

Effect of impurities on the superheating field of type-II superconductors

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We consider the effect of nonmagnetic and magnetic impurities on the superheating field H_s in a type-II superconductor. We solved the Eilenberger equations, which take into account the nonlinear pairbreaking of Meissner screening currents, and calculated $H_s(T)$ for arbitrary temperatures and impurity concentrations in a single-band s -wave superconductor with a large Ginzburg-Landau parameter. At low temperatures, nonmagnetic impurities suppress a weak maximum in $H_s(T)$, which has been predicted for the clean limit, resulting, instead, in a maximum of H_s as a function of impurity concentration in a moderately clean limit. It is shown that nonmagnetic impurities weakly affect H_s even in the dirty limit, while magnetic impurities suppress both H_s and the critical temperature T_c . The density of quasiparticle states $N(\epsilon)$ is strongly affected by an interplay of impurity scattering and current pairbreaking. We show that a clean superconductor at $H = H_s$ is in a gapless state, but a quasiparticle gap ϵ_g in $N(\epsilon)$ at $H = H_s$ appears as the concentration of nonmagnetic impurities increases. As the nonmagnetic scattering rate α increases above $\alpha_c = 0.36$, the quasiparticle gap $\epsilon_g(\alpha)$ at $H = H_s$ increases, approaching $\epsilon_g \approx 0.32\Delta_0$ in the dirty limit $\alpha \gg 1$, where Δ_0 is the superconducting gap parameter at zero field. The effects of impurities on H_s can be essential for the nonlinear surface resistance and superconductivity breakdown by strong RF fields.

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I. INTRODUCTION

Type-II superconductors are in the Meissner state if the applied magnetic field H is smaller than the lower critical magnetic field H_{c1} , above which the Gibbs free energy of a vortex becomes negative. However, the Meissner state can remain metastable at higher magnetic fields, $H > H_{c1}$, up to the superheating field H_s at which the Bean-Livingston surface barrier¹ for penetration of vortices disappears and the Meissner screening currents at the surface become unstable with respect to small perturbations of the order parameter. The field H_s is, therefore, the maximum magnetic field at which a type-II superconductor can remain in a true nondissipative state not altered by dissipative motion of vortices. In addition to the fundamental interest in what limits H_s in superconductors, there is a strong interest in the physics of H_s promoted by recent advances in superconducting Nb cavities for particle accelerators in which the quality factors $Q \sim 10^{10}$ – 10^{11} at 2 K and 2 GHz were observed up to the surface RF fields close to the thermodynamic critical field $H_c \simeq 200$ mT of Nb.² Here the RF frequency is well below the gap frequency $\Delta_0/h \simeq 400$ GHz for Nb where Δ_0 is the superconducting gap, so the current pairbreaking in the cavities at low temperatures may occur under quasistatic conditions.

Calculations of the superheating field H_s and the depairing current density J_c have a long history, starting from the pioneering works by Ginzburg,³ de-Gennes,⁴ and Matricon and Saint-James,⁵ who used the Ginzburg-Landau (GL) theory. It was shown that as the applied field H reaches H_s , the Meissner screening current density at the surface becomes of the order of $J_c \simeq cH_c/4\pi\lambda$, which makes the superconducting state unstable. The value of H_s in the limits of large and small GL parameter κ is given by^{6–11}

$$H_s = (\sqrt{5}/3)H_c, \quad \kappa \gg 1, \quad (1)$$

$$H_s = 2^{-1/4}\kappa^{-1/2}H_c, \quad \kappa \ll 1. \quad (2)$$

Here the factor $\sqrt{5}/3$ in Eq. (1) reflects the suppression of the order parameter by the Meissner currents in the local limit $\kappa \gg 1$, while the enhancement of H_s by the factor $\kappa^{-1/2}$ for $\kappa \ll 1$ results from the proximity effect reduction of the Meissner pairbreaking localized in a narrow surface layer of thickness $\lambda \ll \xi$, where λ is the London penetration depth and ξ is the coherence length. The extensive calculations of H_s based on the GL equations^{6–11} have shown that, for $\kappa \gg 1$, the Meissner state becomes absolutely unstable with respect to small 2D perturbations of current and order parameter with the wavelength $\sim(\xi^3\lambda)^{1/4}$ along the surface and decaying over the length $\sim\sqrt{\lambda\xi}$ perpendicular to the surface. Such perturbations describe the initial stage of penetration of vortex rows with the period $\sim(\phi_0/H_s)^{1/2}$ corresponding to the equilibrium vortex lattice at the field $H = H_s$.

Unlike $H_s(T)$ in the GL region near T_c , the behavior of $H_s(T)$ at low temperatures is not well understood, not least because the calculation of $H_s(T)$ requires solving the nonlinear Gor'kov or Eilenberger equations,¹² which take into account pairbreaking effect of Meissner currents. Manifestations of pairbreaking effects in a clean superconductor at low temperatures can differ from the GL results, as was shown long ago by Parmenter¹³ and Bardeen.¹⁴ The first calculation of $H_s(T)$ for the entire temperature range $0 < T < T_c$ in the clean limit and $\kappa \rightarrow \infty$ was done by Galaiko¹⁵ who obtained $H_s = 0.84H_c$ at $T \rightarrow 0$ and $H_s = (\sqrt{5}/3)H_c = 0.745H_c$ at $T \rightarrow T_c$. Catelani and Sethna¹⁶ solved the Eilenberger equations to calculate the temperature dependence of $H_s(T)$ for $0 < T < T_c$ in the clean limit for $\kappa \rightarrow \infty$ and found a maximum in $H_s(T)$ at low T . Such a nonmonotonic temperature dependence of $H_s(T)$ shows that the behavior of $H_s(T)$ at low T can hardly be extrapolated from the GL results near T_c .

The effect of impurities on $H_s(T)$ outside the GL region has not been addressed. This problem is of interest because the clean limit in s -wave superconductors at $T \ll T_c$ is a

rather singular case: for $T = 0$, the Meissner currents do not affect the superfluid density n_s until the superfluid velocity $v_s = J/n_s e$ reaches the critical value, $v_s = v_c = \Delta_{00}/p_F$, where p_F is the Fermi momentum and $\Delta_{00} = \Delta_0(0)$ is the modulus of the order parameter $\Psi = \Delta \exp(i\varphi)$ at zero superfluid velocity and $T = 0$.^{13,14} For $v_s > v_c$, the gap in the quasiparticle spectrum disappears and $n_s(v_s)$ rapidly drops to zero in a narrow region $v_c < v_s < 1.08v_c$.^{17,18} Thus, unlike the d -wave superconductors, the s -wave superconductors at $T \ll T_c$ do not exhibit the nonlinear Meissner effect caused by the dependence of $n_s(J)$ on the current density,^{19–22} and the superheating field is reached at the superfluid velocity $v_s > v_c$, which corresponds to a *gapless* state.²³ The latter has important consequences for the low-frequency ($\omega \ll \Delta_0$) impedance of clean superconductors because at $H = H_s = 0.84H_c$ the surface resistance R_s becomes of the order of R_s in the normal state, unlike the exponentially small $R_s \sim \omega^2 \exp(-\Delta_0/k_B T)$ in a fully gapped state at $H \ll H_s$.^{24,25} The effect of the RF Meissner currents on the quasiparticle spectrum can result in a strong dependence of the surface resistance on the RF field amplitude with $H > TH_c/\Delta_0$.²⁶

For $\kappa \gg 1$, the field H_g at which the gap in the quasiparticle density of states $N(\epsilon)$ closes at $T = 0$ can be calculated from the London equation $H_g = 4\pi\lambda J_c/c$, where $J_c = nev_c$, $\lambda = (mc^2/4\pi ne^2)^{1/2}$, n is the total electron density, $-e$ is the electron charge, m is the band effective mass, and c is the speed of light. Here the linear London screening of $H(x) = H_0 e^{-x/\lambda}$ remains valid as long as the superfluid density is independent of J , that is, the Meissner current density $J = (c/4\pi)\partial H/\partial x$ is smaller than J_c . This yields $H_g = c\Delta_{00}/e\lambda v_F$, which can be expressed in terms of $H_c = [4\pi N(0)]^{1/2}\Delta_{00}$, where $N(0) = m^2 v_F/2\pi^2 \hbar^3$ is the density of states per one spin orientation for an isotropic parabolic band, and $v_F = (3\pi^2 n)^{1/3} \hbar/m$. Hence,

$$H_g = (2/3)^{1/2} H_c \approx 0.816 H_c. \quad (3)$$

Since $H_s = 0.84H_c$,¹⁵ the gapless state in the clean, large- κ limit occurs in a narrow field range, $0.97H_s \lesssim H < H_s$.

The goal of this work is to address the pairbreaking effect of impurities on H_s . Indeed, while nonmagnetic impurities do not affect T_c , Δ_0 , and H_c at zero superfluid velocity,^{27,28} these impurities become pairbreakers in the current-carrying state,¹⁸ which, for example, manifests itself in the nonlinear Meissner effect.^{19–22} Thus, the extent to which nonmagnetic impurities would affect H_s needs to be understood. The effects of impurities on the quasiparticle density of state $N(\epsilon)$ at $H = H_s$ and the conditions under which impurities can restore the gap in $N(\epsilon)$ at $H = H_s$ are important for the understanding of the nonlinear surface resistance at high fields and the limits of superconductivity breakdown under low-frequency RF fields. We will also consider the superheating field in thin film multilayers consisting of alternating superconducting and dielectric layers thinner than λ . The parallel H_{c1} of such multilayers is greatly enhanced, which enables probing the pairbreaking limits in parallel magnetic fields. It was suggested to use these multilayer coatings to increase the RF breakdown fields of superconducting cavities in particle accelerators.²⁹ The paper is organized as follows.

