Thermalization and dissipation in out-of-equilibrium quantum systems: A perturbative renormalization group approach

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A perturbative renormalization group approach is employed to study the effect of a periodic potential on a system of one-dimensional bosons in a nonequilibrium steady state due to an initial interaction quench. The renormalization group flows are modified significantly from the well-known equilibrium Berezinski-Kosterlitz-Thouless form. They show several new features such as a generation of an effective temperature, generation of dissipation, as well as a change in the location of the quantum critical point separating the weak- and strong-coupling phases. Detailed results on the weak-coupling side of the phase diagram are presented, such as the renormalization of the parameters and the asymptotic behavior of the correlation functions. The physical origin of the generated temperature and friction is discussed.

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I. INTRODUCTION

The high degree of tunability and control associated with cold-atomic gases[1] has motivated an explosion of theoretical activity involving the study of dynamics of interacting quantum systems. Several interesting problems could be studied in these systems, including quantum quenches (see Ref. 2 and references therein) and other classes of steady-state nonequilibrium phenomena such as systems subjected to a time-dependent noise.[3] In all these situations, one of the fundamental questions is what the steady state of the system is and, in particular, whether the system can thermalize.

Not surprisingly in such a difficult problem, exactly solvable models and integrable systems have proven to be a good playground to address these issues. The study of quench dynamics in many integrable systems reveals that the large number of conserved quantities in the system prevent the system from thermalizing. Instead, the long-time behavior of the system is characterized by time-dependent or time-independent nonequilibrium states.[4,5] Often after a time averaging, the resultant steady state can be described by a generalized Gibbs ensemble (GGE) constructed from identifying the conserved quantities of the system.[6–12] However, the generality and applicability of the GGE remains under debate since not all observables can be described by it.[13–18]

Aside from a lack of complete understanding of the long-time behavior of integrable and even some exactly solvable models after a quench, the more complex question as to how the presence of nontrivial interactions that cause scattering and/or break integrability affects the behavior is largely open. Numerical studies on finite systems show that for large enough breaking of integrability, thermalization is associated with a change of the level statistics from Poisson to Wigner-Dyson,[19] with the onset of thermalization occurring via a two-step process, where in the first step the system is trapped for a long time in a nonequilibrium prethermalized state.[20–24] At the same time, fluctuations in a finite-size system also play an important role,[25,26] so that generalization of numerical results to systems in the thermodynamic limit may not be straightforward.

The aim of this paper is to address some of these questions in precisely this limit of long times and infinite system size where numerical studies are hard. In particular, we study an exactly solvable model, which is in a nonequilibrium steady state due to an initial quench, and explore the stability of the resulting athermal state to nontrivial interactions that generate mode coupling. The effects of mode coupling will be treated within a perturbative renormalization group (RG) approach. While there are many candidate models to study this physics, due to experimental relevance and its relative simplicity, we choose to study a one-dimensional system of interacting bosons, leading to the so-called Luttinger-liquid physics.[27] The excitations of such a system can be represented by density modes, which are essentially independent. On such a system, quenches corresponding to a change of the interaction reveal a steady state that still has independent modes, but these are now characterized by a nonequilibrium distribution that does not relax to a thermal state.[7,15,28–31] Note that we use the term "steady state" to reflect the fact that averages of various physical observables reach a time-independent value at long times after the quench.

We study the effect of a mode-coupling term, such as one generated by a periodic potential, on this steady state. We assume that the periodic potential has been switched on very slowly, so that in the absence of the initial quench, the system reaches the ground state in the presence of the periodic potential. The results are presented in the parameter regime where the periodic potential is (dangerously) "irrelevant" so that a perturbative RG approach remains valid at arbitrary length scales. We find that infinitesimally weak potentials or mode coupling can generate an effective temperature and cause the system to *asymptotically* thermalize. In addition, and somewhat unexpectedly, a dissipation is also generated. Thus, we find that the effective low-energy theory at long times after the quench is a quadratic theory of thermal bosons with a finite lifetime. This asymptotic thermalization and dissipation occurs because the low-frequency and momentum modes can transfer energy to the higher-energy modes (which are gradually eliminated in the RG procedure). These high-energy modes thus act as a bath, providing thermalization and dissipation. The dissipation thus shares features with Landau damping where plasmons acquire a finite lifetime.[32] It also shares similarities with turbulent systems that are known to exhibit the well-known Kolmogorov cascades where energy is passed down from large-scale to small-scale structures.[33] We believe that this type of behavior and mechanism for thermalization, unraveled in a controlled way on the particular system we study, is quite general.

A short account of some of these results was given in Ref. 31. In this paper, we give a detailed derivation of the above results and in particular discuss the general derivation of the RG equations using the Keldysh approach, which may be adapted to study other types of out-of-equilibrium bosonic systems. We build on the results of Ref. 31 to derive new results. In particular, we show that the RG flow can be significantly modified by different quench protocols. We also examine the behavior close to the critical point where the external potential becomes relevant. In this limit, various peculiarities such as singular behavior of the expansion in spatial and temporal gradients of the bosonic field are encountered. In addition, we present a detailed discussion of the violation of the fluctuation-dissipation theorem of the post-quench system by studying the frequency dependence of two-point correlation functions, showing that even though the system appears "thermal" in low-energy scales, the crossover from low to high energies is still complicated.

It should be noted that some RG-based approaches to study quenches already exist in the literature. For example, the flow equation method was used to study quenches in a Fermi liquid[21] and in the quantum sine-Gordon model,[34] where for both cases a long-lived prethermalized regime was found. RG was also used to study the a classical two-dimensional sine-Gordon model after a quench.[35] Here, the effective temperature due to the quench was found to generate a dynamical vortex binding-unbinding transition, but unlike the quantum one-dimensional (1D) problem studied in Ref. 31 and this paper, no dissipation was generated. Furthermore, the system that we study also shares many common features with a Luttinger liquid subjected to a nonequilibrium noise source,[3] where the particular form of the noise in our system arises due to the out-of-equilibrium occupation of the bosonic modes after the quench. It is thus interesting to make a connection between the quench problem, where in the asymptotic the system is free, and systems where an external noise source is constantly imposed on the system and is found to exhibit similar properties.[36]

The paper is organized as follows. In Sec. II, the basic model and notation is introduced and the equilibrium properties of the relevant response and correlation functions are discussed. In Sec. III, the interaction quench in the Luttinger liquid and the properties of the resultant nonequilibrium steady state are presented. In Sec. IV, the periodic potential is introduced and the perturbative RG equations are derived. In Sec. V, the solution of the RG equations is presented, and in Sec. VI the resultant low-energy theory near the fixed point is discussed. Finally, in Sec. VII, we present our conclusions and discuss open questions.

II. MODEL

The Hamiltonian for interacting bosons in a periodic potential is

$$H = H_0 + V_{sg}, \tag{1}$$

$$H_0 = \frac{u}{2\pi}\int dx \left(K[\pi\Pi(x)]^2 + \frac{1}{K}[\partial_x\phi(x)]^2\right), \tag{2}$$

$$V_{sg} = -\frac{gu}{\alpha^2}\int dx\,\cos[\gamma\phi(x)], \tag{3}$$

where H_0 is the quadratic part that describes the Luttinger liquid or long-lived sound modes. The density of these modes is $\rho = -\partial_x\phi/\pi$, whereas $\Pi = \partial_x\theta/\pi$ is the variable canonically conjugate to ϕ. V_{sg} represents the periodic potential, the most important effect of which is a source of backscattering that can localize the density modes via the well-known Berezenskii-Kosterlitz-Thouless (BKT) transition.[27]

It is convenient to represent the fields ϕ,θ in terms of bosonic creation and annihilation operators $(b_p,b_p^\dagger)$ (Ref. 27):

$$\phi(x) = -(N_R+N_L)\frac{\pi x}{L} - \frac{i\pi}{L}\sum_{p\neq 0}\left(\frac{L|p|}{2\pi}\right)^{1/2} \times \frac{1}{p}e^{-\alpha|p|/2-ipx}(b_p^\dagger + b_{-p}), \tag{4}$$

$$\theta(x) = (N_R-N_L)\frac{\pi x}{L} + \frac{i\pi}{L}\sum_{p\neq 0}\left(\frac{L|p|}{2\pi}\right)^{1/2} \times \frac{1}{|p|}e^{-\alpha|p|/2-ipx}(b_p^\dagger - b_{-p}), \tag{5}$$

where α^{-1} is an ultraviolet cutoff. Thus,

$$H_0 = \sum_{p\neq 0} u|p|\eta_p^\dagger\eta_p, \tag{6}$$

where

$$\eta_p = \cosh\beta b_p + \sinh\beta b_{-p}^\dagger, \tag{7}$$

$$\eta_{-p}^\dagger = \cosh\beta b_{-p}^\dagger + \sinh\beta b_p, \tag{8}$$

and $e^{-2\beta} = K$, $u = v_F/K$.

Since we are interested in nonequilibrium dynamics, we will use the Keldysh formalism,[37] where $\phi_{-/+}$ will denote fields that are (time/antitime) ordered on the Keldysh axis. Further, it will be convenient to define quantum (ϕ_q) and classical fields (ϕ_{cl})

$$\phi_\pm = \frac{1}{\sqrt{2}}(\phi_{cl} \mp \phi_q). \tag{9}$$

A. Correlation functions in equilibrium

The two-point functions that are directly influenced by the periodic potential, and therefore of interest to us, are

$$C_{\phi\phi}^{\pm,\pm}(xt,yt') = \langle e^{i\gamma\phi_\pm(x,t)}e^{-i\gamma\phi_\pm(y,t')}\rangle. \tag{10}$$

In order to compute the above correlators in the absence of a periodic potential ($g = 0$), or perturbatively in g, some useful identities are

$$\langle \eta_{cl}(p,t)\eta_{cl}^{\dagger}(p',t')\rangle = \delta_{pp'}e^{-iu|p|(t-t')}\coth\left(\frac{u|p|}{2T}\right), \quad (11)$$

$$\langle \eta_{cl}(p,t)\eta_{q}^{\dagger}(p',t')\rangle = \delta_{pp'}\theta(t-t')e^{-iu|p|(t-t')}, \quad (12)$$

$$\langle \eta_{q}(p,t)\eta_{cl}^{\dagger}(p',t')\rangle = -\delta_{pp'}\theta(t'-t)e^{-iu|p|(t-t')}, \quad (13)$$

where T is the temperature of the bosons.

