Symmetry-protected topological phases in noninteracting fermion systems

Xiao-Gang Wen
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
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Symmetry-protected topological (SPT) phases are gapped quantum phases with a certain symmetry, which can all be smoothly connected to the same trivial product state if we break the symmetry. For noninteracting fermion systems with time reversal ($\hat{T}$), charge conjugation ($\hat{C}$), and/or $U(1)$ ($\hat{N}$) symmetries, the total symmetry group can depend on the relations between those symmetry operations, such as $\hat{T}\hat{N}\hat{T}^{-1} = \hat{N}$ or $\hat{T}\hat{N}\hat{T}^{-1} = -\hat{N}$. As a result, the SPT phases of those fermion systems with different symmetry groups have different classifications. In this paper, we use Kitaev's K-theory approach to classify the gapped free-fermion phases for those possible symmetry groups. In particular, we can view the $U(1)$ as a spin rotation. We find that superconductors with the S_z spin-rotation symmetry are classified by Z in even dimensions, while superconductors with the time reversal plus the S_z spin-rotation symmetries are classified by Z in odd dimensions. We show that all 10 classes of gapped free-fermion phases can be realized by electron systems with certain symmetries. We also point out that, to properly describe the symmetry of a fermionic system, we need to specify its full symmetry group that includes the fermion number parity transformation $(-)^{\hat{N}}$. The full symmetry group is actually a projective symmetry group.

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I. INTRODUCTION

We used to believe that all possible phases and phase transitions are described by Landau symmetry-breaking theory.[1–3] However, the experimental discovery of fractional quantum Hall states[4,5] and the theoretical discovery of chiral spin liquids[6,7] indicates that new states of quantum matter without symmetry breaking and without long-range order can exist. Such a new kind of orders is called topological order,[8,9] because their low-energy effective theories are topological quantum field theories.[10] At first, the theory of topological order was developed based on its robust ground-state degeneracy on compact spaces and the associated robust non-Abelian Berry phases.[8,9] Later, it was realized that topological order can be characterized by the boundary excitations,[11,12] which can be directly probed by experiments. One can develop a theory of topological order based on the boundary theory.[13]

Since its introduction, we have been trying to obtain a systematic understanding of topological orders. Some progress has been made for certain simple cases. We found that all two-dimensional (2D) Abelian topological orders can be classified by integer K matrices.[14–16] The 2D nonchiral topological orders (which can be smoothly connected to time-reversal and parity-symmetric states) are classified by spherical fusion category.[17–20] The recent realization of the relation between topological order and long-range entanglement[19] (defined through local unitary transformations[21,22]) allows us to separate another simple class of gapped quantum phases—symmetry-protected topological (SPT) phases. SPT phases are gapped quantum phases with a certain symmetry, which can all be smoothly connected to the same trivial product state if we break the symmetry. A generic construction of bosonic SPT phases in any dimension using the group cohomology of the symmetry group was obtained in Refs. 23 and 24. The constructed SPT phases include interacting bosonic topological insulators and topological superconductors (and much more).

Another type of simple system is the free-fermion system, for which a classification of gapped quantum phases can be obtained through K theory[25–27] or a nonlinear σ model of disordered fermions.[28] They include the noninteracting topological insulators[29–35] and the noninteracting topological superconductors.[36–40] Most gapped quantum phases of free-fermion systems are SPT phases protected by some symmetries, such as topological insulators protected by the time-reversal symmetry. However, some others have intrinsic topological orders (i.e., stable even without any symmetry), such as topological superconductors with no symmetry. Just like the interacting topological ordered phases, the topological phases for free fermions are also characterized by their gapless boundary excitations. The boundary excitations play a key role in the theory and experiments of free-fermion topological phases.

For noninteracting fermion systems with time-reversal (generated by $\hat{T}$), charge-conjugation (generated by $\hat{C}$), and/or $U(1)$ (generated by $\hat{N}$) symmetries, the total symmetry group may not simply be $Z_2^T \times Z_2^C \times U(1)$. The group can take different forms, depending on the different relations between those symmetry operations, such as $\hat{T}\hat{N}\hat{T}^{-1} = \hat{N}$ or $\hat{T}\hat{N}\hat{T}^{-1} = -\hat{N}$. As a result, the gapped phases of those fermion systems with different symmetry groups have different classifications. In this paper, we use Kitaev's K-theory approach to classify the gapped free-fermion phases for those different symmetry groups. In Table I, we list some electron systems and their full symmetry group G_f. In Tables II and III, the ten classes[25,28] of gapped free-fermion phases protected by those many-body symmetry groups (and many other symmetry groups) are listed. Here we have assumed that the fermions form one irreducible representation of the full symmetry group. The result will differ if the fermions contain several distinct irreducible representations of the full symmetry group (see Sec. III E). In Refs. 25 and 28, the ten classes of gapped free-fermion phases are already associated with many different many-body symmetries of electron systems. In this paper, we generalize the results in Refs. 25 and 28 to more symmetry groups.

We note that electron systems, with $\hat{T}\hat{N}\hat{T}^{-1} = \hat{N}$, only realize a subset of the possible symmetry groups. The emergent

TABLE I. Electron systems and their full symmetry groups G_f. The groups are defined in Table IV. The symmetry-group symbols have the following meaning: for example, $G^{++}_{--}(U,T,C)$ is a symmetry group generated by $\hat{N}$ [the $U(1)$ fermion number conservation or spin rotation], $\hat{T}$ (the time reversal), and $\hat{C}$ (the charge conjugation or 180° spin rotation). The $\pm$ subscripts and superscripts describe the relations between the transformations $\hat{N}$, $\hat{T}$, and/or $\hat{C}$ (see Table IV). Sometimes, when we describe the symmetry of a fermion system, we do not include the fermion number parity transformation $(-)^{\hat{N}}$ in the symmetry group G. Here the full symmetry group G_f does include the fermion number parity transformation $(-)^{\hat{N}}$. So the full symmetry group of a fermion system with no symmetry is $G_f = Z_2^f$ generated by the fermion number parity transformation. G_f is a Z_2^f extension of G: $G = G_f/Z_2^f$. Free-electron systems with symmetry $G^{++}_{--}(U,T,C)$ actually have a higher symmetry $G[SU(2),T]$. Similarly, free-electron systems with symmetry $G_-(U,C)$ actually have a higher symmetry $SU(2)$.

Electron systems	Full symmetry group G_f
Insulators with spin-orbital coupling and spin order (or non-coplanar spin order) $(i\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_{i'} + i\hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_{j'} + i\hat{c}_k^\dagger \mathbf{n_3}\cdot\sigma\hat{c}_{k'}) + (\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_i + \hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_j + \hat{c}_k^\dagger \mathbf{n_3}\cdot\sigma\hat{c}_k)$	$U(1)$
Superconductors with spin-orbital coupling and spin order (or non-coplanar spin order) $\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_i + \hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_j + \hat{c}_k^\dagger \mathbf{n_3}\cdot\sigma\hat{c}_k + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$	none $= Z_2^f$
Insulators with spin-orbital coupling $i\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_{i'} + i\hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_{j'} + i\hat{c}_k^\dagger \mathbf{n_3}\cdot\sigma\hat{c}_{k'}$ (*symmetry*: charge-conservation and time-reversal symmetries)	$G^-_-(U,T)$
Superconductors with spin-orbital coupling and real pairing $i\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_{i'} + i\hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_{j'} + i\hat{c}_k^\dagger \mathbf{n_3}\cdot\sigma\hat{c}_{k'} + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: time-reversal symmetry)	$G_-(T) = Z_4$
Superconductors with S_z conserving spin-orbital coupling and real pairing $i\hat{c}_i^\dagger\sigma^z\hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: time-reversal and S_z spin-rotation symmetries)	$G^+_-(U,T) = U(1)\times Z_2^T$
Superconductors with coplanar spin order and real pairing $\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_i + \hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: a combined time-reversal and 180° spin-rotation symmetry)	$G_+(T) = Z_2^T\times Z_2^f$
Superconductors with real pairing and collinear spin order $\hat{c}_i^\dagger\sigma^z\hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: S_z spin rotation and a combined time-reversal and 180° S_y spin-rotation symmetry)	$G^-_+(U,T) = U(1)\rtimes Z_2^T$
Insulators with coplanar spin order $\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_i + \hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_j$ (*symmetry*: charge-conservation and a combined time-reversal and 180° spin-rotation symmetries)	$G^-_+(U,T) = U(1)\rtimes Z_2^T$
Superconductors with real triplet $S_z = 0$ pairing $\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} + \hat{c}_{\downarrow i}\hat{c}_{\uparrow j}$ (*symmetry*: a combined 180° S_y spin-rotation and time-reversal symmetry, a combined 180° S_y spin-rotation and charge-rotation symmetry, and S_z spin-rotation symmetry)	$G^{--}_{++}(U,T,C)$
Superconductors with time-reversal, 180° S_y spin-rotation, and S_z spin-rotation symmetries	$G^{++}_{--}(U,T,C)$
Superconductors with real singlet pairing $\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j}$ (*symmetry*: time-reversal and $SU(2)$ spin-rotation symmetries)	$G[SU(2),T]$
Superconductors with 180° S_y spin-rotation and S_z spin-rotation symmetries	$G_-(U,C)$
Superconductors with complex singlet pairing $e^{i\theta_{ij}}(\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$ (*symmetry*: $SU(2)$ spin-rotation symmetry)	$SU(2)$
Insulators with spin-orbital coupling and intersublattice hopping $i\hat{c}_{i_A}^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_{i_B} + i\hat{c}_{j_A}^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_{j_B} + i\hat{c}_{k_A}^\dagger \mathbf{n_3}\cdot\sigma\hat{c}_{k_B}$ (*symmetry*: charge-conservation, time-reversal, and charge-conjugation symmetries)	$G^{-+}_{--}(U,T,C)$

fermion (such as the spinon in spin liquid) may realize other possible symmetry groups, since their symmetries are described by projective symmetry groups which can be different for different topologically ordered states.[41]

The $p = 0$ line in Table II classifies two types of electron systems: (1) insulators with only fermion number conservation (which includes integer quantum Hall states) and (2) superconductors with only S_z spin-rotation symmetry, which can be realized by superconductors with collinear spin order. The $p = 1$ line in Table II classifies superconductors with only time-reversal and S_z spin-rotation symmetry [full symmetry group $G^+_-(U,T)$], which can be realized by superconductors with real pairing and S_z conserving spin-orbital coupling.

In Table III, the $p = 0$ column classifies electronic insulators with coplanar spin order [full symmetry group $G^-_+(U,T)$, which contains the charge conservation and a time-reversal symmetry]. The $p = 1$ column classifies electronic superconductors with coplanar spin order and real

TABLE II. Classification of the gapped phases of noninteracting fermions in d-dimensional space, for some symmetries. The space of the gapped states is given by $C_{p+d \bmod 2}$, where p depends on the symmetry group. The distinct phases are given by $\pi_0(C_{p+d \bmod 2})$. "0" means that only trivial phases exist. Z means that nontrivial phases are labeled by nonzero integers and the trivial phase is labeled by 0.

