PHYSICAL REVIEW B **85**, 104205 (2012)

Mean field theory of superglasses

Xiaoquan Yu
International School for Advanced Studies (SISSA), via Bonomea 265, 34136 Trieste, Italy

Markus Müller
The Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, 34151 Trieste, Italy

(Received 5 December 2011; published 19 March 2012)

We study the interplay of superfluidity and glassy ordering of hard core bosons with random, frustrating interactions. This is motivated by bosonic systems such as amorphous supersolid, disordered superconductors with preformed pairs, and helium in porous media. We analyze the fully connected mean field version of this problem, which exhibits three low-temperature phases, separated by two continuous phase transitions: an insulating, glassy phase with an amorphous frozen density pattern, a nonglassy superfluid phase, and an intermediate phase, in which both types of order coexist. We elucidate the nature of the phase transitions, highlighting in particular the role of glassy correlations across the superfluid-insulator transition. The latter suppress superfluidity down to $T = 0$, due to the depletion of the low-energy density of states, unlike in the standard BCS scenario. Further, we investigate the properties of the coexistence (superglass) phase. We find anticorrelations between the local order parameters and a nonmonotonous superfluid order parameter as a function of T. The latter arises due to the weakening of the glassy correlation gap with increasing temperature. Implications of the mean field phenomenology for finite dimensional bosonic glasses with frustrating Coulomb interactions are discussed.

DOI: 10.1103/PhysRevB.85.104205 PACS number(s): 61.43.−j, 67.25.dj

I. INTRODUCTION

Bosons in random environments occur in a variety of experimentally relevant systems, ranging from cold atomic gases, superconductors, and quantum liquids. The superfluidity of helium-4 (^{4}He) in porous media was one of the first phenomena observed in this type of system,[1,2] featuring an interesting competition between Bose-Einstein condensation and localization by random potentials.[3]

In more recent years, supersolidity in crystalline ^{4}He[4–6] has been reported. It soon became clear that defects of the crystalline order and amorphous solids sustain more robust supersolidity, spurring the idea that disorder, or even glassy order, may be a crucial element in understanding the superfluid part of those systems.[7–11]

A recent experiment[12] reported indeed that the supersolidity in ^{4}He is accompanied by the onset of very slow glassy relaxation. This suggested that an amorphous glass with a superfluid component is forming, a state of matter that was dubbed a "superglass." These experimental results have motivated several theoretical investigations into the possibility and nature of such amorphously ordered and yet supersolid systems.[13–18]

Similar questions as to the coexistence and interplay of glassy density ordering and superfluidity arises in disordered superconducting films, which feature disorder- or field-driven superconductor-to-insulator quantum phase transitions.[19] In several experimental materials, this transition appears to be driven by phase fluctuations of the order parameter rather than by the depairing of electrons, suggesting that the transition can be described in terms of bosonic degrees of freedom only.[20] This had led to the dirty boson model[21] and the notion of the Bose glass,[22] in which disorder and interactions lead to the localization of the bosons, while the system remains compressible.[23,24] In this context, the term "glass" refers mostly to the amorphous nature of the state rather than to the presence of slow relaxation and out-of-equilibrium phenomena due to frustrated interactions. However, if superconductivity develops in a highly disordered environment, such frustration may appear in the form of Coulomb interactions between the charged carriers, which may become important, as screening is not very effective. It is well known that in more insulating regimes, strong disorder and Coulomb interactions may induce a glassy state of electrons (the Coulomb glass).[25–27] It is therefore an interesting question whether such glassy effects can persist within the superconducting state of disordered films. A memory dip in resistance versus gate voltage (similar to the conductance dip in insulators, which is considered a smoking gun of electron glassiness[28]), was reported as a possible indication of such a superglassy state in Ref. 29, even though doubts about its intrinsic nature were raised later on.[30]

The recent developments in ultracold atoms[31] open new ways to studying bosonic atoms in the presence of both interactions and disorder. Those can exhibit superfluid or localized, and potentially also glassy phases, especially if the interactions are sufficiently long ranged and frustrated, as is possibly the case for dipolar interactions.

Motivated by these experimental systems, we study a solvable model of bosons subject to disorder and frustrating interactions, as proposed previously in Ref. 15. This solvable case provides insight into the possibility of coexistence of superfluidity and glassy density order, as well as into the nature of the coexistence phase (the superglass). In particular, for the considered mean field model we prove the existence of a superglass phase. This complements the numerical evidence for such phases provided by quantum Monte Carlo investigations in finite dimensions[15] and on random graphs.[16] Those were, however, limited to finite temperature, and could thus not elucidate the structure of the phases at $T = 0$. In contrast, the analytical approach allows us to understand the quantum phase transition between glassy superfluid and insulator, and

the nontrivial role played by glassy correlations. Indeed it is an interesting feature of the considered model that despite its infinite connectivity, it features an insulating, Anderson localized phase at sufficiently small but finite hopping strength. This is in contrast to nonglassy models where the localization and insulating behavior is lost in the large connectivity limit, unless the hopping is downscaled logarithmically with the connectivity.[32]

The model studied here is an interesting type of a quantum glass.[33] Like other canonical quantum glass models of mean field type, such as the random exchange Heisenberg model,[34] the Ising spin glass in transverse field,[35,36] and frustrated rotor models,[37–39] the glass phase of the superglass model breaks ergodicity, and will exhibit a large number of metastable states and associated slow relaxation and out-of-equilibrium dynamics. It has recently been pointed out that such mean field glasses can faithfully be realized by atoms in laser cavities, which are coupled at long distances through interactions via discrete photon modes.[40,41]

In the above-mentioned canonical models the nonglassy phase is usually quantum disordered, having no broken symmetry. However, a new situation arises in the model studied here: The glassy ordering competes with a different order: the superfluidity, or the transverse (XY) ferromagnetism in a magnetic analogon. The microscopic coexistence of these two types of order, which we demonstrate to exist in our model, is rather nontrivial, as one might instead expect a first-order transition between phases with either of the two orders. At zero temperature, the quantum glass transition is induced by frustrated interactions, which win over weak quantum fluctuations. While the quantum phase transition of such models is understood relatively well,[42] the deep glass phase with its collective gapless modes has not been much explored.[43,44]

Let us finally remark that the glassy, amorphous supersolid, which the superglass phase constitutes, is quite different from the type of supersolid proposed theoretically in the early 1970s.[45] In those scenarios the bosons organize spontaneously on a lattice, which breaks translational symmetry, but is incommensurate with the boson density, allowing for vacancies to move through the solid. Our model considers instead bosons on a predefined lattice, on which an inhomogeneous density pattern establishes in the glass phase.

The remainder of the paper is organized as follows: In Sec. II we introduce the mean field model of a superglass. We obtain the effective action for a single spin with both the replica method and the cavity approach in Sec. III, and introduce the key concept of the distribution of local fields. The self-consistency equations of the mean field theory are solved under static approximation, which is argued to be exact in parts of the phase diagram. Section IV determines the instabilities toward forming glassy and superfluid order, and establishes the phase diagram featuring two quantum phase transitions at $T = 0$. We also discuss the robustness of the phase diagram to random potential disorder. In Sec. V, we study the bulk of the superglass phase in more detail. We show that the glass and superfluid order parameters are locally anticorrelated. Moreover, we find the superfluid order parameter to have an interesting nonmonotonous behavior as a function of T. The implications of this mean field analysis for realistic finite dimensional models, e.g., with frustrating long-range Coulomb interactions, will be discussed in Sec. VI. Some detailed derivations are collected in the Appendix.

II. MODEL

We consider the fully connected model of hard-core bosons with random pairwise interactions between all bosons,

$$H = -\sum_{i<j} V_{ij} n_i n_j + \sum_i \epsilon_i n_i - \frac{t_b}{N} \sum_{i<j} (b_i^\dagger b_j + b_j^\dagger b_i). \quad (1)$$

Here, $n_i = b_i^\dagger b_i$ is the number operator on site i, and the hard-core constraint limits n_i to assume values 0 or 1. $b_i (b_i^\dagger)$ denote the annihilation (creation) operators for a hard-core boson at site i. V_{ij} is a quenched disorder with Gaussian distribution of zero mean and variance V^2/N, ϵ_i describes a quenched disorder potential for the bosons, and t_b/N is the unfrustrated hopping strength between any pair of sites. The scaling of the couplings with N is chosen so as to yield a nontrivial thermodynamic limit for $N \to \infty$.

In the absence of hopping, the model becomes classical and is equivalent to the Sherrington-Kirkpatrick spin-glass (SK) model[46] in a random field. It is well known that except for lowering the transition temperature, the random fields do not alter the low-temperature properties of the glass phase. We thus restrict our attention mostly to a slightly simpler model proposed in Ref. 15, which corresponds to a special choice of the ϵ_i:

$$H = -\sum_{i<j} V_{ij} (n_i - 1/2)(n_j - 1/2) - \frac{t_b}{N} \sum_{i<j} (b_i^\dagger b_j + b_j^\dagger b_i). \quad (2)$$

Similar fermionic mean field models have been studied in Refs. 47 and 48. The identification $2n_i - 1 = s_i^z \in \{\pm 1\}$, $b_i^\dagger = s_i^+, b_i = s_i^-$, allows us to map this model into a fully connected spin-glass model with quantum fluctuations arising from nonrandom spin-flip terms,

$$H = -\sum_{i<j} J_{ij} s_i^z s_j^z - \frac{t}{N} \sum_{i<j} \left(s_i^x s_j^x + s_i^y s_j^y \right), \quad (3)$$

with the simple dictionary

$$J_{ij} = \frac{V_{ij}}{4}, \quad t = \frac{t_b}{2}. \quad (4)$$

For $t = 0$, this Hamiltonian reduces to the SK model, which possesses a spin-glass phase at low temperature, $T < T_g = J$. Without the Ising interactions, $J_{ij} = 0$, the Hamiltonian turns into the mean field XY model, which has a superfluid (or XY ferromagnetic) phase at low temperatures ($T < T_s = t$). In this paper we establish the phase diagram and study the properties of the bulk phases resulting from the competition of random density-density interactions and boson hopping (bosonic language) or equivalently random Ising interactions and ferromagnetic transverse coupling (spin language).