In Sec. II we solve the Eilenberger equations for a superconductor with uniform current and analyzed the nonlinear dependencies of Δ and J on the superfluid velocity in the presence of impurities. In Sec. III we obtained equations for $H_s(T)$ in the entire temperature range $0 < T < T_c$ for $\kappa \gg 1$ and arbitrary concentrations of nonmagnetic and magnetic impurities. We solved the equations for H_s numerically and analyzed the dependencies of H_s on the magnetic and nonmagnetic scattering rates. The range of parameters in which H_s can be optimized by varying the impurity concentration was found. In Sec. IV we consider the effect of impurities on the quasiparticle density of states at $0 < H < H_s$, particularly the emergence of the gap in $N(\epsilon)$ as the concentration of nonmagnetic impurities increases. Section V is devoted to the calculation of nonlinear screening of strong DC field and H_s in multilayers. In Sec. VI we discuss our results and their implications for a nonlinear surface resistance at high RF fields and superconductivity breakdown in cavities.

II. THEORY

A. Eilenberger equations

In this work we use the Eilenberger equations for a single-band s -wave superconductor with magnetic and nonmagnetic impurities:¹²

$$(2\omega_n + \mathbf{v} \cdot \mathcal{D})f = 2\Psi g + \frac{1}{\tau_-} g \langle f \rangle - \frac{1}{\tau_+} f \langle g \rangle, \quad (4)$$

where Ψ is the order parameter, \mathbf{v} is the Fermi velocity, $\omega_n = \pi T(2n + 1)$ are Matsubara frequencies, T is the temperature, $\mathcal{D} = \nabla + 2\pi i \mathbf{A}/\phi_0$, ϕ_0 is the flux quantum, and \mathbf{A} is the vector potential. We use the units for which $\hbar = k_B = 1$ unless stated otherwise. The time constants τ_+ and τ_- are defined by

$$\frac{1}{\tau_{\pm}} = \frac{1}{\tau} \pm \frac{1}{\tau_m}, \quad (5)$$

where τ is the electron scattering time on nonmagnetic impurities and τ_m is the spin-flip scattering time on magnetic impurities. The angular brackets $\langle \cdots \rangle = \int_{S_F} d^2 k_F$ mean angular averaging over the Fermi surface. The quasiclassical Green functions $f(\mathbf{v}, \mathbf{r}, \omega_n)$ and $g(\mathbf{v}, \mathbf{r}, \omega_n)$ are normalized by

$$g^2 + f f^\dagger = 1, \quad (6)$$

where $f^\dagger(\mathbf{v}, \mathbf{r}, \omega_n) = f^*(-\mathbf{v}, \mathbf{r}, \omega_n)$ and the asterisk means complex conjugation. The self-consistency equation for the superconducting order parameter $\Psi(\mathbf{r}) = \Delta \exp(i\varphi)$ is given by

$$\Psi = 2\pi T N(0) |V| \sum_{\omega_n=0}^{\omega_D} \langle f \rangle, \quad (7)$$

where $N(0)$ is the normal density of states per one spin at the Fermi surface, V is the BCS coupling constant, and the cut-off frequency ω_D is of the order of Debye frequency. Here V can be expressed in terms of the critical temperature T_c of a superconductor without nonmagnetic impurities: $|V|^{-1} = N(0) \ln(2\omega_D \gamma / \pi T_c)$, where $\gamma = 1.78$. Equations (4)–(6) are supplemented by the Maxwell equation for \mathbf{A} and the

supercurrent density \mathbf{J} :

$$\nabla \times \nabla \times \mathbf{A} = \frac{4\pi}{c} \mathbf{J}, \quad (8)$$

$$\mathbf{J} = -4\pi T e N(0) \text{Im} \sum_{\omega_n=0}^{\infty} \langle g \mathbf{v} \rangle. \quad (9)$$

B. Geometry and assumptions

To study the stability of the Meissner state with respect to small perturbations, we consider a planar type-II superconductor occupying the region $x > 0$ with magnetic field \mathbf{H} applied along the z axis. For a large- κ superconductor, the previous calculations based on the GL^{7-9,11} and Eilenberger¹⁶ theories have shown that the instability at $H = H_s$ is driven by coupled fluctuations of $\delta\Psi(x, y)$ and $\delta\mathbf{J}(x, y)$ that rapidly oscillate on the scale $\sim \xi \kappa^{1/4}$ parallel to the surface and decay on a longer length $\sim \xi \kappa^{1/2}$ perpendicular to the surface. Taking these inhomogeneous unstable modes into account is essential for the calculation of small corrections $\sim H_c \kappa^{-1/2}$ to H_s . However, the main contribution to $H_s \sim H_c$ is determined by the condition that the superfluid velocity $v(0)$ at the surface reaches the critical pairbreaking value v_c for which $J(v)$ is maximum and the superconducting state becomes unstable. It is the same condition that defines the uniform depairing current density.³⁰

For uniform current flow, the solution of the Eilenberger equation can be sought in the form

$$\Psi(\mathbf{r}) = \Delta e^{iqy}, \quad (10)$$

$$f(\mathbf{r}, \omega_n, \theta) = f(\theta, \omega_n) e^{iqy}, \quad (11)$$

where q is the wave vector of the condensate, θ is the angle between the Fermi velocity and the current, and the amplitudes Δ and $f(\theta, \omega_n)$ are independent of the coordinates. Equations (10) and (11) approximate the solution necessary for the calculation of H_s , taking into account the difference in characteristic lengthscales in an extreme type-II superconductor with $\kappa \gg 1$. In this case H_s is reached as the current density at the surface becomes of the order of the depairing current density, which corresponds to the wavelength $2\pi/q \simeq \xi$.³⁰ Slow decrease of the Meissner screening currents over the London penetration depth increases H_s , which is now limited by perturbations $\delta\mathbf{J}(x, y)$ and $\delta f(\omega_n, \theta, x, y)$ oscillating on the scale $2\pi/k \sim (\xi^3 \lambda)^{1/4}$ along the x axis and decaying over $\sim (\xi \lambda)^{1/2}$ along the y axis.^{7-9,11} Thus, Eqs. (10) and (11) give an asymptotically exact solution for the lower bound of H_s in the limit of $\kappa \rightarrow \infty$ which will be addressed in this paper. The effect of London screening resulting in the finite- k instability produces small corrections in $H_s = H_c(\sqrt{5}/3)(1 + 1/\sqrt{2\kappa})$ in the GL region.^{7,11}

Equations (10) and (11) can also be applicable for a wider range of κ in a multilayer system in which superconducting layers are separated by thin dielectric layers as shown in Fig. 1. For thin superconducting layers of thickness $d \ll \lambda$, the Meissner current is nearly uniform across the film, while for $d < \sqrt{\lambda \xi}$, the vortex instability is suppressed as the perpendicular components of $\mathbf{J}(x, y)$ cannot cross the dielectric layers.⁸ In this case a uniform pairbreaking instability at $H = H_s$ develops, for which Eqs. (10) and (11) are exact solutions

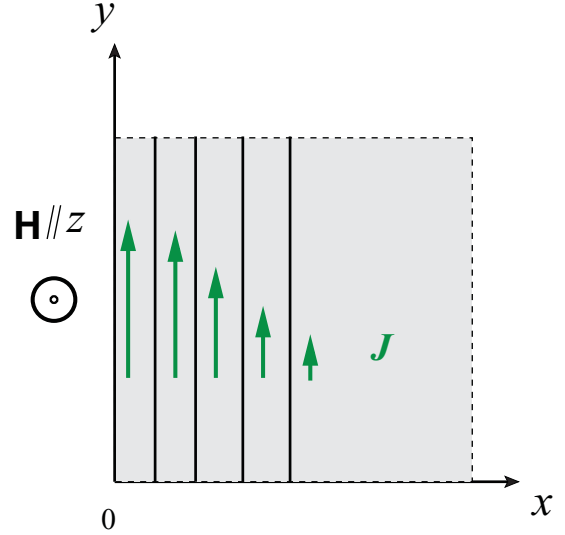


FIG. 1. (Color online) Schematic diagram of a multilayered superconductor: Superconducting layers (gray) separated by insulating layers (dark lines). Applied magnetic field applied along the z direction induces currents flowing along y direction.

of the Eilenberger equations. These solutions will be used to calculate nonlinear screening of magnetic field in multilayers.