At $T = 0$ and $g = 0$, we find

$$C_{\phi\phi}^{--}(xt,yt') = e^{-\frac{\gamma^2 K}{4}[\ln\frac{\sqrt{[(x-y)+u(t-t')]^2+\alpha^2}}{\alpha}+\ln\frac{\sqrt{[x-y-u(t-t')]^2+\alpha^2}}{\alpha}]} \times e^{-\frac{\gamma^2 K}{4}i\,\mathrm{sign}(t-t')[\tan^{-1}\frac{u(t-t')+x-y}{\alpha}+\tan^{-1}\frac{u(t-t')-(x-y)}{\alpha}]}, \quad (14)$$

$$C_{\phi\phi}^{++}(xt,yt') = e^{-\frac{\gamma^2 K}{4}[\ln\frac{\sqrt{[x-y+u(t-t')]^2+\alpha^2}}{\alpha}+\ln\frac{\sqrt{[x-y-u(t-t')]^2+\alpha^2}}{\alpha}]} \times e^{\frac{\gamma^2 K}{4}i\,\mathrm{sign}(t-t')[\tan^{-1}\frac{u(t-t')+x-y}{\alpha}+\tan^{-1}\frac{[u(t-t')-(x-y)]}{\alpha}]}, \quad (15)$$

$$C_{\phi\phi}^{-+}(xt,yt') = e^{-\frac{\gamma^2 K}{4}[\ln\frac{\sqrt{[x-y+u(t-t')]^2+\alpha^2}}{\alpha}+\ln\frac{\sqrt{[x-y-u(t-t')]^2+\alpha^2}}{\alpha}]} \times e^{\frac{\gamma^2 K}{4}i[\tan^{-1}\frac{u(t-t')+x-y}{\alpha}+\tan^{-1}\frac{u(t-t')-(x-y)}{\alpha}]}. \quad (16)$$

Thus, all correlators exhibit the typical power-law decay of a Luttinger liquid, with an exponent $K_{\rm eq}$ where

$$K_{\rm eq} = \frac{\gamma^2 K}{4}. \quad (17)$$

The oscillating factors in the above equations arise due to the Keldysh time ordering and in equilibrium have the right structure so that the well-known fluctuation-dissipation theorem (FDT) is obeyed. To see this, let us define the correlation function

$$C_{\phi\phi}^{K}(xt,yt') = \frac{-i}{2}\left(C_{\phi\phi}^{--}(xt,yt') + C_{\phi\phi}^{++}(xt,yt')\right) \quad (18)$$

$$= -ie^{-K_{\rm eq}[\ln\frac{\sqrt{[x-y+u(t-t')]^2+\alpha^2}}{\alpha}+\ln\frac{\sqrt{[x-y-u(t-t')]^2+\alpha^2}}{\alpha}]} \times \cos\left[K_{\rm eq}\tan^{-1}\left(\frac{u(t-t')+x-y}{\alpha}\right) + K_{\rm eq}\tan^{-1}\left(\frac{u(t-t')-(x-y)}{\alpha}\right)\right] \quad (19)$$

and the response function

$$C_{\phi\phi}^{R}(xt,yt') = \frac{-i}{2}\left(C_{\phi\phi}^{--}(xt,yt') - C_{\phi\phi}^{-+}(xt,yt')\right) \quad (20)$$

$$= -\theta(t-t') \times e^{-K_{\rm eq}[\ln\frac{\sqrt{[x-y+u(t-t')]^2+\alpha^2}}{\alpha}+\ln\frac{\sqrt{[x-y-u(t-t')]^2+\alpha^2}}{\alpha}]} \times \sin\left[K_{\rm eq}\tan^{-1}\left(\frac{u(t-t')+x-y}{\alpha}\right) + K_{\rm eq}\tan^{-1}\left(\frac{u(t-t')-(x-y)}{\alpha}\right)\right]. \quad (21)$$

In Fourier space, $C(q,\omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx e^{i\omega t - iqx} C(x,t)$ are in general complicated to compute. However, we will, in the subsequent section, only highlight how the FDT works, so that it is easy to follow how it is violated in the post-quench situation.

B. Temperature from the fluctuation-dissipation theorem

The FDT implies that

$$C_{\phi\phi}^{K}(q,\omega) = \coth\left(\frac{\omega}{2T}\right) 2\,\mathrm{Im}\left[C_{\phi\phi}^{R}(q,\omega)\right] \quad (22)$$

$$\xrightarrow{T=0} \mathrm{sign}(\omega) 2\,\mathrm{Im}\left[C_{\phi\phi}^{R}(q,\omega)\right]. \quad (23)$$

Using $\ln[\alpha + i(ut \pm r)] = \ln[\sqrt{\alpha^2 + (ut\pm r)^2}] + i\tan^{-1}\frac{ut\pm r}{\alpha}$ (placing the branch cut of the logarithm on the negative real axis), we may write

$$C_{\phi\phi}^{K}(q,\omega) = -2i\int_0^{\infty} dt \int_{-\infty}^{\infty} dr\, \cos(\omega t)\cos(qr) \times \frac{1}{2}\left[e^{-K_{\rm eq}\ln\frac{\alpha+i(ut+r)}{\alpha} - K_{\rm eq}\ln\frac{\alpha+i(ut-r)}{\alpha}} + \text{c.c.}\right], \quad (24)$$

$$2\,\mathrm{Im}\left[C_{\phi\phi}^{R}(q,\omega)\right] = 2i\int_0^{\infty} dt \int_{-\infty}^{\infty} dr\, \sin(\omega t)\cos(qr) \times \frac{1}{2i}\left[e^{-K_{\rm eq}\ln\frac{\alpha+i(ut+r)}{\alpha} - K_{\rm eq}\ln\frac{\alpha+i(ut-r)}{\alpha}} - \text{c.c.}\right]. \quad (25)$$

Thus, the FDT $C^K = 2\,\mathrm{sign}(\omega)\mathrm{Im}[C^R]$ implies

$$\mathrm{sign}(\omega)\int_0^{\infty} dt \int_{-\infty}^{\infty} dr\, \sin(\omega t)\cos(qr) \times \frac{1}{2i}\left[e^{-K_{\rm eq}\ln\frac{\alpha+i(ut+r)}{\alpha} - K_{\rm eq}\ln\frac{\alpha+i(ut-r)}{\alpha}} - \text{c.c.}\right] + \int_0^{\infty} dt \int_{-\infty}^{\infty} dr\, \cos(\omega t)\cos(qr) \times \frac{1}{2}\left[e^{-K_{\rm eq}\ln\frac{\alpha+i(ut+r)}{\alpha} - K_{\rm eq}\ln\frac{\alpha+i(ut-r)}{\alpha}} + \text{c.c.}\right] = 0. \quad (26)$$

The above implies that the following ought to be true:

$$\mathrm{Re}\left[\int_0^{\infty} dt \int_{-\infty}^{\infty} dr\, \cos(qr) e^{-i|\omega|t} e^{-K_{\rm eq}\ln\frac{\alpha+i(ut+r)}{\alpha} - K_{\rm eq}\ln\frac{\alpha+i(ut-r)}{\alpha}}\right] = 0. \quad (27)$$

By analytically continuing $it \to \tau$, it is straightforward to see that the expression in the square brackets is purely imaginary, thus proving the FDT at $T = 0$.

In the next section, when we study the long-time behavior of the response and correlation after an interaction quench in the Luttinger liquid, we will find that the FDT is violated due to the appearance of a new nonequilibrium exponent $K_{\rm neq}$, which governs the power-law decay, while the oscillating factors are still associated with the equilibrium exponent $K_{\rm eq}$.

III. INTERACTION QUENCH IN THE LUTTINGER LIQUID: PROPERTIES OF THE QUADRATIC THEORY

Let us suppose that the system at time $t < 0$ is a Luttinger liquid with interaction parameter K_0 and velocity u_0, and therefore described by the Hamiltonian

$$H_i = \frac{u_0}{2\pi} \int dx \left[K_0[\pi \Pi(x)]^2 + \frac{1}{K_0}[\partial_x \phi(x)]^2 \right] \tag{28}$$

$$= \sum_{p \neq 0} u_0 |p| \eta_p^\dagger \eta_p. \tag{29}$$

We will consider the case where at $t = 0$ there is an interaction quench from $K_0 \to K$ so that the time evolution from $t > 0$ is due to

$$H_f = \frac{u}{2\pi} \int dx \left[K[\pi \Pi(x)]^2 + \frac{1}{K}[\partial_x \phi(x)]^2 \right]$$
$$= \sum_{p \neq 0} u |p| \gamma_p^\dagger \gamma_p. \tag{30}$$

To preserve Galilean invariance (which is not necessary for the formalism), we assume $u = v_F/K,\ u_0 = v_F/K_0$. Note that

$$\begin{pmatrix} b_p \\ b_{-p}^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \gamma_p \\ \gamma_{-p}^\dagger \end{pmatrix}, \tag{31}$$

$$\begin{pmatrix} b_p \\ b_{-p}^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\beta_0 & -\sinh\beta_0 \\ -\sinh\beta_0 & \cosh\beta_0 \end{pmatrix} \begin{pmatrix} \eta_p \\ \eta_{-p}^\dagger \end{pmatrix}, \tag{32}$$

where $e^{-2\beta_0} = K_0,\ \ e^{-2\beta} = K$. The quench for $K_0 = 1$ was studied in Ref. 15, and more general interaction quenches were studied in Refs. 29 and 31. However, the distinction between response and correlation functions were only first identified in Ref. 31. Here, we give more details of the results that appear in Ref. 31, and in addition discuss the crossover behavior from low to high frequencies.

Let us define the functions

$$f(pt) = \cos(u|p|t)\cosh\beta_0 - i\sin(u|p|t)\cosh(2\beta - \beta_0), \tag{33}$$

$$g(pt) = \cos(u|p|t)\sinh\beta_0 + i\sin(u|p|t)\sinh(2\beta - \beta_0), \tag{34}$$

which determine the time evolution after the quench ($t > 0$)

$$b_p^\dagger(t) + b_{-p}(t) = [f^*(pt) - g(pt)]\eta_p^\dagger(0) + [f(pt) - g^*(pt)]\eta_{-p}(0), \tag{35}$$

$$b_p^\dagger(t) - b_{-p}(t) = [f^*(pt) + g(pt)]\eta_p^\dagger(0) - [f(pt) + g^*(pt)]\eta_{-p}(0). \tag{36}$$

Using the above, the basic expectation value for the ϕ fields after the quench can be easily worked out to give

$$-i\langle\phi(xt)\phi(yt')\rangle \xrightarrow{t+t'\to\infty} = -\frac{i}{4}\int_{-\infty}^{\infty}\frac{dp}{|p|}e^{-\alpha|p|}\cos[p(x-y)]\left[\frac{K_0}{2}\left(1+\frac{K^2}{K_0^2}\right) \times \cos u|p|(t-t')\coth\frac{u|p|}{2T} - iK\sin u|p|(t-t')\right], \tag{37}$$

where T denotes the temperature of the Luttinger liquid before the quench. Above, terms that oscillate as $e^{-iu|p|(t+t')}$ have been dropped. This is because such oscillating terms give an overall decay to the correlators of interest defined in Eq. (10). Since we are ultimately interested in the long-time limit rather than the transients, these terms are not important for us. Further, this approximation also leads to a significant simplification as one is now dealing with a nonequilibrium steady-state problem, defined by the following basic retarded and Keldysh Green's functions:

$$G_R(xt,yt') = -i\theta(t-t')\langle[\phi(xt),\phi(yt')]\rangle = -i\langle\phi_{cl}(xt)\phi_q(yt')\rangle, \tag{38}$$

$$G_A(xt,yt') = i\theta(t'-t)\langle[\phi(xt),\phi(yt')]\rangle = -i\langle\phi_q(xt)\phi_{cl}(yt')\rangle, \tag{39}$$

$$G_K(xt,yt') = -i\langle\{\phi(xt),\phi(yt')\}\rangle = -i\langle\phi_{cl}(xt)\phi_{cl}(yt')\rangle, \tag{40}$$

which in Fourier space acquire the following form at $T = 0$:

$$G_R(q,\omega) = \frac{\pi K u}{(\omega + i\delta)^2 - u^2 q^2} = \frac{\pi K}{2|q|}\left[\frac{1}{\omega - u|q| + i\delta} - \frac{1}{\omega + u|q| + i\delta}\right], \tag{41}$$

$$G_K(q,\omega) = -i\frac{\pi^2}{2}K_0\left(1+\frac{K^2}{K_0^2}\right)\frac{\text{sign}(\omega)}{|q|} \times [\delta(\omega - u|q|) - \delta(\omega + u|q|)] \tag{42}$$

$$= \frac{K_0}{2K}\left(1+\frac{K^2}{K_0^2}\right)\text{sign}(\omega)[G_R - G_A]. \tag{43}$$

Note that the retarded Green's function G_R depends only on the final Hamiltonian and therefore is not sensitive to the quench. The Keldysh Green's function G_K, on the other hand, contains information about the occupation probabilities of the bosonic modes, which can be far from thermal equilibrium due to the quench. Thus, G_K depends on the properties of both the initial Hamiltonian (via K_0) and the final Hamiltonian (via K). In equilibrium $K = K_0$, and the FDT (for $T = 0$) $G_K = \text{sign}(\omega)[G_R - G_A]$ is recovered.

