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Electromagnetic and gravitational responses and anomalies in topological insulators and superconductors

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One of the defining properties of the conventional three-dimensional ("$\mathbb{Z}_2$" or "spin-orbit") topological insulator is its characteristic magnetoelectric effect, as described by axion electrodynamics. In this paper, we discuss an analog of such a magnetoelectric effect in the thermal (or gravitational) and magnetic dipole responses in all symmetry classes that admit topologically nontrivial insulators or superconductors to exist in three dimensions. In particular, for topological superconductors (or superfluids) with time-reversal symmetry, which lack SU(2) spin rotation symmetry (e.g., due to spin-orbit interactions), such as the B phase of ^{3}He, the thermal response is the only probe that can detect the nontrivial topological character through transport. We show that, for such topological superconductors, applying a temperature gradient produces a thermal- (or mass-) surface current perpendicular to the thermal gradient. Such charge, thermal, or magnetic dipole responses provide a definition of topological insulators and superconductors beyond the single-particle picture. Moreover, we find, for a significant part of the "tenfold" list of topological insulators found in previous work in the absence of interactions, that in general dimensions, the effective field theory describing the space-time responses is governed by a field theory anomaly. Since anomalies are known to be insensitive to whether the underlying fermions are interacting, this shows that the classification of these topological insulators is robust to adiabatic deformations by interparticle interactions in general dimensionality. In particular, this applies to symmetry classes DIII, CI, and AIII in three spatial dimensions, and to symmetry classes D and C in two spatial dimensions.

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I. INTRODUCTION

The considerable recent progress in understanding topological insulating phases in three dimensions was initiated by studies of single-particle Hamiltonians describing electrons with time-reversal invariance.[1–5] In both two and three dimensions, time-reversal invariant Fermi systems that have topological invariants of $\mathbb{Z}_2$ type are known to exist: insulators can be classified as "ordinary" or "topological" by band-structure integrals similar to the integer-valued integrals that appear in the integer quantum Hall effect.[6,7] These invariants survive when disorder is added to the system. In fact, stability to disorder is one of the defining properties of topological insulating phases (and also topological superconductors). The complete classification of topological insulators and topological superconductors in any dimension has been obtained in Refs. 8 and 9, and in every dimension, five of the ten Altland-Zirnbauer symmetry classes[11,12] of single-particle Hamiltonians (including some describing the Bogoliubov quasiparticles of superconductors or superfluids, rather than ordinary electrons) contain topological insulating phases with topologically protected gapless surface states.

An important question is, how can these various phases be defined in terms of a physical response function? Aside from aiding in experimental detection, such definitions also indicate that the phase is well-defined in the presence of interactions. The best studied example is the conventional three-dimensional ("$\mathbb{Z}_2$" or "spin-orbit") topological insulator with no symmetries beyond time-reversal, which has been recently observed in various materials, including Bi_xSb_{1-x} alloys,[13] Bi_2Se_3, and Bi_2Te_3.[14–17] Such materials support a quantized magnetoelectric response generated by the orbital motion of the electrons, i.e., the phase can be defined by the response of the bulk polarization to an applied magnetic field.[18,19] The possibility of such a bulk response was discussed some time ago as a condensed-matter realization of "axion electrodynamics."[20]

The first goal of this paper is to find, for all three-dimensional topological insulators and superconductors, the corresponding responses that result from the coupling of the theory to gauge and gravitational[21] fields. The second goal of this paper is to understand to what extent the classification scheme found previously for topological insulators of noninteracting fermions can be stable to fermion interactions. This addresses the question of whether certain topological insulators that describe distinct topological phases in the absence of fermion interactions (connected only by quantum phase transitions at which the bulk gap closes) can be adiabatically deformed into each other when interactions are included (without closing the bulk gap). We find that this cannot happen, e.g., in symmetry classes DIII, CI, and AIII in three spatial dimensions, and in symmetry classes D and C in two dimensions. More generally, in the final (more technical) section of this paper, we provide an answer to this question in general dimensionalities for a significant part of the list of topological insulators (superconductors) within the "tenfold" classification scheme, obtained for noninteracting particles.[8–10,22] In particular, we relate the topological features of these topological insulators to the appearance of a topological term in the effective field theory describing

TABLE I. Electromagnetic and gravitational (thermal) responses for five out of ten Altland-Zirnbauer symmetry classes (AII, CI, CII, DIII, and AIII). The assumptions made in the first four classes are that U(1) conserved currents arise from electrical charge and that SU(2) conserved currents arise from spin. In class AIII (as indicated by asterisks), the U(1) conservation law may arise either from charge or one component of spin.

