 23). The statistical phase θ can be controlled *in situ* by modifying the relative phase of the external driving fields.

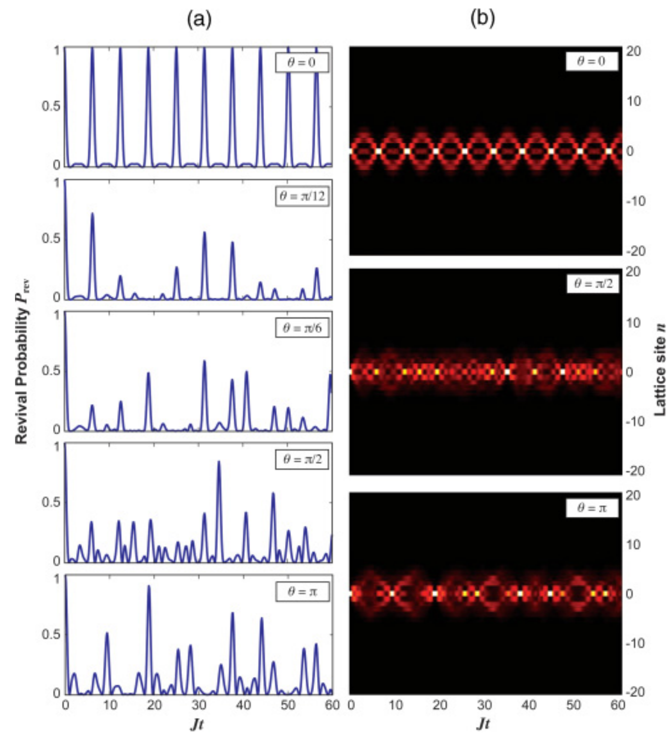

FIG. 1. (Color online) Evolution of (a) the revival probability and (b) the probability density function for two correlated anyons undergoing BOs on a 1D lattice in the absence of on-site interaction for $F/J = 1$ and for increasing values of the statistical phase exchange θ ($\theta = 0$ corresponds to noninteracting bosons, whereas $\theta = \pi$ corresponds to pseudofermions). Note that for $\theta = 0$ BOs with period $T_B = 2\pi/F$ are observed, whereas for $\theta \neq 0$ the BOs are degraded.

III. THE NONINTERACTING LIMIT

In this section we will focus our analysis to study BOs of a couple of anyons in the case $U = 0$, which at best highlights the distinct features of anyons as nonlocal quasiparticles showing many-body effects even in the absence of on-site interaction. The onset of BOs has been investigated by direct numerical analysis of the coupled equations (6) assuming as an initial condition $c_{n,m}(0) = \delta_{n,0}\delta_{m,0}$, corresponding to the two particles initially being localized at the same lattice site $l = 0$. As simple dynamical variables capable of capturing the onset of BOs, we assume here the revival probability $P_{\text{rev}}(t) = |\langle\psi(0)|\psi(t)\rangle|^2$ and the probability density function (PDF) $P_k(t) = (1/2)\langle\psi(t)|\hat{n}_k|\psi(t)\rangle$ $(k = 0,\pm1,\pm2,\ldots)$. A sufficiently wide lattice (typically formed by $N = 40$ or 50 sites) has been assumed to avoid boundary effects. For $\theta = 0$, i.e., for two noninteracting bosons, it is known that BOs manifest as a periodic breathing dynamics of the PDF with period $T_B = 2\pi/\omega_B = 2\pi/F$, as shown as an example in the upper panels of Figs. 1(b) and 2(b). For a nonvanishing value of θ, the BOs turn out to be generally degraded. This is shown, as an example, in Figs. 1 and 2, which depict typical evolutions of the PDF and revival probability for $F/J = 1$ and $F/J = 0.3$, respectively. Note that, for a nonvanishing

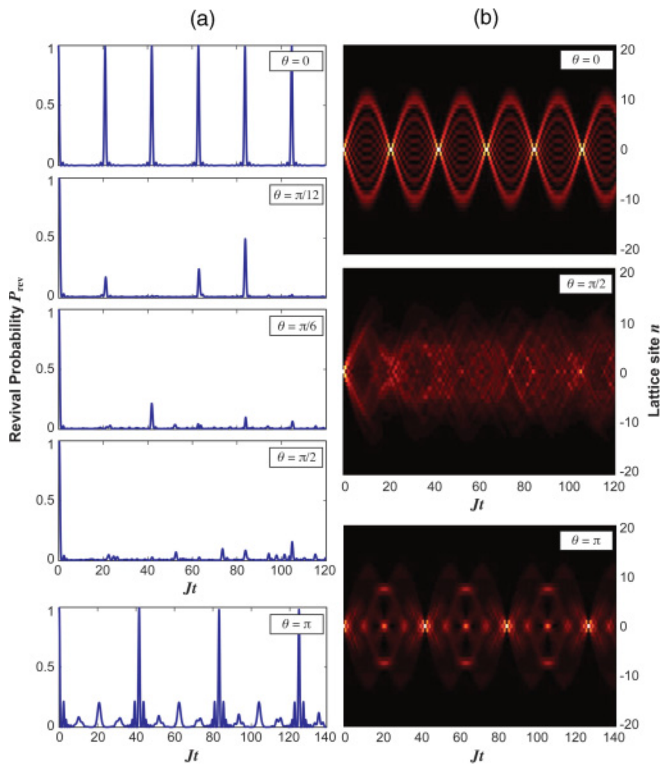

FIG. 2. (Color online) As Fig. 1, but for $F/J=0.3$. Note that in this case for pseudofermions ($\theta=\pi$) long-lived BOs, with a period $2T_B=4\pi/F$, are observed.