A. Correlation functions after the quench

It is convenient to define a nonequilibrium exponent

$$K_{\text{neq}} = \frac{\gamma^2}{8}K_0\left(1 + K^2/K_0^2\right), \tag{44}$$

which we will show represents the new power-law decay of the correlations in Eq. (10). Interestingly, both the equilibrium exponent K_{eq} defined in Eq. (17) and the exponent K_{neq} affect the unequal time correlators, which are found to be

$$C_{\phi\phi}^{--}(xt,yt') = e^{-K_{\text{neq}}\left[\ln\frac{\sqrt{[(x-y)+u(t-t')]^2+\alpha^2}}{\alpha} + \ln\frac{\sqrt{[x-y-u(t-t')]^2+\alpha^2}}{\alpha}\right]} \times e^{-iK_{\text{eq}}\text{sign}(t-t')\left[\tan^{-1}\frac{u(t-t')+x-y}{\alpha} + \tan^{-1}\frac{u(t-t')-(x-y)}{\alpha}\right]}, \tag{45}$$

$$C^{++}_{\phi\phi}(xt,yt') = e^{-K_{\rm neq}[\ln \frac{\sqrt{[x-y+u(t-t')]^2+\alpha^2}}{\alpha} + \ln \frac{\sqrt{[x-y-u(t-t')]^2+\alpha^2}}{\alpha}]} \times e^{iK_{\rm eq}{\rm sign}(t-t')[\tan^{-1}\frac{u(t-t')+x-y}{\alpha} + \tan^{-1}\frac{[u(t-t')-(x-y)]}{\alpha}]}, \tag{46}$$

$$C^{-+}_{\phi\phi}(xt,yt') = e^{-K_{\rm neq}[\ln \frac{\sqrt{[x-y+u(t-t')]^2+\alpha^2}}{\alpha} + \ln \frac{\sqrt{[x-y-u(t-t')]^2+\alpha^2}}{\alpha}]} \times e^{iK_{\rm eq}[\tan^{-1}\frac{u(t-t')+x-y}{\alpha} + \tan^{-1}\frac{u(t-t')-(x-y)}{\alpha}]}. \tag{47}$$

The above agrees with the equal time ($t = t'$) correlators for $K_0 = 1$ studied in Ref 15.

The power-law decay is determined by $K_{\rm neq}$, which is a memory-dependent exponent as it explicitly depends on the interaction parameter K_0 before the quench. On the other hand, $K_{\rm eq}$ depends only on the interaction parameter of the final Hamiltonian, and characterizes the equilibrium $T = 0$ properties of the system. Further, $K_{\rm neq} > K_{\rm eq}$, so that the power-law decay is always somewhat faster in the nonequilibrium steady state. The faster decay occurs both for $\langle e^{i\phi(1)}e^{-i\phi(2)}\rangle$ and the dual $\langle e^{i\theta(1)}e^{-i\theta(2)}\rangle$. In that sense, the effect of a quench is similar to a temperature, however, the system remains in a critical state. In the next section, we will find that as a consequence of this, the periodic potential is always less relevant for the nonequilibrium steady-state problem, with the critical point shifting to smaller values of K. Similar power-law decays with nonequilibrium exponents can also arise in open systems subjected to a nonequilibrium noise source such as $1/f$ noise.[3]

It is interesting to observe that two different quench protocols can lead to the same nonequilibrium steady state, at least for a case where the steady state is determined by the behavior of the $C_{\phi\phi}$ correlators. To see this, for simplicity, set $\gamma = 2$ (so that $K_{\rm eq} = K$). Then,

$$K_{\rm neq} = K + \frac{(K-K_0)^2}{2K_0}. \tag{48}$$

From the above equation, one may see that two different K_0 may lead to the same $K_{\rm eq}$ and $K_{\rm neq}$. For example, $K_{\rm eq} = 1$, $K_{\rm neq} = 2$ can be obtained for a quench from $K_0 \to K$ where $K = 1$, whereas the initial interaction parameter K_0 can take two different values $K_0 = 2 \pm \sqrt{3}$. This behavior simply reflects the fact that when $K = K_0$, the system being in equilibrium $K_{\rm neq} = K$. On the other hand, $K_{\rm neq}$ increases with respect to K for both types of quenches, one where $K_0 > K$ and the other where $K_0 < K$. For both these cases, the system is driven out of equilibrium, giving rise to a faster decay than in equilibrium.

As before, we now discuss the FDT ratio for the $C_{\phi\phi}$ two-point functions. (Note that often our convention will be to express length scales in units of α and energy scales in units of u/α.) The Keldysh correlation function [defined in Eq. (18)] is found to be

$$C^K_{\phi\phi}(r,t) = -i\cos\left[K_{\rm eq}\tan^{-1}\left(\frac{ut+r}{\alpha}\right) + K_{\rm eq}\tan^{-1}\left(\frac{ut-r}{\alpha}\right)\right] \times \left(\sqrt{\frac{\alpha^2}{\alpha^2+(ut-r)^2}}\right)^{K_{\rm neq}}\left(\sqrt{\frac{\alpha^2}{\alpha^2+(ut+r)^2}}\right)^{K_{\rm neq}}, \tag{49}$$

and the retarded correlation function [defined in Eq. (20)] is found to be

$$C^R_{\phi\phi}(r,t) = -\theta(t)\sin\left[K_{\rm eq}\tan^{-1}\left(\frac{ut+r}{\alpha}\right) + K_{\rm eq}\tan^{-1}\left(\frac{ut-r}{\alpha}\right)\right] \times \left(\sqrt{\frac{\alpha^2}{\alpha^2+(ut-r)^2}}\right)^{K_{\rm neq}}\left(\sqrt{\frac{\alpha^2}{\alpha^2+(ut+r)^2}}\right)^{K_{\rm neq}}. \tag{50}$$

In general, the Fourier transform of the above expressions for $C^{K,R}$ needs to be calculated numerically, and was briefly discussed in Ref. 31. In the $q = 0$, $\omega \to 0$ limit, however, analytic expressions for $C_{\phi\phi}$ can again be obtained. In particular,

$$C^K_{\phi\phi}(q=0,\omega=0) = -iI_{T0}, \tag{51}$$

where (setting $u = 1$)

$$I_{T0} = 2\alpha^2\left[\frac{\pi}{2^{K_{\rm neq}-1}(K_{\rm neq}-1)B\left(\frac{K_{\rm eq}+K_{\rm neq}}{2},\frac{K_{\rm neq}-K_{\rm eq}}{2}\right)}\right]^2, \tag{52}$$

with $B(x,y)$ being the beta function. Similarly, we find

$$2\,{\rm Im}\left[C^R_{\phi\phi}(q=0,\omega\to 0)\right] = -i\omega I_{\eta 0}, \tag{53}$$

where (setting $u = 1$)

$$I_{\eta 0} = \alpha^3\left[\frac{\pi}{2^{K_{\rm neq}-1}(K_{\rm neq}-1)B\left(\frac{K_{\rm eq}+K_{\rm neq}}{2},\frac{K_{\rm neq}-K_{\rm eq}}{2}\right)}\right] \times \left[\frac{\pi}{2^{K_{\rm neq}-2}(K_{\rm neq}-2)B\left(\frac{K_{\rm eq}+K_{\rm neq}-2}{2},\frac{K_{\rm neq}-K_{\rm eq}}{2}\right)} - \frac{\pi}{2^{K_{\rm neq}-2}(K_{\rm neq}-2)B\left(\frac{K_{\rm eq}+K_{\rm neq}}{2},\frac{K_{\rm neq}-K_{\rm eq}-2}{2}\right)}\right]. \tag{54}$$

Note that, in equilibrium, $K_{\rm eq} = K_{\rm neq}$ and $I_{T,\eta 0} = 0$.

B. Violation of the quantum FDT and zero-frequency effective temperature

Even though the quantum FDT is not obeyed, one may define an effective temperature $T_{\rm eff,0}$ in the low-frequency limit as follows:

$$\frac{C^K_{\phi\phi}(q=0,\omega=0)}{2\,{\rm Im}\left[C^R_{\phi\phi}(q=0,\omega\to 0)\right]} = \frac{2T_{\rm eff,0}}{\omega}, \tag{55}$$

where we find that the effective temperature (in dimensions of u/α) is

$$T_{\rm eff,0} = \frac{\alpha I_{T0}}{2I_{\eta 0}} = \frac{K_{\rm neq}-2}{2K_{\rm eq}}. \tag{56}$$

As is typical of nonequilibrium systems, this effective temperature depends on the correlation function being studied as it certainly does not characterize the low-frequency properties of

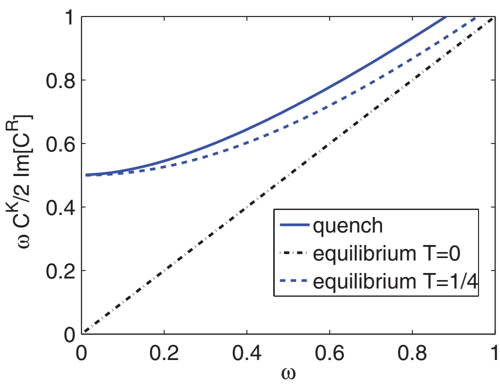

FIG. 1. (Color online) The ratio $\frac{\omega C^K(q=0,\omega)}{2\,\mathrm{Im}[C^R(q=0,\omega)]}$ for a quench where $K_{\rm eq}=2$, $K_{\rm neq}=3$. This is compared with the equilibrium expression $\omega\coth\frac{\omega}{2T}$.

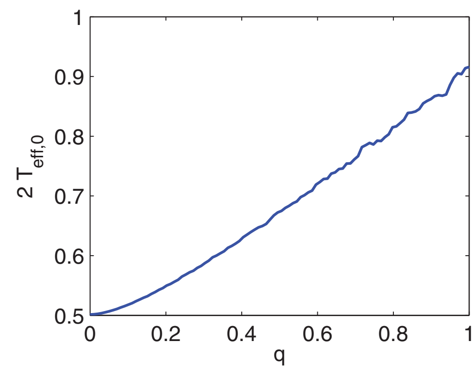

FIG. 2. (Color online) q dependence of $2T_{\rm eff,0}$ for a quench where $K_{\rm eq}=2$ and $K_{\rm neq}=3$.

the simpler correlators in Eq. (43). Moreover, this temperature or, equivalently, the noise correlator has a complicated frequency dependence. However, as we will show by doing RG, the low-frequency limit of the noise or the temperature has important physical consequences, as it changes the long-time and distance behavior of correlation functions by causing them to decay exponentially fast (rather than as a power law with exponent $K_{\rm neq}$).