III. FREE-ENERGY AND SELF-CONSISTENT EQUATIONS

A. General formalism

The disorder average of the free energy of the model (3) can be obtained using the replica method,[49]

$$\langle \ln Z\rangle_J = \lim_{n\to 0}\frac{\langle Z^n\rangle_J - 1}{n}, \tag{5}$$

where Z is the partition function and $\langle\cdots\rangle_J$ indicates an average over the couplings J_{ij}.

Following a method introduced by Bray and Moore[34] it is useful to represent the partition function as an imaginary time path integral:

$$Z^n = \mathrm{Tr}\mathcal{T}\exp\left\{\beta\int_0^1 d\tau\sum_{a=1}^{n}\sum_{i<j}\left[J_{ij}s_{ia}^z(\tau)s_{ja}^z(\tau) + \frac{t}{N}\left(s_{ia}^x(\tau)s_{ja}^x(\tau)+s_{ia}^y(\tau)s_{ja}^y(\tau)\right)\right]\right\}, \tag{6}$$

where $\mathcal{T}$ orders the operators in decreasing order of their argument $\tau\in[0,1]$. This "time" argument of $s(\tau)$ merely serves us to define the time ordering, while $s(\tau)$ denotes always the same Pauli matrix, independently of time.

Averaging over disorder and decoupling the spins on different sites using a Hubbard-Stratonovich transformation with the order-parameter fields Q_{ab}, M_a^x, M_a^y, we obtain

$$\langle Z^n\rangle_J \propto \int\prod_a dQ_{aa}(\tau,\tau')\,dM_a^x(\tau)\,dM_a^y(\tau) \times\prod_{a<b}dQ_{ab}(\tau,\tau')\exp\left(-N\mathcal{F}\right) \tag{7}$$

with

$$\mathcal{F} = \frac{J^2\beta^2}{4}\int_0^1\int_0^1 d\tau d\tau'\left[\sum_{a\neq b}Q_{ab}^2(\tau,\tau')+\sum_a Q_{aa}^2(\tau,\tau')\right] + \frac{t\beta}{2}\int_0^1 d\tau\sum_a\left[M_a^x(\tau)^2+M_a^y(\tau)^2\right]-\ln\mathcal{Z}, \tag{8}$$

$$\mathcal{Z} = \mathrm{Tr}\mathcal{T}\exp\left(-S_{\mathrm{eff}}\right), \tag{9}$$

$$S_{\mathrm{eff}} = -\frac{J^2\beta^2}{2}\int_0^1\int_0^1 d\tau d\tau'\left[\sum_{a\neq b}Q_{ab}(\tau,\tau')s_a^z(\tau)s_b^z(\tau') + \sum_a Q_{aa}(\tau,\tau')s_a^z(\tau)s_a^z(\tau')\right] - t\beta\sum_a\int_0^1 d\tau\left(M_a^x(\tau)s_a^x(\tau)+M_a^y(\tau)s_a^y(\tau)\right). \tag{10}$$

In the limit $N\to\infty$, the functional integral (7) is dominated by the saddle point of the replicated free energy $\mathcal{F}$, which satisfies

$$0 = \frac{\delta\mathcal{F}}{\delta Q_{ab}(\tau,\tau')}\Rightarrow Q_{ab}(\tau,\tau') = \left\langle\mathcal{T}s_a^z(\tau)s_b^z(\tau')\right\rangle_{\mathrm{eff}} = \left\langle s_a^z s_b^z\right\rangle_{\mathrm{eff}}\equiv Q_{ab}, \tag{11}$$

$$0 = \frac{\delta\mathcal{F}}{\delta Q_{aa}(\tau,\tau')}\Rightarrow Q_{aa}(\tau,\tau') = \left\langle\mathcal{T}s_a^z(\tau)s_a^z(\tau')\right\rangle_{\mathrm{eff}} \equiv R(\tau,\tau'), \tag{12}$$

$$0 = \frac{\delta\mathcal{F}}{\delta M_a^x(\tau)}\Rightarrow M_a^x(\tau) = \left\langle s_a^x(\tau)\right\rangle_{\mathrm{eff}}\equiv M^x, \tag{13}$$

$$0 = \frac{\delta\mathcal{F}}{\delta M_a^y(\tau)}\Rightarrow M_a^y(\tau) = \left\langle s_a^y(\tau)\right\rangle_{\mathrm{eff}}\equiv M^y, \tag{14}$$

where $\langle\cdots\rangle_{\mathrm{eff}}$ denotes the average with respect to the effective action S_{eff} of a single site. We have used that, as usual, the saddle-point values of $Q_{a\neq b}$ and $M_a^{x,y}$ are independent of imaginary time, while $Q_{aa}(\tau,\tau')$ depends only on the imaginary time difference.[50] Furthermore, Q_{aa} and $M_a^{x,y}$ do not depend on the replica index a. For Q_{ab} we make the standard ultrametric ansatz, parametrized by a monotonous function $q(x)$ on the interval $x\in[0,1]$, which is well known to describe successfully the SK model and other mean field glasses.[49] We are free to choose coordinates in the x,y plane such that the spontaneous magnetization $\vec{M}$ points in the x direction, and thus we set $M^y=0$.

Note that $M^x\neq 0$ signals the presence of transverse (XY) order of the spins, that is, superfluidity of the hard-core bosons, which breaks the U(1) symmetry spontaneously. On the other hand, a nonconstant value of $Q_{a\neq b}$ implies the spontaneous breaking of the replica symmetry, and thus the presence of a glass phase with many metastable states and nontrivially broken ergodicity. As long as we do not consider random-field disorder, the breaking of replica symmetry coincides with the breaking of the Ising symmetry and is signalled by a nonzero value of $Q_{a\neq b}$. We will see below that the U(1) and the replica symmetries can be broken simultaneously in a what has been called a "superglass phase" in Refs. 15 and 16.

To find the location of a (continuous) glass transition, we expand the free energy to second order in Q_{ab}. We find an instability toward replica symmetry breaking, and thus the emergence of a glassy density ordering of bosons, when

$$\beta J\int_0^1\int_0^1 d\tau d\tau'\left\langle\mathcal{T}s_a^z(\tau)s_a^z(\tau')\right\rangle_{\mathrm{eff}} = \beta J\int_0^1 d\tau R(\tau) = 1, \tag{15}$$

or

$$J\chi^{\parallel} = 1, \tag{16}$$

where $\chi^{\parallel}\equiv\chi_{zz}(\omega=0)$ is the zero-frequency limit of the longitudinal susceptibility. This condition is of course to be evaluated at $Q_{a\neq b}=0$.

On the other hand, a second-order phase transition from the high-temperature phase toward a superfluid state is indicated by the instability condition, which follows from $\partial^2\mathcal{F}/\partial M^2 = 0$:

$$\beta t\int_0^1\int_0^1 d\tau d\tau'\left\langle\mathcal{T}s_a^x(\tau)s_a^x(\tau')\right\rangle_{\mathrm{eff}} = 1, \tag{17}$$

or

$$t\chi^{\perp} = 1, \tag{18}$$

where $\chi^{\perp} \equiv \chi_{xx}(\omega = 0)$ is the static transverse susceptibility. These expressions must be calculated in the nonsuperfluid phase where $M = 0$. In this regime the effective action $S_{\rm eff}$ is classical, which entails the further simplification $R(\tau) = 1$. This feature is due to the suppression of quantum fluctuations in the nonsuperfluid phase by factors of $1/N$, due to the scaling of the transverse coupling. It allows us to find the superfluid-insulator transition analytically, even at zero temperature, without solving a full quantum impurity problem. In particular, we immediately find that the transition from the disordered high-temperature phase to a glassy phase is given by

$$T_g = J, \tag{19}$$

exactly as in the classical SK model. However, the glass transition line will be modified if it is preceded by a superfluid transition at higher temperature.

B. Solution of the saddle-point equations

A full solution of the saddle-point equations involves the solution of the problem of interacting replica as well as the evaluation of dynamical correlation functions with the effective action $S_{\rm eff}$, if $M \neq 0$ and the replica symmetry is broken as well. Here we describe what steps an exact solution involves, and then discuss the approximations we will use to study parts of the phase diagram, especially the bulk of the superglass phase.

To describe a nonglassy superfluid phase, the replica structure is trivial, and one needs to solve the self-consistency equations

$$M = \langle s^x \rangle_{\rm eff}, \quad R(\tau) = \langle \mathcal{T} s^z(\tau) s^z(0) \rangle_{\rm eff}, \tag{20}$$

with effective action

$$S_{\rm eff} = -\frac{\beta^2 J^2}{2} \int_0^1 \int_0^1 d\tau d\tau' s^z(\tau) R(\tau - \tau') s^z(\tau') - \beta t M \int_0^1 d\tau s^x(\tau). \tag{21}$$

These can be solved using techniques as used in dynamical mean field theory.[51]

In a glassy phase the replica structure has to be taken into account. Assuming the standard ultrametric structure of the saddle-point matrix Q_{ab}, the above single-replica scheme has to be generalized to include a self-consistent distribution of frozen longitudinal fields $P(y)$ acting on a given replica. This captures the distribution of random frozen fields y_i created by the exchange of sites i with the frozen magnetization pattern with a spin-glass state.[52] In practice this requires the simultaneous solution of

$$\begin{aligned} m(y) &= \langle s^z \rangle_{S_{\rm eff}(y)}, \quad m_x(y) = \langle s^x \rangle_{S_{\rm eff}(y)}, \\ M &= \int dy P(y) m_x(y), \\ R(\tau) &= \int dy P(y) \langle \mathcal{T} s^z(\tau) s^z(0) \rangle_{S_{\rm eff}(y)}, \end{aligned} \tag{22}$$

where the effective single replica action in a frozen field y reads

$$S_{\rm eff}(y) = -\frac{\beta^2 J^2}{2} \int_0^1 \int_0^1 d\tau d\tau' s^z(\tau) [R(\tau - \tau') - q_{\rm EA}] s^z(\tau') - \beta t M \int_0^1 d\tau s^x(\tau) - \beta y \int_0^1 d\tau s^z(\tau). \tag{23}$$