C. Solution of the Eilenberger equations

Substituting Eqs. (10) and (11) into Eq. (4) gives the following algebraic equations for the amplitudes $g(\theta, \omega_n)$ and $f(\theta, \omega_n)$:

$$\frac{g}{f} = \frac{1}{2\Delta + \langle f \rangle / \tau_-} \left(2\omega_n + \frac{\langle g \rangle}{\tau_+} + iuv \cos \theta \right), \quad (12)$$

where we assumed a spherical Fermi surface for which $\langle f \rangle = \frac{1}{2} \int_0^\pi f(\theta) \sin \theta d\theta$, where θ is the angle between the Fermi velocity and the direction of current. The gauge-invariant wave vector $u = q + 2\pi A_y / \phi_0$ is proportional to the superfluid velocity $v_s = \hbar u / 2m$, where m is the effective electron mass. Solving Eqs. (6) and (12) yields

$$f = \frac{b}{\sqrt{b^2 + (a + i \cos \theta)^2}}, \quad (13)$$

$$g = \frac{a + i \cos \theta}{\sqrt{b^2 + (a + i \cos \theta)^2}}, \quad (14)$$

where a and b depend on $\langle g \rangle$ and $\langle f \rangle$,

$$a = (2\omega_n + \langle g \rangle / \tau_+) / uv, \quad (15)$$

$$b = (2\Delta + \langle f \rangle / \tau_-) / uv. \quad (16)$$

Integrating Eqs. (13) and (14) over θ , we obtain two self-consistent equations for $\langle f \rangle = -b \text{Im} \sinh^{-1}[(a + i)/b]$ and $\langle g \rangle = \text{Im} \sqrt{b^2 + (a + i)^2}$. The equations for $\langle f \rangle$ and $\langle g \rangle$ can be recasted in the following form:

$$\langle f \rangle = b \tan^{-1} \frac{\langle g \rangle}{a}, \quad (17)$$

$$\langle g \rangle^4 + \langle g \rangle^2 (a^2 + b^2 - 1) - a^2 = 0. \quad (18)$$

It is convenient to introduce a new variable X such that

$$X = \frac{\langle f \rangle uv\tau_-}{2\Delta\tau_- + \langle f \rangle}, \quad (19)$$

$$\langle g \rangle = \frac{2\omega_n\tau_+}{uv\tau_+ \cot X - 1}. \quad (20)$$

Substituting this into Eqs. (17) and (18) gives the following equation for X :

$$\left(\frac{\Delta \sin X \tau_-}{uv\tau_- - X} \right)^2 + \left(\frac{\omega_n \tau_+}{uv\tau_+ \cot X - 1} \right)^2 = \frac{1}{4}. \quad (21)$$

The gap equation and the current density $J = |\mathbf{J}|$ in Eqs. (7) and (9) can be expressed in terms of X as follows:

$$\ln \frac{T}{T_c} + 2\pi T \sum_{n=0}^{\infty} \left\{ \frac{1}{\omega_n} - \frac{2X\tau_-}{uv\tau_- - X} \right\} = 0, \quad (22)$$

$$J = 4\pi T e v N(0) \tau_-^2 \Delta^2 \sum_{n=0}^{\infty} \frac{2X - \sin 2X}{(uv\tau_- - X)^2}. \quad (23)$$

Equations (19)–(23) implicitly define $\langle f \rangle$, $\langle g \rangle$, and Δ as functions of J in a superconductor with arbitrary concentration of nonmagnetic and magnetic impurities in the entire range $0 < T < T_c$. Generally, Eqs. (19)–(23) can be solved only numerically, but analytical solutions can be obtained in some limiting cases.

To make the effect of impurities on H_s more clear, we, first, discuss the instructive case of a clean superconductor $(\tau_+, \tau_-) \rightarrow \infty$.^{13,14,18} In the absence of current $u \rightarrow 0$, Eqs. (19) and (21) yield the Gor'kov result, $\langle f \rangle = \Delta/\sqrt{\Delta^2 + \omega_n^2}$. Solving Eqs. (19) and (21) for a current-carrying state, $u > 0$, gives

$$\langle f \rangle = \frac{2\Delta}{uv} \tan^{-1} \sqrt{z/2}, \quad (24)$$

$$z = \frac{u^2 v^2}{4\omega_n^2} - \frac{\Delta^2}{\omega_n^2} - 1 + \left[\left(\frac{u^2 v^2}{4\omega_n^2} - \frac{\Delta^2}{\omega_n^2} - 1 \right)^2 + \frac{u^2 v^2}{\omega_n^2} \right]^{1/2}. \quad (25)$$

The dependence of Δ on u and T is then determined by Eqs. (7), (24), and (25).

Let us discuss the clean limit at $T = 0$ in more detail. For $T \rightarrow 0$, the summation in Eqs. (22) and (23) can be replaced by integration. Substituting Eq. (24) in those equations gives:^{18,30}

$$\ln \frac{\Delta}{\Delta_0} = \begin{cases} 0, & w \leq 1 \\ -\cosh^{-1} w + \sqrt{1 - 1/w^2}, & w > 1 \end{cases}, \quad (26)$$

where $w = uv/2\Delta$, Δ is function of u , and Δ_0 is the order parameter at $u = 0$ and $T = 0$. Calculating $J(u)$ in Eq. (23) yields

$$\frac{J}{J_0} = \begin{cases} 1, & w \leq 1 \\ 1 - (1 - 1/w^2)^{3/2}, & w > 1 \end{cases}. \quad (27)$$

Here the current density

$$J_0 = \frac{1}{3} e N(0) v^2 u = env_s$$

corresponds to superconducting flow of all electrons.

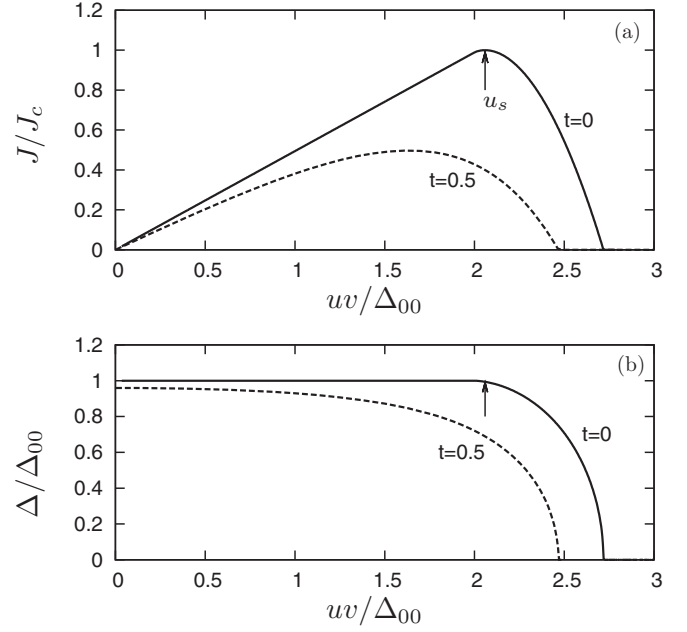


FIG. 2. (a) Current density and (b) gap as functions of u at $T \rightarrow 0$ (solid line) and at $T = 0.5T_c$ (dashed line). For $T = 0$, $J(u)$ increases linearly with u , reaches J_c at u_s and decreases to zero. The gap stays constant for $u \leq 2\Delta_0/v$ and then drops. The arrow shows the location of the departing momentum $u_s = 2.059\Delta_0/v$, where current reaches the critical current J_c . At $T = 0.5T_c$, u_s is shifted to lower u .

The calculated dependencies of $J(u)$ and $\Delta(u)$ on u for a clean superconductor at $T \rightarrow 0$ are shown in Fig. 2. The current $J = env_s$ increases linearly as $u = 2mv_s/\hbar$ increases up to $u = 2\Delta_0/v$, then $J(u)$ becomes nonlinear and reaches the maximum value J_c at the pair-breaking momentum $u_s = 2.059\Delta_0/v$, and then drops to zero. At the same time, the gap $\Delta(u) = \Delta_0$ remains unaffected by current at $0 < u < 2\Delta_0/v$ but rapidly drops to zero for $u > 2\Delta_0/v$. Here Δ_s at $u = u_s$ is slightly smaller than Δ_0 .

A clean s -wave superconductor at $T = 0$ exhibits no current pairbreaking in the broad range of $0 < u < 2\Delta_0/v$ where the linear London electrodynamics is applicable. For finite temperatures, this anomalous feature no longer holds, as illustrated by Fig. 2 which shows $J(u, T)$ and $\Delta(u, T)$ calculated numerically from Eqs. (22)–(25) for $T = 0.5T_c$. Here $\Delta(u)$ decreases as u increases, and $J(u)$ deviates from linearity for u smaller than Δ_0/v , similarly to the nonlinearity of $J(u) \propto u(1 - u^2\xi^2)$ due to current pairbreaking in the GL theory. Such a difference in the behaviors of $\Delta(u)$ at $T = 0$ and $T \sim T_c$ is due to thermally activated quasiparticles that are negligible at $T \ll T_c$ but become essential at higher temperatures. In any case, $J(u)$ reaches the maximum at the critical superfluid momentum $u \sim v/\Delta_0 \sim 1/\xi_0$, where $\xi_0 = \hbar v/\pi\Delta_0$ is the superconducting coherence length in the clean limit. The maximum of $J(u)$ at the critical momentum u_s defines the superheating field, which will be calculated in the next section.