Symmetry	$C_p\|_{\text{for } d=0}$	$p\backslash d$	0	1	2	3	4	5	6	7	Example
$U(1)$ $G_-(C)$	$\frac{U(l+m)}{U(l)\times U(m)} \times \mathbb{Z}$	0	Z	0	Z	0	Z	0	Z	0	(Chern) insulator; Superconductor with collinear spin order
$G^{+}_{\pm}(U,T)$ $G^{+}_{--}(T,C)$ $G^{+}_{+-}(T,C)$	$U(n)$	1	0	Z	0	Z	0	Z	0	Z	Superconductor with realpairing and S_z conserving spin-orbital coupling

pairing [full symmetry group $G_+(T)$, which contains a time-reversal symmetry]. The $p = 2$ column classifies electronic superconductors with non-coplanar spin order (full symmetry group "none"). The $p = 3$ column classifies electronic superconductors with spin-orbital coupling and real pairing [full symmetry group $G_-(T)$, which contains the time-reversal symmetry]. The $p = 4$ column classifies electronic insulators with spin-orbital coupling [full symmetry group $G^{-}_{-}(U,T)$, which contains the charge-conservation and the time-reversal symmetry). The $p = 5$ column classifies electronic insulators on bipartite lattices with spin-orbital coupling and only intersublattice hopping [full symmetry group $G^{-+}_{--}(U,T,C)$, which contains the charge-conservation, the time-reversal, and a charge-conjugation symmetry). The $p = 6$ column classifies electronic spin-singlet superconductors with complex pairing [full symmetry group $SU(2)$]. The $p = 7$ column classifies electronic spin-singlet superconductors with real pairing (full symmetry group $G[SU(2),T]$, which contains the $SU(2)$ spin rotation and time-reversal symmetry).[42]

In this paper, we first discuss a simpler case where fermion systems have only $U(1)$ symmetry. Then we discuss a more complicated case where fermion systems can have time-reversal, charge-conjugation, and/or $U(1)$ symmetries. The classification of the gapped phases with translation symmetry and the classification of nontrivial defects with protected gapless excitations are also studied.

II. GAPPED FREE FERMION PHASES: THE COMPLEX CLASSES

A. The $d = 0$ case

Let us first consider a zero-dimensional free fermion system with one orbital. How many different gapped phases do we have for such a system? The answer is two. The two different gapped phases are labeled by $m = 0,1$: the $m = 0$ gapped phase corresponds to the empty orbital, while the $m = 1$ gapped phase corresponds to the occupied orbital. But "two" is not the complete answer. We can always add occupied and empty orbitals to the system and still regard the extended system as in the same gapped phase. So we should consider a system with n orbitals in the $n \to \infty$ limit. In this case, the zero-dimensional gapped phases are labeled by an integer m in Z, where m (with a possible constant shift) still corresponds to the number of occupied orbitals.

TABLE III. Classification of gapped phases of noninteracting fermions in d spatial dimensions, for some symmetries. The space of the gapped states is given by $R_{p-d \bmod 8}$, where p depends on the symmetry group. The phases are classified by $\pi_0(R_{p-d \bmod 8})$. Here $\mathbb{Z}_2$ means that there is one nontrivial phase and one trivial phase labeled by 1 and 0.

Symmetry	$G^{-}_{+}(U,T)$ $G^{-}_{+-}(T,C)$	$G_+(T)$ $G^{+}_{++}(T,C)$ $G^{--}_{++}(U,T,C)$ $G^{-+}_{++}(U,T,C)$ $G^{+-}_{-+}(U,T,C)$ $G^{++}_{++}(U,T,C)$	None $G_+(C)$ $G^{-}_{++}(T,C)$ $G^{-}_{-+}(T,C)$ $G_+(U,C)$	$G_-(T)$ $G^{+}_{-+}(T,C)$ $G^{--}_{-+}(U,T,C)$ $G^{-+}_{-+}(U,T,C)$ $G^{+-}_{++}(U,T,C)$ $G^{++}_{-+}(U,T,C)$	$G^{-}_{-}(U,T)$ $G^{-}_{--}(T,C)$	$G^{--}_{--}(U,T,C)$ $G^{-+}_{--}(U,T,C)$ $G^{+-}_{--}(U,T,C)$ $G^{++}_{+-}(U,T,C)$	$G_-(U,C)$ $SU(2)$	$G^{--}_{+-}(U,T,C)$ $G^{-+}_{+-}(U,T,C)$ $G^{+-}_{+-}(U,T,C)$ $G^{++}_{--}(U,T,C)$ $G[SU(2),T]$
$R_p\|_{\text{for } d=0}$	$\frac{O(l+m)}{O(l)\times O(m)} \times \mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)} \times \mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$
$d = 0$	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0	0	0
$d = 1$	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0	0
$d = 2$	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0
$d = 3$	0	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z
$d = 4$	Z	0	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$d = 5$	0	Z	0	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$d = 6$	$\mathbb{Z}_2$	0	Z	0	0	0	Z	$\mathbb{Z}_2$
$d = 7$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0	0	0	Z
Example	Insulator with coplanar spin order	Superconductor with coplanar spin order	Superconductor	Superconductor with time reversal	Insulator with time reversal	Insulator with time reversal and intersublattice hopping	Spin singlet superconductor	Spin singlet superconductor with time reversal

Now let us obtain the above result using a fancier mathematical setup. The single-body Hamiltonian of the n-orbital system is given by the $n \times n$ Hermitian matrix H. If the orbitals below a certain energy are filled, we can deform the energies of those orbitals to -1 and deform the energies of other orbitals to $+1$ without closing the energy gap. So, without losing generality, we can assume H to satisfy

$$H^2 = 1. \tag{1}$$

Such a Hermitian matrix has the form

$$H = U_{n\times n}\begin{pmatrix} I_{l\times l} & 0 \\ 0 & -I_{m\times m}\end{pmatrix}U^\dagger_{n\times n}, \tag{2}$$

where $n = l + m$ and $U_{n\times n} in U(n)$ is an $n \times n$ unitary matrix. But $U_{n\times n}$ is not a one-to-one labeling of the Hermitian matrix satisfying $H^2 = 1$. To obtain a one-to-one labeling, we note that $\left(\begin{smallmatrix} I_{l\times l} & 0 \\ 0 & -I_{m\times m}\end{smallmatrix}\right)$ is invariant under the unitary transformation $\left(\begin{smallmatrix} V_{l\times l} & 0 \\ 0 & W_{m\times m}\end{smallmatrix}\right)$ with $V_{l\times l} \in U(l)$ and $W_{m\times m} \in U(m)$. Thus, the space C_0 of the Hermitian matrix satisfying $H^2 = 1$ is given by $\cup_m U(l + m)/U(l) \times U(m)$, which, in the $n \to \infty$ limit, has the form

$$C_0 \equiv \frac{U(l+m)}{U(l)\times U(m)} \times \mathbb{Z}. \tag{3}$$

Clearly $\pi_0(C_0) = \mathbb{Z}$, which recovers the result obtained above using a simple argument: the zero-dimensional gapped phases of free conserved fermions are labeled by integers Z.

B. The properties of classifying spaces

The space C_0 is the complex Grassmannian—the space formed by the subspaces of (infinite-dimensional) complex vector space. It is also the space of the Hermitian matrix satisfying $H^2 = 1$. Actually, C_0 is a part of a sequence. More generally, a space C_p can be defined by first picking p fixed Hermitian matrices γ_i, $i = 1,2,\dots,p$, satisfying

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}. \tag{4}$$

Then C_p is the space of the Hermitian matrix satisfying

$$H^2 = 1, \quad \gamma_i H = -H\gamma_i, \quad i = 1,\dots,p. \tag{5}$$

To find the C_1 space, let us choose $\gamma_1 = I_{n\times n}$ that satisfy $\gamma_1^2 = 1$. But for such a choice, we cannot find any H that satisfies $\gamma_1 H = -H\gamma_1$. Actually, we have a condition on the choice of γ_1. We must choose a γ_1 such that $\gamma_1 H = -H\gamma_1$ and $H^2 = 1$ has a solution. So we should choose $\gamma_1 = \sigma^x \otimes I_{n\times n}$. We note that γ_1 is invariant under the following unitary transformations: $e^{i\sigma^x\otimes A_{n\times n}}e^{i\sigma^0\otimes B_{n\times n}} \in U(n) \times U(n)$ (where $A_{n\times n}$ and $B_{n\times n}$ are Hermitian matrices). Then H satisfying $H^2 = 1$ and $\gamma_1 H + H\gamma_1 = 0$ has the form

$$H = e^{i\sigma^x\otimes A_{n\times n}}e^{i\sigma^0\otimes B_{n\times n}}(\sigma^z \otimes I_{n\times n})e^{-i\sigma^0\otimes B_{n\times n}}e^{-i\sigma^x\otimes A_{n\times n}}, \tag{6}$$

whose positive and negative eigenvalues are paired. We see that the space C_1 is $U(n) \times U(n)/U(n) = U(n)$.

To construct the C_2 space, we can choose $\gamma_1 = \sigma^x \otimes I_{n\times n}$ and $\gamma_2 = \sigma^y \otimes I_{n\times n}$. Then H satisfying $H^2 = 1$ and $\gamma_i H + H\gamma_i = 0$, $i = 1,2$, has the form

$$H = \sigma^0 \otimes U_{n\times n}\left[\sigma^z \otimes \begin{pmatrix} I_{l\times l} & 0 \\ 0 & -I_{m\times m}\end{pmatrix}\right]\sigma^0 \otimes U^\dagger_{n\times n}, \tag{7}$$

where $n = l + m$ and $U_{n\times n} \in U(n)$. We see that the space $C_2 = C_0$.

To construct the C_3 space, we can choose $\gamma_1 = \sigma^x \otimes I_{n\times n}$, $\gamma_2 = \sigma^y \otimes I_{n\times n}$, and $\gamma_3 = \sigma^z \otimes I_{n\times n}$. But for such a choice, the equations $\gamma_i H + H\gamma_i = 0$, $i = 1,2,3$, and $H^2 = 1$ has no solution for H. So we need to impose the following condition on γ_i's:

$$\begin{aligned}&\text{The equations} \quad \gamma_i \mathrm{H} + \mathrm{H}\gamma_i = 0, \quad \mathrm{H}^2 = 1\\ &\text{have a solution for H.}\end{aligned} \tag{8}$$

(Later, we see that such a condition has an amazing geometric origin.) Let us choose $\gamma_1 = \sigma^x \otimes \sigma^x \otimes I_{n\times n}$, $\gamma_2 = \sigma^y \otimes \sigma^x \otimes I_{n\times n}$, and $\gamma_3 = \sigma^z \otimes \sigma^x \otimes I_{n\times n}$ instead. Then H satisfying $H^2 = 1$ and $\gamma_i H + H\gamma_i = 0$, $i = 1,2,3$, has the form

$$\begin{aligned} H = {}& e^{i\sigma^0\otimes\sigma^x\otimes A_{n\times n}}e^{i\sigma^0\otimes\sigma^0\otimes B_{n\times n}}(\sigma^0 \otimes \sigma^z \otimes I_{n\times n})\\ &\times e^{-i\sigma^0\otimes\sigma^0\otimes B_{n\times n}}e^{-i\sigma^0\otimes\sigma^x\otimes A_{n\times n}}. \end{aligned} \tag{9}$$

We find that $C_3 = C_1$.

Now, it is not hard to see that $C_p = C_{p+2}$, which leads to $\pi_d(C_p) = \pi_d(C_{p+2})$. Thus,

$$\pi_0(C_p) = \begin{cases} \mathbb{Z}, & p = 0 \mod 2,\\ \{0\}, & p = 1 \mod 2. \end{cases} \tag{10}$$

C. The $d \neq 0$ cases

Next we consider d-dimensional free conserved fermion systems and their gapped ground states. Note that the only symmetry that we have is the $U(1)$ symmetry associated with the fermion number conservation. We do not have translation symmetry and other symmetries.