The crossover from $\omega \ll T_{\rm eff,0}$ to $\omega \gg T_{\rm eff,0}$ is illustrated in Fig. 1, which plots the ratio $\frac{\omega C^K_{\phi\phi}(q=0,\omega)}{2\,\mathrm{Im}C^R_{\phi\phi}(q=0,\omega)}$. In equilibrium, this ratio takes the value of $\omega\coth\frac{\omega}{2T_{\rm eff,0}}$, which in the high-frequency limit becomes $\omega\coth\frac{\omega}{2T_{\rm eff,0}}\xrightarrow{T_{\rm eff,0}\ll\omega}|\omega|$, and in the low-frequency limit is $\omega\coth\frac{\omega}{2T_{\rm eff,0}}\xrightarrow{T_{\rm eff,0}\gg\omega}2T_{\rm eff,0}$. The plot shows that the nonequilibrium system shows a slower crossover to $|\omega|$ with increasing frequencies than the equilibrium system, indicating that the occupation of higher-energy modes decays slower than exponential. It should be noted that similar noise with a complicated crossover behavior from low to high frequencies was studied in open and driven systems near quantum critical points.[38–41] The low-frequency limit of the noise was found to cut off the power-law decay of critical fluctuations, and to also cause a classical ordering-disordering phase transition.[38,39]

Another measure of the violation of FDT is to extract the momentum dependence of the zero-frequency temperature $2T_{\rm eff,0}(q)=\frac{\omega C^K_{\phi\phi}(q,\omega=0)}{2\,\mathrm{Im}C^R_{\phi\phi}(q,\omega\to 0)}$. This quantity is plotted in Fig. 2 and shows that the shorter the distances, the higher is the effective temperature, unlike in equilibrium where all length scales are associated with the same temperature.

At this stage, a peculiarity of the zero-frequency effective temperature, namely, that it does not vanish as $K_{\rm neq}\to K_{\rm eq}$, should be noted. In this case, the first equality in Eq. (56) shows that the effective temperature is a ratio of two quantities, both of which go to zero. However, the limit approaches a finite value. The origin of this result that the temperature approaches a finite value as the quench becomes smaller and smaller is due to the singular form of the equilibrium distribution function $\coth(\omega/2T)$, which probably persists even for the weakly nonequilibrium problem. The singular form of $\coth(\omega/2T)$ implies that the limits $\omega\to 0$ and $T\to 0$ do not commute. For one order of limits, the answer is divergent, and for the other it is 1. In defining the zero-frequency effective temperature, we have implicitly taken the frequency to zero first. This result also signals that the energy scale $T_{\rm eff,0}$ can not be used as a good measure of the energy stored in the system due to the quench. This energy is expected to be distributed in a rather complicated way among all frequency modes. However, by doing RG, we will show that $T_{\rm eff,0}$ acts like a regular temperature when studying two-point correlation functions, as this energy scale causes the correlations to decay exponentially fast (as compared to a power law) at long times after the quench. The size of the quench $|K_0-K|$, on the other hand, is inversely related to how long one has to wait to see this behavior.

Aside from an effective temperature, perhaps a more surprising result is a generation of friction. This effect appears at this stage as a nonzero slope of $2\,\mathrm{Im}[C^R_{\phi\phi}]\propto -i\eta\omega$. In the next section, when we do RG, we will show that η corresponds to a finite lifetime of the bosonic modes, an effect that is distinct from the generation of an effective temperature. Thus, we will find that even though the system we study is closed, and in equilibrium is characterized by long-lived bosonic modes with $\eta=0$, the quench together with the mode coupling arising due to the periodic potential gives rise to additional scattering, which generates an η. The finite lifetime of low-frequency bosonic modes implies that there is a flow of energy from low- to high-energy scales. An alternate, but simpler, example of this phenomenon is the decay of collective modes of a system of one-dimensional weakly interacting fermions via the creation of particle-hole excitations, where the fermions are in a nonequilibrium state due to an initial quench.[32]

IV. DERIVATION OF THE RG EQUATIONS

In order to derive the RG equations, it is convenient to write the Keldysh action for the steady state as

$$Z_K=\int \mathcal{D}[\phi_{cl},\phi_q]e^{i(S_0+S_{sg})}, \tag{57}$$

where S_0 is the quadratic part, which describes the physics at long times after the interaction quench:

$$S_0 = \sum_{q,\omega} (\phi^*_{cl}(q,\omega) \quad \phi^*_q(q,\omega)) \begin{pmatrix} 0 & G_A^{-1} \\ G_R^{-1} & -G_R^{-1}G_K G_A^{-1} \end{pmatrix} \times \begin{pmatrix} \phi_{cl}(q,\omega) \\ \phi_q(q,\omega) \end{pmatrix}, \tag{58}$$

where

$$-i\left\langle \begin{pmatrix} \phi_{cl} \\ \phi_q \end{pmatrix} (\phi^*_{cl} \quad \phi^*_q) \right\rangle = \begin{pmatrix} G_K & G_R \\ G_A & 0 \end{pmatrix}. \tag{59}$$

Using the expressions for $G_{R,A,K}$ in Eqs. (41) and (42), Eq. (58) may be written in the following manner:

$$S_0 = \sum_{q,\omega} (\phi^*_{cl}(q,\omega) \quad \phi^*_q(q,\omega)) \times \frac{1}{\pi K u} \begin{pmatrix} 0 & (\omega - i\delta)^2 - u^2 q^2 \\ (\omega + i\delta)^2 - u^2 q^2 & 4i|\omega|\delta \frac{K_0}{2K}\left(1 + \frac{K^2}{K_0^2}\right) \end{pmatrix} \times \begin{pmatrix} \phi_{cl}(q,\omega) \\ \phi_q(q,\omega) \end{pmatrix}, \tag{60}$$

where $\delta = 0+$, and the modes are long lived. Note that the above action, although obtained for the specific case of an interaction quench in a Luttinger liquid, has the same generic form as any Luttinger liquid subjected to a nonequilibrium noise source. The details of the noise determine the particular form of the coefficient of the ϕ_q^2 term in the action.[3,36] In fact, nonequilibrium noise can also arise in open systems driven out of equilibrium by current flow.[38]

We now discuss how the above free theory is affected by a mode-coupling term due to a periodic potential. The action corresponding to this is

$$S_{sg} = \frac{gu}{\alpha^2} \int dx \int dt [\cos\gamma\phi_- - \cos\gamma\phi_+]. \tag{61}$$

We split the fields into slow and fast components

$$\phi_{cl,q}(xt) = \phi^<_{cl,q}(xt) + \phi^>_{cl,q}(xt) \tag{62}$$

and integrate out the fast components. We will outline two different procedures for integrating out the fast modes. In one, we impose a hard cutoff, where the fast fields are defined as

$$\phi^<_{cl,q}(xt) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda'}^{\Lambda'} \frac{dq}{2\pi} e^{iqx - i\omega t} \phi_{q,cl}(q,\omega), \tag{63}$$

$$\phi^>_{cl,q}(xt) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{\Lambda > |q| > \Lambda'} \frac{dq}{2\pi} e^{iqx - i\omega t} \phi_{q,cl}(q,\omega), \tag{64}$$

and $\Lambda/\Lambda' = e^{d\ln(l)}$. The results in Ref. 31 were presented far from the critical point using the above RG scheme.

The second way of integrating out fast modes is a scheme outlined by Nozieres and Gallet,[42] where the slow and fast correlators are defined as

$$G_{0,\Lambda} = G^<_{0,\Lambda - d\Lambda} + G^>_{\Lambda - d\Lambda,\Lambda}. \tag{65}$$

Thus, the fast correlator may be obtained from taking a derivative of the slow or full correlator

$$G^>_{\Lambda - d\Lambda,\Lambda} = d\Lambda \frac{dG_\Lambda}{d\Lambda}, \tag{66}$$

where $\Lambda = \alpha^{-1}$. Note that when doing RG in real time, all cutoffs should be imposed only in momentum space, as imposing cutoffs in time and varying them during the RG flow lead to inconsistencies such as a violation of causality.

The Nozieres-Gallet scheme is a more consistent way to deal with the cutoff, and finally the results in this paper will be presented using this method. However, even the hard cutoff scheme outlined in Eqs. (63) and (64) and used in Ref. 31 gives qualitatively similar results. Furthermore, in this paper, we will use the Nozieres-Gallet scheme to study the behavior close to the critical point.

Now our task is to expand Z_K in powers of g, integrate over the fast modes, and rescale the cutoff back to its original value $[dx(dt) \to \frac{\Lambda}{\Lambda'} dx(dt)]$. Up to quadratic order in g, we find

$$Z_K = \int \mathcal{D}[\phi^<_{cl},\phi^<_q] e^{iS_0^<} \Big[1 + i\frac{gu}{\alpha^2}\left(\frac{\Lambda}{\Lambda'}\right)^2 \int dx\, dt \times e^{-\frac{\gamma^2}{4}\langle(\phi^>_{cl})^2\rangle} \{\cos[\gamma\phi^<_-(1)] - \cos[\gamma\phi^<_+(1)]\} \tag{67}$$

$$- \frac{g^2u^2}{\alpha^4} \int dx_1 dt_1 dx_2 dt_2 \theta(t_1 - t_2) \times \langle \cos\{\gamma[\phi^<_-(1) + \phi^>_-(1)]\} \cos\{\gamma[\phi^<_-(2) + \phi^>_-(2)]\}\rangle_> - \frac{g^2u^2}{\alpha^4} \int dx_1 dt_1 dx_2 dt_2 \theta(t_2 - t_1) \times \langle \cos\{\gamma[\phi^<_+(1) + \phi^>_+(1)]\} \cos\{\gamma[\phi^<_+(2) + \phi^>_+(2)]\}\rangle_> + \frac{g^2u^2}{\alpha^4} \int dx_1 dt_1 dx_2 dt_2 [\theta(t_2 - t_1) + \theta(t_1 - t_2)] \times \langle \cos\{\gamma[\phi^<_-(1) + \phi^>_-(1)]\} \cos\{\gamma[\phi^<_+(2) + \phi^>_+(2)]\}\rangle_> \Big]. \tag{68}$$

The above may be reexponentiated using the cumulative expansion $\langle e^V\rangle \simeq e^{\langle V\rangle + \frac{1}{2}\langle V^2\rangle - \frac{1}{2}\langle V\rangle^2}$ to obtain

$$Z_K = \int \mathcal{D}[\phi^<_{cl},\phi^<_q] e^{iS_0^< + i\delta S}. \tag{69}$$

The expression for δS may be simplified by dropping terms that are proportional to $e^{i\phi_\pm(1) + i\phi_\pm(2)}$ as they are more irrelevant than terms such as $e^{i\phi_\pm(1) - i\phi_\pm(2)}$. Furthermore, using the fact that

$$\langle \sin[\gamma\phi^>_a(1) - \gamma\phi^>_b(2)]\rangle = 0, \tag{70}$$

all the terms containing sin functions vanish. Moreover, by using $\cos\phi =: \cos\phi : e^{-\langle\phi^2\rangle/2}$, we find

$$\delta S = \frac{gu}{\alpha^2}\left(\frac{\Lambda}{\Lambda'}\right)^2 \int dx \int dt\, e^{-\frac{\gamma^2}{4}\langle(\phi^>_{cl})^2\rangle} \times \{\cos[\gamma\phi^<_-(1)] - \cos[\gamma\phi^<_+(1)]\} \tag{71}$$

$$+ i\frac{g^2u^2}{2\alpha^4} \int dx_1 \int dt_1 \int dx_2 \int dt_2 \theta(t_1 - t_2) \times :\cos\{\gamma[\phi^<_-(1) - \phi^<_-(2)]\}: e^{-\frac{\gamma^2}{2}\langle[\phi_-(1) - \phi_-(2)]^2\rangle} \times \left[1 - e^{-\frac{\gamma^2}{2}\langle(\phi^>_{cl})^2\rangle + \frac{\gamma^2}{2}\langle[\phi^>_-(1) - \phi^>_-(2)]^2\rangle}\right] \tag{72}$$

$$+ i\frac{g^2u^2}{2\alpha^4}\int dx_1dt_1dx_2dt_2\theta(t_2-t_1)$$
$$\times\ :\cos\{\gamma[\phi_+^<(1)-\phi_+^<(2)]\}: e^{-\frac{\gamma^2}{2}\langle[\phi_+(1)-\phi_+(2)]^2\rangle}$$
$$\times\left[1-e^{-\frac{\gamma^2}{2}\langle(\phi_{cl}^>)^2\rangle+\frac{\gamma^2}{2}\langle[\phi_+^>(1)-\phi_+^>(2)]^2\rangle}\right] \quad (73)$$

$$- i\frac{g^2u^2}{2\alpha^4}\int dx_1dt_1dx_2dt_2[\theta(t_2-t_1)+\theta(t_1-t_2)]$$
$$\times\ :\cos\{\gamma[\phi_-^<(1)-\phi_+^<(2)]\}: e^{-\frac{\gamma^2}{2}\langle[\phi_-(1)-\phi_+(2)]^2\rangle}$$
$$\times\left[1-e^{-\frac{\gamma^2}{2}\langle(\phi_{cl}^>)^2\rangle+\frac{\gamma^2}{2}\langle[\phi_-^>(1)-\phi_+^>(2)]^2\rangle}\right], \quad (74)$$

where $\langle[\phi_\pm(1)-\phi_\pm(2)]^2\rangle$ involves averaging all (both slow and fast) modes.