Impurities do not change qualitatively the behavior of $J(u)$ shown in Fig. 3, but they increase the critical momentum u_s . We calculated the effect of nonmagnetic impurities on u_s at $T \rightarrow 0$ by solving the full Eqs. (21)–(23). The results are shown in Fig. 3 which displays u_s as a function of the scattering

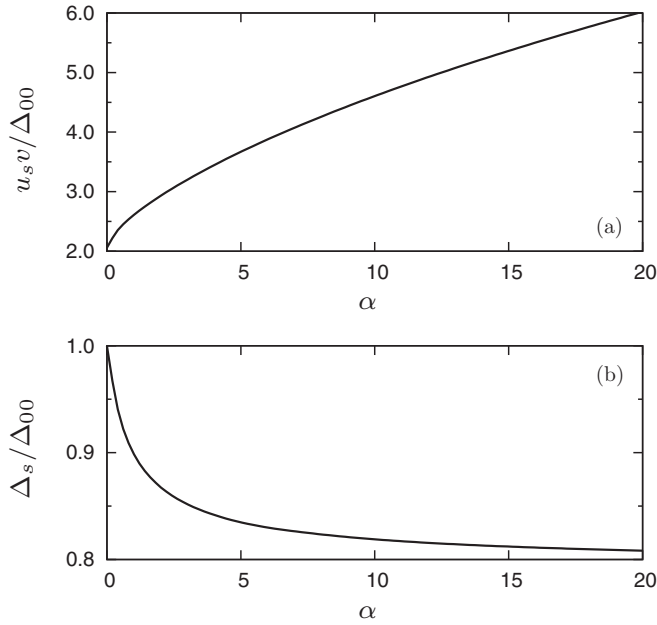


FIG. 3. (a) The pairbreaking momentum $u_s(\alpha)$ as a function of the nonmagnetic scattering rate α at $T = 0$. (b) Δ_s at the critical momentum $u = u_s$ as a function of α . Here Δ_s approaches $0.79\Delta_{00}$ for $\alpha \gg 1$.

parameter $\alpha \equiv 1/\Delta_{00}\tau = \pi\xi_0/\ell$, where $\ell = v\tau$ is the mean free path. The critical momentum u_s increases as α increases, approaching the square-root dependence $u_s \propto \sqrt{\alpha}$ for $\alpha > 1$, consistent with the general result, $u_s \sim 1/\xi$, where $\xi \simeq (\xi_0\ell)^{1/2}$ in the dirty limit. The dependence of $u_s(\alpha)$ can be described by the interpolation formula

$$u_s \approx [1.053(\alpha + 0.655)^{1/2} + 1.146]\Delta_{00}/v, \quad (28)$$

which approximates the calculated $u_s(\alpha)$ for $0 < \alpha < 20$ to an accuracy better than 1%.

Numerical solutions of the equation for Δ at $T = 0$ show that in the presence of impurities $\Delta(u)$ decreases as u increases even if $u\xi \ll 1$. As a result, nonmagnetic impurities become pairbreakers, reducing the superfluid density as J increases.¹⁸ This manifests itself in a nonlinear Meissner effect even at $T = 0$.²² Shown in Fig. 3 are the calculations of Δ_s at the depairing momentum u_s as a function of the scattering parameter α at $T = 0$. Here Δ_s decreases as α increases, approaching $\Delta_s(\alpha \rightarrow \infty) = 0.79\Delta_{00}$ in the dirty limit.

III. SUPERHEATING FIELD

We calculate H_s for a superconductor occupying the half-space $x > 0$ by solving the Maxwell equation

$$-\frac{\partial B}{\partial x} = \frac{4\pi}{c}J, \quad (29)$$

where the field is applied along the z axis, so both the current density $J(u)$ and the vector potential A_y have only the y components. Here $J(u) = -c\delta\mathcal{F}/\delta A_y$ in Eq. (9) can be

obtained by varying the free energy density¹²

$$\begin{aligned} \mathcal{F} = & \Delta^2 N(0) \ln \frac{T}{T_c} + 2\pi N(0)T \\ & \times \sum_{n=0}^{\infty} \left\{ \frac{\Delta^2}{\omega_n} - 2\Delta \langle f \rangle - 2\omega_n (\langle g \rangle - 1) \right. \\ & \left. - iuv \langle g \cos \theta \rangle - \frac{1}{2} \left(\frac{\langle f \rangle^2}{\tau_-} + \frac{\langle g \rangle^2 - 1}{\tau_+} \right) \right\}, \quad (30) \end{aligned}$$

where f and g are the solution of Eqs. (4) and (6). For $\kappa \gg 1$, the amplitudes Δ and f vary slowly on the scale of $u_s \sim 1/\xi$ so the gradient terms ∇f in \mathcal{F} are negligible. In this case the variational derivative $J = -c\delta\mathcal{F}/\delta A$ becomes the partial derivative, $J = -c\partial\mathcal{F}/\partial A$. Moreover, since $d\mathcal{F}/dA = \partial\mathcal{F}/\partial A + (\partial\mathcal{F}/\partial\Delta)(\partial\Delta/\partial A) + (\partial\mathcal{F}/\partial f)(\partial f/\partial A) + (\partial\mathcal{F}/\partial g)(\partial g/\partial A) = 0$ in equilibrium, we have $J = -c\partial\mathcal{F}/\partial A$.

Multiplying both sides of Eq. (29) by $B = dA/dx$ and integrating from $x = 0$ to $x = \infty$, we obtain

$$-\int_0^{\infty} B \frac{dB}{dx} dx = -4\pi \int_0^{\infty} \frac{d\mathcal{F}}{dA} \frac{dA}{dx} dx. \quad (31)$$

Here the boundary conditions are $B \rightarrow 0$ and $\mathcal{F} \rightarrow -H_c^2/8\pi$ at $x \rightarrow \infty$ and $B \rightarrow H$ at $x = 0$. Thus, Eq. (31) reduces to

$$H^2 = H_c^2 + 8\pi\mathcal{F}[u(0)], \quad (32)$$

where H_c is the thermodynamic critical field and $\mathcal{F}[u(0)]$ and $u(0)$ are the free energy density and the momentum of condensate at the surface, respectively. Equation (32) thus determines the dependence of $u(0)$ on the applied field.

As the magnetic field H increases, $J(0)$ and $u(0)$ at the surface increase. However, as $u(0)$ reaches u_s at which $J(u)$ is maximum, further increase of H does not cause any increase of screening current density, making the Meissner state absolutely unstable with respect to small perturbations of the order parameter and initiating penetration of vortices. The condition $u(0) = u_s$ in Eq. (32) defines the superheating field:

$$H_s^2(T) = H_c^2(T) + 8\pi\mathcal{F}(u_s, T). \quad (33)$$

Substituting the gap equation [Eq. (7)] and $2i\langle g \cos \theta \rangle = a(\langle g \rangle + 1/\langle g \rangle) - b\langle f \rangle$ to Eq. (30) yields $\mathcal{F} = 2\pi N(0)T \sum_{\omega_n > 0} \omega_n (2 - \langle g \rangle - 1/\langle g \rangle)$. Hence, we obtain the final expression for H_s ,

$$H_s^2 = H_c^2 + 16\pi^2 N(0)T \sum_{n=0}^{\infty} \omega_n \left\{ 2 - \langle g_s \rangle - \frac{1}{\langle g_s \rangle} \right\}. \quad (34)$$

Here the index s means that the function $\langle g \rangle$ is calculated at the critical momentum $u = u_s$. The thermodynamic critical field H_c is given by

$$H_c^2 = 16\pi^2 N(0)T \sum_{n=0}^{\infty} \left\{ \frac{2\omega_n^2 + \Delta_0^2}{\sqrt{\omega_n^2 + \Delta_0^2}} - 2\omega_n \right\}, \quad (35)$$

where $\Delta_0(T)$ is the superconducting order parameter at $u = 0$. In the clean limit, Eq. (34) reduces to

$$H_s^2 = 16\pi^2 T N(0) \sum_{n=0}^{\infty} \left\{ \frac{\omega_n^2}{\sqrt{\omega_n^2 + \Delta_0^2}} + \sqrt{\omega_n^2 + \Delta_0^2} - \frac{u_s v}{\sqrt{2z(u_s)}} - \frac{\omega_n^2 \sqrt{2z(u_s)}}{u_s v} \right\}, \quad (36)$$

where $z(u)$ is given by Eq. (24).¹⁵

Equations (34) and (35) combined with Eqs. (21)–(23) define H_s as a function of temperature and concentrations of nonmagnetic and magnetic impurities. Calculations of H_s involve solving coupled Eqs. (21)–(23) to obtain Δ and J as functions of u and then finding self-consistently the depairing momentum u_s from the condition $dJ/du = 0$. The results of these calculations of H_s for different concentrations of impurities and temperatures are presented below.

A. Effect of nonmagnetic impurities

We quantify the effect of nonmagnetic impurities on H_s by the dimensionless scattering rate

$$\alpha = \hbar/\tau \Delta_{00} = \pi \xi_0/\ell, \quad (37)$$

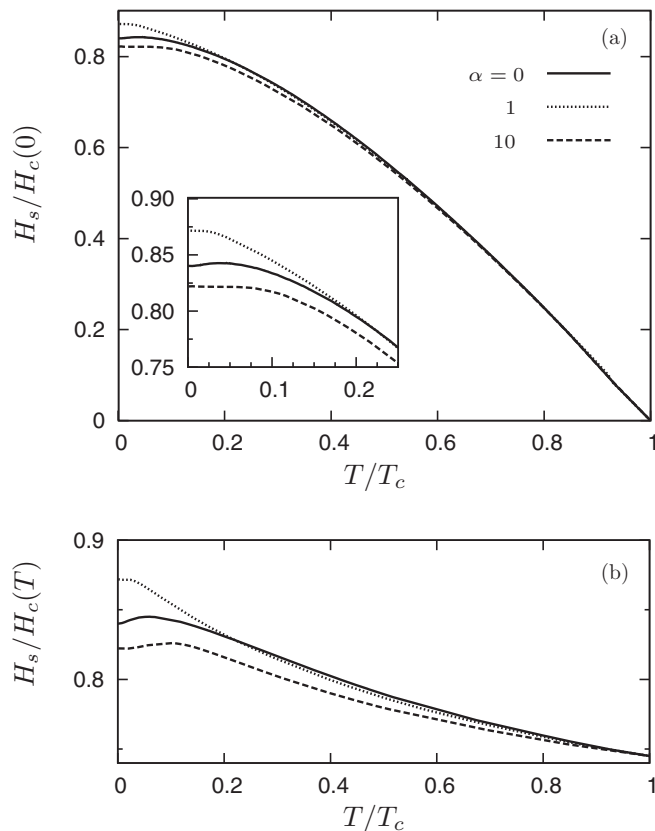


FIG. 4. Temperature dependencies of H_s calculated for $\alpha = 0, 1,$ and 10 . (a) $H_s(T)$ in the units of $H_c(0)$ and (b) the ratio $H_s(T)/H_c(T)$, where $H_c(T)$ is calculated from Eq. (35). At low T , nonmagnetic impurities result in nonmonotonic dependence of H_s on α , eliminating the maximum in $H_s(T)$ at $T = 0.04T_c$ for the clean limit, as shown in the inset. The difference between $H_s(T)$ diminishes at higher T , where the $H_s(T)$ curves approach the GL result given by Eq. (1).