To be more precise, our d-dimensional space is a ball with no nontrivial topology. Since the systems have a boundary, here we can only require that the "bulk" gap of the fermion systems is nonzero. The free-fermion system may have protected gapless excitations at the boundary. (Requiring the fermion systems to be even gapped at the boundary only gives us trivial gapped phases.) We call the free-fermion systems that are gapped only inside of the d-dimensional ball as "bulk" gapped fermion systems. A bulk gapped fermion system may or may not be gapped at the boundary.

Kitaev has shown that the space C_d^H of such bulk gapped free-fermion systems is homotopically equivalent to the space C_d^M of mass matrices of a d-dimensional Dirac equation: $\pi_n(C_d^H) = \pi_n(C_d^M)$.[25] In the following, we give a intuitive explanation of the result.

To start, let us first assume that the fermion system has translation symmetry and charge-conjugation symmetry. We also assume that its energy bands have some Dirac points at zero energy and there are no other zero-energy states in the Brillouin zone. So if we fill the negative energy bands, the single-body gapless excitations in the system are described by the Hermitian matrix H, whose continuous limit has the form

$$H = \sum_{i=1}^{d} \gamma_i \mathrm{i}\partial_i, \tag{11}$$

where we have folded all the Dirac points to the $\mathbf{k} = \mathbf{0}$ point. Without losing generality, we have also assumed that all the Dirac points have the same velocity. Since $\mathrm{i}\partial_i$ is Hermitian, γ_i, $i = 1,\ldots,d$, are the Hermitian γ matrices (of infinite dimension) that satisfy Eq. (4).

When $d = 1$, do we have a system that has $\gamma_1 = I_{n\times n}$? The answer is no. Such a system will have n right-moving chiral modes that cannot be realized by any pure one-dimensional systems with short-ranged hopping. In fact γ_1 must have the form $\gamma_1 = \left(\begin{smallmatrix} I_{l\times l} & 0 \\ 0 & -I_{m\times m}\end{smallmatrix}\right)$ with $l = m$ (the same number of right- and left-moving modes). So the allowed γ_1 always satisfies the condition that $H^2 = 1$ and $\gamma_1 H + H\gamma_1 = 0$ has a solution for H. We see that the extra condition Eq. (8) on γ_i has a very physical meaning.

Now we add perturbations that may break the translation symmetry. We like to know how many different ways there are to gap the Dirac point. The Dirac points can be fully gapped by the "mass" matrix M that satisfies

$$\gamma_i M + M\gamma_i = 0, \quad M^\dagger = M. \tag{12}$$

To fully gap the Dirac points, M must have no zero eigenvalues. Without losing generality, we may also assume that $M^2 = 1$. (Since the Dirac points may have different crystal momenta before the folding to $\mathbf{k} = \mathbf{0}$, we need perturbations that break the translation symmetry to generate a generic mass matrix that may mix Dirac points at different crystal momenta.) The space C_d^M of such mass matrices is nothing but C_d introduced before: $C_d^M = C_d$.

So the different ways to gap the Dirac point form a space C_d. The different disconnected components of C_d represent different gapped phases of the free fermions. Thus, the gapped phases of the conserved free fermions in d dimensions are classified by $\pi_0(C_d)$, which is Z for even d and 0 for odd d. The nontrivial phases at $d = 2$ are labeled by Z, which are the integer quantum Hall states. The results are summarized in Table II.

III. GAPPED FREE-FERMION PHASES: THE REAL CLASSES

When the fermion number is not conserved and/or when there is a time-reversal symmetry, the gapped phases of noninteracting fermions are classified differently. However, using the idea and approaches similar to the above discussion, we can also obtain a classification. Instead of considering Hermitian matrices that satisfy certain conditions, we just need to consider real antisymmetric matrices that satisfy certain conditions.

A. The $d = 0$ case: The symmetry groups

Again, we start with $d = 0$ dimensions. In this case, a free-fermion system with n orbitals is described by the following quadratic Hamiltonian:

$$\hat{H} = \sum_{ij} H_{ij}\hat{c}_i^\dagger \hat{c}_j + \sum_{ij}[G_{ij}\hat{c}_i\hat{c}_j + \text{H.c.}], \quad i,j = 1,\ldots,n. \tag{13}$$

Introducing the Majorana fermion operator $\hat{\eta}_I$, $I = 1,\ldots,2n$,

$$\{\hat{\eta}_I,\hat{\eta}_J\} = 2\delta_{IJ}, \quad \hat{\eta}_I^\dagger = \hat{\eta}_I, \tag{14}$$

to express the complex fermion operator $\hat{c}_i$,

$$\hat{c}_i = \tfrac{1}{2}(\hat{\eta}_{2i} + \mathrm{i}\eta_{2i+1}), \tag{15}$$

we can rewrite $\hat{H}$ as

$$\hat{H} = \frac{\mathrm{i}}{4}\sum_{IJ} A_{IJ}\hat{\eta}_I\hat{\eta}_J, \tag{16}$$

where A is a real antisymmetric matrix. For example, for a one-orbital Hamiltonian $\hat{H} = \epsilon(\hat{c}^\dagger\hat{c} - \frac{1}{2})$, we get $A = \left(\begin{smallmatrix} 0 & -\epsilon \\ \epsilon & 0\end{smallmatrix}\right)$.

If the fermion number is conserved, $\hat{H}$ commutes with the fermion number operator

$$\hat{N} \equiv \sum_i \left(\hat{c}_i^\dagger\hat{c}_i - \frac{1}{2}\right) = \frac{\mathrm{i}}{4}\sum_{IJ} Q_{IJ}\hat{\eta}_I\hat{\eta}_J, \tag{17}$$

where

$$Q = \varepsilon \otimes I, \quad Q^2 = -1, \quad \varepsilon \equiv -\mathrm{i}\sigma^y. \tag{18}$$

$[\hat{H},\hat{N}] = 0$ requires that

$$[A,Q] = 0. \tag{19}$$

Such a matrix A has the form $A = \sigma^0 \otimes H_a + \varepsilon \otimes H_s$, where H_s is symmetric and H_a antisymmetric. We can convert such an antisymmetric matrix A into a Hermitian matrix $H = H_s + \mathrm{i}H_a$ and reduce the problem to the one discussed before.

The time-reversal transformation $\hat{T}$ is antiunitary: $\hat{T}\mathrm{i}\hat{T}^{-1} = -\mathrm{i}$. Since $\hat{T}$ does not change the fermion numbers, therefore, $\hat{T}\hat{c}_i\hat{T}^{-1} = U_{ij}\hat{c}_j$, where U is a unitary matrix. In terms of the Majorana fermions, we have

$$\begin{aligned} \hat{T}\hat{\eta}_{2i}\hat{T}^{-1} &= \mathrm{Re}U_{ij}\hat{\eta}_{2j} - \mathrm{Im}U_{ij}\hat{\eta}_{2j+1}, \\ \hat{T}\hat{\eta}_{2i+1}\hat{T}^{-1} &= -\mathrm{Re}U_{ij}\hat{\eta}_{2j+1} - \mathrm{Im}U_{ij}\hat{\eta}_{2j}. \end{aligned} \tag{20}$$

Therefore, we have

$$\hat{T}\hat{\eta}_i\hat{T}^{-1} = T_{ij}\hat{\eta}_j, \quad T = \sigma^3 \otimes \mathrm{Re}U - \sigma^1 \otimes \mathrm{Im}U. \tag{21}$$

We see that, in the Majorana fermion basis, $U \to \sigma^3 \otimes \mathrm{Re}U - \sigma^1 \otimes \mathrm{Im}U = T$ and $\mathrm{i} \to \epsilon \otimes I$. We indeed have $T(\epsilon \otimes I) = -T(\epsilon \otimes I)$.

For fermion systems, we may have $\hat{T}^2 = s_T^{\hat{N}}$, $s_T = \pm$. In fact $s_T = -$ for electron systems. This implies that $\hat{T}^2\hat{c}_i\hat{T}^{-2} = s_T\hat{c}_i$ and $T^2 = s_T$. The time-reversal invariance $\hat{T}\hat{H}\hat{T}^{-1} = \hat{H}$

implies that $T^\top A T = -A$, where $T^\top$ is the transpose of T. We can show that

$$
\begin{aligned}
T^\top T &= (\sigma^3 \otimes \mathrm{Re}U^\dagger - \sigma^1 \otimes \mathrm{Im}U^\dagger)(\sigma^3 \otimes \mathrm{Re}U + \sigma^1 \otimes \mathrm{Im}U) \\
&= \sigma^0 \otimes (\mathrm{Re}U^\dagger \mathrm{Re}U - \mathrm{Im}U^\dagger \mathrm{Im}U) \\
&\quad - \epsilon \otimes (\mathrm{Re}U^\dagger \mathrm{Im}U + \mathrm{Im}U^\dagger \mathrm{Re}U) \\
&= \sigma^0 \otimes I,
\end{aligned} \tag{22}
$$

where we have used $\mathrm{Re}U^\dagger \mathrm{Re}U - \mathrm{Im}U^\dagger \mathrm{Im}U = I$ and $\mathrm{Re}U^\dagger \mathrm{Im}U + \mathrm{Im}U^\dagger \mathrm{Re}U = 0$ for unitary matrix U. Therefore, $T^\top = T^{-1}$ and

$$
AT = -TA, \quad T^2 = s_T. \tag{23}
$$

Also, for fermion systems, the time-reversal transformation $\hat{T}$ and the $U(1)$ transformation $\hat{N}$ may have a nontrivial relation: $\hat{T} e^{\mathrm{i}\theta \hat{N}} \hat{T}^{-1} = e^{s_{UT} \mathrm{i}\theta \hat{N}}$, $s_{UT} = \pm$, or $\hat{T}\hat{N}\hat{T}^{-1} = -s_{UT}\hat{N}$. This gives us

$$
TQ = s_{UT} QT. \tag{24}
$$

The charge-conjugation transformation $\hat{C}$ is unitary. Since $\hat{C}$ changes $\hat{c}_i$ to $\hat{c}_i^\dagger$, then $\hat{C}\hat{c}_i\hat{C}^{-1} = U_{ij}\hat{c}_j^\dagger$, where U is a unitary matrix. In terms of the Majorana fermions, we have

$$
\begin{aligned}
\hat{C}\hat{\eta}_{2i}\hat{C}^{-1} &= \mathrm{Re}U_{ij}\hat{\eta}_{2j} + \mathrm{Im}U_{ij}\hat{\eta}_{2j+1}, \\
\hat{C}\hat{\eta}_{2i+1}\hat{C}^{-1} &= -\mathrm{Re}U_{ij}\hat{\eta}_{2j+1} + \mathrm{Im}U_{ij}\hat{\eta}_{2j}.
\end{aligned} \tag{25}
$$

Therefore, we have

$$
\hat{C}\hat{\eta}_i\hat{C}^{-1} = C_{ij}\hat{\eta}_j, \quad C = \sigma^3 \otimes \mathrm{Re}U + \sigma^1 \otimes \mathrm{Im}U. \tag{26}
$$

Again, we can show that $C^\top = C^{-1}$.

For fermion systems, we may have $\hat{C}^2 = s_C^{\hat{N}}$, $s_C = \pm$, which implies that $\hat{C}^2 \hat{c}_i \hat{C}^{-2} = s_C \hat{c}_i$ and $C^2 = s_C$. The charge-conjugation invariance $\hat{C}\hat{H}\hat{C}^{-1} = \hat{H}$ implies that A satisfies

$$
CA = CA, \quad C^2 = s_C. \tag{27}
$$

Since $\hat{C}\hat{N}\hat{C}^{-1} = -\hat{N}$, we have

$$
CQ = -QC. \tag{28}
$$

However, the commutation relation between $\hat{T}$ and $\hat{C}$ has two choices: $\hat{T}\hat{C} = s_{TC}^{\hat{N}}\hat{C}\hat{T}$, $s_{TC} = \pm$; we have

$$
CT = s_{TC} TC. \tag{29}
$$

We see that when we say a system has $U(1)$, time-reversal, and/or charge-conjugation symmetries, we still do not know what is the actual symmetry group of the system, since those symmetry operations may have different relations as described by the signs of s_T, s_C, s_{UT}, and s_{TC}, which lead to different full symmetry groups. Because symmetry plays a key role in our classification, we cannot obtain a classification without specifying the symmetry groups. We have discussed the possible relations among various symmetry operations. In Table IV, we list the corresponding symmetry groups.