The Edwards-Anderson order parameter $q_{\rm EA}$ characterizes the glassy freezing in a pure state of the glass and is given by

$$q_{\rm EA} = q(x = 1) = \int dy P(y) m^2(y). \tag{24}$$

As first derived by Sommers and Dupont,[52] the frozen field distribution $P(y) \equiv P(y, x = 1)$ is obtained from a self-consistent solution of the differential equations on the interval $x \in [0,1]$,

$$\dot{m}(y,x) = -\frac{\dot{q}(x)}{2} [m''(y,x) + 2x m(y,x) m'(y,x)], \tag{25}$$

$$\dot{P}(y,x) = \frac{\dot{q}(x)}{2} [P''(y,x) - 2x (m(y,x) P(y,x))'], \tag{26}$$

with

$$m(y, x = 1) = m(y), \quad P(y, x = 0) = \delta(y), \tag{27}$$

where dots and primes denote derivatives with respect to x and y, respectively. The solutions of these differential equations solve the saddle-point equations of the replica free energy (8).[52] The overlap function $q(x)$, which parametrizes the ultrametric matrix Q_{ab} by the distance x between replicas, must obey the self-consistency relation

$$q(x) = \int P(y,x) m(y,x)^2. \tag{28}$$

Notice that Eqs. (25) and (26) are the same as in a classical spin glass. The influence of quantum fluctuations enters through the boundary condition $m(y, x = 1) \equiv m(y)$, where $m(y)$ was defined in Eq. (22) These differential equations provide an elegant way of integrating out all spins except for one.[53]

Once the above scheme has been solved self-consistently, site-averaged observables such as the longitudinal magnetization are given by

$$M^z \equiv \left\langle \frac{1}{N} \sum_i s_i^z \right\rangle = \int dy P(y) \langle s^z \rangle_{S_{\rm eff}(y)} = \int dy P(y) m(y). \tag{29}$$

The properties of the solution of these differential equations are well understood in several classical models exhibiting full replica symmetry breaking with continuous functions $q(x)$.[54–57] The full solution of mean field quantum glasses in the ergodicity broken has not been analyzed in the literature so far. However, an analysis of the transverse field SK model shows that most features of the low-temperature solution of $q(x)$ carry over rather naturally to the quantum case.[43] A salient new feature in the quantum case is the fact that full replica symmetry breaking implies marginal stability of the whole glass phase, which in turn ensures the presence of gapless collective excitations. The latter is very similar to what was

found, e.g., in the threshold states of quantum p-spin models,[58] or in the quantum dynamics of elastic manifolds, approximated with a replica symmetry-breaking variational approach.[59]

C. Alternative derivation by a cavity approach

The replica-diagonal part of the above scheme will become easier to understand, if we derive it in a cavity framework[60] similarly to the derivation of the quantum analog of Thouless-Anderson-Palmer equations by Biroli and Cugliandolo.[58] From a cumulant expansion in the couplings involving site o it is easy to obtain the following effective action for the site o:

$$\begin{aligned} S_o^{\rm eff} &= -\frac{\beta^2}{2}\int_0^1\int_0^1 d\tau d\tau' s_o^z(\tau)\left[\sum_i J_{oi}^2\langle s_i^z(\tau)s_i^z(\tau')\rangle_c^o\right]s_o^z(\tau') \\ &\quad -\beta\int_0^1 d\tau\left(h_o^z(\tau)s_o^z(\tau)+h^x(\tau)s_o^x(\tau)\right). \end{aligned} \tag{30}$$

Here

$$h_o^z(\tau)=\sum_i J_{oi}\langle s_i^z(\tau)\rangle^o=\sum_i J_{oi}\langle s_i^z\rangle^o \tag{31}$$

is the site-dependent longitudinal field, which does not depend on time, however. The index o denotes a "cavity average," i.e. an average over the action of the system, in which the site o has been removed. The subscript c indicates a connected correlator. The effective transverse field,

$$\begin{aligned} h^x(\tau) &= \frac{t}{N}\sum_i\langle s_i^x(\tau)\rangle^o=\frac{t}{N}\sum_i\langle s_i^x\rangle^o \\ &= \frac{t}{N}\sum_i\langle s_i^x\rangle = tM, \end{aligned} \tag{32}$$

does not fluctuate from site to site, and is independent of τ if we neglect subleading terms, which scale as inverse powers of N. Note that for large N

$$\sum_i J_{oi}^2\langle s_i^z(\tau)s_i^z(\tau')\rangle_c^o\to J^2[R(\tau-\tau')-q_{\rm EA}], \tag{33}$$

independently of the site o. The distribution of h_i^z over the sites i is the frozen field distribution,

$$P(y)=N^{-1}\sum_{i=1}^N\delta\left(y-h_i^z\right), \tag{34}$$

computed in the replica formalism. Thus we precisely recover the self-consistency problem for the replica diagonal, while the solution of the replica off-diagonal part furnishes the distribution $P(y)$.

For the study of the phase transition from the insulating glass phase into the superfluid, it will prove crucial to use the full low-temperature solution of the SK model. However, in order to analyze properties of the mixed superglass phase we will restrict ourselves to a one-step approximation, which we discuss in the next section.

D. Static and one-step approximation

In order to avoid solving numerically a full self-consistent quantum problem as outlined in Eqs. (22) above, we will resort to the widely used static approximation. The latter consists in seeking a minimum of the free energy not with respect to the full function space $R(\tau)$ but, instead with respect to a constant value $R(\tau)\to R$.

A further approximation, which we will use in the study of the quantum glassy phases, is the one-step approximation for the structure of replica symmetry breaking. It is equivalent to assuming a step form of $q(x)$,

$$q(x)=\Theta(x-x_1)Q_1+\Theta(x_1-x)Q_0, \tag{35}$$

and optimizing the free energy over x_1, Q_1, Q_0. This is expected to give qualitatively good results, especially at intermediate temperatures and close to the glass transition. Combined with the static approximation for the replica diagonal, short-time part one obtains the free-energy functional per spin:

$$\begin{aligned} \beta f &= \frac{\beta^2J^2}{4}\left[(x_1-1)Q_1^2-x_1Q_0^2+R^2\right]+\frac{\beta t}{2}M^2 \\ &\quad -\frac{1}{x_1}\int Dy_0\log\int Dy_1 \\ &\quad \times\left[\int Dy_R 2\cosh\left(\beta\sqrt{h_y^2+t^2M^2}\right)\right]^{x_1}, \end{aligned} \tag{36}$$

where $h_y=y_0+y_1+y_R$. Dy_0, Dy_1, and Dy_R are Gaussian measures: $Dy_0=\frac{\exp\left(-y_0^2/2Q_0J^2\right)}{\sqrt{2\pi Q_0J^2}}dy_0$, $Dy_1=\frac{\exp[-y_1^2/2(Q_1-Q_0)J^2]}{\sqrt{2\pi(Q_1-Q_0)J^2}}dy_1$, and $Dy_R=\frac{\exp[-y_R^2/2(R-Q_1)J^2]}{\sqrt{2\pi(R-Q_1)J^2}}dy_R$.

Note that $Q_1=q_{\rm EA}$ is the Edwards Anderson order parameter in the one-step approximation, while Q_0 is the overlap between different spin-glass states. We point out that the above free energy differs from the expression given in Ref. 15, where the static approximation was not carried out correctly. This error was at the origin of several strange features of the phase diagram reported there, such as a T-independent transition between superfluid and superglass and a J-independent superfluid transition.

E. 1RSB free-energy and self-consistent equations

Here we rewrite the one-step self-consistency equations with the help of the local-field distribution.

The effective partition function of a single spin is

$$\begin{aligned} Z_{\rm eff}(y) &= {\rm Tr}\mathcal{T}\exp[-S_{\rm eff}(y)] \\ &= {\rm Tr}\mathcal{T}\exp\left(\frac{\beta^2J^2}{2}\int_0^1\int_0^1 d\tau d\tau'\right. \\ &\quad \times s^z(\tau)[R(\tau-\tau')-q_{\rm EA}]s^z(\tau') \\ &\quad \left.+\beta tM\int_0^1 d\tau s^x(\tau)+\beta y\int_0^1 d\tau s^z(\tau)\right). \end{aligned} \tag{37}$$

In the case of one-step replica symmetry breaking (RSB), the frozen field distribution within one pure state can be

obtained by stepwise integration of the flow equations (25) and (26), yielding [cf. Eq. (49)]

$$P(y) = \int Dy_0 \frac{\int Dy_1 \delta[y-(y_0+y_1)] Z_{\rm eff}^{x_1}(y_0+y_1)}{\int D\widetilde{y}_1 Z_{\rm eff}^{x_1}(y_0+\widetilde{y}_1)}, \quad (38)$$

where $D\widetilde{y}_1$ is a Gaussian measure like Dy_1 with variance $(Q_1-Q_0)J^2$.

Under the static approximation, Eq. (37) becomes

$$Z_{\rm eff}(y) = \int Dy_R Z_{\rm stat}(y+y_R), \quad (39)$$

where

$$Z_{\rm stat}(y) = 2\cosh(\beta\sqrt{y^2+M^2t^2}). \quad (40)$$

One can interpret y_R as a random field, which is generated by the thermal fluctuations of the nonfrozen part of the magnetization.

