

Empirical oscillating potentials for alloys from *ab initio* fits and the prediction of quasicrystal-related structures in the Al-Cu-Sc system

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By fitting to a database of *ab initio* forces and energies, we can extract pair potentials for alloys with a simple six-parameter analytic form including Friedel oscillations, which give a remarkably faithful account of many complex intermetallic compounds. Furthermore, such potentials are combined with a method of discovering complex zero-temperature structures with hundreds of atoms per cell, given only the composition and the constraint of known lattice parameters, using molecular-dynamics quenches. We apply this approach to structure prediction in the Al-Cu-Sc quasicrystal-related system.

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Various problems in materials modeling can be addressed only by “empirical” interatomic potentials.¹ If we wish to evaluate a physical property (e.g., total energy) of some material with a complex crystal structure, we cannot directly insert the results of crystallographic refinements, as they (almost always) include sites with mixed or fractional occupancies. To obtain valid results, we must assign those occupancies plausibly based on computed energy differences. A single relaxation with fast *ab initio* codes such as VASP² may be tractable in a cell of 10^3 atoms; however, if (say) 5% of those atoms are uncertain, this must be repeated many times to find the one optimum state out of 2^{50} possibilities.

But with empirical, approximate potentials that can be evaluated in negligible time, this optimization is tractable and can be followed up by *ab initio* relaxation to obtain the most accurate positions and total energies. When combined with effective search algorithms, such as genetic algorithms or the “cell-constrained melt-quench” method (presented in Sec. II), this is a powerful tool for *ab initio* structure discovery. Some other questions that call for empirical potentials are phonon spectra (or other dynamics), and thermodynamic simulations of phase transformations in complex alloys.

A popular framework of empirical potentials is the “embedded-atom method” (EAM),^{3,4} in which the full Hamiltonian contains the usual pair term $V_{ij}(r_i - r_j)$ but also an implicitly many-atom term $\sum_i U(\rho(r_i))$, where $\rho(r_i)$ is a sum of contributions at atom i from nearby atoms. Accurate EAM potentials are straightforward to extract for pure elements but demand patience and skill to obtain even for binary systems; there is no *systematic* recipe for multicomponent systems.

Here we present an alternative approach fitting only pair interactions but incorporating Friedel oscillations. These “empirical oscillating pair potentials” (EOPP) have the form⁵

$$V(r) = \frac{C_1}{r^{\eta_1}} + \frac{C_2}{r^{\eta_2}} \cos(k_* r + \phi_*). \quad (1)$$

All six parameters, including k_* , are taken as independent in the fit for each pair of elements. Equation (1) was inspired by effective potentials (e.g., Refs. 6 and 7) used in previous work on structurally complex metals, e.g. quasicrystals.^{8,9} In such systems, energy differences between competing structures are often controlled by second- and third-neighbor wells due to

Friedel oscillations, which are a consequence (mathematically) of Fourier transforming the Fermi surface or (physically) are equivalent to the Hume-Rothery stabilization by enhancing the strength of structure factors that hybridize states across the Fermi surface.¹⁰ In Pettifor’s framework (Ref. 10, Sec. 6.6), the short-range repulsion is captured by the first term of Eq. (1); the medium-range potential (first-neighbor well) as well as the long-range oscillatory tail are captured by the second term, their relative weights being adjusted by the η_1 and η_2 parameters. Since the second term has this double duty of representing both the nearest-neighbor and long-distance behaviors, the fitted $1/r^{\eta_2}$ decay generally does not agree with analytic asymptotic result of $1/r^3$; also, k_* need not match the Fermi wave vector, and these parameters take different values for the six kinds of pair potentials.

In the rest of this Brief Report, we describe how our potentials are fitted (typically for a particular composition range) and then demonstrate their capabilities through case studies in the alloy system Al-Cu-Sc, which has local order similar to the binary quasicrystal i (CaCd) or to Zn-rich Mg-Zn alloys. Along with this, we also describe a method of “constrained cell quenching” that accesses low-energy structures in surprisingly large cells. Finally, we will summarize other systems where EOPP have been applied and discuss their limitations.