We like to point out that sometimes, when we describe the symmetry of a fermion system, we do not include the fermion number parity transformation $(-)^{\hat{N}}$ in the symmetry group G. However, in this paper, we use the full symmetry group G_f to describe the symmetry of a fermion system. The full symmetry group G_f does include the fermion number parity transformation $(-)^{\hat{N}}$. So the full symmetry group of a fermion system with no symmetry is $G_f = Z_2^f$ generated by the fermion number parity transformation. G_f is actually a Z_2^f extension of G: $G = G_f/Z_2^f$. It is a projective symmetry group discussed in Ref. 41.

TABLE IV. Different relations between symmetry transformations give rise to 36 different groups that contain $U(1)$ (represented by U), time-reversal (T), and/or charge-conjugation (C) symmetries.

Symmetry groups	Relations
$U(1)$, $SU(2)$	
$G_{s_C}(C)$	$\hat{C}^2 = s_C^{\hat{N}}$, $s_C = \pm$
$G_{s_C}(U,C)$	$\hat{C}^2 = s_C^{\hat{N}}$, $\hat{C}e^{\mathrm{i}\theta\hat{N}}\hat{C}^{-1} = e^{-\mathrm{i}\theta\hat{N}}$, $s_C = \pm$
$G_{s_T}(T)$	$\hat{T}^2 = s_T^{\hat{N}}$, $s_T = \pm$.
$G_{s_T}^{s_{UT}}(U,T)$	$\hat{T}e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{s_{UT}\mathrm{i}\theta\hat{N}}$, $\hat{T}^2 = s_T^{\hat{N}}$, $s_{UT}, s_T = \pm$
$G_{s_T s_C}^{s_{TC}}(T,C)$	$\hat{T}^2 = s_T^{\hat{N}}$, $\hat{C}\hat{T} = (s_{TC}^{\hat{N}})\hat{T}\hat{C}$, $s_{TC}, s_T, s_C = \pm$
$G_{s_T s_C}^{s_{UT} s_{TC}}(U,T,C)$	$\hat{C}e^{\mathrm{i}\theta\hat{N}}\hat{C}^{-1} = e^{-\mathrm{i}\theta\hat{N}}$, $\hat{T}e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{s_{UT}\mathrm{i}\theta\hat{N}}$, $\hat{T}^2 = s_T^{\hat{N}}$, $\hat{C}^2 = s_C^{\hat{N}}$, $\hat{C}\hat{T} = (s_{TC}^{\hat{N}})\hat{T}\hat{C}$, $s_T, s_C, s_{UT}, s_{TC} = \pm$

In the following, we study the symmetries of various electron systems to see which symmetry groups listed in Table IV can be realized by electron systems.

For insulators with non-coplanar spin order $\delta H = \hat{c}_i^\dagger \mathbf{n_1} \cdot \sigma \hat{c}_i + \hat{c}_j^\dagger \mathbf{n_2} \cdot \sigma \hat{c}_j + \hat{c}_k^\dagger \mathbf{n_3} \cdot \sigma \hat{c}_k$, the full symmetry group is $G_f = U(1)$ generated by the total charge $\hat{N}_C$.

For superconductors with non-coplanar spin order $\delta H = \hat{c}_i^\dagger \check{n}_1 \cdot \sigma \hat{c}_i + \hat{c}_j^\dagger \mathbf{n_2} \cdot \sigma \hat{c}_j + \hat{c}_k^\dagger \mathbf{n_3} \cdot \sigma \hat{c}_k + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$, the full symmetry group is reduced to $G_f = Z_2^f$ generated by the fermion number parity operator $P_f = (-)^{\hat{N}_C}$. We note that the full symmetry group of any fermion system contains Z_2^f as a subgroup. So we usually use the group G_f/Z_2^f to describe the symmetry of the fermion system, and we say there is no symmetry for superconductors with non-coplanar spin order. But in this paper, we use the full symmetry group G_f to describe the symmetry of fermion systems.

For insulators with spin-orbital coupling $\delta H = \mathrm{i}\hat{c}_i^\dagger \mathbf{n_1} \cdot \sigma \hat{c}_{i'} + \mathrm{i}\hat{c}_j^\dagger \mathbf{n_2} \cdot \sigma \hat{c}_{j'} + \mathrm{i}\hat{c}_k^\dagger \mathbf{n_3} \cdot \sigma \hat{c}_{k'}$, they have the charge-conservation ($\hat{N}_C$) and the time-reversal ($\hat{T}_{\mathrm{phy}}$) symmetries. The time-reversal symmetry is defined by

$$
\hat{T}_{\mathrm{phy}}\hat{c}_{\alpha,i}\hat{T}_{\mathrm{phy}}^{-1} = \epsilon_{\alpha\beta}\hat{c}_{\beta,i}, \quad \hat{T}_{\mathrm{phy}}\hat{c}_{\alpha,i}^\dagger\hat{T}_{\mathrm{phy}}^{-1} = \epsilon_{\alpha\beta}\hat{c}_{\beta,i}^\dagger. \tag{30}
$$

We can show that

$$
\begin{aligned}
\hat{T}_{\mathrm{phy}}\hat{c}_i^\dagger \sigma \hat{c}_j \hat{T}_{\mathrm{phy}}^{-1} &= -\hat{c}_i^\dagger \sigma \hat{c}_j, \\
\hat{T}_{\mathrm{phy}}\hat{N}_C\hat{T}_{\mathrm{phy}}^{-1} &= \hat{N}_C, \quad \hat{T}_{\mathrm{phy}}^2 = (-)^{\hat{N}_C}
\end{aligned} \tag{31}
$$

Thus, δH is invariant under $\hat{T}_{\mathrm{phy}}$ and $e^{\mathrm{i}\theta\hat{N}_C}$. Let $\hat{T} = \hat{T}_{\mathrm{phy}}$ and $\hat{N} = \hat{N}_C$; we find that

$$
\hat{T}e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{-\mathrm{i}\theta\hat{N}}, \quad \hat{T}^2 = (-)^{\hat{N}}, \tag{32}
$$

which define the full symmetry group $G_-^-(U,T)$ of an electron insulator with spin-orbital coupling.

For superconductors with spin-orbital coupling and real pairing $\delta H = \mathrm{i}\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_{i'} + \mathrm{i}\hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_{j'} + \mathrm{i}\hat{c}_k^\dagger \mathbf{n_3}\cdot\sigma\hat{c}_{k'} + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$, they have the time-reversal symmetry $\hat{T}_{\rm phy}$. Setting $\hat{T} = \hat{T}_{\rm phy}$ and $\hat{N} = \hat{N}_C$, we find

$$\hat{T}^2 = (-)^{\hat{N}}, \tag{33}$$

which defines the full symmetry group $G_-^+(U,T) = Z_4$ of superconductors with spin-orbital coupling and real pairing. - For superconductors with S_z conserving spin-orbital coupling and real pairing $\delta H = \mathrm{i}\hat{c}_i^\dagger \sigma^z \hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$, they have the time-reversal $\hat{T}_{\rm phy}$ and S_z spin-rotation symmetries. Setting $\hat{T} = \hat{T}_{\rm phy}$ and $\hat{N} = 2\hat{S}_z$, we find

$$\hat{T} e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{\mathrm{i}\theta\hat{N}}, \quad \hat{T}^2 = (-)^{\hat{N}}, \tag{34}$$

which defines the full symmetry group $G_-^+(U,T) = U(1)\times Z_2$ of superconductors with S_z conserving spin-orbital coupling and real pairing.

For superconductors with real pairing and coplanar spin order $\delta H = \hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_i + \hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$, they have a combined time-reversal and 180° spin-rotation symmetry. The spin rotation is generated by $S_a = \sum_i \frac{1}{2} c_i^\dagger \sigma^a c_i$, $a = x,y,z$. We have

$$\hat{T}_{\rm phy}\hat{S}_a\hat{T}_{\rm phy}^{-1} = -\hat{S}_a. \tag{35}$$

The Hamiltonian δH is invariant under $\hat{T} = e^{\mathrm{i}\pi\hat{S}^y}\hat{T}_{\rm phy}$. Since $\hat{T}^2 = 1$, the full symmetry group of superconductors with real pairing and coplanar spin order is $G_+(T) = Z_2\times Z_2^f$.

For superconductors with real pairing and collinear spin order $\delta H = \hat{c}_i^\dagger \sigma^z \hat{c}_j + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} - \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$, they have the S_z spin rotation and a combined time-reversal and 180° S_y spin-rotation symmetry. The Hamiltonian δH is invariant under $\hat{T} = e^{\mathrm{i}\pi\hat{S}^y}\hat{T}_{\rm phy}$ and S_z spin rotation $\hat{N} = 2\hat{S}_z$. We find that

$$\hat{T} e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{-\mathrm{i}\theta\hat{N}}, \quad \hat{T}^2 = (-)^{\hat{N}}, \tag{36}$$

which define the full symmetry group $G_+^-(U,T)$ of superconductors with real pairing and collinear spin order.

For insulators with coplanar spin order $\hat{c}_i^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_i + \hat{c}_j^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_j$, they have the charge-conservation and a combined time-reversal and 180° spin-rotation symmetries. The Hamiltonian δH is invariant under $\hat{T} = e^{\mathrm{i}\pi\hat{S}^y}\hat{T}_{\rm phy}$ and the charge rotation $\hat{N} = \hat{N}_C$. We find that

$$\hat{T} e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{-\mathrm{i}\theta\hat{N}}, \quad \hat{T}^2 = (-)^{\hat{N}}, \tag{37}$$

which define the full symmetry group $G_+^-(U,T)$ of insulators with coplanar spin order.

For superconductors with real triplet $S_z = 0$ pairing $\delta H = \hat{c}_{\uparrow i}\hat{c}_{\downarrow j} + \hat{c}_{\downarrow i}\hat{c}_{\uparrow j}$, they have a combined time-reversal and charge-rotation symmetry, a combined 180° S_y spin-rotation and charge-rotation symmetry, and the S_z spin-rotation symmetry. The Hamiltonian δH is invariant under $\hat{T} = e^{\mathrm{i}\frac{\pi}{2}\hat{N}_C}\hat{T}_{\rm phy}$, $\hat{C} = e^{\mathrm{i}\frac{\pi}{2}\hat{N}_C}e^{\mathrm{i}\pi\hat{S}_y}$, and the S_z spin rotation $\hat{N} = 2\hat{S}_z$. We find that

$$\begin{aligned}\hat{C} e^{\mathrm{i}\theta\hat{N}}\hat{C}^{-1} &= e^{-\mathrm{i}\theta\hat{N}}, \quad \hat{T} e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{\mathrm{i}\theta\hat{N}},\\ \hat{T}^2 &= 1, \quad \hat{C}^2 = 1, \quad \hat{C}\hat{T} = (-)^{\hat{N}}\hat{T}\hat{C},\end{aligned} \tag{38}$$

which define the full symmetry group $G_{++}^{--}(U,T,C)$ of superconductors with real triplet $S_z = 0$ pairing.