The longitudinal and transverse magnetizations of a spin in a frozen field y introduced in Eqs. (22) are easily seen to be given by

$$m(y) = \langle s^z\rangle_{S_{\rm eff}(y)} = \frac{1}{\beta}\frac{\partial}{\partial y}\ln[Z_{\rm eff}(y)], \quad (41)$$

$$m_x(y) = \langle s^x\rangle_{S_{\rm eff}(y)} = \frac{1}{\beta}\frac{\partial}{\partial(tM)}\ln[Z_{\rm eff}(y)]. \quad (42)$$

The saddle-point equations for the Edwards-Anderson parameter Q_1 and the superfluid order parameter M can now be expressed as

$$Q_1 = \frac{1}{N}\sum_i \langle s_i^z\rangle^2 = \int dy P(y) m^2(y), \quad (43)$$

$$M = \frac{1}{N}\sum_i \langle s_i^x\rangle = \int dy P(y) m_x(y). \quad (44)$$

The saddle-point equation for the parameter R reads

$$\beta(R-Q_1) = \int P(y)\chi_{\rm loc}^{\parallel}(y)dy, \quad (45)$$

which relates the static approximation of the connected s^z correlator, $R-Q_1$, to the average local susceptibility

$$\chi_{\rm loc}^{\parallel}(y) = \frac{\partial m(y)}{\partial y}. \quad (46)$$

The saddle-point equation for the Q_0 can be written in a similar way:

$$Q_0 = \int dy_0 P(y_0;x_1) m^2(y_0;x_1), \quad (47)$$

where

$$P(y_0;x_1) = \frac{1}{\sqrt{2\pi Q_0 J^2}}\exp\left(-\frac{y_0^2}{2Q_0J^2}\right),$$

$$m(y_0;x_1) = \frac{\int Dy_1 Z_{\rm eff}^{x_1}(y_0+y_1) m(y_0+y_1)}{\int D\widetilde{y}_1 Z_{\rm eff}^{x_1}(y_0+\widetilde{y}_1)}, \quad (48)$$

are discrete versions of the continuous functions $P(y,x)$,$m(y,x)$ introduced above.

Optimizing the one-step free energy with respect to Q_1, M, R, and Q_0 yields the saddle-point equations Eqs. (43)–(47). To capture equilibrium states, we should further extremize with respect to the Parisi parameter x_1, i.e., $\frac{\partial f}{\partial x_1}=0$, which yields the further condition

$$-\frac{\beta^2J^2}{4}(Q_1^2-Q_0^2)m^2 = \int Dy_0 \ln\int Dy_1 Z_{\rm eff}^{x_1}(y_0+y_1) - x_1\int Dy_0 \frac{\int Dy_1 Z_{\rm eff}^{x_1}(y_0+y_1)\ln Z_{\rm eff}(y_0+y_1)}{\int Dy_1 Z_{\rm eff}^{x_1}(y_0+y_1)}. \quad (49)$$

It is a useful check that upon imposing $Q_1=Q_0$, the saddle-point equations for Q_0 and Q_1 reduce to the same replica symmetric constraint. When $M=0$, the local-field distribution, the free energy and the saddle-point equations reduce to those of the classical SK model, as it should be.

IV. PHASE DIAGRAM

Let us now study the phase diagram of our model (3). The gross features of the phase diagram we find are similar to the ones found in Refs. 15 and 16: The low-temperature phase exhibits three phases: a nonglassy superfluid at small J/t, an insulating (nonsuperfluid) glass phase at large J/t, and, most interestingly, a phase in between with both glassy order and superfluidity. However, as mentioned before, we find a distinctly different behavior of the phase boundaries than Ref. 15.

Moreover, we are able to analyze the limit $T\to 0$, whose properties were inaccessible in previous works.[15,16] The latter is of particular interest in the context of the superfluid-insulator transition.

The findings of the mean field analysis are in qualitative agreement with Monte Carlo studies in finite dimensions at low but finite temperatures. The analytical approach allows for a detailed analysis of the properties of the mixed phase, and of the glass-to-superglass transition.

A. High-temperature phase

The high-temperature phase is simple to describe. Since $M=0$, the system behaves identically to the paramagnetic phase of the classical SK model, and $R=1$ holds exactly. In this regime the static approximation is of course exact.

At large enough J/t, the leading instability upon lowering the temperature is the classical glass transition at $T_g=J$, as mentioned earlier. However, at small values of J/t, the tendency to form a superfluid wins. The instability condition toward XY symmetry breaking,

$$t\chi^{\perp} = t\left.\frac{\partial m_x(y=0)}{\partial h_x}\right|_{h_x=0} = 1, \quad (50)$$

can be evaluated exactly. In this expression h_x is a uniform transverse field. The transverse susceptibility is easily calculated for the replicated Hamiltonian with a Hubbard-Stratonovich transformation of the quadratic term $R\int d\tau d\tau' s^z(\tau)s^z(\tau')$. This results in the instability criterion

$$\frac{1}{t} = \chi^{\perp} = \beta\frac{\int dh e^{-h^2/2J^2}\sinh(\beta h)/\beta h}{\int dh e^{-h^2/2J^2}\cosh\beta h}. \quad (51)$$

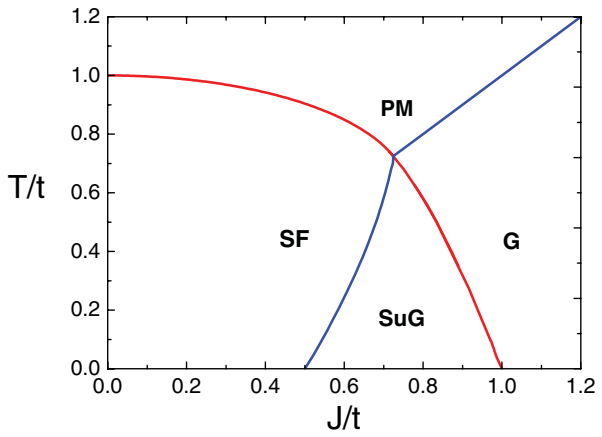

FIG. 1. (Color online) Phase diagram of glassy hard-core bosons. At high temperature, the straight blue line $T = J$ indicates the classical SK glass transition line. The red solid line shows the superfluid phase boundary which is given by the instability condition (51). The two lines cross at the tricritical point $(T/t)_T = (J/t)_T = 0.7248$. At low temperature, the blue line shows the phase boundary of the glass within the superfluid phase, as evaluated within the static approximation, cf. Eq. (16). The glass transition at $T = 0$ occurs at $(J/t)_{g,\mathrm{stat}} = 1/2$. The red solid line indicates the location of the onset of superfluidity within the glass phase, as evaluated within the full breaking of the replica symmetry to the instability condition (63). The superfluid transition at $T = 0$ takes place at $(J/t)_s = 1.00$.

The glass transition and the superfluid transition line cross at the tricritical point,

$$T_T = J_T = t_T \frac{\int dz e^{-z^2/2} \sinh(z)/z}{\int dz e^{-z^2/2} \cosh z} = 0.7248\, t_T. \quad (52)$$

The result (51) does not have the familiar looking form of an average local transverse susceptibility. However, it can indeed be recast in such a way. This furnishes us a better understanding of the interaction effects in the high-temperature phase, and at the same time illuminates the nature of the static approximation in the superfluid phases.

Let us rederive the above result directly from the nonreplicated Hamiltonian:

$$H = -\frac{1}{2}\sum_{ij} J_{ij} s_i^z s_j^z - \frac{t}{2N}\sum_{ij}\left(s_i^x s_j^x + s_i^y s_j^y\right) - \sum_i h_x s_i^x, \quad (53)$$

where h_x is an infinitesimal field. In a given classical Ising configuration, the spin i sees an "instantaneous" local field $h_i^z = \sum_{j\neq i} J_{ij} s_j^z$, while the transverse coupling is negligible in the paramagnetic phase where $N^{-1}\sum_j \langle s_j^x\rangle = M = 0 = N^{-1}\sum_j \langle s_j^y\rangle$. Thus the transverse susceptibility can be calculated as a site and configuration average of the susceptibility of a single spin sitting in an instantaneous field h, $\chi^\perp(h) = \int_0^\beta d\tau \langle s^x(\tau) s^x(0)\rangle = \tanh(\beta h)/h$.

The thermal distribution of instantaneous local fields of the SK model has been well studied,[61] and takes the rather simple form

$$P_{\mathrm{inst}}(h) = \cosh(\beta h)\frac{\exp\left(-\frac{h^2}{2J^2} - \frac{\beta^2 J^2}{2}\right)}{\sqrt{2\pi J^2}} \quad (54)$$

in the paramagnetic phase. Note that the instantaneous field distribution is not a simple Gaussian, but small fields are under-represented. This phenomenon is closely related to the suppression of small fields encountered in the cavity approach to Ising systems,[60] and is a precursor effect of the opening of the pseudogap in glassy phases at low temperatures.[26,56,57]

The total transverse susceptibility is obtained as an average of the local susceptibility $\chi(h)$ over $P_{\mathrm{inst}}(h)$:

$$\chi^\perp = \int dh\, P_{\mathrm{inst}}(h)\chi^\perp(h) = e^{-\beta^2 J^2/2}\int dh \frac{e^{-h^2/2J^2}}{\sqrt{2\pi J^2}}\frac{\sinh \beta h}{h}, \quad (55)$$

which indeed coincides with the replica result (51).

The static approximation for superfluid phases has a completely analogous effect. The approximation replaces the dynamically fluctuating exchange fields on the various sites by a random distribution of quasistatic fields. The latter differs from the distribution of frozen fields (which is Gaussian at high T) by a random Gaussian smearing with variance $J^2(R - q_{\mathrm{EA}})$, and a reweighing factor proportional to $\cosh(\beta h)$, which accounts for the fact that a small instantaneous field is less likely to be observed on a given site, as it implies a positive free-energy fluctuation in the environment.

B. Onset of glassy order within the superfluid

The instability toward forming a glass occurs when $J\chi^\parallel = \beta J \int d\tau d\tau' R(\tau - \tau') = 1$. Within the superfluid phase it is difficult to calculate this susceptibility exactly, and we thus first resort to the static approximation, $R(\tau - \tau') \to R$. The instability of the statically approximated free energy occurs when $J\beta R = 1$, βR being the static approximation for the longitudinal susceptibility $\chi^\parallel$. Within the nonglassy superfluid phase there are no frozen fields, $P(y) = \delta(y)$. Thus, from Eqs. (44) and (45), the two relevant saddle-point equations read

$$M = m_x(y = 0), \quad (56)$$

$$\beta R = \chi^\parallel_{\mathrm{loc}}(y = 0), \quad (57)$$

where $m_x(y = 0)$ and $\chi^\parallel_{\mathrm{loc}}(y = 0)$ are to be evaluated from Eqs. (39)–(42) and (46).