I. DATABASE AND POTENTIAL FITTING

The parameters in Eq. (1) are fitted to an *ab initio* data set, generated using the density functional theory- (DFT) based code VASP,² with PW91 exchange and correlation functional.¹¹ We use the plane-wave basis set energy cutoff $E_{\text{cut}} = 273.2$ eV and k -point meshes with the density of ~ 1.5 points per \AA^{-3} . The database combines both the fully relaxed $T = 0$ structures and molecular dynamics (MD) simulations at high T . Four criteria in choosing structures for the database are (i) to bracket the composition range of interest, (ii) to mix simple and complex structures, (iii) to ensure adequately many contacts of every kind (in particular, nearest neighbors of the least abundant species), and (iv) to have similar atom densities.¹² In *ab initio* MD simulations of the simpler structures, a supercell is always used with dimensions comparable to the fitting potential cutoff radius, which (for the fit procedure) is always 12\AA .

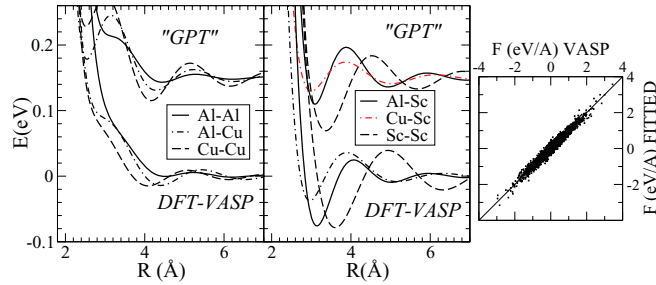


FIG. 1. (Color online) (a) (Top) EOPP form [Eq. (1)] fitted to “GPT” potentials (Ref. 7); the fitted/GPT curves lie exactly on top of each other. The Cu-Sc potential is not available from Ref. 7; the figure shows a fit to *ab initio* data with the other five pairs constrained to GPT form. (Bottom) The EOPP form fitted to VASP force and energy data. The curves fitted to “GPT” are shifted by 0.15 eV/atom for clarity. (b). Scatter plot of the forces for the EOPP fit to Eq. (1) from the fitted Al-Cu-Sc potentials shown in (a), bottom. The pair potential forces \mathbf{F}_j are on the vertical axis and *ab initio* data are on the horizontal axis.

We define each structure’s energy as a difference relative to a coexisting mixture (with the same total composition) of reference phases, chosen to bracket all database compositions. Every structure is used for both forces (from MD at high T) and energy differences¹³ (high- T MD, as well as relaxed at $T = 0$). For the high- T portion, we took one snapshot of each structure at the end of a short *ab initio* MD run. Usually $\gtrsim 10^3$ force components enter the fit, along with ~ 50 energy differences. Typical forces are ~ 3 eV/Å and typical energy differences are ~ 0.3 eV/atom; the fit residuals are $\sim 5\%$ and $\sim 1\%$, respectively [see, e.g., Fig. 1(a)].

Our least-squares fit minimizes (by the Levenberg-Marquardt algorithm) $\chi^2 \equiv \sum \Delta E_i^2 / \sigma_E^2 + \sum |\Delta \mathbf{F}_j|^2 / \sigma_F^2$, where $\{\Delta E_i\}$ and $\{\Delta \mathbf{F}_j\}$ are the energy and force residuals; we found a weighting ratio $\sigma_E / \sigma_F \sim 10^{-3}$ Å was optimal, and so neither energies nor forces dominate the fit.

There is some risk of converging to a false minimum (or not converging at all, from an unreasonable initial guess). Thus, it is important to repeat the fit from several starting guesses. For this we used, e.g., potentials first fitted to pure elements or binary systems and additionally a library of parameter sets previously fitted for a different alloy system. The fitted parameters in Eq. (1) for each of our examples are available as supplementary information;⁵ similar potentials were plotted in Refs. 14 (for Al-Mg) and 15 (for Sc-Zn).