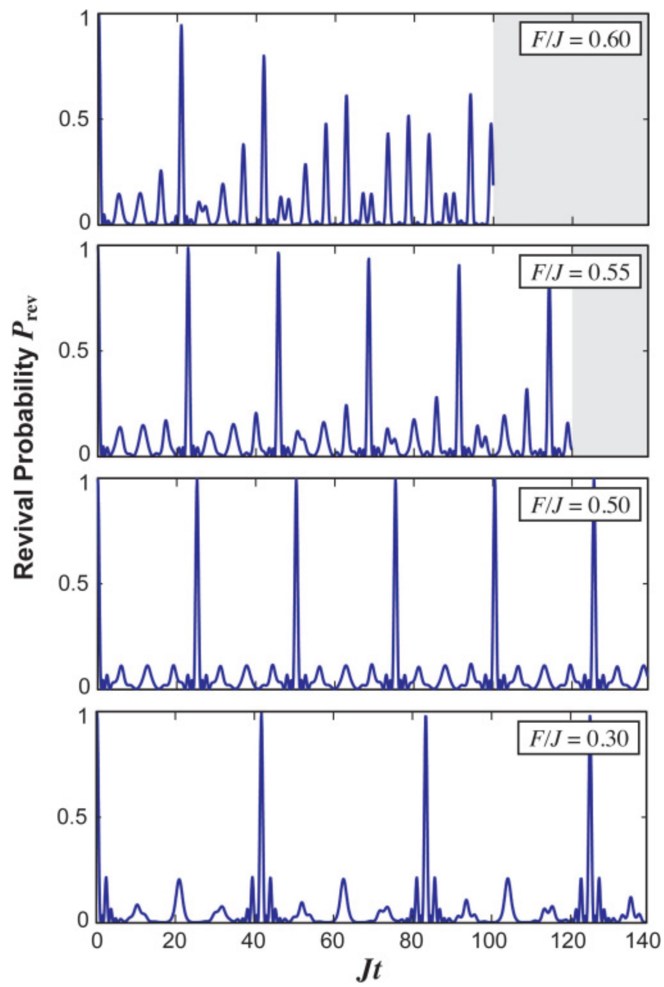

FIG. 3. (Color online) Evolution of the revival probability for $U=0$, $\theta=\pi$ (pseudofermions), and for decreasing values of F/J. Note that for $F/J <\sim 0.5$ long-lived BOs with a period $2T_B$ can be observed.

value of θ, the evolution of $P_{\rm rev}$ and P_k is not periodic. A noticeable feature occurs for $\theta=\pi$, i.e., for pseudofermions, for which long-lived BOs do appear for values F/J smaller than ~ 0.5, but with a period that is *twice* the single-particle value T_B. This is clearly shown in Fig. 3. To explain such behavior and to clarify the roles of θ and F/J in the onset of BOs, we have numerically computed the eigenvalues and corresponding eigenvectors for Eqs. (6), i.e., the energy spectrum of the two-particle anyon-Hubbard Hamiltonian (1). With this aim, we assumed a finite lattice size made of N sites (typically we used $N=40$ or 50) and computed the N^2 eigenvalues of the matrix associated with Eqs. (6); to avoid finite-size effects, we kept only the eigenvalues whose eigenvectors showed the largest overlap with the center of the lattice. Figure 4 shows the behavior of the numerically computed energy spectrum at increasing values of θ, from noninteracting bosons ($\theta=0$) to pseudofermions ($\theta=\pi$), and for two values of F/J, whereas in Fig. 5 the energy spectrum is depicted as a function of F/J for $\theta=\pi/2$ and $\theta=\pi$. At $\theta=0$, the spectrum is given by the well-known Wannier-Stark ladder of two noninteracting bosons, $E_{n,m}=F(n+m)$ ($n,m=0,\pm1,\pm2,\ldots$), with degenerate eigenvalues equally spaced by F. For a nonvanishing value of the statistical phase θ, some degeneracy of the spectrum is lifted, and the eigenvalues are not equidistant any more. However, Figs. 4 and 5 clearly show that for pseudofermions and for F/J smaller than ~ 0.5 the spectrum turns out to be nearly equally spaced, but with spacing $F/2$ rather than F. This explains the doubling of the BO period observed for pseudofermions. It should be noted that the halving of BO frequency is strictly a feature of the correlated dynamics of the two pseudofermions in the absence of on-site particle interaction. This kind of correlation is related to the nonlocal nature of anyons,[21] and its impact on BOs is very different from the correlation of bosons induced by strong on-site interaction. In this case, in fact, for two particles interaction would lead to a doubling of the BO frequency, rather than to a frequency halving.[14,15]

IV. THE STRONG-INTERACTION LIMIT

In the previous section we considered the hopping dynamics of two anyons on a lattice in the absence of on-site interaction and showed that correlations are still observed in the BO dynamics. In particular, we showed that, while nonlocality of anyons generally causes the BO motion to degrade, for pseudofermions a halving of the BO frequency can be observed. On the other hand, for two bosons initially occupying the same site it is known that correlations induced by on-site particle interaction lead to a doubling of the BO frequency, rather than halving.[14,15] This result follows from the circumstance that two strongly interacting bosons initially occupying the same site