They have a relatively simple low-temperature limit. One verifies that it is self-consistent to assume that

$$\beta R = \chi^\parallel \to \frac{r}{t}, \quad M \to 1 - m\frac{T}{t}, \quad (58)$$

with finite numbers r,m, as $T \to 0$.

Injecting this into the above self-consistency equations, and evaluating the Gaussian integral over y_R in Eq. (39) around the stationary point, the equations simplify to

$$r = 1 + \frac{J^2 r}{t^2 - J^2 r} + O(T/t), \quad (59)$$

$$m = \frac{J^2 r}{2(t^2 - r J^2)} + O(T/t). \quad (60)$$

This yields the solution for the susceptibility $J\chi^\parallel = Jr/t = \frac{t}{2J}(1 + \sqrt{1 - 4(\frac{J}{t})^2})$.

The static approximation predicts the quantum glass transition to occur at the critical point,

$$\left(\frac{J}{t}\right)_{g,\mathrm{stat}} = \frac{1}{2}, \quad (T=0), \tag{61}$$

where $J\chi^{\parallel}=1$.

It is difficult to predict whether we over- or underestimate the phase boundary with the static approximation in the superfluid phase. This is because the approximation has two competing effects with respect to the onset of glassy order. On one hand, we approximate the dynamic longitudinal susceptibility by the static one. Since the latter is bigger, we tend to overestimate the stability of the glassy ordering of s^z. This effect is well known from the SK model in a (constant) transverse field Γ.[50,62,63] On the other hand, the static approximation underestimates quantum fluctuations of s_x, at least at low T. Indeed we see above that at $T=0$, the static approximation predicts maximal transverse order, $M=1$, independently of the value of J/t, while it is easy to show that quantum fluctuations around the transverse ferromagnetic state decrease the magnetization as $M=1-O[(J/t)^2]$. The overestimate of M leads to an underestimate of the longitudinal susceptibility, and thus of the tendency to glassy order. In view of these competing tendencies, it is hard to predict on which side with respect to Eq. (61) the exact glass instability will be located.

However, there is a simple way to obtain an upper bound for the quantum critical point. In the superfluid phase our model (3) is very similar to the SK model in a constant transverse field Γ,[63] with the difference that the effective transverse field Mt is self-generated and has to be determined self-consistently. However, it is clear that the effective transverse field is always smaller than t. From quantum Monte Carlo results for the transverse field SK model, one knows that a quantum glass phase is obtained for $J/\Gamma \geqslant 0.76$.[50] This implies that the model studied in the present work must certainly be in a glassy phase if $J/t \geqslant 0.76$. The latter value is thus an upper bound for $(J/t)_g$. Approaching from large values of J/T we will find below in Eq. (66) that the nonsuperfluid glass phase becomes unstable toward superfluidity already at $(J/t)_s = 1.00$. Hence we conclude that a phase with both superfluid and glassy order parameters exists for a substantial range of parameters covering at least the interval $0.76 \leqslant J/t \leqslant 1.00$.

C. Superfluid instability within the insulating glass phase

1. Instability criterion

Our discussion of the phase boundaries will be complete, once we have addressed the superfluid instability with in the glass phase at large J/t. The instability condition reads

$$t\int dy P(y) \left.\frac{\partial m_x(y)}{\partial h_x}\right|_{h_x=0} = 1, \tag{62}$$

where $P(y)$ is the nontrivial distribution of frozen local fields in the classical glass phase of the SK model. The properties of $P(y)$ are well studied, and turn out to be crucial to understand the low-temperature behavior of the phase boundary and the physics of the glassy superfluid-to-insulator quantum phase transition.

We recall that in the nonsuperfluid glass phase the static approximation is exact with $R=1$, so that the instability criterion can be expressed in the form

$$t\int dy P(y) \frac{\int Dy_R \sinh(\beta(y+y_R))\frac{1}{y+y_R}}{\int Dy_R \cosh[\beta(y+y_R)]} = 1, \tag{63}$$

where $Dy_R = \frac{1}{\sqrt{2\pi(1-q_{\mathrm{EA}})J^2}} \exp(-\frac{y_R^2}{2(1-q_{\mathrm{EA}})J^2})$. This condition can be expressed in terms of the instantaneous field distribution as

$$t\int dh\, P_{\mathrm{inst}}(h) \frac{\tanh(\beta h)}{h} = 1, \tag{64}$$

where

$$P_{\mathrm{inst}}(h) = \int P(y)dy \frac{\cosh\beta h}{\cosh\beta y} \frac{\exp\left(-\frac{\beta(h-y)^2}{2h_O} - \frac{\beta h_O}{2}\right)}{\sqrt{2\pi h_O/\beta}} \tag{65}$$

is the instantaneous field distribution, which was first derived in Ref. 61. The term $h_O = \beta J^2(1-q_{\mathrm{EA}})$ is known as Onsager's back reaction. Equation (64) can be recognized as a BCS equation, where the instantaneous field distribution $P_{\mathrm{inst}}(h)$ takes the role of the density of states.

The temperature-dependent local-field distribution can be obtained from a numerical solution of the self-consistent set of full RSB equations (25)–(28), from which the phase boundary of the insulator-to-superfluid transition is deduced. This yields the solid (red) line in Fig. 1. For comparison we also evaluate the phase boundary within a one-step approximation, which works well at moderate temperatures. However, it fails badly at low T where a nonphysical reentrance of the superfluid instability would be predicted, and the quantum phase transition at $T \to 0$ is completely missed.

We note in passing that the thermodynamics of the insulating phase is essentially classical because of the scaling of the transverse coupling as t/N. If instead t were random and scaled as $1/\sqrt{N}$, the glass phase would also exhibit quantum fluctuations and would not reduce to the purely classical SK model. In that case, the analysis of the transition would become much more complicated. However, even though the thermodynamics can be obtained by a purely classical saddle-point computation, one should not conclude that excitations do not have any quantum dynamics.

2. Low temperatures and quantum phase transition

At low temperatures, the most prominent feature of the local-field distribution $P(y)$ is a linear pseudogap which opens at small fields. The latter is required to assure the stability of the glass phase,[64,65] in a very similar manner as the Efros-Shklovskii Coulomb gap arises in electron glasses with unscreened, long-range $1/r$ interactions.[27,57] More precisely, it is known that $P(y)=\alpha|y|+O(T)$ with $\alpha=0.301$ for fields in the range $T \ll |y| \ll J$, while the distribution decays like a Gaussian for $|y| \gg J$. At zero temperature the pseudogap extends down to $y=0$ (i.e., the chemical potential in the terminology of hard-core bosons), while at finite but low temperatures $T \ll J$, $P(y)$ assumes a scaling form $P(y)=Tp(y/T)$ with $P(0)=\mathrm{const}$ and $p(x \gg 1) = \alpha|x| + \mathrm{const}$.[56,57]

This scaling form asserts that only a fraction of $(T/J)^2$ is thermally active. Therefore the Edwards-Anderson parameter tends to 1 as $1 - q_{\rm EA} \sim (T/J)^2$. Accordingly, as $T \to 0$ there is no difference between the distribution of frozen and instantaneous fields, $P_{\rm inst}(h)$, since no thermal fluctuations are left. In this limit the instability condition (63) for onset of superfluidity then takes the form

$$t_s \int dy \frac{P(y; T = 0)}{|y|} = 1. \tag{66}$$

Using the above-mentioned features of $P(y)$ at low T one can easily obtain a rough estimate for the superfluid-insulator transition point as $(J/t)_s \simeq 1.05 \pm 0.1$. However, since the precise value is also sensitive to the part of $P(y)$ at high fields, $y \geqslant J$, a full numerical evaluation of the condition (66) is necessary to obtain the exact location of the quantum critical point. Using high precision data for $P(y; T)$ at low T from Ref. 66, we find $(J/t)_s \simeq 1.00 \pm 0.01$.

We emphasize an important difference between the quantum phase transition we have found here and a standard BCS transition. The latter, in the presence of a constant low-energy density of states always yields a finite T_c, even though it becomes exponentially small in $1/t$ for small t. In our glassy system the situation is fundamentally different in that the frustrated interactions suppress the density of states around the chemical potential with $P(y \to 0) \to 0$. This quenches the tendency for superfluidity and allows for a superfluid-to-insulator transition at a finite value of t, even in the mean field limit of $N \to \infty$, which we consider here.

This has important consequences for the nature of excitations and transport properties across the superfluid-insulator transition. In particular, the transition to the Bose insulator is accompanied by the Anderson localization of lowest energy excitations, whereas higher energy excitations remain delocalized relatively far into the insulator.[67] We believe that the physics revealed by this mean field model is relevant for Coulomb frustrated bosonic systems which undergo a transition from a superfluid to a Bose glass state in finite dimensions. This will be discussed in detail elsewhere.[68]

It is interesting to compare our mean field predictions for the phase diagram with the three-dimensional (3D) quantum Monte Carlo (QMC) simulation results reported in Ref. 15. The mean field predictions for the quantum critical points actually match the numerical results surprisingly well. The latter were done for the Hamiltonian

$$H = -\sum_{\langle i,j \rangle} V'_{ij}(n_i - 1/2)(n_j - 1/2) - t' \sum_{\langle i,j \rangle} (b_i^\dagger b_j + \text{H.c.}), \tag{67}$$

with binary disorder, $V'_{ij} = \pm V'$ with equal probability. Contact with the mean field model (2) is made by replacing the coordination number with $N \to z = 6$ for the 3D cubic lattice, and taking a Gaussian disorder with the same variance, $V^2/z = V'^2$, as well as a hopping $t_b/z = t'$.