where $\xi_0 = \hbar v/\pi \Delta_{00}$ is the clean limit coherence length and $\ell = v\tau$ is the mean free path. For $u = 0$, impurities do not affect Δ , H_c , and T_c of s -wave superconductors,^{27,28} but in a current state both Δ and u_s are affected by impurities,¹⁸ as shown in Fig. 3.

We, first, consider the effect of nonmagnetic impurities on the temperature dependence of $H_s(T)$. Shown in Fig. 4 are the superheating fields calculated for the clean ($\alpha = 0$), moderately dirty ($\alpha = 1$), and dirty ($\alpha = 10$) limit. The first conclusion apparent from Fig. 4 is that nonmagnetic impurities have a rather weak effect on H_s , despite their pairbreaking nature in the presence of current. Near T_c the $H_s(T)$ curves coincide, approaching $H_s(T) = (\sqrt{5}/3)H_c$ predicted by the GL theory, where H_c is independent of α according to the Anderson theorem.²⁷ At lower temperatures impurities result in different behaviors of $H_s(T)$ for different α . For instance, one of the features of the clean limit at low temperatures is a maximum in $H_s(T)$ at $T = 0.04T_c$ pointed out by Catelani and Sethna.¹⁶ As follows from Fig. 4, this maximum disappears in the moderately dirty limit ($\alpha = 1$), as well as the dirty limit ($\alpha = 10$). As will be shown below, this change in the behaviors of $H_s(T)$ at low temperatures results from the impurity-induced change in the quasiparticle density of states at $H = H_s$ from the gapless state in the clean limit to a gapped state for $\alpha > \alpha_c$.

We now turn to the dependence of H_s on the nonmagnetic scattering rate α . Shown in Fig. 5 are $H_s(\alpha)$ calculated for different temperatures. These $H_s(\alpha)$ curves exhibit surprising maxima at low temperatures, resulting in a $\simeq 4.2\%$ enhancement of $H_s(\alpha)$ at $T = 0$ as compared to the clean limit. Here the arrows in the figure mark the maxima in H_s , the location of which depends nonmonotonically on temperature as described in the caption to Fig. 5. Thus, there is an optimum mean free path $\ell_{\max} = 5.32\xi_0$ at $T = 0$ for which the superheating field is maximum. At higher temperatures, $T \gtrsim 0.3T_c$, the maximum in $H_s(\alpha)$ disappears.

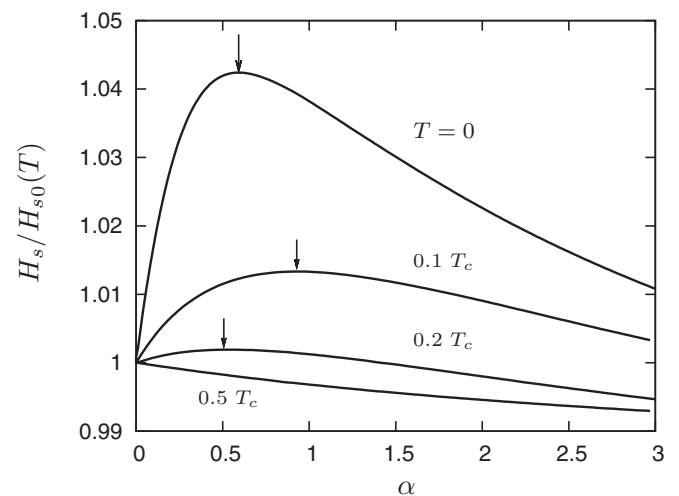


FIG. 5. Enhancement of H_s by nonmagnetic impurities at low temperature where $H_s(\alpha, T)$ is normalized to $H_{s0}(T)$ for a clean superconductor. Here the position of the maximum $\alpha_{\max}(T)$ in $H_s(\alpha)$ depends on temperature: $\alpha_{\max} = 0.59$ for $T = 0$, $\alpha_{\max} = 0.91$ for $T = 0.1T_c$, and $\alpha_{\max} = 0.53$ for $T = 0.2T_c$. At higher temperatures, the maximum disappears.

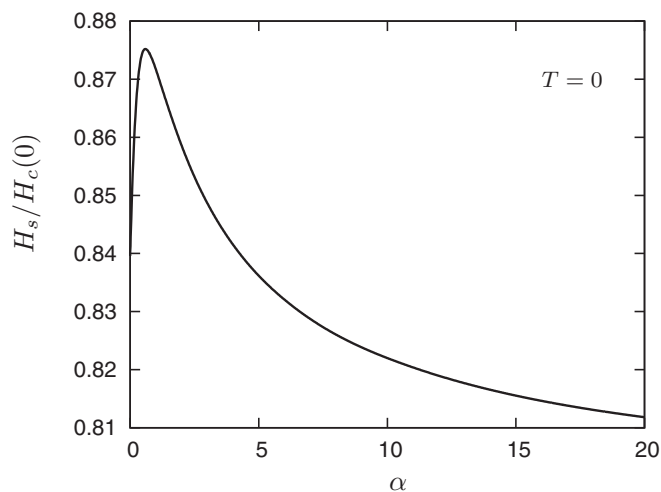


FIG. 6. Dependence of H_s at $T = 0$ on the nonmagnetic scattering rate α .

The results of this section show that the effect of nonmagnetic impurities on H_s is most pronounced at low temperatures where impurities can eliminate the maximum in the temperature dependence of H_s characteristic of the clean limit. At the same time, impurities can cause a new maximum in H_s as a function of the scattering rate α at low temperatures, although the overall effect of nonmagnetic scattering on H_s turns out to be comparatively weak. This results from two opposite effects that nearly cancel each other: the increase of H_s due to the increase of the pairbreaking momentum u_s shown in Fig. 3 and the decrease of superfluid density as α increases. Here H_s roughly scales like the thermodynamic critical field H_c in which these opposite trends cancel out exactly, as prescribed by the Anderson theorem. Yet our results show that this cancellation is not exact for H_s , which depends weakly on the scattering rate α , as illustrated by Fig. 6, which shows the calculated $H_s(0, \alpha)$ at $T = 0$. For instance, $H_s \approx 0.82H_c$ at $\alpha = 10$ is rather close to $H_s = 0.84H_c$ in the clean limit.

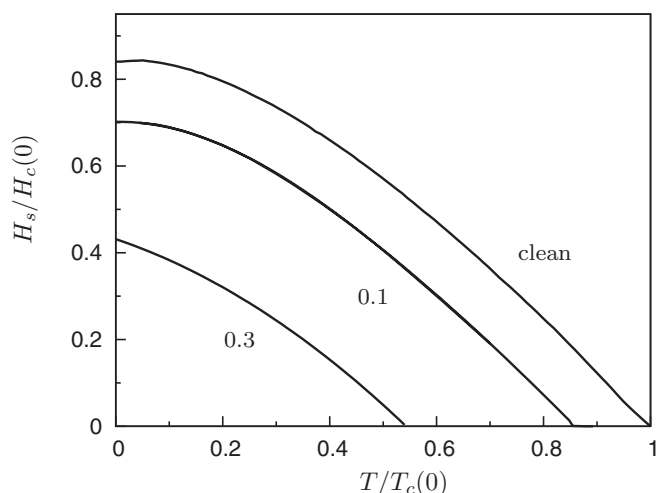


FIG. 7. Temperature dependencies of $H_s(T)$ calculated for $\alpha = 0$ and different magnetic scattering rates α_m . Here $T_c(0)$ is the critical temperature for clean superconductors.