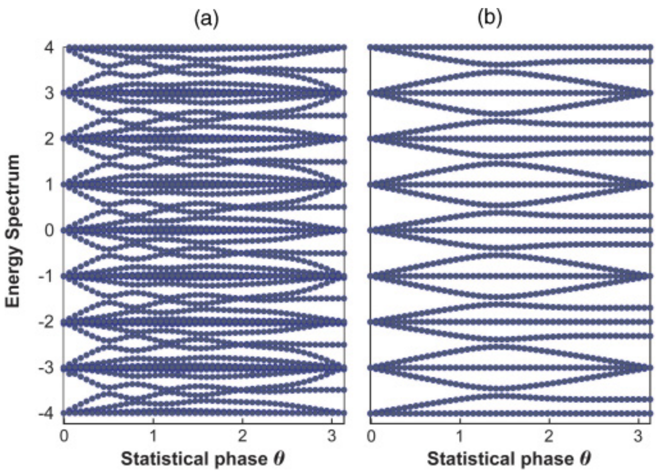

FIG. 4. (Color online) Numerically computed energy spectrum of the two-particle anyon-Hubbard Hamiltonian for $U = 0$ versus the statistical phase exchange θ for (a) $F/J = 0.3$, and (b) $F/J = 1$. The energies are given in units of F. For the sake of clearness only the energies in the range $(-4F, 4F)$ are plotted.

form a bound pair and tunnel together on the lattice.[24] It is thus worth investigating the dynamics of two correlated anyons, initially occupying the same lattice site, when the on-site interaction is switched on. As an example, in Fig. 6 we show the numerically computed evolution of the revival probability and PDF for increasing values of U/J and for $F/J = 0.3$ in the case of pseudofermions (i.e., $\theta = \pi$). Note that, as U/J is increased, a transition from an oscillatory behavior at frequency $\omega_B/2$ to an oscillatory motion at frequency $2\omega_B$ is observed. More generally, for an arbitrary statistical phase θ one observes BOs at frequency $2\omega_B$ in the strong-interaction limit $|U|/J \gg 1$. This behavior can be explained by an asymptotic analysis of Eqs. (6) in the limit $J/U \to 0$. Let us assume as an initial condition $c_{n,m}(0) = A_n(0)\delta_{n,m}$, where $A_n(0)$ is the amplitude probability to find both anyons at site n of the lattice, and let us search for a solution to Eqs. (6) as a power expansion in the smallness parameter $J/|U| \ll 1$. An approximate solution to Eqs. (6) can be derived by a

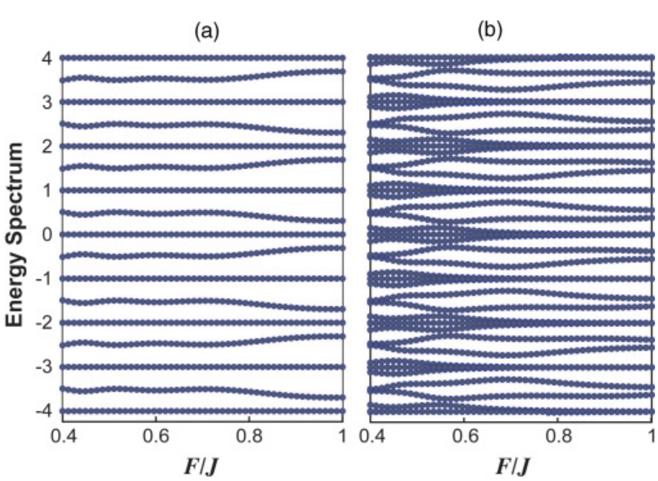

FIG. 5. (Color online) Numerically computed energy spectrum (in units of the forcing F) for $U = 0$ versus F/J for (a) $\theta = \pi$ and (b) $\theta = \pi/2$.

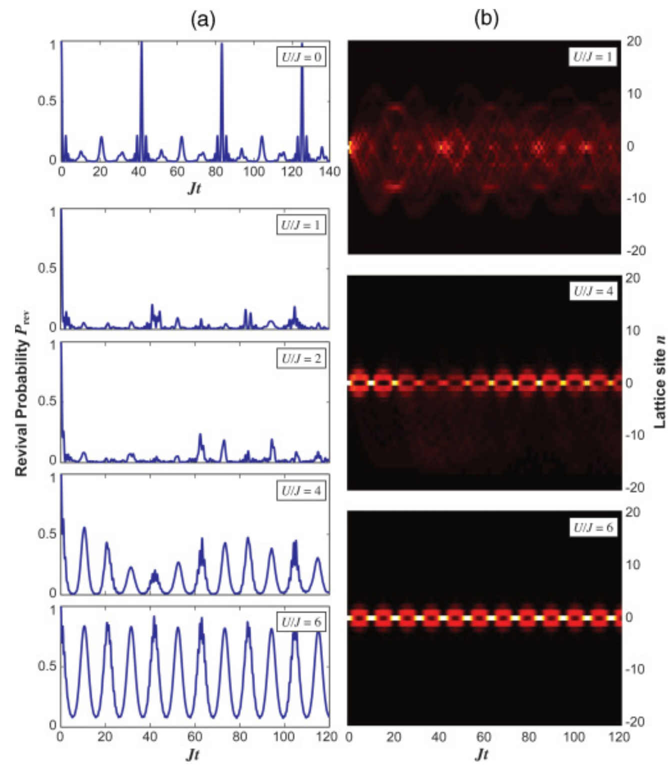

FIG. 6. (Color online) Evolution of (a) the revival probability and (b) the probability density function for two correlated antifermions ($\theta = \pi$) for $F/J = 0.3$ and for increasing values of the normalized on-site interaction U/J. Note a transition from BOs with frequency $\omega_B/2 = F/2$ at $U = 0$ to BOs with frequency $2\omega_B = 2F$ at $U/J = 6$.