Recalling the dictionary (4), the mean field estimate of the superglass-to-glassy insulator quantum phase transition is

$$\left(\frac{V'}{t'}\right)_s^{\rm MF} = \left(\frac{V/\sqrt{z}}{t_b/z}\right)_s = \left(\frac{4J/\sqrt{z}}{2t/z}\right)_s = 2\sqrt{z}\left(\frac{J}{t}\right)_s \simeq 4.9 \approx \left(\frac{V'}{t'}\right)_s^{\rm QMC} \simeq 5, \tag{68}$$

which comes close to the extrapolation of QMC results to $T = 0$. The transition point between superglass and nonglassy superfluid is estimated from the static approximation as

$$\left(\frac{V'}{t'}\right)_g^{\rm MF} = 2\sqrt{z}\left(\frac{J}{t}\right)_{g,\rm stat} \simeq 2.45 \approx \left(\frac{V'}{t'}\right)_g^{\rm QMC} \simeq 3.2. \tag{69}$$

This indicates that the static approximation overestimates the stability of the superglass phase, similarly as what is known from the mean field version of the transverse field Ising spin glass.

The mean field prediction (with static approximation) for the interaction-to-hopping ratio $(V'/t')_T$ at the tricritical point is rather good, too,

$$\left(\frac{V'}{t'}\right)_T^{\rm MF} = 2\sqrt{z}\left(\frac{J}{t}\right)_{T,\rm stat} \simeq 3.55 \approx \left(\frac{V'}{t'}\right)_T^{\rm QMC} \simeq 3.8. \tag{70}$$

While the tricritical ordering temperature is overestimated by a factor of 2 (similarly as in the classical Ising spin glass),[69]

$$\left(\frac{T}{t'}\right)_T^{\rm MF} = \frac{z}{2}\left(\frac{T}{t}\right)_T \simeq 2.2, \tag{71}$$

$$\left(\frac{T}{t'}\right)_T^{\rm QMC} \simeq 1.1. \tag{72}$$

D. Robustness of the phase diagram to random-field disorder

In the previous sections we have seen that the model (3) possesses an intermediate phase which is simultaneously superfluid and glassy. We have determined the phase boundaries as instability lines, assuming second-order phase transitions. Indeed it seems unlikely that any of the instabilities could be preempted by a first-order transition. Since the superfluid-to-insulator transition at $(J/t)_s$ is of particular interest, we provide further arguments in this section that the parts of the phase diagram related to the phase boundary of the nonsuperfluid glass remain robust when disorder potentials, i.e., random fields ϵ_i of variance W^2, are restituted to the model. In particular we will show that glass and superfluid transition lines meet at a tricritical point at finite temperature T_T/J and $(J/t)_T$. Further, we determine the superfluid instability of the glass phase at $T = 0$ and show that it always occurs at a larger ratio (J/t) than the tricritical point, $(J/t)_s > (J/t)_T$. This suggests that for any W the transition line between nonsuperfluid and superfluid glass is not reentrant

as a function of temperature. The absence of reentrance in turn suggests that the quantum phase transition out of the insulating glass remains second order, independent of the strength of the disorder potential.

The Hamiltonian with a disorder potentials reads

$$H = -\sum_{i<j} J_{ij} s_i^z s_j^z - \frac{t}{N}\sum_{i<j}\left(s_i^x s_j^x + s_i^y s_j^y\right) + \sum_i \epsilon_i s_i^z. \quad (73)$$

The disorder potential breaks the Z_2 symmetry, therefore $Q_{a\neq b} \neq 0$ already in the high-temperature phase, where it assumes a constant replica symmetric value Q_0. The glass phase occurs at the Almeida-Thouless instability, which is given by[57]

$$\beta^2 J^2 \int dy \frac{P_W(y)}{\cosh^4(\beta y)} = 1, \quad (74)$$

where

$$P_W(y) = \frac{\exp\left(-\frac{y^2}{2(W^2+J^2Q_0)}\right)}{\sqrt{2\pi(W^2+J^2Q_0)}}, \quad (75)$$

and Q_0 satisfies the self-consistent equation

$$Q_0 = \int P_W(y)\tanh^2(\beta y)dy. \quad (76)$$

The instability toward the superfluid phase is instead determined by

$$1/t = \int dh\, P_{\rm inst}(h)\frac{\tanh(\beta h)}{h}, \quad (77)$$

where

$$P_{\rm inst}(h) = \int dy\, P_W(y)\frac{\cosh\beta h}{\cosh\beta y}\frac{\exp\left(-\frac{\beta(h-y)^2}{2h_O} - \frac{\beta h_O}{2}\right)}{\sqrt{2\pi h_O/\beta}}, \quad (78)$$

with the Onsager field $h_O = \beta J^2(1-Q_0)$.[57,61]

The glass and superfluid transition lines meet at a tricritical point at $(T/J)_T$ and $(J/t)_T$, which are to be evaluated from Eqs. (74)–(78). In the limit $W/J \gg 1$, one finds the tricritical temperature $(T/J)_T = \frac{4}{3\sqrt{2\pi}}\frac{J}{W} + O(\frac{J^2}{W^2})$ and $\beta_T h_O \to 3/2$, as $W/J \to \infty$.

The superfluid transition at $T = 0$ is given by the condition

$$(1/t)_s = \int dh \frac{P_{\rm inst}(h;T=0)}{|h|}. \quad (79)$$

In the limit $W/J \gg 1$, $P_{\rm inst}(h;T=0)$ is known to have a simple structure:

$$P_{\rm inst}(h;T=0) = \begin{cases} \alpha|h|/J^2, & |h| \ll h^\star, \\ \frac{\exp\left(\frac{-(h-\gamma J^2/W)^2}{2W^2}\right)}{\sqrt{2\pi W^2}}, & |h| \gg h^\star, \end{cases} \quad (80)$$

with a smooth crossover between the two limiting forms around $\frac{h^\star}{J} = \frac{1}{\alpha\sqrt{2\pi}}\frac{J}{W} + O(\frac{J^2}{W^2})$. The value of the constant $\gamma = O(1)$ can be estimated by the normalization condition $\int dh\, P_{\rm inst}(h;T=0) = 1$, but will be irrelevant below.

For $W/J \gg 1$, $(J/t)_s$ and $(J/t)_T$ both behave as $\frac{4}{\sqrt{2\pi}}\frac{\ln(W/J)}{W/J}$ to leading order. Their difference scales like cJ/W.

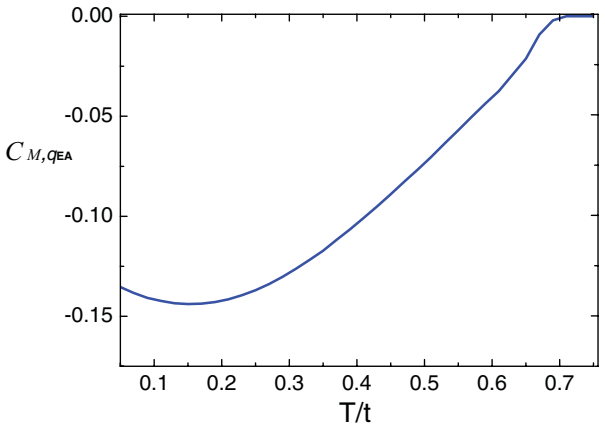

FIG. 2. (Color online) Cross correlation between the local order parameters for superfluidity and glassy order, respectively. The correlations are evaluated from Eq. (85) in the temperature range $0.05 < T/t < 0.75$ at fixed disorder $J/t = 0.724$, close to the ratio corresponding to the tricritical point. The local order parameters are anticorrelated, the maximal anticorrelation occurs at intermediate temperatures.

The coefficient c can be evaluated easily by rescaling the variables $\beta_T h = \hat{h}$ and $\beta_T y = \hat{y}$,

$$c = \lim_{W/J\to\infty}\frac{W}{J}\left[(J/t)_s - (J/t)_T\right] = \sqrt{\frac{2}{3}}\int\frac{d\hat{h}}{\hat{h}}[\hat{p}(\hat{h};T=0) - f(\hat{h})], \quad (81)$$

where

$$\hat{p}(\hat{h};T=0) = \lim_{W/J\to\infty}\sqrt{\frac{3}{2}}W P_{\rm inst}(\hat{h}/\beta_T;T=0) \quad (82)$$

and

$$f(\hat{h}) = \int\frac{d\hat{y}}{\sqrt{2\pi}}\frac{\sinh(\hat{h})}{\cosh(\hat{y})}\frac{\exp\left(\frac{-(\hat{h}-\hat{y})^2}{3} - \frac{3}{4}\right)}{\sqrt{2\pi}}, \quad (83)$$

and we have used $\beta_T h_O \to 3/2$. We approximate Eq. (80) by extending the formula all the way to $h^\star$ and neglecting the shift of field h and we get

$$\hat{p}(\hat{h};T=0) \approx \begin{cases} \frac{2\alpha\hat{h}}{\sqrt{3\pi}}, & |\hat{h}| \leqslant \frac{3}{4\alpha}, \\ \sqrt{\frac{3}{4\pi}}, & |\hat{h}| \geqslant \frac{3}{4\alpha}. \end{cases} \quad (84)$$

Evaluating Eq. (81) numerically, using the estimate Eq. (84), one obtains $c = 0.231 > 0$, establishing that $(J/t)_s > (J/t)_T$ even in the presence of strong disorder. We point out that Eq. (84) overestimates Eq. (82), but this overestimation should be much smaller than $c = 0.231$.

V. PROPERTIES OF THE SUPERGLASS PHASE

Having established the phase diagram of the model, we now focus on the properties of the bulk of the superglass phase. There the interplay between temperature, glassy order, and superfluid order induce several interesting phenomena, which potentially survive also in finite dimensional models of frustrated bosons. In the following, we investigate how the

glassy and superfluid orders evolve with temperature, and how they are locally correlated.