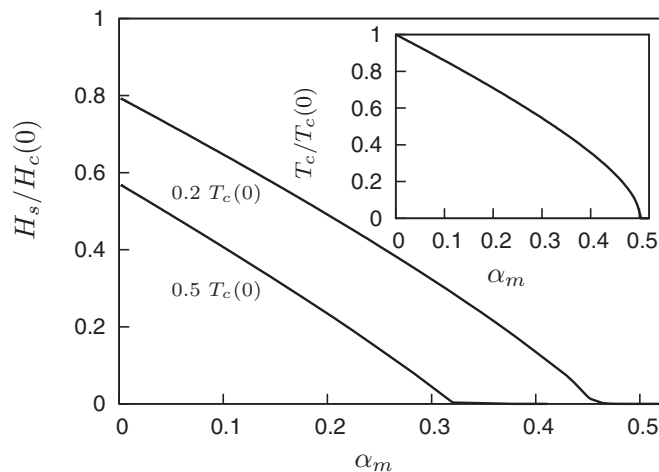


FIG. 8. Dependencies of H_s on α_m calculated for at $T = 0.2T_c(0)$ and $T = 0.5T_c(0)$. Insert shows the dependence of T_c on α_m calculated from the Abrikosov-Gor'kov theory.²⁸

B. Effect of magnetic impurities

Unlike nonmagnetic impurities, magnetic impurities strongly suppress T_c ,²⁸ resulting in a significant reduction of H_s as well. We quantify this effect by the dimensionless magnetic scattering rate α_m similar to that was used above for nonmagnetic impurities,

$$\alpha_m = \hbar/\tau_m \Delta_{00} = \pi \xi_0/\ell_m, \quad (38)$$

where τ_m and $\ell_m = v\tau_m$ are the spin-flip scattering time and mean free path, respectively.

Shown in Fig. 7 are the temperature dependencies $H_s(T)$ calculated for $\alpha = 0$ and different values of α_m . The superheating field is significantly reduced by magnetic scattering, the H_s suppression is stronger at higher temperatures. One can see that for $\alpha_m \simeq 0.1$, which corresponds to $\ell_m \sim 30\xi_0$, the superheating field is roughly reduced by half as compared to the case of $\alpha_m = 0$.

Figure 8 shows the reduction $H_s(\alpha_m)$ by magnetic impurities at a given temperature. Unlike the effect of nonmagnetic impurities which is most pronounced at low temperatures, the

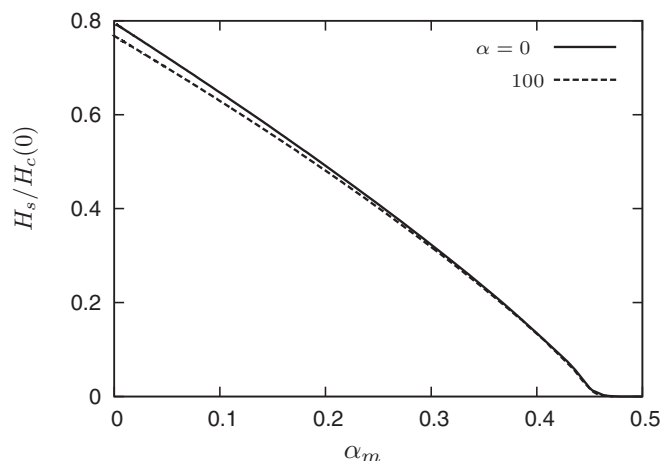


FIG. 9. $H_s(\alpha_m)$ calculated for $T = 0.2T_c(0)$ and different nonmagnetic scattering rates α .

effect of magnetic impurities is stronger at higher temperature. Here $H_s(\alpha_m)$ decreases as α_m decreases, vanishing at a critical scattering rate $\alpha_{mc}(T)$.

The combined effect of magnetic and nonmagnetic impurities on H_s is shown in Fig. 9. From the curves $H_s(\alpha_m)$ calculated for different α at $T = 0.2T_c(0)$, we can see that there is practically no interplay between the effects of magnetic and nonmagnetic scattering on H_s . Similar to the case of $\alpha_m = 0$, the effect of nonmagnetic impurities at $\alpha_m > 0$ is most noticeable at low temperatures, where it remains rather weak even for very dirty superconductors with $\alpha \sim 100$.

IV. QUASIPARTICLE DENSITY OF STATES IN THE MEISSNER STATE

As was discussed in the Introduction, the effect of current on the quasiparticle density of states $N(\epsilon)$ in the Meissner state is rather nontrivial because a clean superconductor at $H = H_s$ is in a gapless state. The appearance of quasiparticle at the Fermi level at H_s can have important consequences for the breakdown of superconductivity under strong RF fields, as will be discussed below. This feature of the s -wave clean limit results from the spectrum of quasiparticles in a uniform current-carrying state,^{13,14}

$$\epsilon_{\mathbf{k}} = \sqrt{\Delta_0^2 + v^2(p - p_F)^2} + \hbar \mathbf{u} \cdot \mathbf{v}/2, \quad (39)$$

where the last term describes the effect of current with the superfluid velocity $v_s = \hbar \mathbf{u}/2m$. The quasiparticle gap ϵ_g corresponds to the minimum of $\epsilon_{\mathbf{k}}$, giving $\epsilon_g = \Delta_0$ for $u = 0$. However, in the presence of current, $\Delta_0(T)$ is no longer the gap in the quasiparticle spectrum. For instance, Δ_{00} at $T = 0$ remains independent of J in the entire interval $0 < u < 2\Delta_{00}/\hbar v$, as shown in Fig. 2, while the anisotropic quasiparticle gap $\epsilon_g(\mathbf{u}) = |\Delta_{00}| + \frac{1}{2}\hbar u v \cos \varphi$, obtained by setting $p = p_F$ in Eq. (39), depends on the angle φ between the current and quasiparticle momentum. The minimum gap $\epsilon_g = |\Delta_{00}| - \hbar u v/2$ vanishes at the condensate momentum $u > u_g = 2\Delta_{00}/\hbar v$ slightly smaller than the pairbreaking momentum u_s corresponding to $H = H_s$. The question is then how the density of states at $H = H_s$ is altered by impurity scattering. To address this issue we use the approach developed by Maki^{17,18} and Fulde²³ to calculate $N(\epsilon, u)$ as functions of energy ϵ and the superfluid velocity. For the sake of simplicity, we only consider here the effect of nonmagnetic impurities on $N(\epsilon, u)$.

The density of state $N(\epsilon)$ is given by the imaginary part of the real frequency Eilenberger function $g(\epsilon)$ obtained by analytic continuation of the thermodynamic function $g(\omega_n)$ from the imaginary Matsubara axis onto the real energy axis ϵ . We consider here the limit $T = 0$ for which this procedure yields^{17,23}

$$\nu(\epsilon) = \text{Im} \frac{2uv\tau^2\epsilon}{uv\tau - \tan \chi}, \quad (40)$$

where $\nu(\epsilon) = N(\epsilon)/N(0)$ is the normalized density of states and χ is the solution of the real-frequency version of Eq. (21) in which the substitution $\omega \rightarrow -i\epsilon$ is made,

$$\Delta^2 \left(\frac{uv}{\sin \chi} - \frac{1}{\tau \sin \chi} \right)^{-2} - \epsilon^2 \left(\frac{uv}{\tan \chi} - \frac{1}{\tau} \right)^{-2} = \frac{1}{4}. \quad (41)$$

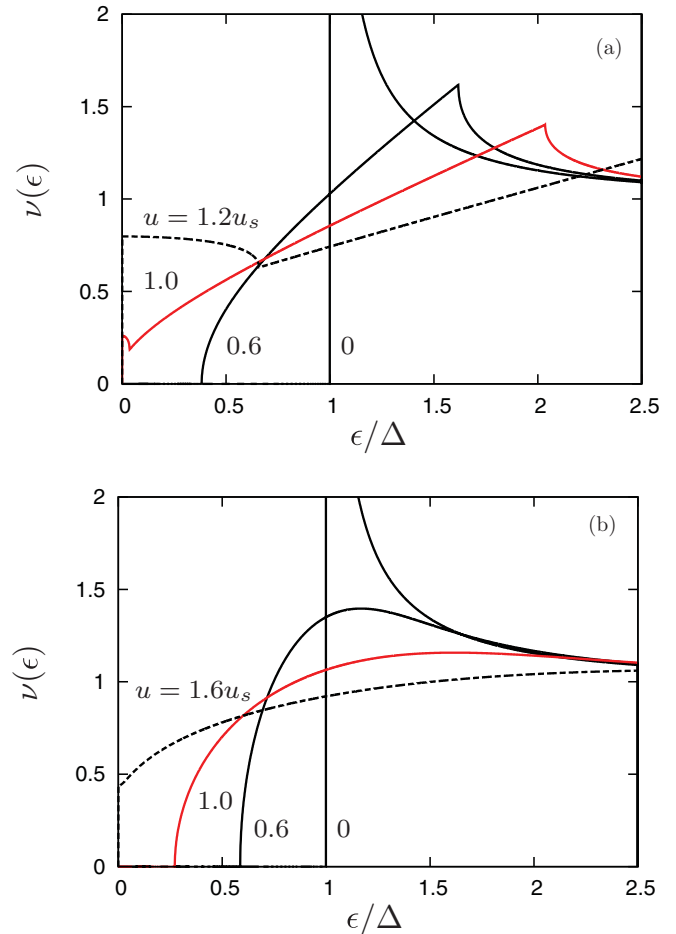


FIG. 10. (Color online) Density of state $\nu(\epsilon)$ in the current-carrying states for (a) clean limit, $\alpha = 0$; (b) moderate dirty limit, $\alpha = 3.6$. Here Δ is understood as $\Delta(u)$ at $T = 0$ for a given α . The red lines show $\nu(\epsilon)$ at $u = u_s$. Solid lines show $\nu(\epsilon)$ for $u < u_s$, while dashed lines correspond to $u > u_s$. For $\alpha = 0$, the gap in the spectrum closed at $u > 0.970u_s$. For $\alpha = 3.6$, we obtained $\epsilon_s = 0.211\Delta$.