multiple-scale asymptotic analysis using standard techniques (see, for instance, Ref. 25). The solution at leading order can be written as $c_{n,m}(t) \simeq A_n(t)\delta_{n,m}\exp(-i\sigma t - iUt)$, where the slow evolution of the amplitudes $A_n(t)$ is governed by the coupled equations (see the Appendix for technical details)

$$i\frac{dA_n}{dt} = J_{\text{eff}}^* A_{n+1} + J_{\text{eff}} A_{n-1} + 2FnA_n \tag{8}$$

and where we have set

$$\sigma = \frac{4J^2}{U}, \quad J_{\text{eff}} = \frac{2J^2\exp(i\theta)}{U}. \tag{9}$$

Equations (8) clearly show that, in the strong-interaction limit, the two anyons tunnel together with an effective hopping rate J_{eff} given by Eq. (9), and behave effectively like a single bound particle which is subjected to a dc force $2F$ rather than F. This behavior is analogous to that previously found for correlated tunneling and BOs of two bosons,[14,15] which is retrieved from the previous equations after setting $\theta = 0$. Note that the statistical phase θ of anyons enters solely in the phase of the effective hopping rate J_{eff}, which nevertheless does not alter the dynamics of correlated BOs. Hence, in contrast to the noninteracting limit studied in the previous section, where the BOs are deeply sensitive to the anyon phase, the correlated dynamics of two strongly interacting anyons is not sensitive to their statistical phase θ. This is shown in Fig. 7, which depicts the evolution of revival probability for $U/J = 4$, $F/J = 0.3$, and a few values of the statistical phase θ.

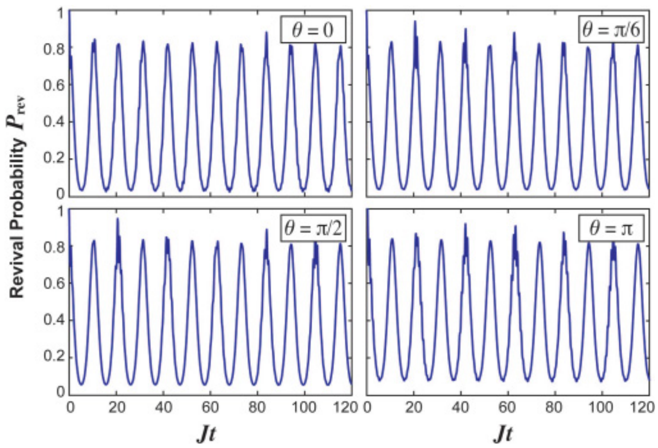

FIG. 7. (Color online) Evolution of the revival probability in the strong-interaction regime ($U/J = 6$) for $F/J = 0.3$ and for a few values of the statistical phase θ.

It should be finally noticed that the doubling of the BO frequency in the strong-correlation regime, shown in Figs. 6 and 7, is observed, provided that the two particles occupy the same lattice site at the initial time. In this case, the two particles form a bound state and they tunnel together along the lattice with an effective hopping rate $J_{\rm eff}$ which decreases toward zero as the on-site interaction U is increased, according to Eq. (9). In this regime, the initial condition basically excites bound states of the two-dimensional lattice model, defined by Eq. (6) with $F = 0$, which are localized along the defect diagonal $n = m$.[13,14] Such bound states form a narrow band of width $4|J_{\rm eff}|$, which shrinks as $U/J \to 0$.[13,14] This explains why the amplitude of frequency-doubled BOs, observed in Fig. 6(b), decreases as $|U|/J$ increases. However, if the initial condition corresponds to a nonvanishing probability of the particles to occupy distinct sites, both bound and unbound states of the two-dimensional lattice (6) are initially excited. While the wave packet component exciting the localized states on the diagonal $n = m$ is responsible for the doubling of the BOs, the wave packet component that excites delocalized states of the two-dimensional lattice (6) basically undergoes BOs without frequency doubling, corresponding to uncorrelated BOs of the two particles. Since the amplitude of the frequency-doubled BOs vanishes as the particle-particle interaction U is increased, we expect that in the strong-interaction regime the oscillation of the mean single-particle position induced by the forcing shows the single-frequency component solely of BOs, regardless of the statistical phase θ. This behavior, which is analogous to that observed for two fermions or bosons,[14] is shown in Fig. 8, which depicts as an example the typical evolution of the probability density function P_k corresponding to the initial wave packet condition $c_{n,m}(0) \propto \exp[-(n^2 + m^2)/\sigma^2]$ for $\sigma = 3$ and in the case of pseudofermions ($\theta = \pi$). For such an initial condition, the probability of finding the two anyons at distinct sites does not vanish, and excitation of both bound and delocalized states of the two-dimensional lattice (6) is attained in this case. The superposition of the two kinds of states yields a dynamical evolution comprising the interference of uncorrelated (fundamental-frequency) and correlated (frequency-doubled) BOs, which is clearly visible in the evolution of the particle density function shown in Fig. 8. In the absence of particle interaction, frequency doubling of BOs is mainly absent [Fig. 8(a)]; however, as compared to bosons or fermions, even for $U = 0$ some correlation can be seen, i.e., the two particles do not exactly undergo uncorrelated dynamics and BOs are not exactly periodic, as discussed in the previous section. As the particle-particle interaction U is switched on, frequency-doubled BOs clearly appear [see Figs. 8(a) and 8(b)], but their amplitude vanishes as $|U|/J$ increases [see Fig. 8(d)]. Hence, in the strong-interaction regime, the oscillation of the single-particle mean position basically arises from the unbound states, and thus the phenomenon of frequency doubling of BOs disappears. This behavior is fully analogous to that observed for a pair of bosons or fermions,[14] and it is not substantially influenced by the anyonic nature of the particles, i.e., by the statistical phase θ.