A. Competition between glassy and superfluid order

While the effective transverse field $h_i^x = Mt$ is uniform for every site, the frozen longitudinal field h_i^z depends on the site (and on the pure state in which the system is frozen). Therefore the magnetization of the local spin s_i due to the local field $\overrightarrow{h_i} = (h_i^x, h_i^z)$ fluctuates from site to site. It is interesting to study the correlation of the local magnetization, whose components are the local order parameters of the glassy and the superfluid order, respectively. More precisely, we investigate the following correlation function:

$$C_{M,q_{\rm EA}} \equiv \frac{\frac{1}{N}\sum_i \langle s_i^x\rangle\langle s_i^z\rangle^2 - \left(\frac{1}{N}\sum_i \langle s_i^x\rangle\right)\left(\frac{1}{N}\sum_i \langle s_i^z\rangle^2\right)}{\left(\frac{1}{N}\sum_i \langle s_i^x\rangle\right)\left(\frac{1}{N}\sum_i \langle s_i^z\rangle^2\right)} = \frac{\int dy P(y) m_x(y) m^2(y) - M q_{\rm EA}}{M q_{\rm EA}}. \quad (85)$$

We have evaluated the correlation function (85) within the static one-step RSB approximation in the center of the superglass phase ($J/t = 0.724$) as a function of temperature ($0.05 < T/t < 0.75$); see Fig. 2. Not surprisingly, the correlation is negative, since glassy and superfluid orders compete with each other. Indeed, one easily checks that for every pair of sites (i,j) it holds that if $\langle s_i^z\rangle^2 < \langle s_j^z\rangle^2$ then $\langle s_i^x\rangle > \langle s_j^x\rangle$. The maximal amplitude of the normalized correlation $C_{M,q_{\rm EA}}$ is only of order ≈ 0.1, suggesting that in the superfluid phase the nonuniformity of the two local order-parameter fields is actually not very strong. It may be that the one-step approximation underestimates these correlations a bit. The relative weakness of the anticorrelations might be the reason they have not been noticed in the quantum Monte Carlo studies of Refs. 15 and 16.

B. Nonmonotonicity of the superfluid order

In the superglass phase, the glass order parameter $q_{\rm EA} = Q_1$ monotonously decreases with increasing temperature, as one should expect. However, surprisingly, the superfluid order parameter M exhibits nonmonotonic behavior with a maximum at an intermediate crossover temperature T_m, as shown in Fig. 3. Below T_m, the superfluid order parameter M decreases, anomalously, when lowering the temperature. Above T_m, M decreases with increasing temperature as usual in a standard superfluid.

This phenomenon is related to the anticorrelation between glassy and superfluid order discussed in the previous section. While on one hand, thermal fluctuations tend to diminish both glassy and superfluid order, there appears to be a low-temperature regime $T < T_m$, where quantum fluctuations of the superfluid order are dominant. Due to the competition between the glassy and the superfluid order, the thermally induced decrease of the glassy order enhances the superfluid order. This effect dominates over the direct thermal effects on the superfluidity.

It seems natural that it is the superfluid order, which undergoes such nonmonotonic behavior, rather than the glassy order. Indeed, we expect the latter to react less sensitively to the diminution of quantum fluctuations due to decreasing transverse fields.

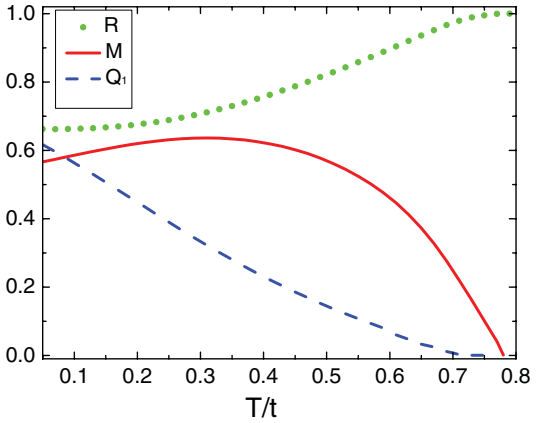

FIG. 3. (Color online) The order parameters in the superglass phase as a function of temperature $0.05 < T/t < 0.79$ at the fixed disorder $J/t = 0.724$. The blue dashed line indicates the Edwards-Anderson order parameter Q_1 (in one-step approximation), which monotonously decreases with increasing temperature. The superfluid order parameter M (red solid line) exhibits nonmonotonic behavior. The long-time (static) on-site charge correlation R (green dotted line) becomes 1 in the disordered high-T phase.

We note that also the local order-parameter correlations $C_{M,q_{\rm EA}}$ exhibit a nonmonotonous behavior within our static one-step approximation, as shown in Fig. 2. The absolute value of $C_{M,q_{\rm EA}}$ increases with temperature at very low temperatures, and decreases at higher temperatures. This can be seen again as a consequence of the nonmonotonicity of the superfluid order. At fixed T, the larger h^x the stronger the normalized anticorrelation $C_{M,q_{\rm EA}}$. Since $h^x = tM$ initially increases with T, it is natural to expect an increasing $C_{M,q_{\rm EA}}$ until eventually thermal fluctuations become dominant and diminish $C_{M,q_{\rm EA}}$.

VI. DISCUSSION

In this paper we have analyzed a fully connected mean field model. The full connectivity is not a real limitation, however. Indeed, one can generalize the model to a highly connected Cayley tree. While this does not affect the thermodynamics of the model, this generalization allows for the study of localization and delocalization of excitations, since this model is endowed with a notion of distance. The analysis of localization properties is of particular interest in the vicinity of the superglass-to-insulating-glass quantum phase transition, where the boson system collectively delocalizes into a superfluid at low energies. The nature of higher energy excitations in the insulator are crucially affected by the suppression of low-energy states in the glass, leading to a nontrivial excitation spectrum at the glassy SI transition. The details of this analysis will be reported elsewhere.[68]

What features of the mean field model should be expected to carry over to finite dimensions? In the present model we find a genuine insulating phase at $T = 0$, which suppresses the superfluidity, due to the strong self-generated on-site disorder within the glassy phase. A crucial ingredient for the

suppression of superfluidity is the linear pseudogap within the glass phase. A very similar pseudogap is known to occur in disordered Coulomb interacting systems, where it is due to unscreened $1/r$ interactions between charged particles. This Coulomb gap may well be of importance in strongly disordered superconductors and play a significant role in the competition between glassy insulating behavior and superconductivity. In particular, in materials with strong negative U centers, one may think of preformed electron pairs constituting hard core bosons which interact with Coulomb repulsions.[70] The power law suppression of the low-energy density of states makes it likely that the superfluid condensate is entirely destroyed once the hopping becomes too small. For short-range interactions the density of states is merely reduced at low energy, but does not tend to zero. On a Cayley tree of very large connectivity this will always lead to delocalization, unless the hopping t is scaled down logarithmically with the connectivity. In finite dimensions, however, sufficiently strong disorder is known to suppress superfluidity,[23] and thus one may expect that at sufficiently large ratios J/t, the disordered boson model will localize due to *spontaneously* created, frozen-in local fields. Such a conclusion may be suggestive from a straight extrapolation of the quantum Monte Carlo results of Ref. 15 to $T = 0$, but it seems difficult to exclude a scenario in which T_c becomes merely exponentially small with J/t. A more careful analysis will be necessary to settle this question in finite dimensional, short-range interacting glasses.

As for the coexistence phase, the superglass, the numerical data[15,16] provide evidence that it exists also in finite dimensions. It would be interesting to confirm and quantify the local anticorrelation of order parameters in such simulations. From our mean field analysis one expects that the anticorrelation is in fact relatively weak. A further nontrivial prediction with measurable consequences is the nonmonotonicity of the superfluid order parameter, which should translate into an equivalent nonmonotonicity of the superfluid stiffness as a function of temperature. This nonmonotonicity has its origin in the softening of the glassy order at low T, a feature which may potentially survive in finite dimensions, especially when the lattice connectivity is large, or the interactions are not too short ranged. We should caution though that we obtained this effect by employing a static approximation and a replica symmetry breaking at the one-step level only. However, we believe that it is a real feature of the model.

As discussed earlier, a number of experiments have already shown promising indications of possible coexistence of glassy order with superfluidity. We hope that our analysis will help to unambiguously identify such phases in experiments. Note that finding an experimental system exhibiting a glassy superfluid-insulator transition might also be of great interest to study the intricate interplay of interactions and disorder with respect to glassy ergodicity breaking, and quantum ergodicity breaking, i.e., Anderson localization.

ACKNOWLEDGMENTS

We thank L. Leuzzi for providing us the high precision data on the local-field distribution $P(y)$ of the SK model at finite temperature. We thank L. Foini, M. Gingras, F. Zamponi, and A. W. Sandvik for useful discussions.

APPENDIX: DERIVATION OF RS FREE ENERGY

With standard replica trick[49] we can get the RS free energy:

$$\begin{aligned}\beta f =& -\frac{J^2\beta^2}{4}Q^2+\frac{t\beta}{2}M^2+\frac{J^2\beta^2}{4}\int_0^1\int_0^1 d\tau d\tau' R^2(\tau,\tau')\\ &-\lim_{n\to 0}\frac{1}{n}\ln \mathrm{Tr}\mathcal{T}\exp\left[\frac{J^2\beta^2}{2}Q\left(\sum_a\int_0^1 ds_a^z(\tau)\right)^2\right.\\ &+\frac{J^2\beta^2}{2}\sum_a\int_0^1 d\tau\int_0^1 d\tau' R(\tau,\tau')s_a^z(\tau)s_a^z(\tau')\\ &-\frac{J^2\beta^2}{2}Q\sum_a\left(\int_0^1 ds_a^z(\tau)\right)^2\\ &\left.+t\beta\sum_a\int_0^1 d\tau M_a^x s_a^x(\tau)\right].\end{aligned}\tag{A1}$$

Under static approximation: $R(\tau,\tau') = R$, we have

$$\begin{aligned}\beta f =& -\frac{J^2\beta^2}{4}Q^2+\frac{t\beta}{2}M^2+\frac{J^2\beta^2}{4}R^2\\ &-\lim_{n\to 0}\frac{1}{n}\ln \mathrm{Tr}\mathcal{T}\exp\left[\frac{J^2\beta^2}{2}Q\left(\sum_a\int_0^1 ds_a^z(\tau)\right)^2\right.\\ &+\frac{J^2\beta^2}{2}(R-Q)\sum_a\left(\int_0^1 ds_a^z(\tau)\right)^2\\ &\left.+t\beta\sum_a\int_0^1 dt M_a^x s_a^x(\tau)\right].\end{aligned}\tag{A2}$$

According to Hubbard-Stratonovich transformations, we linearize the quadratic terms $(\sum_a\int_0^1 ds_a^z(\tau))^2$ and $(\int_0^1 ds_a^z(\tau))^2$ by introducing extra fields y_0 and y_R:

$$\begin{aligned}\beta f =& -\frac{J^2\beta^2}{4}Q^2+\frac{J^2\beta^2}{4}R^2+\frac{t\beta}{2}M^2\\ &-\int Dy_0 \ln\int Dy_R \mathrm{Tr}\exp[\beta(y_0 s^z+y_R s^z+tMs^x)]\\ =& -\frac{J^2\beta^2}{4}Q^2+\frac{J^2\beta^2}{4}R^2+\frac{t\beta}{2}M^2\\ &-\int Dy_0\ln\int Dy_R\cosh(\beta\sqrt{(y_0+y_R)^2+t^2M^2}).\end{aligned}\tag{A3}$$

One can get Eq. (36) following the similar steps above.