For our specific case of Al-Cu-Sc, our database had 84 relaxed energies at $T = 0$ K from (in addition to the pure elements) the binaries Al₃Sc.cP4, Cu₂Sc.tI6, Al₂Sc.cF24, and Al₂Cu.cF12 and ternaries AlCuSc.hP12, Mg₂Cu₆Ga₅.cP39 (with Mg \rightarrow Sc, Ga \rightarrow Al), and AlCrCu₂.cF16 (with Cr \rightarrow Sc); structures are identified by Pearson labels. The database also had 7428 force points at $T > 0$ K, taken from all those structures and from Al₂Sc.hP12 and Sc(Al_{1-x}Cu_x)₆.cI168. The EOPP potentials give very good agreement with this database, as shown in Fig. 1(a): the rms deviation of forces was 0.11 eV/Å and that of energy differences was 21.7 meV/atom, with relative weights set to maximize accuracy of the force data.¹⁶

Our empirical potentials can be compared with Moriarty’s “GPT” potentials,¹⁷ derived from a systematic expansion, which are known for all but one of the six Al-Cu-Sc pair potentials [see Fig. 1(b)]. First, when the GPT potentials are fitted to the EOPP form [Eq. (1)], the curves virtually lie on top of each other: thus, the EOPP form *can* represent all features of the GPT potentials. Second, the GPT and EOPP potentials show a strong similarity; the main mismatches are (i) the EOPP has no first-neighbor well for Al-Cu and Cu-Cu, (ii) the oscillation wavelength for Sc-Sc differs, and (iii) the overall energy scale of the GPT pair potential is too small by $\sim 50\%$.¹⁸ In practice, empirically fitted potentials account for some of the many-body contributions by modifying their pair terms and, hence, work better than truncating a systematic expansion (e.g., GPT)¹⁷ after the pair terms.

II. CONSTRAINED-CELL MELT QUENCHING

We now turn to the second method which, together with fitted potentials, has enabled the present structure study and others. In many alloy systems, with no structural information known except the composition and the unit cell, structures with > 100 atoms per cell may be predicted from careful “melt-quenching” (MQ) simulations. The relation to the EOPP notion is that the larger systems—particularly when supercells are used—are too large for direct *ab initio* calculations, so empirical potentials are crucial for melt quenching. This method has been applied with GPT potentials to improve known structures of the decagonal Al-Co-Ni quasicrystals.¹⁹

In most cases, annealing requires “tempering Monte Carlo (MC)”²⁰ wherein ~ 10 samples are annealed simultaneously at equally spaced temperatures spanning across the melting temperature. Each tempering cycle includes a short MD run (~ 1 ps with 1-fs steps) followed by lattice-gas MC annealing (~ 200 attempts per atom) in which the chemical identities of two randomly selected atoms may be swapped. Then, pairs of samples may be swapped using a Metropolis-like criterion. (For the Al-Cu-Sc structures we studied, our temperatures spanned 600–1700 K with spacing $\Delta T = 100$ K, and the interaction cutoff was set to $r_{\text{cut}} = 9$ Å.) The resulting structure may be tested subsequently by *ab initio* calculations of the total energy, for which we used VASP.²

A key diagnostic in a tempering simulation is the (time-dependent) energies $E_m(t)$ of all samples, as specific heat is approximately $\Delta E / \Delta T$, where $\Delta E = E_{m+1} - E_m$. For our structures ΔE peaked around $T = 1400$ K, indicating the melting point. In our cases with 52 or 84 atoms per primitive cell, several independently initialized runs yielded identical low- T structures, which took 100–200 cycles.

III. TARGET STRUCTURES

We now apply empirical pair potentials and cell-constrained melt quenching to realistic structure prediction in the Al-Cu-Sc system for two recently discovered phases^{21,22} in which lattice parameters were known from electron microscopy but no single-grain structures were available for structure determination. We label each phase by its Pearson symbol. These belong to a family of phases in which atom size is the salient attribute: Ca, Mg, Sc, and rare earths are “large” atoms,