Equation (71) implies

$$g(\Lambda') = g(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^2 e^{-\frac{\gamma^2}{4}\langle(\phi_{cl}^>)^2\rangle} \quad (75)$$

$$= g(\Lambda)\left(\frac{\Lambda}{\Lambda'}\right)^2 \exp\left[-\frac{\gamma^2}{8}K_0\left(1+K^2/K_0^2\right)\ln(\Lambda/\Lambda')\right]. \quad (76)$$

Note that

$$\left[1-e^{-\frac{\gamma^2}{2}\langle(\phi_{cl}^>)^2\rangle+\frac{\gamma^2}{2}\langle[\phi_a^>(1)-\phi_b^>(2)]^2\rangle}\right]\simeq O\left(\ln\frac{\Lambda}{\Lambda'}\right). \quad (77)$$

Next, we introduce the center of mass $R=\frac{x_1+x_2}{2}$, $T_m=\frac{t_1+t_2}{2}$, and relative coordinates $r=x_1-x_2$, $\tau=t_1-t_2$, and expand the $O(g^2)$ terms in powers of $r=x_1-x_2$, $\tau=t_1-t_2$. Using

$$:\cos\{\gamma[\phi_-^<(1)-\phi_-^<(2)]\}:$$
$$\simeq 1-\frac{\gamma^2}{4}\left[\left(r\partial_R+\tau\partial_{T_m}\right)\phi_{cl}+\left(r\partial_R+\tau\partial_{T_m}\right)\phi_q\right]^2, \quad (78)$$

$$:\cos\{\gamma[\phi_+^<(1)-\phi_+^<(2)]\}:$$
$$\simeq 1-\frac{\gamma^2}{4}\left[\left(r\partial_R+\tau\partial_{T_m}\right)\phi_{cl}-\left(r\partial_R+\tau\partial_{T_m}\right)\phi_q\right]^2, \quad (79)$$

$$:\cos\{\gamma[\phi_-^<(1)-\phi_+^<(2)]\}:$$
$$\simeq 1-\frac{\gamma^2}{4}\left[\left(r\partial_R+\tau\partial_{T_m}\right)\phi_{cl}+2\phi_q\right]^2. \quad (80)$$

We note that the terms with 1 and purely classical fields cancel. We regroup the remaining terms and find the following corrections to the quadratic part of the action [the correction to the cosine term is already given above in Eq. (76)]:

$$S_0 = \int dR\int d(uT_m)\frac{1}{\pi K}\Big[\phi_q\left(\partial_R^2-\partial_{uT_m}^2\right)\phi_{cl}$$
$$+\phi_{cl}\left(\partial_R^2-\partial_{uT_m}^2\right)\phi_q+\frac{\delta u}{u}\phi_q\left(\partial_R^2+\partial_{uT_m}^2\right)\phi_{cl}$$
$$+\frac{\delta u}{u}\phi_{cl}\left(\partial_R^2+\partial_{uT_m}^2\right)\phi_q-2\frac{\eta}{u}\left(\frac{\Lambda}{\Lambda'}\right)\phi_q\partial_{uT_m}\phi_{cl}$$
$$+i\frac{4T_{\rm eff}\eta}{u^2}\frac{K_0}{2K}\left(1+\frac{K^2}{K_0^2}\right)\left(\frac{\Lambda}{\Lambda'}\right)^2\phi_q^2\Big]. \quad (81)$$

The above implies the following RG equations:

$$\frac{dg}{d\ln l} = \left[2-\frac{\gamma^2}{8}K_0\left(1+K^2/K_0^2\right)\right]g, \quad (82)$$

$$\frac{dK^{-1}}{d\ln l} = \frac{\pi g^2}{4\alpha^4}\left(\frac{\gamma^2}{2}\right)^2\frac{K_0}{2}\left(1+\frac{K^2}{K_0^2}\right)I_K, \quad (83)$$

$$\frac{1}{Ku}\frac{du}{d\ln l} = \frac{\pi g^2}{4\alpha^4}\left(\frac{\gamma^2}{2}\right)^2\frac{K_0}{2}\left(1+\frac{K^2}{K_0^2}\right)I_u, \quad (84)$$

$$\frac{d\eta}{d\ln l} = \eta+\frac{\pi g^2Ku}{2\alpha^4}\left(\frac{\gamma^2}{2}\right)^2\frac{K_0}{2}\left(1+\frac{K^2}{K_0^2}\right)I_\eta, \quad (85)$$

$$\frac{d(T_{\rm eff}\eta)}{d\ln l} = 2T_{\rm eff}\eta+\frac{\pi g^2u^2K^2}{4\alpha^4}\left(\frac{\gamma^2}{2}\right)^2 I_T, \quad (86)$$

where [defining $\mathrm{Re}(x)=(x+x^*)/2$, $\mathrm{Im}(x)=(x-x^*)/(2i)$]

$$I_T = \int_{-\infty}^{\infty}dr\int_{-\infty}^{\infty}dt\,\mathrm{Re}\left[e^{-\frac{\gamma^2}{2}\langle[\phi_-(t,r)-\phi_+(0,0)]^2\rangle}F^{a/ng}\right], \quad (87)$$

$$I_\eta = \int_{-\infty}^{\infty}dr\int_{-\infty}^{\infty}dt\,t\,\mathrm{Im}\left[e^{-\frac{\gamma^2}{2}\langle[\phi_-(t,r)-\phi_+(0,0)]^2\rangle}F^{a,ng}\right], \quad (88)$$

$$I_K = \int_{-\infty}^{\infty}dr\int_0^{\infty}dt\,(r^2-t^2)\mathrm{Im}\left[e^{-\frac{\gamma^2}{2}\langle[\phi_-(t,r)-\phi_+(0,0)]^2\rangle}F^{a/ng}\right], \quad (89)$$

$$I_u = \int_{-\infty}^{\infty}dr\int_0^{\infty}dt\,(r^2+t^2)\mathrm{Im}\left[e^{-\frac{\gamma^2}{2}\langle[\phi_-(t,r)-\phi_+(0,0)]^2\rangle}F^{a/ng}\right], \quad (90)$$

where F^a arises due to the hard cutoff scheme and F^{ng} is due to the Nozieres-Gallet scheme. In particular,

$$e^{-\frac{\gamma^2}{2}\langle[\phi_-(r,t)-\phi_+(0,0)]^2\rangle} = e^{-\frac{K_{\rm neq}}{2}\ln\left(\frac{\alpha^2+(t+r)^2}{\alpha^2}\right)-\frac{K_{\rm neq}}{2}\ln\left(\frac{\alpha^2+(t-r)^2}{\alpha^2}\right)}$$
$$\times e^{i[K_{\rm eq}\tan^{-1}(\frac{t+r}{\alpha})+K_{\rm eq}\tan^{-1}(\frac{t-r}{\alpha})]}, \quad (91)$$

$$F^a = \frac{1}{2}(e^{i\Lambda(t+r)}+e^{i\Lambda(t-r)})+\frac{i}{2}\left(\frac{K_{\rm eq}}{K_{\rm neq}}-1\right)$$
$$\times\{\sin[\Lambda(t+r)]+\sin[\Lambda(t-r)]\}, \quad (92)$$

$$F^{ng} = \frac{1}{2}\left[\frac{\alpha^2}{\alpha^2+(t+r)^2}+\frac{\alpha^2}{\alpha^2+(t-r)^2}\right.$$
$$\left.+i\left(\frac{K_{\rm eq}}{K_{\rm neq}}\right)\left(\frac{\alpha(t+r)}{\alpha^2+(t+r)^2}+\frac{\alpha(t-r)}{\alpha^2+(t-r)^2}\right)\right]. \quad (93)$$

V. RESULTS IN THE GAPLESS PHASE

A. General structure of equations

In this section, we study the consequence of Eqs. (82), (83), (84), (85), and (86). Equation (82) shows that there is a critical point located at $K_{\rm neq}=\frac{\gamma^2}{8}K_0(1+K^2/K_0^2)=2$, and therefore at a value of K that is different from the equilibrium critical point at $K = 8/\gamma^2$. Since $K_{\rm neq} > K_{\rm eq}$, this critical point is always located at a smaller value of K, which is another way of saying that the cosine potential for the post-quench case is more irrelevant.

Equation (83) represents the flow of the interaction constant, however, depending on the quench protocol (and hence the values of $K_{\rm eq}$ and $K_{\rm neq}$), the flow can be significantly different from the equilibrium flow. For example, while $I_K > 0$ in equilibrium, implying a decrease of the interaction parameter in the presence of the periodic potential, for the quench problem I_K can become negative, and also diverge at the critical point. This will be discussed in more detail in the following. Equation (84) is the flow of the velocity that occurs even in equilibrium, and is primarily due to the cutoff procedure employed here, which does not preserve Lorentz invariance. The effects of this are small, and in what follows, we will ignore it.

Equation (85) shows that a dissipation or a finite lifetime of the bosons η is generated, changing the low-frequency properties of the bosonic system qualitatively. Further, I_η (which is the rate at which η increases with flow) diverges at the critical point, implying a diverging dissipation. The origin of this divergence is similar to the divergence of I_K briefly mentioned above. This divergence can have two possible causes: (i) The flow is derived using the original (athermal state) correlation functions. Since a finite temperature and dissipation is generated, this might regularize the divergence close to critical point. (ii) The renormalization of the coefficients η and K results from a gradient expansion of the second-order results. Such a divergence in the correction might indicate that this expansion breaks down. This could mean a nonanalytical behavior at low energy. Unraveling this point is a challenging question.

Finally, Eq. (86) shows that a constant term is generated for the strength of the noise correlator in the zero-frequency limit, which can be interpreted as a product of dissipation and effective temperature. In general, this noise is expected to have a complicated frequency dependence as was highlighted for the $g = 0$ case in the previous section. To determine how this frequency dependence evolves with RG is a daunting task, and in this paper, we will only discuss the effects of g on the low-frequency part of the noise spectrum.