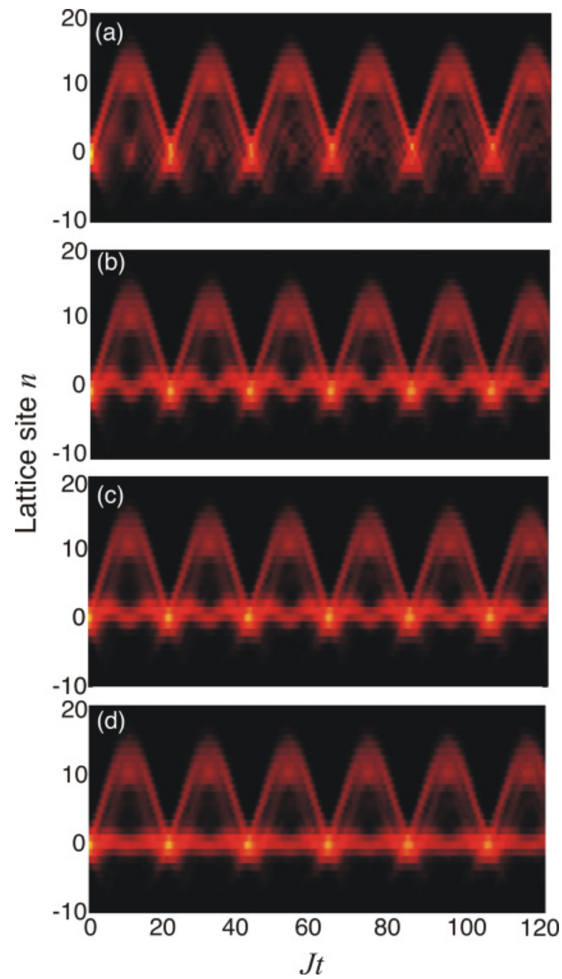

FIG. 8. (Color online) Evolution of the probability density function of pseudofermions ($\theta = \pi$) for $F/J = 0.3$ and for an initial condition corresponding to a Gaussian distribution $c_{n,m}(0) \propto \exp[-(n^2 + m^2)/\sigma^2]$ with $\sigma = 3$. (a) $U/J = 0$; (b) $U/J = 5$; (c) $U/J = 8$; (d) $U/J = 20$.

V. CONCLUSIONS AND DISCUSSION

In this work we have investigated the onset of Bloch oscillations of correlated anyons hopping on a one-dimensional lattice in the framework of the anyon-Hubbard model. Anyons can be regarded as topological states of matter whose quasiparticle excitations obey fractional statistics. Even in the absence of on-site particle interaction, many-body effects exist for anyons.[18,21] As a result, BOs of anyons are generally degraded even in the absence of on-site particle interaction. However, an exception is found for pseudofermions, i.e., particles that, although they are bosons on site, are fermions off site. In this case, if the ratio of forcing to the hopping rate is smaller than $\sim$0.5, in the absence of on-site interaction long-lived BOs are observed at a frequency which is half the BO frequency of single particles. This is remarkably distinct from the results for previously investigated BOs of two strongly correlated bosons or fermions, in which particle correlation leads to a frequency doubling of the BO frequency, rather than to a frequency halving. On the other hand, for strong on-site particle interaction the onset of BOs turns out to be insensitive to the statistical phase exchange of anyons, and bound particle states undergoing BOs at a frequency twice the single-particle BO frequency are attained, as in previous studies.[14,15] The present numerical results show that BOs of correlated anyons are very different from those of bosons in the noninteracting limit and that a special dynamics is found for pseudofermions. It is envisaged that such interesting and previously unexplored results should motivate further studies on BOs of anyons, addressing a few questions that remain open. For example, is it possible to analytically calculate the energy spectrum of the anyon-Hubbard Hamiltonian (1) in the noninteracting limit, at least in the two-particle case? What happens to BOs of more than two anyons in the noninteracting limit? Are long-lived BOs for pesudofermions still observed, and at what frequency? Our results are also expected to stimulate experimentalists to devise quantum or classical simulators aimed at observing BOs of correlated anyons, for example using trapped bosons in optical lattices with occupation-dependent hopping amplitudes as suggested in Ref. 21.

ACKNOWLEDGMENT

The authors acknowledge financial support by the Italian MIUR (Grant No. PRIN-2008-YCAAK).