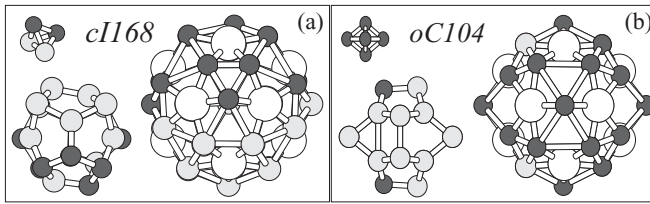


FIG. 2. (a) Tsai cluster in the $cI168$ structure and (b) “pseudo-Tsai” cluster found in the $oC104$ structure. View along twofold direction. In each panel, the successive shells are (i) tetrahedron Al_2Cu_2 or octahedron Cu_6 , (ii) dodecahedron $(Al,Cu)_{20}$, and (iii) icosidodecahedron $(Al,Cu)_{30}$ plus icosahedron Sc_{12} . Large white circles represent Sc, dark smaller circles Cu, and light smaller circles Al.

constituting $\sim 1/6$ by number, whereas Al, Zn, Cu, and Cd are the majority “small” atoms.

Our first target is “ $cI168$ ” with 84 atoms per primitive cell, hypothesized to be isostructural with the $ScZn_6$ phase. No single-crystal data are available; a preliminary Rietveld refinement of powder data technique²² confirmed the $ScZn_6$ type structure, with refined lattice parameter $a = 13.52 \text{ \AA}$. The chemical ordering and occupancies of Al/Cu could not be determined reliably prior to our modeling.²²

The $ScZn_6$ phase is an “approximant” of the recently discovered thermodynamically stable, icosahedral quasicrystal $i(ScZn)$, meaning its unit cell contents are identical to a fragment of the quasicrystal structure. $i(ScZn)$ has the same atomic structure as $i(CaCd)$,²³ which (along with similar rare earth-Cd quasicrystals) are the only known stable binary quasicrystals. Each site is specific to either a small atom (Cd or Zn) or a large atom (Sc, Ca, or rare earth).

Both $cI168$ - $ScZn_6$ and the related quasicrystals are understood to be packings of a three-shell, icosahedrally symmetric cluster called the “Tsai cluster.” Its outermost shell has an icosahedron of 12 large atoms plus 30 small atoms (forming an icosidodecahedron) on the midpoints of the large-large bonds [see Fig. 2(a)], and inside that is a dodecahedron of 20 small atoms. At the cluster center is a *tetrahedron* of small atoms; as this breaks the icosahedral symmetry, it has many degenerate orientations, which leads to interesting slow dynamics of reorientations²⁴ and to structural ordering transitions at low T in the related crystals.

Our second target is “ $oC104$,” of composition $Al_{38.8}Cu_{45.7}Sc_{15.5}$. Initially²¹ the lattice parameters $a = 8.32 \text{ \AA}$, $b = 8.36 \text{ \AA}$, $c = 21.99 \text{ \AA}$ and C-centered orthorhombic Bravais lattice were identified from powder data but that was insufficient to refine the structure.

IV. RESULTS: PHASE STABILITY AND PSEUDO-TSAI CLUSTERS

Constrained-cell melt quenching using the same database-fitted EOPP potentials converged to low-temperature structures in both cases, the energies of which were subsequently evaluated by VASP. In the $cI168$ case this was the known structure. In the $oC104$ case, a previously unknown structure was obtained,⁵ computed unstable by only 5 meV/atom with respect to $cI168$; its validity was confirmed by a subsequent refinement of the powder data²¹ with this as the starting point.

This predicted structure had space group $Amm2$, in which there were 11 Cu, 7 Al, and 4 Sc Wyckoff sites, with no chemical disorder.

The fundamental motif in both $cI168$ and $oC104$ is an icosahedral cluster whose innermost shell has less symmetry. In $cI168$ this is a Tsai cluster [Fig. 2(a)] in which each of the three Al/Cu shells has composition $Al_{0.5}Cu_{0.5}$. Strikingly, within each shell, the Al and Cu are segregated into hemispheres centered on the fivefold axis of the icosahedron;²⁵ furthermore, the Al parts of each shell overlay the Cu parts of the preceding one and vice versa [see Fig. 2(a)]. Due to this symmetry breaking, one expects the low-temperature phase to have at least several equivalent domains, corresponding to different orientations of the pseudo-Tsai clusters.