Equations (40) and (41) allow us to obtain $N(\epsilon)$ using $\Delta(u)$ and u_s calculated in the previous sections with the use of the thermodynamic Eilenberger equations.

Figure 10(a) shows the evolution of $\nu(\epsilon)$ calculated for a clean superconductor ($\alpha = 0$) as the condensate wave vector u increases. For $u = 0$, we recover the BCS density of states $\nu(\epsilon) = \epsilon/\sqrt{\epsilon^2 - \Delta_0^2}$ for $\epsilon > \Delta_0$ and $\nu(\epsilon) = 0$ for $\epsilon < \Delta_0$. In the presence of current, the singularity in $\nu(\epsilon)$ at $\epsilon = \Delta_0$ disappears, and the energy gap in the spectrum $\epsilon_g(\alpha, u)$ defined as the maximum energy at which $\nu(\epsilon_g)$ vanishes, becomes smaller than Δ_0 . Here the quasiparticle gap $\epsilon_g(u)$ decreases as u increases. At the critical value $u = u_s$ corresponding to the superheating field, the density of state at the Fermi surface in the gapless state equals $\nu(0) = 0.243$.

The effect of impurities on $\nu(\epsilon, u)$ for a moderately dirty superconductor with $\alpha = 3.6$ is shown in Fig. 10(b). For $u = 0$, nonmagnetic impurities do not change the BCS density of states, in accordance with the Anderson theorem. However, $\nu(\epsilon)$ in a current-carrying superconductor with impurities begins to differ markedly from $\nu(\epsilon)$ in a clean superconductor. As it is evident from Figs. 10(a) and 10(b), impurities not only

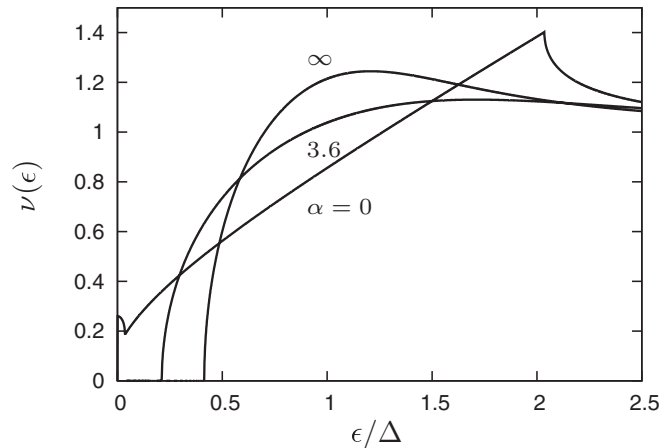


FIG. 11. Density of state $\nu(\epsilon)$ at $u = u_s$ for different scattering rates α . Here $\Delta = \Delta(\alpha)$ at $T = 0$ and $u = u_s$.

smear the cusps in $\nu(\epsilon)$ characteristic of the clean limit but also reduce $N(0)$ at $u = u_s$, eventually restoring the gapped state at $H = H_s$, where $\nu(\epsilon) = 0$ for $\epsilon < \epsilon_g(\alpha)$. For the particular case shown in Fig. 10(b), our calculations give $\epsilon_s(3.6) = 0.211\Delta(u_s) = 0.169\Delta_{00}$. For $\alpha \gg 1$, the gap approaches the limiting value $\epsilon_s = 0.410\Delta(u_s) = 0.323\Delta_{00}$.²³

The different behaviors of $\nu(\epsilon)$ in the clean and dirty limits shown in Fig. 10 suggest that at the superheating field a quasiparticle gap $\epsilon_s(\alpha)$ appears as a superconductor gets dirtier. This is illustrated by Fig. 11, which shows the evolution of $\nu(\epsilon, u_s)$ at the depairing momentum as α increases. One can clearly see the transition from a gapless to a gapped state induced by nonmagnetic impurity scattering, giving, for example, $\epsilon_s = 0.17\Delta_{00}$ at $\alpha = 1$. Therefore, at $u = u_s$, a quasiparticle gap $\epsilon_s(\alpha)$ opens at $\alpha > \alpha_c$, where the critical scattering rate α_c calculated numerically from Eqs. (40) and (41) is

$$\alpha_c = 0.36. \quad (42)$$

The calculated quasiparticle gap $\epsilon_s(\alpha)$ at $u = u_s$ is shown in Fig. 12. Here $\epsilon_s(\alpha)$ monotonically increases as α increases above $\alpha > \alpha_c$, approaching $\epsilon_s(\infty) \approx 0.323\Delta_{00}$ at $\alpha \rightarrow \infty$. The dependence of ϵ_s on α can be approximated by the formula

$$\epsilon_g = 0.566\Delta_{00}[\tan^{-1}(0.626\alpha + 1.345) - 1] \quad (43)$$

to the accuracy better than 1.2%.

V. NONLINEAR SCREENING AND H_s IN MULTILAYERS

In this section, we use the results obtained above to calculate screening of a magnetic field parallel to a multilayer consisting of alternating superconducting (S) and insulating (I) layers, as shown in Fig. 1. Such multilayers stabilize the Meissner state against penetration of vortices up to the superheating field of the material of the thin S layers with $d \ll \lambda$ for which the parallel lower critical magnetic field $H_{c1} = (2\phi_0/\pi d^2) \ln(d/\xi)$ is greatly increased.²⁹ In turn, the suppression of perpendicular currents in thin S layers by nonconducting I layers also suppresses the pairbreaking instability at the finite wave vectors k along the surface,⁸ which initiates penetration of vortices. As a result, the superheating field H_s in thin film

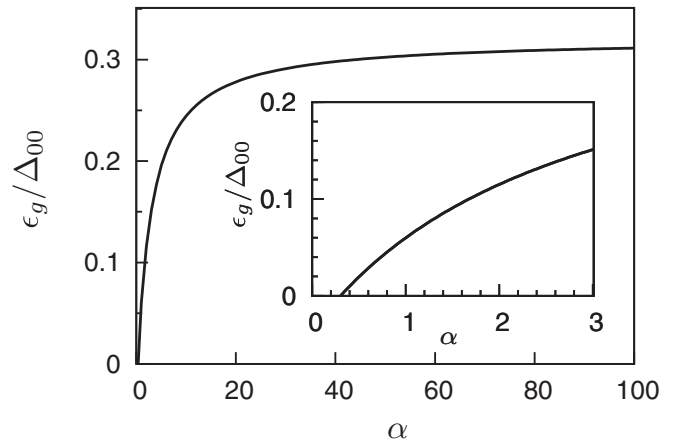


FIG. 12. Gap in the quasiparticle spectrum at the depairing momentum $u = u_s$ as a function of the nonmagnetic scattering rate α . The gap ϵ_s opens at $\alpha \geq 0.36$. (Insert) The behavior of $\epsilon_s(\alpha)$ at small α .

multilayers is defined by the condition that $J(u)$ in the first layer reaches the depairing current density.

Penetration of magnetic field is described by the Maxwell equation $\nabla \times \nabla \times \mathbf{A} = 4\pi\mathbf{J}/c$, where the supercurrent $J(u)$ depends on the gauge-invariant phase gradient $\mathbf{u} = \nabla\varphi + 2\pi\mathbf{A}/\phi_0$, as shown in Fig. 2(a). Here $\varphi(\mathbf{r})$ is the phase of the order parameter. In the Meissner state $\mathbf{B} = \nabla \times \mathbf{A} = (\phi_0/2\pi)\nabla \times \mathbf{u}$, the Maxwell equation for the planar multilayer geometry in which $\mathbf{u} = [0, u(x), 0]$ can be written in terms of the y component $u(x)$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{8\pi^2}{\phi_0 c} J(u). \quad (44)$$

We assume specular scattering of quasiparticles at the S-I interfaces, so there is no suppression of the order parameter due to surface scattering in the S layers. In this case $J(u)$ is nearly uniform across each S layer, a slight decrease of $u(x)$ as x increases resulting from the London screening over the penetration length $\lambda \gg d$. The solution of Eq. (44) gives the distribution of magnetic field $B(x) = (\phi_0/2\pi)\partial_x u(x)$ across the multilayer.

The boundary conditions to Eq. (44) are as follows: $\partial_x u(0) = 2\pi H/\phi_0$ at $x = 0$, $u(\infty) = 0$ at $x \rightarrow \infty$, constant B in the I layers, and the continuity of $u(x)$ at the S-I interfaces. Shown in Fig. 13 is the distribution of $H(x)$ calculated from Eq. (44) in which $J(u)$ given by Eq. (23) was calculated using the solution of the Eilenberger equations for $\alpha = 1$ and $T = 0.5T_c$. As a comparison, we also show $H(x)$ (dashed line) calculated by solving the linear London equation $\lambda^2 \partial_{xx} u - u = 0$ for the same parameters. One can see that the Eilenberger theory, which takes current pairbreaking into account, gives slower penetration of the magnetic field in the first few layers at the surface as compared to the London model, which disregards current pairbreaking effects.