APPENDIX: BLOCH OSCILLATIONS OF BOUND ANYONS IN THE STRONG-INTERACTION LIMIT

In this Appendix we derive Eq. (8) from an asymptotic analysis of Eq. (6) in the strong-interaction limit $|U| \gg J$ and assuming that the two anyons initially occupy the same lattice site. In this case, the two anyons form a bound particle which undergoes BOs at a frequency twice the BO frequency of a single particle, regardless of the value of the statistical phase θ. To derive Eq. (8), let us introduce in Eq. (6) the scaled time variable $\tau = Ut$ (Ref. 26) and the amplitudes

$$b_{n,m} = c_{n,m} \exp[iUt\delta_{n,m} + iF(n+m)t], \tag{A1}$$

so that the amplitude probabilities $b_{n,m}$ satisfy the coupled equations

$$\begin{aligned} i\frac{db_{n,m}}{d\tau} = & -\frac{J}{U} b_{n+1,m} \exp[i\tau(\delta_{n,m} - \delta_{n+1,m}) - i(F/U)\tau] \\ & -\frac{J}{U} b_{n-1,m} \exp[i\tau(\delta_{n,m} - \delta_{n-1,m}) + i(F/U)\tau] \\ & -\frac{J}{U} b_{n,m-1} \exp[i\tau(\delta_{n,m} - \delta_{n,m-1}) + i(F/U)\tau \\ & - i\varphi_{n,m-1}] - \frac{J}{U} b_{n,m+1} \exp[i\tau(\delta_{n,m} - \delta_{n,m+1}) \\ & - i(F/U)\tau + i\varphi_{n,m}]. \end{aligned} \tag{A2}$$

Let us introduce the smallness parameter $\epsilon = J/U$, which measures the ratio between the hopping rate and the on-site interaction energy, and let us assume a small forcing F such that $F/U \sim \epsilon^2$, i.e., let us set $(F/U) = f_0\epsilon^2$, with $f_0 \sim 1$. Let us then search for a solution to Eq. (A2) as a power series expansion in ϵ:

$$b_{n,m}(\tau) = b^{(0)}_{n,m}(\tau) + \epsilon b^{(1)}_{n,m}(\tau) + \epsilon^2 b^{(2)}_{n,m}(\tau) + \cdots . \tag{A3}$$

To ensure that the expansion (A3) is uniformly valid as τ grows, multiple time scales

$$T_0 = \tau, \;\; T_1 = \epsilon\tau, \;\; T_2 = \epsilon^2\tau, \ldots \tag{A4}$$

have to be introduced to avoid the occurrence of secular growing terms in the asymptotic expansion. Equation (A2) can then be written in the form

$$\begin{aligned} i\frac{db_{n,m}}{d\tau} = & -\epsilon b_{n+1,m} \exp[iT_0(\delta_{n,m} - \delta_{n+1,m}) - if_0T_2] \\ & -\epsilon b_{n-1,m} \exp[iT_0(\delta_{n,m} - \delta_{n-1,m}) + if_0T_2] \\ & -\epsilon b_{n,m-1} \exp[iT_0(\delta_{n,m} - \delta_{n,m-1}) + if_0T_2 \\ & - i\varphi_{n,m-1}] - \epsilon b_{n,m+1} \exp[iT_0(\delta_{n,m} - \delta_{n,m+1}) \\ & - if_0T_2 + i\varphi_{n,m}], \end{aligned} \tag{A5}$$

which is suited for an asymptotic analysis. Using the derivative rule

$$\frac{d}{d\tau} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots , \tag{A6}$$

substitution of Eqs. (A3) and (A6) into Eq. (A5), and after equating terms of the same power in ϵ, a hierarchy of equations for successive corrections to $b_{n,m}$ is obtained. At leading order $\sim\epsilon^0$ one simply obtains

$$i\frac{\partial b^{(0)}_{n,m}}{\partial T_0} = 0, \tag{A7}$$

i.e., $b^{(0)}_{n,m}$ does not evolve on the time scale T_0. If we assume as an initial condition that the two anyons occupy the same lattice site n, one then has

$$b^{(0)}_{n,m} = B_n(T_1,T_2,\ldots)\delta_{n,m}, \tag{A8}$$

where the amplitudes B_n vary on the slow time scales $T_1,T_2,\ldots$. At the order $\sim\epsilon^l$ ($l \geqslant 1$), one generally obtains an equation of the form

$$i\frac{\partial b^{(l)}_{n,m}}{\partial T_0} = -i\frac{\partial b^{(0)}_{n,m}}{\partial T_l} - G^{(l)}_{n,m}\left(b^{(0)}_{n,m}, b^{(1)}_{n,m}, \ldots, b^{(l-1)}_{n,m}\right), \tag{A9}$$

where the driving term $G^{(l)}_{n,m}$ on the right-hand side in the equation depends on the lower-order approximations to $b_{n,m}$. To avoid the occurrence of secular growing terms, the dc term in T_0 on the right-hand side of Eq. (A9) should vanish. Such a solvability condition enables one to determine the evolution of the amplitudes B_n on the slow time scale T_l, namely, one has

$$i\frac{\partial B_n}{\partial T_l} = \bar{G}^{(l)}_{n,n}, \tag{A10}$$

where the overbar denotes the time average with respect to the T_0 variable. In particular, at order $\sim\epsilon$ one has

$$\begin{aligned} G^{(1)}_{n,m} &= 0 \quad \text{for } n = m \text{ or } |n-m| > 1, \\ G^{(1)}_{n,n+1} &= -B_{n+1}\exp(-iT_0 - if_0T_2) \\ &\quad - B_n \exp(-iT_0 + if_0T_2 + i\theta), \\ G^{(1)}_{n,n-1} &= -B_{n-1}\exp(-iT_0 + if_0T_2) \\ &\quad - B_n \exp(-iT_0 - if_0T_2 - i\theta). \end{aligned} \tag{A11}$$