[1] Cao Lie-zhao, D. F. Brewer, C. Girit, E. N. Smith, and J. D. Reppy, Phys. Rev. B **33**, 106 (1986).

[2] J. D. Reppy, J. Low Temp. Phys. **87**, 205 (1992).

[3] K. Yamamoto, Y. Shibayama, and K. Shirahama, Phys. Rev. Lett. **100**, 195301 (2008).

[4] E. Kim and M. H. W. Chan, Science **305**, 1941 (2004).

[5]E. Kim and M. H. W. Chan, Nature (London) **427**, 225 (2004).
[6]Ann Sophie C. Rittner, J. D. Reppy, Phys. Rev. Lett. **97**, 165301 (2006).
[7]S. Balibar and F. Caupin, J. Phys.: Condens. Matter **20**, 173201 (2008).
[8]M. Boninsegni, N. Prokof'ev, and B. Svistunov, Phys. Rev. Lett. **96**, 105301 (2006).
[9]B. Svistunov, Physica B **404**, 521 (2009).
[10]L. Dang, M. Boninsegni, and L. Pollet, Phys. Rev. B **79**, 214529 (2009).
[11]J. West, O. Syshchenko, J. Beamish, and M. H. W. Chan, Nat. Phys. **5**, 598 (2009).
[12]B. Hunt, E. Pratt, V. Gadagkar, M. Yamashita, A. V. Balatsky, and J. C. Davis, Science **324**, 632 (2009).
[13]G. Biroli, C. Chamon, and F. Zamponi, Phys. Rev. B **78**, 224306 (2008).
[14]G. Biroli, B. Clark, L. Foini, and F. Zamponi, Phys. Rev. B **83**, 094530 (2011).
[15]K. M. Tam, S. Geraedts, S. Inglis, M. J. P. Gingras, and R. G. Melko, Phys. Rev. Lett. **104**, 215301 (2010).
[16]G. Carleo, M. Tarzia, and F. Zamponi, Phys. Rev. Lett. **103**, 215302 (2009).
[17]L. Foini, G. Semerjian, and F. Zamponi, Phys. Rev. B **83**, 094513 (2011).
[18]D. Larson and Ying-Jer Kao, e-print arXiv:1202.3908 (to be published).
[19]A. Goldman and N. Markovic, Phys. Today **51**(11), 39 (1998).
[20]A. F. Hebard and M. A. Paalanen, Phys. Rev. Lett. **65**, 927 (1990); M. A. Paalanen, A. F. Hebard, and R. R. Ruel, *ibid.* **69**, 1604 (1992).
[21]M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).
[22]T. Giamarchi and H. J. Schulz, Phys. Rev. B **37**, 325 (1988).
[23]M. Ma and P. A. Lee, Phys. Rev. B **32**, 5658 (1985); M. Ma, B. I. Halperin, and P. A. Lee, *ibid.* **34**, 3136 (1986).
[24]W. Krauth, N. Trivedi, and D. Ceperley, Phys. Rev. Lett. **67**, 2307 (1991).
[25]J. H. Davies, P. A. Lee, and T. M. Rice, Phys. Rev. Lett. **49**, 758 (1982).
[26]S. Pankov and V. Dobrosavljević, Phys. Rev. Lett. **94**, 046402 (2005).
[27]M. Müller and L. B. Ioffe, Phys. Rev. Lett. **93**, 256403 (2004).
[28]M. Ben-Chorin, Z. Ovadyahu, and M. Pollak, Phys. Rev. B **48**, 15025 (1993).
[29]G. Martinez-Arizala, D. E. Grupp, C. Christiansen, A. M. Mack, N. Marković, Y. Seguchi, and A. M. Goldman, Phys. Rev. Lett. **78**, 1130 (1997).
[30]N. Marković, C. Christiansen, G. Martinez-Arizala, and A. M. Goldman, Phys. Rev. B **65**, 012501 (2001).
[31]B. Deissler, M. Zaccanti, G. Roati, C. D'Errico, M. Fattori, M. Modugno, G. Modugno, and M. Inguscio, Nat. Phys. **6**, 354 (2010).
[32]R. Abou-Chacra, P. W. Anderson, and D. J. Thouless, J. Phys. C **6**, 1734 (1973).
[33]R. Bhatt, in *Spin Glasses and Random Fields*, Directions in Condensed Matter Physics Vol. 12, edited by A. P. Young (World Scientific, Singapore, 1998).
[34]A. J. Bray and M. A. Moore, J. Phys. C **13**, L655-60 (1980).
[35]Y. Y. Goldschmidt and P. Y. Lai, Phys. Rev. Lett. **64**, 2467 (1990).
[36]*Quantum Ising Phases and Transitions in Transverse Ising Models*, B. K. Chakrabarti, A. Dutta, and P. Sen (Springer-Verlag, Heidelberg, 1996).
[37]J. Ye, S. Sachdev, and N. Read, Phys. Rev. Lett. **70**, 4011 (1993).
[38]N. Read, S. Sachdev, and J. Ye, Phys. Rev. B **52**, 384 (1995).
[39]A. Georges, O. Parcollet, and S. Sachdev, Phys. Rev. Lett. **85**, 840 (2000).
[40]S. Gopalakrishnan, B. L. Lev, and P. M. Goldbart, Phys. Rev. Lett. **107**, 277201 (2011).
[41]P. Strack and S. Sachdev, Phys. Rev. Lett. **107**, 277202 (2011).
[42]S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).
[43]A. Andreanov and M. Müller (unpublished).
[44]M. Müller and L. B. Ioffe, e-print arXiv:0711.2668 (to be published).
[45]A. Leggett, Phys. Rev. Lett. **25**, 1543 (1970); G. Chester, Phys. Rev. A **2**, 256 (1970); A. F. Andreev and I. M. Lifshitz, Sov. Phys. JETP **29**, 1107 (1969) [Zh. Eksp. Teor. Fiz. **56**, 2057 (1969)]; H. Matsuda and T. Tsuneto, Prog. Theor. Phys. Suppl. **46**, 411 (1970).
[46]D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975); S. Kirkpatrick and D. Sherrington, Phys. Rev. B **17**, 4384 (1978).
[47]A. A. Pastor and V. Dobrosavljević, Phys. Rev. Lett. **83**, 4642 (1999).
[48]D. Dalidovich and V. Dobrosavljević, Phys. Rev. B **66**, 081107 (2002).
[49]M. Mézard, G. Parisi, and M. A. Virasoro, *Spin-Glass Theory and Beyond*, Lecture Notes in Physics Vol. 9 (World Scientific, Singapore, 1987).
[50]M. J. Rozenberg and D. R. Grempel, Phys. Rev. Lett. **81**, 2550 (1998).
[51]A. Georges, G. Kotliar, W. Krauth, and M. Rozenberg, Rev. Mod. Phys. **68**, 13 (1996).
[52]H.-J. Sommers and W. Dupont, J. Phys. C **17**, 5785 (1984).
[53]B. Duplantier, J. Phys. A **14**, 283 (1981).
[54]G. Parisi and G. Toulouse, J. Phys. (Paris) Lett. **41**, L361 (1980).
[55]A. Crisanti and T. Rizzo, Phys. Rev. E **65**, 046137 (2002).
[56]S. Pankov, Phys. Rev. Lett. **96**, 197204 (2006).
[57]M. Müller and S. Pankov, Phys. Rev. B **75**, 144201 (2007).
[58]G. Biroli and L. F. Cugliandolo, Phys. Rev. B **64**, 014206 (2001).
[59]T. Giamarchi and P. Le, Phys. Rev. B **52**, 1242 (1995).
[60]M. Mézard, G. Parisi, and M. A. Virasoro, Europhys. Lett. **1**, 77 (1986).
[61]M. Thomsen, M. F. Thorpe, T. C. Choy, D. Sherrington, and H.-J. Sommers, Phys. Rev. B **33**, 1931 (1986).
[62]G. Büttner and K. D. Usadel, Phys. Rev. B **42**, 6385 (1990).
[63]J. Miller and D. A. Huse, Phys. Rev. Lett. **70**, 3147 (1993).
[64]D. J. Thouless, P. W. Anderson, and R. G. Palmer, Philos. Mag. **35**, 593 (1977).
[65]R. G. Palmer and C. M. Pond, J. Phys. F **9**, 1451 (1979).
[66]A. Crisanti, L. Leuzzi, G. Parisi, and T. Rizzo, Phys. Rev. B **70**, 064423 (2004).
[67]The notion of delocalization requires a straightforward extension of the present model to a lattice model, which is not fully connected but maintains a large coordination number N.
[68]X. Yu and M. Müller (unpublished).
[69]K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).
[70]J. Mitchell, A. Gangopadhyay, V. Galitski, and M. Müller, e-print arXiv:1110.0074 (to be published).