For superconductors with real triplet $S_z = 0$ pairing and collinear spin order $\delta H = \hat{c}_i\sigma^z c_i + (\hat{c}_{\uparrow i}\hat{c}_{\downarrow j} + \hat{c}_{\downarrow i}\hat{c}_{\uparrow j})$, they have a combined time-reversal and 180° S_y spin-rotation symmetry, and the S_z spin-rotation symmetry. The Hamiltonian δH is invariant under $\hat{T} = e^{\mathrm{i}\pi\hat{S}_y}\hat{T}_{\rm phy}$ and the S_z spin rotation $\hat{N} = 2\hat{S}_z$. We find that

$$\hat{T} e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{-\mathrm{i}\theta\hat{N}}, \quad \hat{T}^2 = 1, \tag{39}$$

which define the full symmetry group $G_+^-(U,T)$ of superconductors with real triplet $S_z = 0$ pairing and collinear spin order.

For superconductors with the time-reversal, the 180° S_y spin-rotation, and the S_z spin-rotation symmetries, the Hamiltonian is invariant under $\hat{T} = \hat{T}_{\rm phy}$, $\hat{C} = e^{\mathrm{i}\pi\hat{S}_y}$, and $\hat{N} = 2\hat{S}_z$. We find that

$$\begin{aligned}\hat{C} e^{\mathrm{i}\theta\hat{N}}\hat{C}^{-1} &= e^{-\mathrm{i}\theta\hat{N}}, \quad \hat{T} e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{\mathrm{i}\theta\hat{N}},\\ \hat{T}^2 &= \hat{C}^2 = (-)^{\hat{N}}, \quad \hat{C}\hat{T} = \hat{T}\hat{C},\end{aligned} \tag{40}$$

which define the full symmetry group $G_{--}^{++}(U,T,C)$ of superconductors with the time-reversal, the 180° S_y spin-rotation, and the S_z spin-rotation symmetries. For free electrons with the 180° S_y spin rotation, and the S_z spin-rotation symmetries, they actually have the full $SU(2)$ spin-rotation symmetry. So the above systems are also superconductors with real pairing and $SU(2)$ spin-rotation symmetry. Similarly, for superconductors with complex pairing and $SU(2)$ spin-rotation symmetry, the symmetry group is $SU(2)$, or $G_-(U,C)$.

For insulators with spin-orbital coupling and only intersublattice hopping $H = \mathrm{i}\hat{c}_{i_A}^\dagger \mathbf{n_1}\cdot\sigma\hat{c}_{i_B} + \mathrm{i}\hat{c}_{j_A}^\dagger \mathbf{n_2}\cdot\sigma\hat{c}_{j_B} + \mathrm{i}\hat{c}_{k_A}^\dagger \mathbf{n_3}\cdot\sigma\hat{c}_{k_B}$, they have charge-conservation, time-reversal, and deformed charge-conjugation symmetries. The charge-conjugation transformation $\hat{C}_{\rm phy}$ is defined as

$$\hat{C}_{\rm phy}\hat{c}_{\alpha,i}\hat{C}_{\rm phy}^{-1} = \epsilon_{\alpha\beta}\hat{c}_{\beta,i}^\dagger, \quad \hat{C}_{\rm phy}\hat{c}_{\alpha,i}^\dagger\hat{C}_{\rm phy}^{-1} = \epsilon_{\alpha\beta}\hat{c}_{\beta,i}. \tag{41}$$

We find that

$$\begin{aligned}\hat{C}_{\rm phy}\hat{c}_i^\dagger\sigma\hat{c}_j\hat{C}_{\rm phy}^{-1} &= \hat{c}_i^\dagger\sigma\hat{c}_j, \quad \hat{C}_{\rm phy}\hat{T}_{\rm phy} = \hat{T}_{\rm phy}\hat{C}_{\rm phy},\\ \hat{C}_{\rm phy}\hat{N}_C\hat{C}_{\rm phy}^{-1} &= -\hat{N}_C, \quad \hat{C}_{\rm phy}^2 = (-)^{\hat{N}_C}\end{aligned} \tag{42}$$

The above Hamiltonian is invariant under $\hat{T} = \hat{T}_{\rm phy}$, $\hat{N} = \hat{N}_C$, and $\hat{C} = (-)^{\hat{N}_B}\hat{C}_{\rm phy}$, where $\hat{N}_B$ is the number of electrons on the B sublattice. We find

$$\begin{aligned}\hat{C} e^{\mathrm{i}\theta\hat{N}}\hat{C}^{-1} &= e^{-\mathrm{i}\theta\hat{N}}, \quad \hat{T} e^{\mathrm{i}\theta\hat{N}}\hat{T}^{-1} = e^{-\mathrm{i}\theta\hat{N}},\\ \hat{T}^2 &= (-)^{\hat{N}}, \quad \hat{C}^2 = (-)^{\hat{N}}, \quad \hat{C}\hat{T} = \hat{T}\hat{C},\end{aligned} \tag{43}$$

which define the full symmetry group $G_{--}^{-+}(U,T,C)$ of insulators with spin-orbital coupling and only intersublattice hopping. The above results for electron systems and their full symmetry groups G_f are summarized in Table I.

B. The $d = 0$ case: The classifying spaces

The Hermitian matrix iA describes single-body excitations above the free-fermion ground state. We note that the eigenvalues of iA are $\pm\epsilon_i$. The positive eigenvalues $|\epsilon_i|$ correspond to the single-body excitation energies above the many-body ground state. The minimal $|\epsilon_i|$ represents the excitation-energy gap of $\hat{H}$ above the ground state (the ground state is the lowest energy state of $\hat{H}$). So, if we are considering gapped systems, $|\epsilon_i|$ is always nonzero. We can shift all ϵ_i to ± 1 without closing the gap and change the (matter) phase of the state. Thus, we can set $A^2 = -1$.

In the presence of symmetry, A should also satisfy some other conditions. The space formed by all those A's is called the classifying space. Clearly, the classifying space is determined by the full symmetry group G_f. In this section, we calculate the classifying spaces for some simple groups.

If there is no symmetry, then the real antisymmetric matrix A satisfies

$$A^2 = -1. \tag{44}$$

The space of those matrices is denoted as R_0^0, which is the classifying space for a trivial symmetry group.

If there is only charge-conjugation symmetry [full symmetry group = $G_{s_C}(C)$], then the real antisymmetric matrix A satisfies

$$A^2 = -1, \quad AC = CA, \quad C^2 = s_C. \tag{45}$$

For $s_C = +$, since C commutes with A and C is symmetric, we can always restrict ourselves in an eigenspace of C and C can be dropped. Thus, the space of the matrices is R_0^0, the same as before. For $s_C = -$, we can assume $C = \varepsilon \otimes I$. In this case, A has the form $A = \sigma^0 \otimes H_a + \varepsilon \otimes H_s$, where $H_s = H_s^T$ and $H_a = -H_a^T$. Thus, we can convert A into a Hermitian matrix $H = H_s + \mathrm{i}H_a$, and the space of the matrices is C_0.

If there are $U(1)$ and charge-conjugation symmetries [full symmetry group = $G_{s_C}(U,C)$], then the real antisymmetric matrix A satisfies

$$A^2 = -1, \quad AQ = QA, \quad AC = CA, \quad QC = -CQ,$$
$$Q^2 = -1, \quad C^2 = s_C. \tag{46}$$

For $s_C = +$, we can assume $C = \sigma^z \otimes I$ and $Q = \varepsilon \otimes I$. Since Q and C commute with A, we find that A must have the form $A = \sigma^0 \otimes \tilde{A}$, with $\tilde{A}^2 = -1$. Thus, the space of the matrices $\tilde{A}$, and hence A, is R_0^0.

For $s_C = -$, we can assume $C = \varepsilon \otimes \sigma^x \otimes I$ and $Q = \varepsilon \otimes \sigma^z \otimes I$. We find that A must have the form $A = \sigma^0 \otimes \sigma^0 \otimes H_0 + \varepsilon \otimes \sigma^0 \otimes H_1 + \sigma^z \otimes \varepsilon \otimes H_2 + \sigma^x \otimes \varepsilon \otimes H_3$, where $H_0 = -H_0^T$ and $H_i = H_i^T$, $i = 1,2,3$. Now, we can view $\sigma^0 \otimes \sigma^0$ as 1, $\varepsilon \otimes \sigma^0$ as i, $\sigma^z \otimes \varepsilon$ as j, and $\sigma^x \otimes \varepsilon$ as k. We find that i,j,k satisfy the quaternion algebra. Thus, A can be mapped into a quaternion matrix $H = H_0 + \mathrm{i}H_1 + \mathrm{j}H_2 + \mathrm{k}H_3$ satisfying $H^\dagger = -H$ and $H^2 = -1$. The quaternion matrices that satisfy the above two conditions have the form

$$H = e^{X_{n\times n}} \mathrm{i} I_{n\times n} e^{-X_{n\times n}}, \tag{47}$$

where $X_{n\times n}^\dagger = -X_{n\times n}$ is a quaternion matrix. $e^{X_{n\times n}}$ form the group $Sp(n)$. However, the transformations $e^{A_{n\times n}+\mathrm{i}B_{n\times n}}$ keeps $\mathrm{i}I_{n\times n}$ unchanged, where $A_{n\times n}$ is a real antisymmetric matrix and $B_{n\times n}$ is a real symmetric matrix. $e^{A_{n\times n}+\mathrm{i}B_{n\times n}}$ form the group $U(n)$. Thus, the space of the quaternion matrices that satisfy the above two conditions is given by $Sp(n)/U(n)$. Such a space is the space R_6, which is introduced later.

When $s_C = -1$, we can view Q as the generator of $\hat{S}_z$ spin rotation, and C as the generator of $\hat{S}_x$ spin rotation acting on spin-1/2 fermions. In fact $e^{\theta_z Q}$ and $e^{\theta_x C}$ in this case generate the full $SU(2)$ group. So when $s_C = -$, the free spin-1/2 fermions with $U(1) \rtimes Z_2^C$ symmetry actually have the full $SU(2)$ spin-rotation symmetry. Therefore, $G_-(U,C) \sim SU(2)$.

If there is only the time-reversal symmetry [full symmetry group = $G_{s_T}(T)$], then A satisfies

$$A^2 = -1, \quad A\rho_1 + \rho_1 A = 0, \quad \rho_1^2 = s_T, \quad \rho_1 = T. \tag{48}$$

The space of those matrices is denoted as R_0^1 for $s_T = -1$ and R_1^0 for $s_T = 1$.

If there are time-reversal and $U(1)$ symmetries [full symmetry group = $G_{s_T}^{s_{UT}}(U,T)$], then for $s_{UT} = -$, A satisfies

$$A^2 = -1, \quad A\rho_i + \rho_i A = 0, \quad \rho_1^2 = s_T, \quad \rho_2^2 = s_T,$$
$$\rho_1 = T, \quad \rho_2 = TQ. \tag{49}$$

The space of those matrices is denoted as R_2^0 for $s_T = +$ and R_0^2 for $s_T = -$.

For $s_{UT} = +$, Q commutes with both A and T. Since $Q^2 = -1$, we can treat Q as the imaginary number i and convert both A and T to complex matrices. To see this, let us choose a basis in which Q has a form $Q = \epsilon \otimes I$. In this basis A and T become $A = \sigma^0 \otimes A_2 + \epsilon \otimes A_1$ and $T = \sigma^0 \otimes T_1 + \epsilon \otimes T_2$, where A_1 is symmetric and A_2 is antisymmetric. Let us introduce complex matrices $H = -A_1 + \mathrm{i}A_2$ and $\tilde{T} = T_1 + \mathrm{i}T_2$ for $s_T = +$ or $\tilde{T} = -T_2 + \mathrm{i}T_1$ for $s_T = -$. From $A^2 = -1$, $T^2 = s_T$, and $AT = -TA$, we find

$$H^2 = 1, \quad H\tilde{T} + \tilde{T}H = 0, \quad \tilde{T}^2 = 1. \tag{50}$$

Also $A^T = -A$ allows us to show $H^\dagger = H$. For a fixed $\tilde{T}$, the space formed by H's that satisfy the above conditions is C_1 introduced before. This allows us to show that the space of the corresponding matrices A is C_1 for $s_T = \pm$, $s_{UT} = +$.