In both the hard cutoff as well as the Nozieres-Gallet schemes, the important result that $I_{T,\eta} = 0$ when $K_{\rm eq} = K_{\rm neq}$ is recovered. Further, both schemes reveal the peculiarity of diverging I_K and I_η at the critical point $K_{\rm neq} = 2$ for the nonequilibrium problem. It should be noted that in deriving the above RG equations, we have used the correlators for $T_{\rm eff} = \eta = 0$. The more consistent way to treat the problem is to evaluate the correlators for nonzero $T_{\rm eff}$ and η. However, this makes the problem quite difficult, all the more so because the full frequency dependence of $T_{\rm eff}$ is needed. We argue that taking these effects into account only has a minor influence on the RG flows far from the critical point. However, the divergences in $I_{\eta,K}$ indicate that near the critical point, a more consistent computation may be needed. This is clearly something left for future studies.

In the Nozieres-Gallet scheme, the expressions $I_{T,\eta}$ are found to be related in a rather simple way to $I_{T,\eta 0}$, i.e., to the $q = 0$, $\omega \to 0$ limits of the C^K and ImC^R evaluated for the quadratic theory in Sec. III. In particular,

$$I_T = \frac{8\alpha^2}{K_{\rm neq}(K_{\rm neq}-1)}\left[\frac{\pi}{2^{K_{\rm neq}}B\left(\frac{K_{\rm neq}+K_{\rm eq}}{2},\frac{K_{\rm neq}-K_{\rm eq}}{2}\right)}\right]^2 = \left(\frac{K_{\rm neq}-1}{K_{\rm neq}}\right)I_{T0}, \tag{94}$$

$$I_\eta = 8\alpha^3\left[\frac{\pi}{2^{K_{\rm neq}}B\left(\frac{K_{\rm neq}+K_{\rm eq}}{2},\frac{K_{\rm neq}-K_{\rm eq}}{2}\right)}\right]^2 \times\left(\frac{K_{\rm eq}}{K_{\rm neq}}\right)\frac{(K_{\rm neq}-3/2)}{(K_{\rm neq}-1)^2(K_{\rm neq}-2)} = \left(\frac{K_{\rm neq}-3/2}{K_{\rm neq}}\right)I_{\eta 0}, \tag{95}$$

where $I_{T\eta 0}$ are given in Eqs. (52) and (54). The ratio of these quantities leads to an effective temperature

$$T_{\rm eff}^{ng} = \frac{\alpha I_T}{2I_\eta} = \frac{(K_{\rm neq}-1)(K_{\rm neq}-2)}{2K_{\rm eq}(K_{\rm neq}-3/2)} \tag{96}$$

$$= \left(\frac{K_{\rm neq}-1}{K_{\rm neq}-3/2}\right)T_{\rm eff,0} \tag{97}$$

with $T_{\rm eff,0}$ being the noninteracting expression for the effective temperature. Note that when $K_{\rm neq} \gg 1$, the cosine potential is more irrelevant, and the above expressions approach the noninteracting values. The above analytic expressions also show that I_η is divergent at the critical point as $1/(K_{\rm neq}-2)$.

The expression for I_K is more complex and given by

$$I_K = -\alpha^4\left[\int_0^{\pi/2} dx \tan x\cos(K_{\rm eq}x)(\cos x)^{K_{\rm neq}} \times\int_0^x dy\tan y\sin(K_{\rm eq}y)(\cos y)^{K_{\rm neq}-2}\right. \tag{98}$$

$$+\int_0^{\pi/2} dx\tan x\cos(K_{\rm eq}x)(\cos x)^{K_{\rm neq}-2} \times\int_0^x dy\tan y\sin(K_{\rm eq}y)(\cos y)^{K_{\rm neq}} \tag{99}$$

$$+\left(\frac{K_{\rm eq}}{K_{\rm neq}}\right)\int_0^{\pi/2} dx\tan x\cos(K_{\rm eq}x)(\cos x)^{K_{\rm neq}-2} \times\int_0^x dy(\tan y)^2\cos(K_{\rm eq}y)(\cos y)^{K_{\rm neq}} \tag{100}$$

$$\left.-\left(\frac{K_{\rm eq}}{K_{\rm neq}}\right)\int_0^{\pi/2} dx(\tan x)^2\sin(K_{\rm eq}x)(\cos x)^{K_{\rm neq}} \times\int_0^x dy\tan y\sin(K_{\rm eq}y)(\cos y)^{K_{\rm neq}-2}\right]. \tag{101}$$

I_K shows complicated structure, and in particular depending on the quench protocol can be positive, negative, or zero. For example, $I_K(K_{\rm eq} = 2,\ K_{\rm neq} = 7) = -\alpha^4 0.0022$, showing that for this quench protocol, the periodic potential increases the interaction parameter, making the system more delocalized

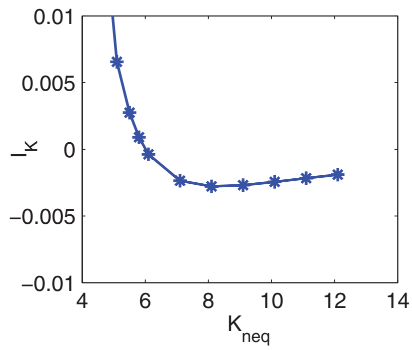

FIG. 3. (Color online) Change of sign of I_K at a certain value of $K_{\rm neq}$. Here, $K_{\rm eq} = 2$.

as compared to the ground state of a Luttinger liquid with interaction parameter K. A plot of I_K for $K_{\rm eq} = 2$ and different $K_{\rm neq}$ is shown in Fig. 3 where a change of sign of I_K is found.

Further, I_K generically diverges at the critical point $K_{\rm neq} = 2$ for all values of $K_{\rm eq}$ except $K_{\rm eq} = 1$ and 2. This divergence is seen from observing that when $K_{\rm neq} = 2$, Eqs. (99) and (100) above are in general divergent because $\tan(\pi/2) = \infty$ and $[\cos(x)]^{K_{\rm neq}-2} = 1$. However, these divergences cancel in equilibrium when $K_{\rm eq} = K_{\rm neq} = 2$, and we find $I_K(K_{\rm eq} = K_{\rm neq} = 2) = \alpha^4(8\pi/32)$. In contrast, at the special point $K_{\rm eq} = 1$, $K_{\rm neq} = 2$, $I_K(K_{\rm eq} = 1, K_{\rm neq} = 2) = 0$.

In the subsequent sections, we will study the RG flows in the gapless phase for two cases, one where the system is far from the critical point and the other where it is close to it. For the latter, we will highlight the effect of the above divergences.

B. Flow far from the critical point

The RG equations can be solved numerically. The flow of the various quantities for a quench from $K_0 = 3$ to $K = 5$ and for $\gamma = 2$ are shown in Figs. 4, 5, 6, and 7. For these values, I_K remains positive, so that K decreases during the flow (Fig. 5). Since g is (dangerously) irrelevant, it flows to zero (Fig. 4); however, for the nonequilibrium problem, g generates new terms such as a dissipation (Fig. 6) and a zero-frequency component of the noise (Fig. 7). The latter may be identified from the zero-frequency limit of the following term in equilibrium:

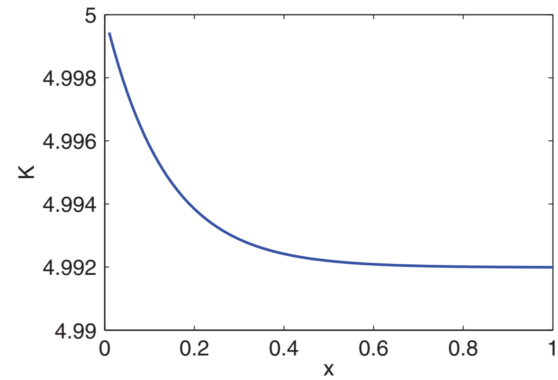

FIG. 5. (Color online) Flow of K with $x = \ln l$ for $K_0 = 3$, $K = 5$, $g = 0.1$, and $\gamma = 2$.

$$\eta\omega\coth(\omega/2T) \to 2\eta T \qquad (102)$$

as dissipation × effective temperature. Note that since we have, from the RG, a quadratic action for the fixed point, this allows us to unambiguously define a temperature for the *low-energy* modes. Indeed, with such a quadratic action, the fluctuation-dissipation theorem would be obeyed for all correlation functions involving the field ϕ with the same temperature T as defined by (102). One thus sees that the RG procedure indicates that the system does thermalize. It is important to remember, however, that this statement only concerns the low-energy modes. The frequency dependence of the Keldysh term is quite different from the standard $\eta\omega\coth(\omega/2T)$. This means that as the frequency is increased, there will be a complicated crossover between this thermal state at low energy and the athermal original state that controls the high-energy behavior of the system. Another interesting question is what will happen for the correlation functions of the dual field θ. The most naive expectation would be that such correlations are also controlled by the same temperature T. However, since in the above action θ has been integrated out, this would need explicit calculations of the θ-θ correlations, a quite complicated calculation. It will thus be interesting to see whether this is indeed the case or whether the dual field can have a different behavior.

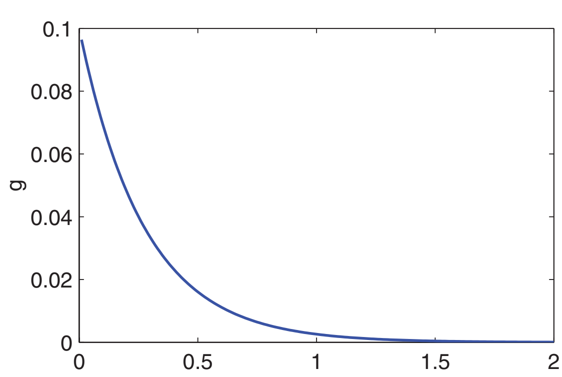

FIG. 4. (Color online) Flow of g with $x = \ln l$ for $K_0 = 3$, $K = 5$, $g = 0.1$, and $\gamma = 2$.

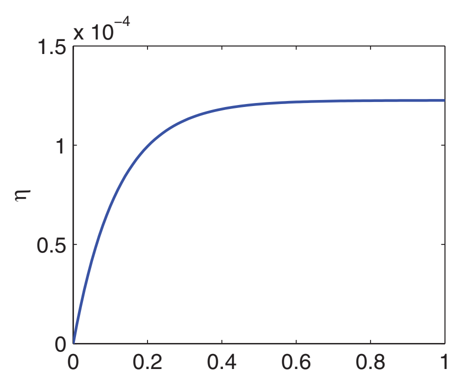

FIG. 6. (Color online) Flow of $e^{-x}\eta(x)$ with $x = \ln l$ for $K_0 = 3$, $K = 5$, $g = 0.1$, and $\gamma = 2$.

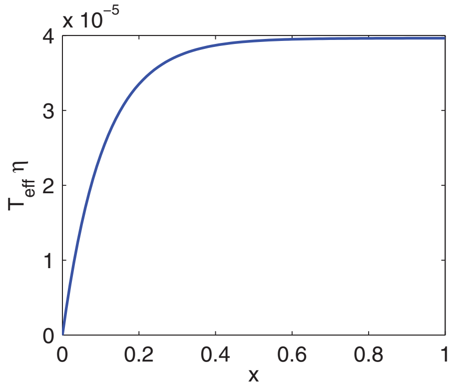

FIG. 7. (Color online) Flow of $e^{-2x}T_{\rm eff}\eta$ with $x = \ln l$ for $K_0 = 3$, $K = 5$, $g = 0.1$, and $\gamma = 2$.