The solvability condition at order $\sim\epsilon$ is thus

$$\frac{\partial B_n}{\partial T_1} = 0, \tag{A12}$$

and the solution at this order can be readily calculated from Eq. (A9) and reads

$$\begin{aligned} b^{(1)}_{n,m} &= -B_{n+1}\exp(-if_0T_2)[\exp(-iT_0) - 1] \\ &\quad - B_n \exp(i\theta + if_0T_2)[\exp(-iT_0) - 1] \quad \text{for } m = n+1, \\ b^{(1)}_{n,m} &= -B_{n-1}\exp(if_0T_2)[\exp(-iT_0) - 1] \\ &\quad - B_n \exp(-i\theta - if_0T_2)[\exp(-iT_0) - 1] \quad \text{for } m = n-1, \\ b^{(1)}_{n,m} &= 0 \quad \text{otherwise.} \end{aligned} \tag{A13}$$

To determine the slow evolution of the amplitudes B_n, we have to push the asymptotic analysis to the order $\sim\epsilon^2$. At this order, one obtains

$$\begin{aligned} G^{(2)}_{n,n} &= -b^{(1)}_{n+1,n}\exp(iT_0 - if_0T_2) - b^{(1)}_{n-1,n}\exp(iT_0 + if_0T_2) \\ &\quad - b^{(1)}_{n,n-1}\exp(iT_0 + if_0T_2 + i\theta) \\ &\quad - b^{(1)}_{n,n+1}\exp(iT_0 - if_0T_2 - i\theta), \end{aligned} \tag{A14}$$

where $b^{(1)}_{n+1,n}$, $b^{(1)}_{n-1,n}$, $b^{(1)}_{n,n+1}$, and $b^{(1)}_{n,n-1}$ are given by Eq. (A13). The solvability condition [Eq. (A10)] at order $\sim\epsilon^2$ thus yields

$$\begin{aligned} i\frac{\partial B_n}{\partial T_2} &= 4B_n + 2B_{n+1}\exp(-i\theta - 2if_0T_2) \\ &\quad + 2B_{n-1}\exp(i\theta + 2if_0T_2). \end{aligned} \tag{A15}$$

If we stop the asymptotic analysis at this order and reintroduce the original variables, from Eqs. (A1), (A3), and (A8) one then obtains

$$c_{n,m}(t) = B_n(t)\delta_{n,m}\exp(-2inFt - iUt) + O(\epsilon), \tag{A16}$$

where the amplitude B_n evolves according to the equation

$$\begin{aligned} i\frac{dB_n}{dt} &= iU\frac{dB_n}{d\tau} = iU\left(\frac{\partial B_n}{\partial T_0} + \epsilon\frac{\partial B_n}{\partial T_1} + \epsilon^2\frac{\partial B_n}{\partial T_2}\right) \\ &= i\frac{J^2}{U}\frac{\partial B_n}{\partial T_2}. \end{aligned} \tag{A17}$$

Substitution of Eq. (A15) into Eq. (A17) yields

$$i\frac{dB_n}{dt} = J^*_{\text{eff}}B_{n+1}\exp(-2iFt) + J_{\text{eff}}B_{n-1}\exp(2iFt) + \sigma B_n, \tag{A18}$$

where J_{eff} and σ are defined by Eq. (9) given in the text. Finally, after setting $A_n(t) = B_n(t)\exp(-2inFt + i\sigma t)$, one has

$$c_{n,m}(t) = A_n(t)\delta_{n,m}\exp(-iUt - i\sigma t) + O(\epsilon), \tag{A19}$$

where the slowly varying amplitudes A_n evolve according to Eq. (8) given in the text.

[1] F. Bloch, Z. Phys. **52**, 555 (1928); C. Zener, Proc. R. Soc. London, Ser. A **145**, 523 (1934).

[2] C. Waschke, H. G. Roskos, R. Schwedler, K. Leo, H. Kurz, and K. Köhler, Phys. Rev. Lett. **70**, 3319 (1993); K. Leo, Semicond. Sci. Technol. **13**, 249 (1998).

[3] M. BenDahan, E. Peik, J. Reichel, Y. Castin, and C. Salomon, Phys. Rev. Lett. **76**, 4508 (1996); S. R. Wilkinson, C. F. Bharucha, K. W. Madison, Q. Niu, and M. G. Raizen, *ibid.* **76**, 4512 (1996).

[4] B. P. Anderson and M. A. Kasevich, Science **282**, 1686 (1998); O. Morsch, J. H. Müller, M. Cristiani, D. Ciampini, and E. Arimondo, Phys. Rev. Lett. **87**, 140402 (2001); G. Roati, E. de Mirandes, F. Ferlaino, H. Ott, G. Modugno, and M. Inguscio, *ibid.* **92**, 230402 (2004).

[5] For recent reviews, see M. Glück, A. R. Kolovsky, and H. J. Korsch, Phys. Rep. **366**, 103 (2002); A. R. Kolovsky and H. J. Korsch, Int. J. Mod. Phys. **18**, 1235 (2004).

[6] R. Morandotti, U. Peschel, J. S. Aitchison, H. S. Eisenberg, and Y. Silberberg, Phys. Rev. Lett. **83**, 4756 (1999); T. Pertsch, P. Dannberg, W. Elflein, A. Bräuer, and F. Lederer, *ibid.* **83**, 4752 (1999); H. Trompeter, T. Pertsch, F. Lederer, D. Michaelis, U. Streppel, A. Bräuer, and U. Peschel, *ibid.* **96**, 023901 (2006); N. Chiodo, G. Della Valle, R. Osellame, S. Longhi, G. Cerullo, R. Ramponi, P. Laporta, and U. Morgner, Opt. Lett. **31**, 1651 (2006); H. Trompeter, W. Krolikowski, D. N. Neshev, A. S. Desyatnikov, A. A. Sukhorukov, Yu. S. Kivshar, T. Pertsch, U. Peschel, and F. Lederer, Phys. Rev. Lett. **96**, 053903 (2006).