If there are time-reversal and charge-conjugation symmetries [full symmetry group = $G_{s_T s_C}^{s_{TC}}(T,C)$], then for $s_{TC} = -$, A satisfies

$$A^2 = -1, \quad A\rho_i + \rho_i A = 0, \quad \rho_1^2 = s_T, \quad \rho_2^2 = -s_T s_C,$$
$$\rho_1 = T, \quad \rho_2 = TC. \tag{51}$$

The space of the matrices A is R_1^1 for $s_T = +, s_C = +$; R_2^0 for $s_T = +, s_C = -$; R_1^1 for $s_T = -, s_C = +$; and R_0^2 for $s_T = -$, $s_C = -$. For $s_{TC} = +$, C will commute with both A and T. We find space of the matrices A to be R_1^0 for $s_T = +, s_C = +$; R_0^1 for $s_T = -, s_C = +$; and C_1 for $s_T = \pm, s_C = -$.

If there are $U(1)$, time-reversal, and charge-conjugation symmetries [full symmetry group = $G_{s_T s_C}^{s_{UT} s_{TC}}(U,T,C)$], then for $s_{TC} = s_{UT} = -$, A satisfies

$$A^2 = -1, \; A\rho_i + \rho_i A = 0, \; \rho_1^2 = \rho_2^2 = s_T, \; \rho_3^2 = -s_T s_C,$$
$$\rho_1 = T, \quad \rho_2 = TQ, \quad \rho_3 = TC. \tag{52}$$

The space of the matrices A is R_2^1 for $s_T = +, s_C = +$; R_3^0 for $s_T = +, s_C = -$; R_1^2 for $s_T = -, s_C = +$; and R_0^3 for $s_T = -$, $s_C = -$.

For $s_{UT} = -$, $s_{TC} = +$, A satisfies

$$A^2 = -1, \; A\rho_i + \rho_i A = 0, \; \rho_1^2 = \rho_2^2 = s_T, \; \rho_3^2 = -s_T s_C, \quad \rho_1 = T, \quad \rho_2 = TQ, \quad \rho_3 = TQC. \tag{53}$$

The space of the matrices A is R_2^1 for $s_T = +, s_C = +$; R_3^0 for $s_T = +, s_C = -$; R_1^2 for $s_T = -, s_C = +$; and R_0^3 for $s_T = -$, $s_C = -$.

For $s_{UT} = +$, $s_{TC} = -$, A satisfies

$$A^2 = -1, \; A\rho_i + \rho_i A = 0, \; \rho_1^2 = s_T, \; \rho_2^2 = \rho_3^2 = -s_T s_C, \quad \rho_1 = T, \quad \rho_2 = TC, \quad \rho_3 = TCQ. \tag{54}$$

The space of the matrices A is R_1^2 for $s_T = +, s_C = +$; R_3^0 for $s_T = +, s_C = -$; R_2^1 for $s_T = -, s_C = +$; and R_0^3 for $s_T = -$, $s_C = -$.

For $s_{UT} = +$, $s_{TC} = +$, we find that A satisfies

$$A^2 = -1, \; A\rho_i + \rho_i A = 0, \; \rho_1^2 = -s_T, \; \rho_2^2 = \rho_3^2 = s_T s_C, \quad \rho_1 = TQ, \quad \rho_2 = TC, \quad \rho_3 = TCQ. \tag{55}$$

We see that the matrices A form a space R_2^1 for $s_T = +, s_C = +$; R_0^3 for $s_T = +, s_C = -$; R_1^2 for $s_T = -, s_C = +$; and R_3^0 for $s_T = -, s_C = -$.

C. The properties of classifying spaces

In general, we can consider a real antisymmetric matrix A that satisfies (for fixed real matrices ρ_i, $i = 1, \dots, p+q$)

$$A = \rho_{p+q+1}, \quad \rho_j\rho_i + \rho_i\rho_j = |_{i\neq j}\, 0, \quad \rho_i^2 = |_{i=1,\dots,p}\, 1, \quad \rho_i^2 = |_{i=p+1,\dots,p+q+1}\, -1. \tag{56}$$

The space of those A matrices is denoted as R_p^q.

Let us show that

$$R_p^q = R_{p+1}^{q+1}. \tag{57}$$

From $\tilde{A} \in R_p^q$ that satisfies the following Clifford algebra $Cl(p,q+1)$,

$$\tilde{A} = \tilde{\rho}_{p+q+1}, \quad \tilde{\rho}_j\tilde{\rho}_i + \tilde{\rho}_i\tilde{\rho}_j = |_{i\neq j}\, 0, \quad \tilde{\rho}_i^2 = |_{i=1,\dots,p}\, 1, \quad \tilde{\rho}_i^2 = |_{i=p+1,\dots,p+q+1}\, -1, \tag{58}$$

we can define

$$\rho_i = |_{i=1,\dots,p}\tilde{\rho}_i \otimes \sigma^z, \quad \rho_{p+1} = I \otimes \sigma^x, \quad \rho_i = |_{i=p+2,\dots,p+q+2}\tilde{\rho}_{i-1} \otimes \sigma^z, \quad \rho_{p+q+3} = I \otimes \varepsilon. \tag{59}$$

We can check that such ρ_i satisfy the following Clifford algebra $Cl(p+1,q+2)$,

$$\rho_j\rho_i + \rho_i\rho_j = |_{i\neq j}\, 0, \quad \rho_i^2 = |_{i=1,\dots,p+1}\, 1, \quad \rho_i^2 = |_{i=p+2,\dots,p+q+3}\, -1. \tag{60}$$

If we fix ρ_i, $i \neq p+q+2$, then the space formed by $A = \rho_{p+q+2}$ satisfying the above condition is given by R_{p+1}^{q+1}. The above construction gives rise to a map from $R_p^q \to R_{p+1}^{q+1}$. Since $A = \rho_{p+q+2}$, satisfying Eq. (60) must has the form $\tilde{A} \otimes \sigma^z$, with $\tilde{A}$ satisfying Eq. (58). This gives us a map $R_{p+1}^{q+1} \to R_p^q$. Thus,$R_{p+1}^{q+1} = R_p^q$.

We can also consider a real symmetric matrix A that satisfies (for fixed real matrices ρ_i, $i = 1, \dots, p$)

$$A = \rho_{p+1}, \quad \rho_j\rho_i + \rho_i\rho_j = |_{i\neq j}\, 0, \quad \rho_i^2 = |_{i=1,\dots,p+1}\, 1. \tag{61}$$

The space of those matrices is denoted as R_p.

In the following, we show that

$$R_0^q = R_{q+2}. \tag{62}$$

From $\tilde{A} \in R_0^q$ that satisfies the Clifford algebra $Cl(0,q+1)$,

$$\tilde{A} = \tilde{\rho}_{q+1}, \quad \tilde{\rho}_j\tilde{\rho}_i + \tilde{\rho}_i\tilde{\rho}_j = |_{i\neq j}\, 0, \quad \tilde{\rho}_i^2 = |_{i=1,\dots,q+1}\, -1, \tag{63}$$

we can define

$$\rho_i = |_{i=1,\dots,q+1}\tilde{\rho}_i \otimes \varepsilon, \quad \rho_{q+2} = I \otimes \sigma^z, \quad \rho_{q+3} = I \otimes \sigma^x. \tag{64}$$

We can check that ρ_i form the Clifford algebra $Cl(q+3,0)$,

$$\rho_j\rho_i + \rho_i\rho_j = |_{i\neq j}\, 0, \quad \rho_i^2 = |_{i=1,\dots,q+3}\, 1. \tag{65}$$

If we fix ρ_i, $i \neq q+1$, then the space formed by $A = \rho_{q+1}$ satisfying the above condition is given by R_{q+2}. The above construction gives rise to a map from $R_0^q \to R_{q+2}$. Since $A = \rho_{q+1}$, satisfying Eq. (65) must have the form $\tilde{A} \otimes \varepsilon$, with $\tilde{A}$ satisfying Eq. (63). This gives us a map $R_{q+2} \to R_0^q$. Thus, $R_0^q = R_{q+2}$.

In addition we also have the following periodic relations:

$$R_p^q = R_p^{q+8} = R_{p+8}^q, \quad R_p = R_{p+8}. \tag{66}$$

This can be shown by noticing the following 16-dimensional real symmetric representation of Clifford algebra $Cl(0,8)$:

$$\begin{aligned} &\theta_1 = \varepsilon \otimes \sigma^z \otimes \sigma^0 \otimes \varepsilon, \quad \theta_2 = \varepsilon \otimes \sigma^z \otimes \varepsilon \otimes \sigma^x, \\ &\theta_3 = \varepsilon \otimes \sigma^z \otimes \varepsilon \otimes \sigma^z, \quad \theta_4 = \varepsilon \otimes \sigma^x \otimes \varepsilon \otimes \sigma^0, \\ &\theta_5 = \varepsilon \otimes \sigma^x \otimes \sigma^x \otimes \varepsilon, \quad \theta_6 = \varepsilon \otimes \sigma^x \otimes \sigma^z \otimes \varepsilon, \\ &\theta_7 = \varepsilon \otimes \varepsilon \otimes \sigma^0 \otimes \sigma^0, \quad \theta_8 = \sigma^x \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0, \end{aligned} \tag{67}$$

which satisfy

$$\theta_i\theta_j + \theta_j\theta_i = |_{i\neq j}\, 0, \quad \theta_i^2 = |_{i=0,\dots,8}\, 1. \tag{68}$$

We find that $\theta = \theta_1\theta_2\theta_3\theta_4\theta_5\theta_6\theta_7\theta_8 = \sigma^z \otimes \sigma^0 \otimes \sigma^0 \otimes \sigma^0$ anticommutes with θ_i. From $\tilde{A} \in R_p^q$ that satisfies Eq. (58), we can define

$$\rho_i = |_{i=1,\dots,p}\tilde{\rho}_i \otimes \theta, \quad \rho_{p+i} = |_{i=1,\dots,8} I \otimes \theta_i, \quad \rho_i = |_{i=p+9,\dots,p+q+9}\tilde{\rho}_{i-8} \otimes \sigma^z. \tag{69}$$

We can check that such ρ_i satisfy

$$\rho_j\rho_i + \rho_i\rho_j = |_{i\neq j} 0, \quad \rho_i^2 = |_{i=1,\dots,p+8} 1, \quad \rho_i^2 = |_{i=p+9,\dots,p+q+9} - 1. \tag{70}$$

If we fix ρ_i, $i \neq p+q+9$, then the space formed by $A = \rho_{p+q+9}$ satisfying the above condition is given by R_{p+8}^q. The above construction gives rise to a map from $R_p^q \to R_{p+8}^q$. On the other hand, the matrix that anticommutes with all θ_i's must be proportional to θ. Thus, $A = \rho_{p+q+9}$ satisfying Eq. (70)

TABLE V. The spaces R_p and their homotopy groups $\pi_d(R_p)$.

p mod 8	0	1	2	3	4	5	6	7
R_p	$\frac{O(l+m)}{O(l)\times O(m)}\times\mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)}\times\mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
$\pi_0(R_p)$	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0	0	0
$\pi_1(R_p)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0	0	0	Z
$\pi_2(R_p)$	$\mathbb{Z}_2$	0	Z	0	0	0	Z	$\mathbb{Z}_2$
$\pi_3(R_p)$	0	Z	0	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_4(R_p)$	Z	0	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
$\pi_5(R_p)$	0	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z
$\pi_6(R_p)$	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0
$\pi_7(R_p)$	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0	0

must have the form $\tilde{A}\otimes\theta$, with $\tilde{A}$ satisfying Eq. (58). This gives us a map $R^q_{p+8}\to R^q_p$. Thus, $R^q_{p+8}=R^q_p$. Using a similar approach, we can show $R_p=R_{p+8}$. Equations (57), (62), and (66) allow us show

$$R^q_p = R_{q-p+2 \bmod 8}. \tag{71}$$

So we can study the space R^q_p via the space $R_{q-p+2 \bmod 8}$.