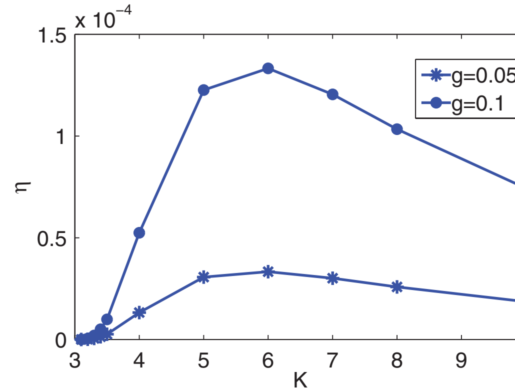

FIG. 8. (Color online) Strength of the dissipation η for $K_0 = 3$, $\gamma = 2$, and $g = 0.05$ and 0.1.

The flow of the effective temperature can be studied most easily by substituting Eq. (85) in Eq. (86). We obtain (after defining dimensionless variables $T_{\rm eff} \to \alpha T_{\rm eff}/u$, $\eta \to \alpha\eta/u$)

$$\frac{dT_{\rm eff}}{d\ln l} = T_{\rm eff} + \frac{\pi g^2}{4}\left(\frac{\gamma^2}{2}\right)^2 \frac{K^2 I_T}{\eta} \times \left[1 - 2T_{\rm eff}\left(\frac{K_0}{2K}\right)\left(1+\frac{K^2}{K_0^2}\right)\frac{I_\eta}{I_T}\right]. \quad (103)$$

Note that the physical temperature $T_{\rm eff}e^{-\ln l}$ obeys the differential equation

$$\frac{d(T_{\rm eff}e^{-\ln l})}{d\ln l} = e^{-\ln l}\frac{\pi g^2}{4}\left(\frac{\gamma^2}{2}\right)^2 \frac{K^2 I_T}{\eta} \times \left[1 - 2T_{\rm eff}\left(\frac{K_0}{2K}\right)\left(1+\frac{K^2}{K_0^2}\right)\frac{I_\eta}{I_T}\right]. \quad (104)$$

The above shows that the steady-state solution corresponds to a temperature

$$T^*_{\rm eff} = \left(\frac{K^*_{\rm neq}-1}{K^*_{\rm neq}-3/2}\right)\left(\frac{K^*_{\rm neq}-2}{2K^*_{\rm neq}}\right). \quad (105)$$

Note that $T^*_{\rm eff}$ differs from the value of $I_T/(2I_\eta)$ in Eq. (97) only because of the way we have defined it. The energy scale that determines the decay of the correlations is the prefactor of the ϕ_q^2 term in the action Eq. (81), and is the combination $T^*_{\rm eff}K^*_{\rm neq}/K^*_{\rm eq}$.

The renormalized dissipation $e^{-\ln l}\eta$ for a quench from $K_0 = 3$ to $K > K_0$ is shown in Fig. 8. The behavior is nonmonotonic as the size of the quench becomes larger and larger. The reason for this is that when $K = K_0$, $\eta = 0$. At the same time, when $K \gg K_0$, the cosine potential being more irrelevant decays faster to zero, so that the renormalized η is also smaller for larger K. These two behaviors for $|K - K_0| \ll 1$ and $K \gg K_0$ imply a maxima in-between.

The generation of a temperature and dissipation in the low-energy theory via the RG procedure is shown schematically in Fig. 9. For the out-of-equilibrium system, there is a gradual flow of energy from the low-energy (long-wavelength) degrees of freedom and the high-energy (short-wavelength) degrees of freedom. Thus, when the latter are integrated out in the RG procedure, this flow of energy appears as a dissipation in the low-energy sector. Such a dissipation is also accompanied by a temperature such that a low-frequency classical FDT is obeyed. The above mechanism for thermalization arising due to an exchange of energy between long- and short-wavelength modes appears to be rather generic, and may even be recovered in a quench involving fermions.[32] In particular, in Ref. 32, the effect of weak interactions on a system of one-dimensional fermions that are in a nonequilibrium state due to an initial quench was studied using the random phase approximation (RPA). The RPA analysis revealed that the highly nonequilibrium fermion distribution generated by the quench results in an enhanced particle-hole continuum. Thus, for attractive interactions between fermions, the collective modes were found to lie within this continuum and were therefore found to be overdamped.

It would be interesting to study other models with nonlinearities to see if a similar generic fixed point is reached, and also how the temperature might depend on the type of nonlinearities that exist in the Hamiltonian. Among the various nonlinearities that one could think of, there exists in

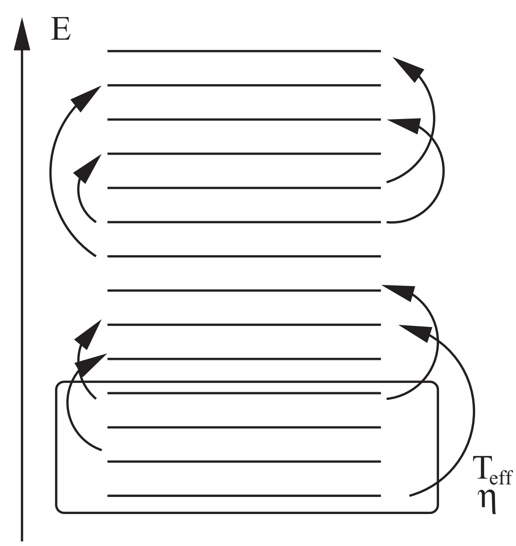

FIG. 9. A schematic of the RG procedure. The resultant dissipation and temperature generated in the low-energy theory (box) arises due to an exchange of energy with the high-energy degrees of freedom that act as a bath.

particular the band curvature that is inherent to a realistic one-dimensional system. Such a band curvature leads in particular to $(\nabla\phi)^3$ terms. Whether such terms could lead to similar effects is an interesting question. Note that, in equilibrium, because of conservation laws, the cosine terms are more efficient in mixing certain modes in contrast to other nonlinearities. In particular, the curvature terms were found not to be able to fully relax a current in Ref. 43. Whether similar differences also exist out of equilibrium is clearly a challenging question.

Independently of these subtle points, we think that the mechanism that we could obtain in a controlled way for this particular model using the RG procedure is quite generic. Figure 9 indicates that provided enough mode coupling exists in a system, the system itself can act as a reservoir and bath for the low-energy part of the degrees of freedom. This subpart will thus acquire a classical behavior in the sense that it will get a finite dissipation and a finite temperature. It is important to contrast this with a standard equilibrium quantum system for which the temperature is the same irrespectively of the energy of the mode considered. Here, the temperature can only be defined if the limit $\omega \to 0$ is taken. The frequency dependence will in general be complicated and correspond to the crossover between the athermal and the (low-energy) thermal state.

C. Flow in the vicinity of the critical point

In this section, we study the RG equations near the critical point and, in particular, highlight the effects of diverging $I_{K,\eta}$. Figure 10 shows the renormalized dissipation for $\gamma = 2$ and for a quench from $K_0 = 1$ and a K chosen to be $K > \sqrt{3}$. Thus, these quenches include those that go across the equilibrium critical point at $K = 2$, and are always on the gapless side of the new nonequilibrium critical point located at $K = \sqrt{3}$. Figure 10 shows that as K is varied such that one approaches this new critical point, the dissipation diverges due to diverging I_η. This could either signal a breakdown in the gradient expansion for the nonequilibrium problem and/or a drawback of our approximation of setting $T_{\rm eff} = \eta = 0$ in the two-point functions used in the evaluation of the RG equations.

Neglecting the effects of temperature and dissipation, it is interesting to study the BKT flow near the quantum critical point. It is of course understood that the flow would be eventually cut off by the temperature and/or the dissipation. Note that close to the quantum critical point $K_{\rm neq} = 2$, the temperature (105) becomes parametrically small while η diverges, so that the product $\eta T_{\rm eff}$ is well defined. Yet, the parametrically small temperature probably signals that as one approaches the Mott phase, the low-energy theory is no longer described by gapless thermalized modes. This is an important open question, which we do not address in this paper.

Near the critical point, we may expand the RG equations for g and K [Eqs. (82) and (83)] about $K_{\rm neq} = 2 + \epsilon$. For concreteness, let us set $K_{\rm eq} = 3/2$. Then, the complicated expression for I_K in Eqs. (98), (99), (100), and (101) reduces to

$$\begin{aligned} I_K(K_{\rm eq} = 3/2, K_{\rm neq} = 2+\epsilon) &= -\alpha^4\bigg[-0.54 + \frac{4}{7}\int_0^{\pi/2} dx \tan x\,(\cos x)^\epsilon \left(\sin\frac{x}{2}\right)^3 \\ &\times \left(\cos\frac{3x}{2}\right)(3 + 3\cos x + \cos 2x) \\ &+ \frac{1}{7}\int_0^{\pi/2} dx \tan x\,(\cos x)^\epsilon \left(\sin\frac{x}{2}\right)^3 \\ &\times \left(\cos\frac{3x}{2}\right)(2 + 9\cos x + 3\cos 2x)\bigg], \end{aligned} \tag{106}$$

where the first smooth numerical term comes from Eqs. (98) and (101), while the rest are terms that diverge as $\epsilon \to 0$. Thus, for $\epsilon \ll 1$, we may approximate

$$I_K \simeq \frac{c}{\epsilon}, \tag{107}$$

where c is a positive number. Thus, after a redefinition of g, the flow of g and ϵ are found to be

$$\frac{dg}{d\ln l} = -\epsilon g, \tag{108}$$

$$\frac{d\epsilon}{d\ln l} = -\frac{g^2}{\epsilon}. \tag{109}$$

The above imply that the flow equations are along the following lines:

$$\frac{g^2}{2} - \frac{\epsilon^3}{3} = A, \tag{110}$$

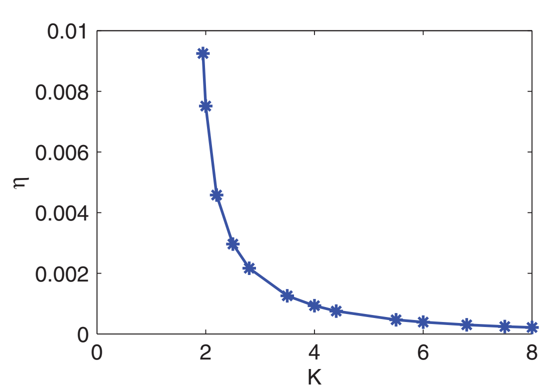

FIG. 10. (Color online) Strength of the dissipation η for $g = 0.05$, $K_0 = 1$, and $\gamma = 2$ so that $K_{\rm neq} = \frac{1}{2}(1 + K^2)$. The critical point is located at $K = \sqrt{3}$.

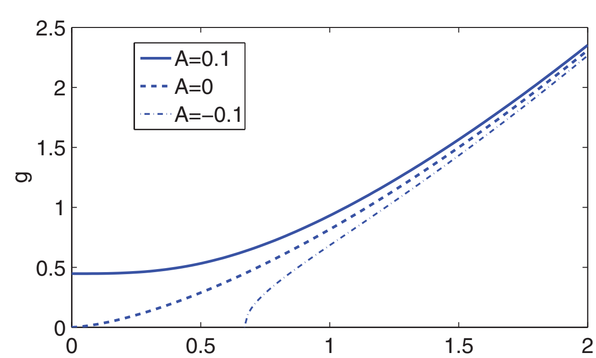

FIG. 11. (Color online) Flow near the critical point $K_{\rm neq} = 2 + \epsilon$, neglecting the effects of temperature and dissipation. Note that the boundary separating the regions in which the cosine is relevant or not is quite different from the equilibrium BKT one.

where A is a constant. Some examples for different choices of A are shown in Fig. 11 and should be contrasted with the equilibrium BKT flow, which are along the lines $g^2 - \epsilon^2 =$ constant.