In the $oC104$ case we find a variant motif²⁶ that we call the “pseudo-Tsai” cluster, in which the innermost shell is a Cu_6 octahedron [Fig. 2(b)]. This cluster was described in $Mg_2Zn_{11}cP39$,²⁶ where the outer shell is a Zn_8Mg_{12} dodecahedron plus a Zn_{12} icosahedron.²⁷

The experimental (powder data) refinement²¹ of $oC104$ is an *average* structure, with higher symmetry (space group $Cmmm$) than our model, accommodating 3 Sc Wyckoff sites and 13 mixed Al/Cu sites (one of the latter being half-occupied). Each experimental site derives from one or two sites in our model, displaced, on average, by $\sim 0.17 \text{ \AA}$ (Al), $\sim 0.12 \text{ \AA}$ (Cu), and 0.08 \AA (Sc).²⁸ The mean deviation of the refined Al/Cu content from the averaged model content is 16%.

The electronic density of states for $cI168$ has a deep, narrow pseudogap at the Fermi energy, which tends to stabilize a unique composition. In contrast, $oC104$ has a shallower and wider pseudogap, suggesting a range of degenerate compositions, so substitutional entropy might stabilize this phase at higher temperatures.

The pseudo-Tsai clusters in $oC104$ (and also the less complex $cP39$) adjoin by sharing atoms such that their centers are closer by a factor ~ 1.618 (golden ratio) than Tsai clusters would be. Could pseudo-Tsai clusters be the basis of the recently reported $i(AlCuSc)$ quasicrystal?²⁹

V. CONCLUSION

We have shown that empirical potentials with the simple oscillating form of Eq. (1), fitted from *ab initio* data and combined with a “cell constrained” brute-force quenching, allows detailed predictions of fairly complex low-temperature optimal structures in Al-Cu-Sc alloys, based on the very limited input of known lattice parameters and composition. Finding the correct structure depends sensitively on having a quantitatively realistic potential, which is achievable only if that potential is constructed or fitted from *ab initio* calculations. One would expect that the oscillating analytical form of Eq. (1) is natural only for simple metals (e.g., Al or Mg, for which it does very well). But, in fact, the EOP potentials sometimes work quite well even when angular or many-body interactions are important, e.g., the transition-metal neighbors in Al-Cu-Sc. But, not surprisingly, they do poorly for elemental Zn or Ga. Of course, *any* pair potential fails when the electron density has large variations in space (as at vacancies, edge dislocations, or surfaces) or in the (many) cases where bond directionality (due to covalent bonding) is prominent.

We believe the EOPP potentials are quite broadly applicable to mimic the atomic interactions of many metallic systems with sufficient accuracy to stand in for *ab initio* energies when those would be computationally prohibitive. Although the EOP potentials were not formally presented before this paper, they have already been applied to a variety of intermetallics: (i) the site contents in complex structures, e.g., Al-Mg¹⁴ or Al-Zn-Mg;³⁰ (ii) solving the complete structures of complex Mg-rich Mg-Pd phases (with >400 atoms/cell) in conjunction with diffraction, when the latter alone is insufficient;³¹ (iii) quantitatively accurate phonon spectra in complex structures of Sc-Zn¹⁵ and Mg-Zn²⁶ alloys,³² and (iv) the dynamics

of the Tsai cluster tetrahedra in the *cI*168 structure ScZn₆,²⁴ as well as the arrangements of the asymmetric inner Al₁₀ shell in the pseudo-Mackay icosahedral clusters in quasicrystal-related Al-Ir, Al-Pd-Mn, and Al-Cu-Fe phases.³³

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¹“Empirical potentials” are explicit, classical functions of coordinates that do not require an eigenvalue calculation.

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¹²For a successful fit, the database must be limited to a relatively narrow range of (electron) densities, so the Fermi wave vector [hence k_* in Eq. (1)] has a consistent value. Even our high- T MD samples were constrained to have the *same* density as at $T = 0$ rather than the physical zero-pressure values.

¹³Fits from forces alone were insufficiently constrained, including energy differences, even with low weight, improved the fit.

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²⁵Properly “pseudo-fivefold,” as this is in a cubic crystal.

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