Let us construct some of the R_p spaces. R_0 is formed by real symmetric matrices A that satisfy $A^2=1$. Thus, A has the form $O\begin{pmatrix} I_{l\times l} & 0\\ 0 & -I_{m\times m}\end{pmatrix}O^{-1}$, $O\in O(l+m)$. We see that $R_0=\cup_m O(l+m)/O(l)\times O(m)=\frac{O(l+m)}{O(l)\times O(m)}\times\mathbb{Z}$.

R_1 is formed by real symmetric matrices A that satisfy $A^2=1$ and $A\rho_1=-\rho_1 A$ with $\rho_1=\sigma^z\otimes I_{n\times n}$. Thus, A has the form

$$\begin{aligned} A &= e^{\sigma^z\otimes M_{n\times n}} e^{\sigma^0\otimes L_{n\times n}}[\sigma^x\otimes I_{n\times n}]e^{-\sigma^0\otimes L_{n\times n}}e^{-\sigma^z\otimes M_{n\times n}}\\ &= e^{\sigma^z\otimes M_{n\times n}}[\sigma^x\otimes I_{n\times n}]e^{-\sigma^z\otimes M_{n\times n}}, \end{aligned} \tag{72}$$

where $e^{\sigma^0\otimes L_{n\times n}}\in O(n)$ and $e^{\sigma^z\otimes M_{n\times n}}\in O(n)$ are the transformations that leave ρ_1 unchanged. We see that $R_1=O(n)$. The other spaces R_p and $\pi_0(R_p)$ are listed in Table V. Note that for space $S\times\mathbb{Z}$, we have $\pi_0(S\times Z)=\pi_0(S)\times\mathbb{Z}$. Also $O(n)$ in the dividend usually leads to $\mathbb{Z}_2$ in π_0. Otherwise, $\pi_0=\{0\}$. $O(n)$ in the dividend can give rise to $\mathbb{Z}_2$ because $O\in O(n)$ with $\det(O)=1$ and $\det(O)=-1$ cannot be smoothly connected. $O(l+m)$ in $\frac{O(l+m)}{O(l)\times O(m)}$ does not lead to $\mathbb{Z}_2$ because for $O\in O(l+m)$ we can change the sign of $\det(O)$ by multiplying O with an element in $O(l)$ [or $O(m)$].

For free-fermion systems in zero dimensions with no symmetry and no fermion number conservation, the classifying space is R^0_0. Since $\pi_0(R^0_0)=\pi_0(R_2)=\mathbb{Z}_2$, such free-fermion systems have two possible gapped phases. One phase has even numbers of fermions in the ground state and the other phase has odd numbers of fermions in the ground state. (Note that the fermion number mod 2 is still conserved even without any symmetry.)

For free-electron systems in zero dimensions with time-reversal symmetry and electron number conservation [the symmetry group $G^-_-(U,T)$], the classifying space is R^2_0. Since $\pi_0(R^2_0)=\pi_0(R_4)=\mathbb{Z}$, the possible gapped phases are labeled by an integer n. The ground state has $2n$ fermions. The electron number in the ground state is always even due to the Kramer degeneracy.

If we drop the electron number conservation [the symmetry group becomes $G_-(T)$], then the ground state will have uncertain but even numbers of electrons. The ground state cannot have odd numbers of electrons. This implies that free-electron systems with only time-reversal symmetry in zero dimensions have only one possible gapped phase. This agrees with $\pi_0(R^1_0)=\pi_0(R_3)=\{0\}$, where R^1_0 is the classifying space for symmetry group $G_-(T)$.

D. The $d\neq 0$ cases

Now let us consider the $d\neq 0$ cases. Again, let us first assume that the fermion system described by $\hat{H}=\frac{\mathrm{i}}{4}\sum_{IJ}A_{IJ}\hat{\eta}_I\hat{\eta}_J$ has translation symmetry, as well as time-reversal symmetry and fermion number conservation. We also assume that the single-body energy bands of antisymmetric Hermitian matrix $\mathrm{i}A$ have some Dirac points at zero energy and there are no other zero-energy states in the Brillouin zone. The gapless single-body excitations in the system are described by the continuum limit of $\mathrm{i}A$:

$$\mathrm{i}A = \mathrm{i}\sum_{i=1}^{d}\gamma_i\partial_i, \tag{73}$$

where we have folded all the Dirac points to the $\mathbf{k}=\mathbf{0}$ point. Without losing generality, we have also assumed that all the Dirac points have the same velocity. Since ∂_i is real and antisymmetric, γ_i, $i=1,\dots,d$, are real symmetric γ matrices (of infinite dimension) that satisfy

$$\gamma_i\gamma_j+\gamma_j\gamma_i=2\delta_{ij},\quad \gamma_i^*=\gamma_i. \tag{74}$$

Again, the allowed γ_i's always satisfy the condition that $M^2=-1$ and $\gamma_i M+M\gamma_i=0$ has a solution for M. Since the time reversal and the $U(1)$ transformations do not affect ∂_i, the symmetry conditions on A, $AT+TA=0$ and $AQ-QA=0$, become the symmetry conditions on the γ matrices:

$$\gamma_i T+T\gamma_i=0,\quad \gamma_i Q-Q\gamma_i=0. \tag{75}$$

Now we add perturbations that may break the translation, time-reversal, and $U(1)$ symmetries, and we ask: how many different ways are there to gap the Dirac points? The Dirac points can be fully gapped by real antisymmetric mass matrices M that satisfy

$$\gamma_i M+M\gamma_i=0. \tag{76}$$

The resulting single-body Hamiltonian becomes $\mathrm{i}A = \mathrm{i}\sum_{i=1}^{d}[\gamma_i\partial_i + M]$.

If there is no symmetry, we only require the real antisymmetric mass matrix M to be invertible [in addition to Eq. (76)]. Without losing generality, we can choose the mass matrix to also satisfy

$$M^2 = -1. \tag{77}$$

The space of those mass matrices is given by R_d^0.

If there are some symmetries, the real antisymmetric mass matrix M also satisfies some additional condition, as discussed before: M anticommutes with a set of $p+q$ matrices ρ_i that anticommute among themselves with p of them square to 1 and q of them square to -1. The number of p,q depends on full symmetry group G_f. Since γ_i do not break the symmetry, just like M, γ_i also anticommutes ρ_i. So, in total, M anticommutes with a set of $p+q+d$ matrices ρ_i and γ_i that anticommute among themselves with $p+d$ of them square to 1 and q of them square to -1. Those mass matrices form a space R_{p+d}^q.

The different disconnected components of R_{p+d}^q represent different "bulk" gapped phases of the free fermions. Thus, the bulk gapped phases of the free fermions in d dimensions are classified by $\pi_0(R_{p+d}^q) = \pi_0(R_{q-p-d+2 \bmod 8})$, with (p,q) depending on the symmetry. The results are summarized in Table III.

E. A general discussion

Now, let us give a general discussion of the classifying problem of free-fermion systems. To classify the gapped phases of the free-fermion Hamiltonian we need to construct the space of antisymmetric mass matrix M that satisfies

$$M^2 = -1. \tag{78}$$

The mass matrices A always anticommute with γ matrices γ_i, $i = 1,2,\dots,d$. When the mass matrices M have some symmetries, then the mass matrices satisfy more linear conditions. Let us assume that all those conditions can be expressed in the following form:

$$\begin{aligned} M\rho_i &= -\rho_i M, \quad \rho_i\rho_j = -\rho_j\rho_i,\\ MU_I &= U_I M, \quad U_I\rho_i = \rho_i U_I, \end{aligned} \tag{79}$$

where ρ_i and U_I are real matrices labeled by i and I, and $\gamma_1,\dots,\gamma_d$ are included in the ρ_i's. If we have another symmetry condition W such that $MW = -WM$ and $W\rho_{i_0} = \rho_{i_0}W$ for a particular i_0, then $U = W\rho_{i_0}$ will commute with M and ρ_i and will be part of U_I.

U_I will form some algebra. Let us use α to label the irreducible representations of the algebra. Then the one-fermion Hilbert space has the form $\mathcal{H} = \oplus_\alpha \mathcal{H}_\alpha \otimes \mathcal{H}_\alpha^0$, where the space $\mathcal{H}_\alpha^0$ forms the αth irreducible representations of the algebra. For such a decomposition of the Hilbert space, M has the following block-diagonal form:

$$M = \oplus_\alpha (M^\alpha \otimes I^\alpha), \tag{80}$$

where I^α acts within $\mathcal{H}_\alpha^0$ as an identity operator, and M^α acts within $\mathcal{H}_\alpha$. The ρ_i's have a similar form,

$$\rho_i = \oplus_\alpha(\rho_i^\alpha \otimes I_\alpha), \tag{81}$$

where ρ_i^α acts within $\mathcal{H}_\alpha$. So within the Hilbert space $\mathcal{H}_\alpha$, we have

$$M^\alpha\rho_i^\alpha = -\rho_i^\alpha M^\alpha, \quad \rho_i^\alpha\rho_j^\alpha = -\rho_j^\alpha\rho_i^\alpha. \tag{82}$$

What we are trying to do in this paper is actually to construct the space of M^α matrices that satisfy the condition of Eq. (82).

If fermions only form one irreducible representation of the U_I algebra, then the classifying space of M_α and M will be the same. The results of this paper (such as Tables II and III) are obtained under such an assumption.

If fermions form n distinct irreducible representations of the U_I algebra, then the classifying space of M will be R^n, where R is the classifying space of M^α constructed in this paper. Note that the classifying spaces of M^α are the same for different irreducible representations and hence R is independent of α. So if M_α's are classified by $\mathbb{Z}_k$, $k = 1,2$, or ∞, then M's are classified by $\mathbb{Z}_k^n$.

To illustrate the above result, let us use the symmetry $G_+(C) = Z_2^C \times Z_2^f$ as an example. If the fermions form one irreducible representation of Z_2^C, for example, $\hat{C}c_i\hat{C}^{-1} = -c_i$, then the noninteracting symmetric gapped phases are classified by

$$\begin{array}{rcccccccc} d: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7,\\ \text{gapped phases}: & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0, \end{array} \tag{83}$$

which is the result in Table III. If the fermions form both the irreducible representations of Z_2^C, $\hat{C}c_{i+}\hat{C}^{-1} = +c_{i+}$ and $\hat{C}c_{i-}\hat{C}^{-1} = -c_{i-}$ (i.e., one type of fermions carries Z_2^C charge 0 and another type of fermions carries Z_2^C charge 1), then the noninteracting symmetric gapped phases are classified by

$$\begin{array}{rcccccccc} d: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7,\\ \text{gapped phases}: & \mathbb{Z}_2^2 & \mathbb{Z}_2^2 & \mathbb{Z}^2 & 0 & 0 & 0 & \mathbb{Z}^2 & 0. \end{array} \tag{84}$$

The four $d=0$ phases correspond to the ground state with even or odd Z_2-charge-0 fermions and even or odd Z_2-charge-1 fermions. The four $d=1$ phases correspond to the phases where the Z_2-charge-0 fermions are in the trivial or nontrivial phases of the Majorana chain and the Z_2-charge-1 fermions are in the trivial or nontrivial phases of the Majorana chain. The $d=2$ phase labeled by two integers $(m,n)\in\mathbb{Z}^2$ corresponds to the phase where the Z_2-charge-0 fermions have m right-moving Majorana chiral modes and the Z_2-charge-1 fermions have n right-moving Majorana chiral modes. (If m and/or n are negative, we then have the corresponding number of left-moving Majorana chiral modes.)