Along the separatrix $A = 0$, the solution of the RG equations gives

$$\epsilon(l) = \frac{\epsilon_0}{1 + \frac{2}{3}\epsilon_0 \ln l}, \tag{111}$$

$$g(l) = g_0 \left[\frac{1}{1 + \left(\frac{2g_0}{3}\right)^{2/3} \ln l} \right]^{3/2} \tag{112}$$

(denoting ϵ_0, g_0 as the initial bare values).

VI. PROPERTIES OF THE FIXED POINT ACTION

In this section, we discuss the general properties of the steady state resulting from the RG. The renormalized action is a quadratic theory of thermal bosons with a finite lifetime. In the low-frequency limit, the effective action is given by

$$S_0 = \sum_{q,\omega} (\phi_{cl}^*(q,\omega) \quad \phi_q^*(q,\omega)) \times \frac{1}{\pi K u} \begin{pmatrix} 0 & \omega^2 - i\eta\omega - u^2q^2 \\ \omega^2 + i\eta\omega - u^2 q^2 & 4iT_{\rm eff}\eta \frac{K_0}{2K}\left(1 + \frac{K^2}{K_0^2}\right) \end{pmatrix} \times \begin{pmatrix} \phi_{cl}(q,\omega) \\ \phi_q(q,\omega) \end{pmatrix}. \tag{113}$$

Thus, the $\langle\phi\phi\rangle$ correlators are

$$G^R(q,\omega) = \frac{\pi K u}{\omega^2 - u^2q^2 + i\eta\omega}, \tag{114}$$

$$G^K(q,\omega) = (-2i\pi)\left(\frac{K_0}{2K}\right)\left(1 + \frac{K^2}{K_0^2}\right) \times (2T_{\rm eff}\eta)\frac{uK}{\eta^2\omega^2 + (\omega^2 - u^2q^2)^2}. \tag{115}$$

The above correlators should be contrasted with those of the nonequilibrium Luttinger liquid discussed in Sec. III. While both effective theories are quadratic, the combined effect of a quench and the periodic potential is to give rise to inelastic scattering that broadens the bosonic modes by an amount given by η. For the Luttinger liquid, on the other hand (Sec. III), the bosonic modes are long lived, but are characterized by a nonequilibrium occupation probability. Note that a finite lifetime η may also be generated for interacting bosons in a periodic potential, which is in equilibrium but at a nonzero temperature, provided there are no special conservation laws.[44,45]

In what follows, we neglect the ω^2 term relative to the $\eta\omega$ term as we are primarily interested in the long-distance and long-time limits. Then, the retarded correlator is found to be

$$-i\langle\phi_{cl}(1)\phi_q(2)\rangle = \int \frac{dq}{2\pi}\int\frac{d\omega}{2\pi} e^{iq(x_1-x_2)-i\omega(t_1-t_2)}G^R(q,\omega) = -\theta(t_1 - t_2)\frac{\sqrt{\pi}}{2}\frac{K}{\sqrt{\eta(t_1-t_2)}}e^{-\frac{\eta(x_1-x_2)^2}{4u^2|t_1-t_2|}}. \tag{116}$$

Similarly, the advanced correlator is

$$-i\langle\phi_q(1)\phi_{cl}(2)\rangle = \int \frac{dq}{2\pi}\int\frac{d\omega}{2\pi} e^{iq(x_1-x_2)-i\omega(t_1-t_2)}G^A(q,\omega) = -\theta(t_2 - t_1)\frac{\sqrt{\pi}}{2}\frac{K}{\sqrt{\eta(t_2-t_1)}}e^{-\frac{\eta(x_1-x_2)^2}{4u^2|t_1-t_2|}}, \tag{117}$$

while the Keldysh correlator is found to be

$$-i\left[\langle\phi_{cl}(1)\phi_{cl}(2)\rangle - \langle\phi_{cl}^2\rangle\right] = \int\frac{dq}{2\pi}\int\frac{d\omega}{2\pi}\left[e^{iq(x_1-x_2)-i\omega(t_1-t_2)} - 1\right]G^K(q,\omega) = i\left(\frac{K_0}{2K}\right)\left(1 + \frac{K^2}{K_0^2}\right)(2T_{\rm eff}K)\left[\sqrt{\pi\frac{|t_1-t_2|}{\eta}}e^{-\frac{\eta(x_1-x_2)^2}{4u^2|t_1-t_2|}} - \frac{\pi}{2}\left(\frac{x_1-x_2}{u}\right)\mathrm{Erf}\left(\frac{\sqrt{\eta}|(x_1-x_2)|}{2u\sqrt{|t_1-t_2|}}\right)\right]. \tag{118}$$

For $t_1 = t_2$, the above reduces to

$$-i\left[\langle\phi_{cl}(x_1,t)\phi_{cl}(x_2,t)\rangle - \langle\phi_{cl}^2\rangle\right] = i\left(\frac{K_0}{2K}\right)\left(1 + \frac{K^2}{K_0^2}\right)(2T_{\rm eff}K)\left[-\frac{\pi}{2}\left(\frac{x_1-x_2}{u}\right)\right]. \tag{119}$$

Equation (119) shows that the equal-time two-point function $C_{\phi\phi}^K$ decays exponentially in position, with a decay rate given by the effective temperature $T_{\rm eff}K_{\rm neq}/K_{\rm eq}$:

$$\langle e^{i\phi_{cl}(x)}e^{-i\phi_{cl}(y)}\rangle \simeq e^{-\frac{T_{\rm eff}K_{\rm neq}}{K_{\rm eq}}\frac{K}{u}|x-y|}. \tag{120}$$

An interesting question concerns the relation between the generated dissipation and the effective temperature. For a Fermi liquid, for example, $\eta \sim g^2T$. It is interesting to explore to what extent our system mimics such a behavior. Figure 12 shows the plot of η/g^2 as a function of T^2 where $T = T_{\rm eff}^*K_{\rm neq}^*/K_{\rm eq}^*$, the latter being the appropriate energy scale that determines the decay of the two-point functions. The $*$ indicates that the values at the fixed point have been taken. Figure 12 corresponds to $\gamma = 2$ and a quench from $K_0 = 3$ to $K \geqslant K_0$. Notice the coincidence of the plots for two different values of g, indicating that the dissipation scales as g^2. However, for small quenches, the behavior is not Fermi-liquid-like as the dissipation increases as T^β with $\beta > 2$. For larger quenches, the relation between η and T

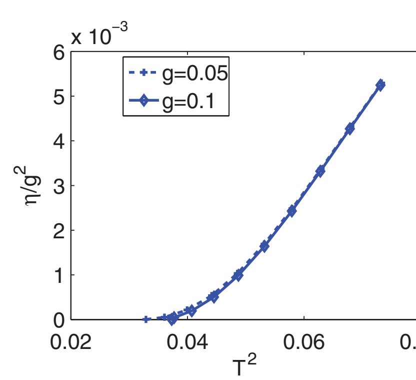

FIG. 12. (Color online) A plot of η/g^2 vs T^2 where $T = T_{\rm eff}^*K_{\rm neq}^*/K_{\rm eq}^*$ and for quenches where $\gamma = 2$, $K_0 = 3$, and $K > 3$.

becomes nonmonotonic (not shown) as T always increases with K,whereas η eventually decreases as g becomes more and more irrelevant.

VII. CONCLUSIONS AND OUTLOOK

In this paper, we have explored how out-of-equilibrium quantum systems can thermalize in the presence of mode-coupling terms. We have analyzed in detail a situation in which a system of interacting bosons is set out of equilibrium by a sudden interaction quench. In Ref. 31 and in this paper, we have shown, using a controlled RG procedure, how mode coupling or interactions that give rise to nontrivial scattering between modes affect this nonequilibrium state.

The main result is that even when the mode coupling is "irrelevant," it generates an effective temperature and a dissipation. The generation of a dissipation indicates that thermalization eventually sets in by the exchange of energy between low- modes and high-energy modes. The flow of energy across different length scales is also found in nonequilibrium classical systems such as turbulence[33] and the classic Fermi-Pasta-Ulam problem.[46] Such a flow of energy can lead to cascades and universal power-law behavior in the distribution function, a result that has also been recovered in recent studies involving nonequilibrium Bose-Einstein condensates.[47] It is clear that exploring further the connection between these systems should prove to be a very fruitful line of study. This also points to quite a general mechanism for thermalization in which mode coupling allows the system itself to act as a reservoir for the low-energy part of the degrees of freedom (see Fig. 9). This subpart thus acquires a classical behavior in the sense that it is characterized by a finite dissipation and a finite temperature. It is important to contrast this to a standard equilibrium quantum system for which the temperature is the same, irrespective of the energy of the mode considered; here, the temperature can only be defined if the limit $\omega \to 0$ is taken. The frequency dependence will in general be complicated and will correspond to a crossover between the athermal and the (low-energy) thermal states.

It is also interesting to compare the approach we have employed with more traditional methods to study dynamics. When studying fermions, a natural route is to write a kinetic equation that systematically takes into account two- and, if needed, three-particle scattering processes. However, for the Luttinger liquid, which has a linearized spectrum, a description in terms of a kinetic theory fails. In particular, a naive perturbation theory about the Luttinger-liquid fixed point leads to divergent results. This has been attempted in the past (see Ref. 48 for a discussion on this point). While our approach is perturbative in the periodic potential, since the basic unit of our perturbation theory involves the correlators $\langle e^{ia\phi} e^{-ia\phi} \rangle$, we have explicitly taken into account multiple scattering between bosons.

There are of course many open problems. It is, in particular, important to extend the analysis of this paper close to the critical point and also to the region in which the mode-coupling term is perturbatively relevant. Although we could obtain in this paper some results close to the critical point, the whole RG procedure has to be made fully consistent by taking into account the temperature and dissipation to obtain a more complete description. Treating these effects in a self-consistent way is hard. Once such a scheme is developed, it would be interesting to study the strong-coupling part of the phase diagram. It is clear that numerical studies of these questions would also be extremely helpful in this regime.

Given the close analogies between fermions and bosons in 1D, exploring the above physics in fermionic systems is an important direction. A first step was undertaken in Ref. 32, where a RPA study of a system of fermions that are out of equilibrium due to an initial quench was done and overdamped collective modes were recovered. However, a more complete study is needed that takes into account backscattering interactions (which are not included in the RPA). Further, we have assumed that the term that generates mode coupling was switched on very slowly. How the physics is affected by the rate at which the mode coupling is turned on is also interesting to explore.

Finally, checking the generality of this mechanism, by introducing other nonlinear terms, such as those arising due to band-curvature, is of course a very interesting question. This is also directly relevant for a test of the present mechanism either in numerical simulations or in experiments. On a practical point, there is also the question of time scales and as to which nonlinear couplings are more efficient than others. In experiments in Weiss's group,[49] where there is currently no optical potential, a prethermalized GGE-type state is found to persist for long times despite nonlinearities arising due to effects such as band curvature. This might be because the time scales for the onset of dissipative and thermal effects due to these couplings are too long for experimental relevance. However, we expect that application of a periodic potential will induce these effects more efficiently as a periodic potential is (dangerously) irrelevant. Once an experiment with an optical lattice is realized, a probe of the density-density response functions, which directly correspond to the correlators $\langle e^{ia\phi} e^{-ia\phi} \rangle$ that we have evaluated, should exhibit dissipative and thermal effects.

ACKNOWLEDGMENTS

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