[7] R. Sapienza, P. Costantino, D. Wiersma, M. Ghulinyan, C. J. Oton, and L. Pavesi, Phys. Rev. Lett. **91**, 263902 (2003); V. Agarwal, J. A. del Rio, G. Malpuech, M. Zamfirescu, A. Kavokin, D. Coquillat, D. Scalbert, M. Vladimirova, and B. Gil, *ibid.* **92**, 097401 (2004).

[8] H. Sanchis-Alepuz, Y. A. Kosevich, and J. Sanchez-Dehesa, Phys. Rev. Lett. **98**, 134301 (2007); Z. He, S. Peng, F. Cai, M. Ke, and Z. Liu, Phys. Rev. E **76**, 056605 (2007).

[9] D. Cai, A. R. Bishop, N. Grønbech-Jensen, and M. Salerno, Phys. Rev. Lett. **74**, 1186 (1995); F. Dominguez-Adame, V. A. Malyshev, F. A. B. F. de Moura, and M. L. Lyra, *ibid.* **91**, 197402 (2003); E. Diaz, F. Dominguez-Adame, Yu. A. Kosevich, and V. A. Malyshev, Phys. Rev. B **73**, 174210 (2006); F. A. B. F. de Moura, L. P. Viana, M. L. Lyra, V. A. Malyshev, and F. Dominguez-Adame,

Phys. Lett. A **372**, 6694 (2008); T. Schulte, S. Drenkelforth, G. K. Büning, W. Ertmer, J. Arlt, M. Lewenstein, and L. Santos, Phys. Rev. A **77**, 023610 (2008); S. Walter, D. Schneble, and A. C. Durst, *ibid.* **81**, 033623 (2010); S. Longhi, Phys. Rev. B **81**, 195118 (2010); C. Gaul, E. Diaz, R. P. A. Lima, F. Dominguez-Adame, and C. A. Müller, Phys. Rev. A **84**, 053627 (2011).

[10] M. Gustavsson, E. Haller, M. J. Mark, J. G. Danzl, G. Rojas-Kopeinig, and H.-C. Nägerl, Phys. Rev. Lett. **100**, 080404 (2008); D. O. Krimer, R. Khomeriki, and S. Flach, Phys. Rev. E **80**, 036201 (2009); C. Gaul, R. P. A. Lima, E. Diaz, C. A. Müller, and F. Dominguez-Adame, Phys. Rev. Lett. **102**, 255303 (2009).

[11] A. Buchleitner and A. R. Kolovsky, Phys. Rev. Lett. **91**, 253002 (2003); A. R. Kolovsky, Phys. Rev. A **70**, 015604 (2004).

[12] J. K. Freericks, Phys. Rev. B **77**, 075109 (2008); M. Eckstein and P. Werner, Phys. Rev. Lett. **107**, 186406 (2011).

[13] F. Claro, J. F. Weisz, and S. Curilef, Phys. Rev. B **67**, 193101 (2003); D. Souza and F. Claro, *ibid.* **82**, 205437 (2010).

[14] W. S. Dias, E. M. Nascimento, M. L. Lyra, and F. A. B. F. de Moura, Phys. Rev. B **76**, 155124 (2007); W. S. Dias, M. L. Lyra, and F. A. B. F. de Moura, Phys. Lett. A **374**, 4554 (2010).

[15] R. Khomeriki, D. O. Krimer, M. Haque, and S. Flach, Phys. Rev. A **81**, 065601 (2010).

[16] K. Kudo, T. Boness, and T. S. Monteiro, Phys Rev A **80**, 063409 (2009); K. Kudo and T. S. Monteiro, *ibid.* **83**, 053627 (2011).

[17] J. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977); F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982); F. D. M. Haldane, *ibid.* **67**, 937 (1991).

[18] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).

[19] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983); F. E. Camino, Wei Zhou, and V. J. Goldman, Phys. Rev. B **72**, 075342 (2005); S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005); P. Bonderson, A. Kitaev, and K. Shtengel, *ibid.* **96**, 016803 (2006); A. Stern and B. I. Halperin, *ibid.* **96**, 016802 (2006).

[20] C. Vitoriano and M. D. Coutinho-Filho, Phys. Rev. Lett. **102**, 146404 (2009).

[21] T. Keilmann, S. Lanzmich, I. McCulloch, and M. Roncaglia, Nat. Commun. **2**, 361 (2011).

[22] D. Jaksch and P. Zoller, New J. Phys. **5**, 56 (2003).

[23] G. Juzeliunas, J. Ruseckas, P. Ohberg, and M. Fleischhauer, Phys. Rev. A **73**, 025602 (2006); A. R. Kolovsky, Europhys. Lett. **93**, 20003 (2011).

[24] The first observation of such an exotic bound state, comprised of a pair of ultracold atoms in an optical lattice, was reported by K. Winkler, G. Thalhammer, F. Lang, R. Grimm, J. H. Denschlag, A. J. Daley, A. Kantian, H. P. Büchler, and P. Zoller, Nature (London) **441**, 853 (2006).

[25] S. Longhi, Phys. Rev. B **77**, 195326 (2008); S. Longhi and G. Della Valle, Opt. Lett. **36**, 4743 (2011).

[26] For the sake of definiteness, we assume $U > 0$, but a similar analysis can be done for $U < 0$.