Some of the above gapped phases have intrinsic fermionic topological orders. So only a subset of them are noninteracting fermionic SPT phases:

$$\begin{array}{rcccccccc} d_{sp}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7,\\ \text{SPT phases}: & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} & 0. \end{array} \tag{85}$$

The two $d_{sp}=0$ phases correspond to the ground states with even numbers of fermions and 0 or 1 Z_2 charges. The two $d_{sp}=1$ phases correspond to the phases where the Z_2-charge-0 fermions and the Z_2-charge-1 fermions are both in the trivial or nontrivial phases of the Majorana chain. The $d_{sp}=2$ phase labeled by one integer $n\in\mathbb{Z}$ corresponds to the phase where the Z_2-charge-0 fermions have n right-moving

Majorana chiral modes and the Z_2-charge-1 fermions have n left-moving Majorana chiral modes.

IV. CLASSIFICATION WITH TRANSLATION SYMMETRY

For the free-fermion systems with certain internal symmetry G_f, we have shown that their gapped phases are classified by $\pi_0(R_{p_G-d})$ or $\pi_0(C_{p_G-d})$ in d dimensions, where the value of p_G is determined from the full symmetry group G_f. We know that π_0 is an Abelian group with commuting group multiplication "+": $a,b \in \pi_0$ implies that $a+b \in \pi_0$. The $+$ operation has a physical meaning. If two "bulk" gapped fermion systems are labeled by a and b in π_0, then stacking the two systems together will give us a new bulk gapped fermion system labeled by $a+b \in \pi_0$.

We note that the classification by $\pi_0(R_{p_G-d})$ or $\pi_0(C_{p_G-d})$ is obtained by assuming there is no translation symmetry. In the presence of translation symmetry, the gapped phases are classified differently.[32–34,43] However, the new classification can be obtained from $\pi_0(R_{p_G-d})$. For the free-fermion systems with internal symmetry G_f and translation symmetry, their gapped phases are classified by[25]

$$\prod_{k=0}^{d}[\pi_0(R_{p_G-d+k})]^{\binom{d}{k}}, \tag{86}$$

where $\binom{d}{k}$ is the binomial coefficient. The above is for the real classes. For the complex classes, we have a similar classification:

$$\prod_{k=0}^{d}[\pi_0(C_{p_G-d+k})]^{\binom{d}{k}}. \tag{87}$$

Such a result is obtained by stacking the lower-dimensional topological phases to obtain higher-dimensional ones. For one-dimensional free-fermion systems with internal symmetry G_f, their gapped phases are classified by $\pi_0(R_{p_G-1})$. We can also have a zero-dimensional gapped phase on each unit cell of the one-dimensional system if there is a translation symmetry. The zero-dimensional gapped phases are classified by $\pi_0(R_{p_G})$. Thus, the combined gapped phases (with translation symmetry) are classified by $\pi_0(R_{p_G-1}) \times \pi_0(R_{p_G})$. In two dimensions, the gapped phases are classified by $\pi_0(R_{p_G-2})$. The gapped phases on each unit cell are classified by $\pi_0(R_{p_G})$. Now we can also have one-dimensional gapped phases on the lines in the x direction, which are classified by $\pi_0(R_{p_G-1})$. We have the same thing for the lines in the y direction. So the combined gapped phases (with translation symmetry) are classified by $\pi_0(R_{p_G-2}) \times [\pi_0(R_{p_G-1})]^2 \times \pi_0(R_{p_G})$. In three dimensions, the translation-symmetric gapped phases are classified by $\pi_0(R_{p_G-3}) \times [\pi_0(R_{p_G-2})]^3 \times [\pi_0(R_{p_G-1})]^3 \times \pi_0(R_{p_G})$.

V. DEFECTS IN d-DIMENSIONAL GAPPED FREE-FERMION PHASES WITH SYMMETRY G_f

For the d-dimensional free-fermion systems with internal symmetry G_f, we have shown that their bulk gapped Hamiltonians (or the mass matrices) form a space R_{p_G-d} or C_{p_G-d}. (More precisely, the gapped Hamiltonians or the mass matrices form a space that is homotopically equivalent to R_{p_G-d} or C_{p_G-d}.) From the space R_{p_G-d} or C_{p_G-d}, we find that the point defects that have symmetry G_f are classified by $\pi_{d-1}(R_{p_G-d})$ or $\pi_{d-1}(C_{p_G-d})$.

Physically, there is another way to classify point defects: we can simply add a segment of the 1D bulk gapped free-fermion hopping system with the same symmetry to the d-dimensional system. Since the translation symmetry is not required, the new d-dimensional system still belongs to the same symmetry class. There are finite bulk gaps away from the two ends of the added 1D segment. So the new d-dimensional system may contain two nontrivial defects. The defects are classified by the classes of the added 1D bulk gapped free-fermion hopping system. So we find that the point defects that have the symmetry G_f are also classified by $\pi_0(R_{p_G-1})$ or $\pi_0(C_{p_G-1})$.

Similarly, the line defects that have the symmetry G_f are classified by $\pi_{d-2}(R_{p_G-d})$ or $\pi_{d-2}(C_{p_G-d})$. Again, we can also create line defects by adding a disk of the 2D bulk gapped free-fermion hopping system to the original d-dimensional system. This way, we find that the line defects that have the symmetry G_f are also classified by $\pi_0(R_{p_G-2})$ or $\pi_0(C_{p_G-2})$.

In general, the defects with dimension d_0 are classified by $\pi_{d-d_0-1}(R_{p_G-d})$ or $\pi_{d-d_0-1}(C_{p_G-d})$, or equivalently by $\pi_0(R_{p_G-d_0-1})$ or $\pi_0(C_{p_G-d_0-1})$. In order for the above physical picture to be consistent, we require that

$$\pi_n(R_{p_G}) = \pi_0(R_{p_G+n}), \quad \pi_n(C_{p_G}) = \pi_0(C_{p_G+n}). \tag{88}$$

The classifying spaces indeed satisfy the above highly nontrivial relation. This is the Bott periodicity theorem. The theorem is obtained by the following observation: the space

TABLE VI. Classification of point defects and line defects that have some symmetries in gapped phases of noninteracting fermions. "0" means that there is no nontrivial topological defects. Z means that topological nontrivial defects plus the topological trivial defect are labeled by the elements in Z. Nontrivial topological defects have protected gapless excitations, while trivial topological defects have no protected gapless excitations.

Symmetry	C_{p_G}	p_G	Point defect	Line defect	Example phases
$U(1)$ $G_-(C)$	$\frac{U(l+m)}{U(l)\times U(m)} \times \mathbb{Z}$	0	0	Z	(Chern) insulator; Superconductor with collinear spin order
$G^+_\pm(U,T)$ $G^+_{--}(T,C)$ $G^+_{+-}(T,C)$	$U(n)$	1	Z	0	Superconductor with real pairing and S_z conserving spin-orbital coupling

TABLE VII. Classification of point defects and line defects that have some symmetries in gapped phases of noninteracting fermions. "0" means that there is no nontrivial topological defects. $\mathbb{Z}_n$ means that topological nontrivial defects plus the topological trivial defect are labeled by the elements in $\mathbb{Z}_n$. Nontrivial defects have protected gapless excitations in them.

Symm.	$G_+^-(U,T)$ $G_{+-}^-(T,C)$	$G_+(T)$ $G_{++}^+(T,C)$ $G_{++}^{--}(U,T,C)$ $G_{++}^{-+}(U,T,C)$ $G_{-+}^{+-}(U,T,C)$ $G_{++}^{++}(U,T,C)$	None $G_+(C)$ $G_{++}^-(T,C)$ $G_{-+}^-(T,C)$ $G_+(C)$ $G_+(U,C)$	$G_-(T)$ $G_{-+}^+(T,C)$ $G_{-+}^{--}(U,T,C)$ $G_{-+}^{-+}(U,T,C)$ $G_{++}^{+-}(U,T,C)$ $G_{-+}^{++}(U,T,C)$	$G_-^-(U,T)$ $G_{--}^-(T,C)$	$G_{--}^{--}(U,T,C)$ $G_{--}^{-+}(U,T,C)$ $G_{--}^{+-}(U,T,C)$ $G_{+-}^{++}(U,T,C)$	$G_-(U,C)$ $SU(2)$	$G_{+-}^{--}(U,T,C)$ $G_{+-}^{-+}(U,T,C)$ $G_{+-}^{+-}(U,T,C)$ $G_{--}^{++}(U,T,C)$ $G[SU(2),T]$
R_{p_G}	$\frac{O(l+m)}{O(l)\times O(m)}\times\mathbb{Z}$	$O(n)$	$\frac{O(2n)}{U(n)}$	$\frac{U(2n)}{Sp(n)}$	$\frac{Sp(l+m)}{Sp(l)\times Sp(m)}\times\mathbb{Z}$	$Sp(n)$	$\frac{Sp(n)}{U(n)}$	$\frac{U(n)}{O(n)}$
p_G:	0	1	2	3	4	5	6	7
Point defect	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0	0
Line defect	0	0	Z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	Z	0
Example phases	Insulator with co-planar spin order	Superconductor with co-planar spin order	Superconductor	Superconductor with time reversal	Insulator with time reversal	Insulator with time reversal and intersublattice hopping	Spin singlet superconductor	Spin singlet superconductor withtime reversal

C_{p+1} can be viewed (in a homotopic sense) as ΩC_p—the space of loops in C_p. So we have $\pi_1(C_p) = \pi_0(\Omega C_p) = \pi_0(C_{p+1})$. Similarly, the space R_{p+1} can be viewed (in a homotopic sense) as ΩR_p—the space of loops in R_p. So we have $\pi_1(R_p) = \pi_0(\Omega R_p) = \pi_0(R_{p+1})$. As a result of the Bott periodicity theorem, the classification of defects is independent of spatial dimensions. It only depends on the dimension and the symmetry of the defects. If the defects lower the symmetry, then we should use the reduced symmetry to classify the defects. In Tables VI and VII, we list the classifications of those symmetric point and line defects for gapped free-fermion systems with various symmetries. We would like to point out that the line defects classified by Z in superconductors without symmetry *do not* correspond to the vortex lines (which usually belong to the trivial class under our classification). The nontrivial line defect here should carry chiral modes that only move in one direction along the defect line. In general, nontrivial defects have protected gapless excitations in them.

VI. SUMMARY

In this paper, we study different possible full symmetry groups G_f of fermion systems that contain $U(1)$, time-reversal (T), and/or charge-conjugation (C) symmetry. We show that each symmetry group G_f is associated with a classifying space C_{p_G} or R_{p_G} (see Tables II and III). We classify d-dimensional gapped phases of free-fermion systems that have those full symmetry groups. We find that the different gapped phases are described by $\pi_0(C_{p_G-d})$ or $\pi_0(R_{p_G-d})$. Those results, obtained using the K-theory approach, generalize the results in Refs. 25 and 28.

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