A curious manifestation of pairbreaking effects in the Meissner magnetization of moderately clean superconductors occurs due to the transition from the gapped to the gapless state as the field H increases above H_g defined by Eq. (3). Consider,

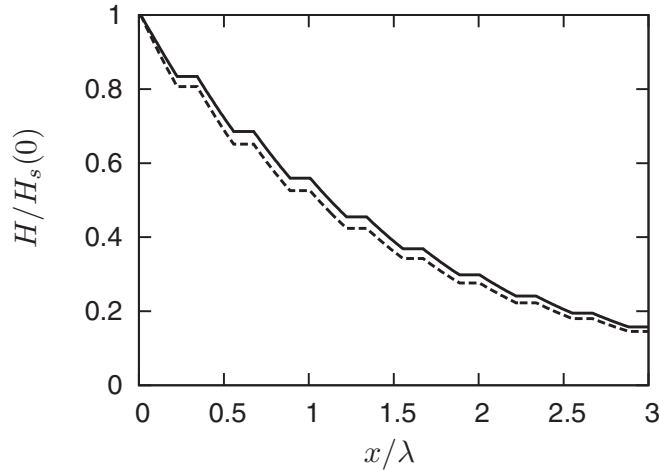


FIG. 13. Distribution of the magnetic field $H(x)$ in a multilayer obtained by numerical solution of Eq. (44) for $\alpha = 1$ and $T = 0.5T_c$ (solid line). The result of the London model for the same parameters is shown by the dashed lines where λ is the penetration depth for the material of the S layer.

for example, the magnetization $m(H)$ of a long cylinder of radius $R \gg \lambda$ in a parallel magnetic field,

$$m = \frac{1}{c} \int_0^R J(v_s) dr, \quad (45)$$

where we use the dependence of $J(v_s)$ on the superfluid velocity $v_s = v_c H(x)/H_g$ for $T = 0$ in the clean limit given by Eq. (27). For $H < H_g$, the dc magnetization $m_0(H) = -H/4\pi$ corresponds to the linear Meissner effect in the entire field region $0 < H < H_g$. However, as H exceeds H_g the magnetization exhibits singularities in higher-order derivatives, indicating a phase transition from the gapped to the gapless state. Indeed, let H be slightly above H_g so that $m(H) = m_0(H) + m_a(H)$, where $m_0(H) = -H/4\pi$ is the ideal Meissner magnetization and $m_a(H)$ is the nonlinear contribution due to the second term in Eq. (27),

$$m_a = \frac{J_c}{c} \int_0^L (w^2 - 1)^{3/2} \frac{dx}{w^3}, \quad (46)$$

where $w(x) = H(x)/H_g$, $H_g = 4\pi J_c \lambda / c$, $H(x) \approx H - Hx/\lambda$ is the field profile at the surface $x = 0$ and $L = (H - H_g)\lambda/H_g \ll \lambda$ is the depth of the gapless layer. Equation (46) at $\varepsilon = (H - H_g)/H_g \ll 1$ then yields $m_a = (\sqrt{2}/5\pi) H_g \varepsilon^{5/2}$, which means a square-root singularity in the third derivative of $m_a(H)$ at $H = H_g + 0$,

$$\frac{d^3 m_a}{dH^3} = \frac{3}{4\pi \sqrt{2} H_g^2} \left(\frac{H}{H_g} - 1 \right)^{-1/2}. \quad (47)$$

The singularity in the thermodynamic quantity $d^3 m/dH^3$ implies the singular discontinuity in the fourth derivative of the thermodynamic potential, indicating a field-induced fourth-order phase transition. Although Eq. (47) was obtained for the clean limit $\alpha = 0$, this singularity remains in the moderately clean limit as well if the scattering rate $\alpha < 0.36$ is smaller than the critical value given by Eq. (42) so the transition from the gapped to the gapless state occurs below the superheating field $H = H_g < H_s$. Experimentally, this rather

weak singularity may be smeared out by local inhomogeneities of impurity concentration that result in a distribution of the local fields $H_g(\mathbf{r})$.

VI. DISCUSSION

In this paper, we use the Eilenberger equations to calculate the effect of nonmagnetic and magnetic impurities on the superheating field of type-II superconductors. Unlike magnetic impurities that strongly suppress H_s , nonmagnetic impurities affect H_s weakly, although they can cause a nonmonotonic dependence of H_s on the scattering rate α at low temperatures. For instance, at $T = 0$, nonmagnetic impurities can increase H_s by $\simeq 4\%$ at $\alpha \simeq 0.6$ as compared to the clean limit. As the scattering rate α further increases, $H_s(\alpha)$ decreases and levels off at $H_s(\infty) \approx H_s(0)$, as shown in Fig. 6. This decrease of H_s at $\alpha \gg 1$ is consistent with the decrease of the gap parameter Δ_s at the pairbreaking momentum $u = u_s$ shown in Fig. 3. Yet, unlike the decrease of the depairing current density $J_c \propto \alpha^{-1/2}$ in the dirty limit,³⁰ our results show that the superheating field roughly scales like H_c even at low temperatures, so H_s is weakly affected by nonmagnetic impurity scattering.

Our results obtained in the limit of $\kappa \gg 1$ give the lower bound of $H_s(T)$. The effect of finite κ increases $H_s(T)$ since the condition $v_s(0) = v_c$ is no longer sufficient to cause the instability of the Meissner state in the surface layer of thickness ξ where the superfluid velocities $v_s(x) \simeq (1 - x/\lambda)v_s(0)$ decreases below v_c because of the London screening. The GL calculations^{7-9,11} have shown that the finite- κ effects increase $H_s(\kappa) \simeq (1 + 0.7\kappa^{-1/2})H_s(\infty)$, giving a correction $\simeq 7-16\%$ as compared to $H_s(\infty)$ at $\kappa \rightarrow \infty$ for $\kappa = 20-100$. These effects also make $H_s(\kappa)$ dependent on α since $\kappa(\alpha)$ increases as α increases, approaching $\kappa \sim \alpha\kappa_0$ in the dirty limit. Thus, $H_s(T)$ generally decreases as the scattering rate α increases, although this effect is comparatively weak for high- κ materials. Addressing the dependence of H_s on α due to the finite- κ effects at low temperatures requires the instability analysis of the Meissner state with respect to 2D perturbations of $\delta f(x, y)$ and $\delta \mathbf{J}(x, y)$ described by the linearized Eilenberger equations.

The nonmonotonic dependence of H_s on α obtained in this work results from interplay of Meissner currents and impurity scattering, and their effect on the quasiparticle density of states $N(\epsilon, J)$. Our calculations revealed the disorder-induced transition from the gapless to the gapped state at $H = H_g$, which can have important implications for the low-temperature surface resistance R_s at high-RF fields, $H(t) \sim H_c$. The BCS surface resistance R_s at small RF fields, $H \ll H_c T/\Delta_0$, low frequencies ($\omega \ll \Delta_0$), and $T \ll T_c$ is given by^{24,25}

$$R_s = \omega^2 \frac{A(\ell, \omega)}{T} \exp\left(-\frac{\Delta_0}{T}\right), \quad (48)$$

where the factor A depends on the mean free path ℓ and (weakly) on the RF frequency ω . The main Boltzmann factor $\exp(-\Delta_0/T)$ accounts for the exponentially small density of thermally activated quasiparticles due to the zero density of states $N(\epsilon)$ for the energies $\epsilon < \Delta_0$. In the presence of Meissner current, the quasiparticle gap $\epsilon_g(J)$ shifts to smaller energies, giving rise to a highly nonlinear dependence of

the surface resistance on the RF field amplitude at low temperatures,

$$R_s \propto \exp \left[- \frac{\epsilon_g(H)}{T} \right]. \quad (49)$$

As the field increases, $R_s(T, H)$ becomes essentially dependent on H if the field-induced change of $\epsilon_g(H)$ is of the order of T .²⁶ In the clean limit for which $\epsilon_g(H) = (1 - H/H_g)\Delta_0$, this condition takes the form

$$H \gtrsim TH_c/\Delta_0. \quad (50)$$

Therefore, the dc superheating field H_s in the clean limit has no direct relevance to the maximum RF magnetic field at which the Meissner state can exist. Moreover, even at fields H_ω smaller than $H_g < H_c$, the RF field starts generating quasiparticles as the gap $\epsilon_g(H)$ becomes smaller than the RF frequency: $\epsilon_g(H_\omega) = \hbar\omega$. Hence,

$$H_g - H_\omega \gtrsim \hbar\omega H_g/\Delta_0. \quad (51)$$

As the magnitude of $H(t)$ approaches H_ω , the surface resistance increases strongly, becoming of the order of $R_s(T)$ in the normal state. Calculation of the nonlinear surface impedance requires solving equations for the nonequilibrium Keldysh functions that take into account not only current pairbreaking but also the effect of RF field on the quasiparticle distribution function determined by collisions of electrons with impurities and phonons.³¹ We will not discuss here this complex problem

in detail but only make a few qualitative remarks based on our solutions of the dc Eilenberger equations that capture the essential effect of impurities on the quasiparticle density of states at high-RF fields.

Nonmagnetic impurities can strongly affect the field dependence of R_s , because they reduce H_s by only a few percentages but restore the gap ϵ_g in the quasiparticle spectrum at $H = H_s$, as shown in Figs. 11 and 12. In the dirty limit, $\alpha > 1$, the gap $\epsilon_g \simeq 0.3\Delta_{00}$ may, therefore, be big enough to ensure both the exponentially small surface resistance in Eq. (49) at low temperatures and the lack of quasiparticles generated by the RF field with $\omega < \epsilon_g$. As a result, nonmagnetic impurities can drastically reduce the field-induced increase of R_s as compared to the clean limit. Moreover, the dc superheating field $H_s \approx 0.83H_c$ at $\alpha > 1$ can now be regarded as a true maximum field amplitude at which the Meissner state can survive under low-frequency RF fields. This conclusion may be essential for the materials optimization for superconducting cavities used in particle accelerators.

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