Symmetry	Charge	Gravitational	Dipole
AII	√	√	
CI		√	√
CII		√	√
DIII		√	
AIII	*	√	*

space-time-dependent responses. Alternately, we relate these topological terms to what are known as "anomalies" appearing in the theories describing the responses. Since the "anomalies" are known to be insensitive to whether the underlying fermions are interacting or not, our so-obtained description of the topological features demonstrates the insensitivity of these topological insulators (superconductors) to adiabatic deformations by interactions.

The general picture emerging from the results presented in this paper is that the topological insulators (superconductors) appearing in the "ten-fold list" can be viewed as generalizations of the $d = 2$ Integer Quantum Hall Effect to systems in different dimensions d and with different ("anti-unitary") symmetry properties.[8] While the "ten-fold classification scheme" was originally established in Refs. 8 and 9 for noninteracting fermions, the characterization in terms of anomalies implies that this extends also to all those *interacting* systems which can be adiabatically connected to noninteracting topological insulators (superconductors) without closing the bulk gap. (This may include fairly strong interactions, albeit typically not expected to exceed the noninteracting bulk gap.) One may expect that to any of the topological insulators (superconductors) in the "ten-fold list" (viewed as generalizations of the Integer Quantum Hall Effect) corresponds a set of "fractional" topological insulators (superconductors) *not* adiabatically connected to a noninteracting one, in analogy to the case of the two-dimensional Quantum Hall Effect. This includes, e.g., a recently proposed three-dimensional "fractional" topological insulator Ref. 23. One expects a description in terms of anomalies to carry over to all such systems and to play a role in a (future) perhaps comprehensive characterization of such "fractional" topological insulators (superconductors). In the present paper, however, we focus on those interacting topological insulators (superconductors) which can be adiabatically connected to a noninteracting system of fermions.

Let us focus now on the topological insulators (superconductors) in $d = 3$ spatial dimension (see also Table I). From a conceptual point of view, it is the *surface responses* that are simplest to describe, and they are quantized (but they may not necessarily be the most easily accessible experimentally; therefore, we also discuss the *bulk responses* further below).

Charge surface response. This is, in particular, relevant for the ("$\mathbb{Z}_2$" or "spin-orbit") topological insulator, which is time-reversal-invariant. Upon subjecting its surface to a weak time-reversal symmetry-breaking perturbation (in the zero-temperature limit), the surface turns into a quantum Hall insulator whose electrical surface Hall conductance takes on the quantized value[24]

$$\sigma_{xy}/(e^2/h) = \frac{n}{2} \tag{1}$$

(a multiple of half the conductance quantum) as the strength of the symmetry-breaking perturbation is reduced to zero (always at zero temperature). Here, $n = 0$ and 1 for the "$\mathbb{Z}_2$" (or "spin-orbit") topological insulator[18,24] (in the so-called symmetry class AII), in the topologically trivial and nontrivial phase, respectively. While the surface of $\mathbb{Z}_2$ topological insulators in class AII may exhibit any odd (even) number Dirac cones in the topologically nontrivial (trivial) phase at the microscopic level, only the odd-even parity, $n = 1$ and 0 of that number, is topologically protected. For the less familiar topological insulator in symmetry class AIII a relation analogous to Eq. (1) applies.[22,26,45]

Spin surface response. Analogous effects are known[29] for the time-reversal-invariant topological (spin-singlet) superconductor in symmetry class CI in $d = 3$ spatial dimension. Subjecting its surface, as above, to a weak time-reversal symmetry-breaking perturbation (in the zero-temperature limit), the surface turns into what is known as the "spin quantum Hall insulator."[27,28] Due to spin-singlet pairing, this superconductor has SU(2) Pauli-spin rotation symmetry, which permits the definition of the "surface spin conductivity." In particular,[27] a gradient of magnetic field within the surface (say in the z direction of spin space) leads to a spin current perpendicular to the gradient (and within the surface). This defines the "surface spin-Hall conductance," which, similar to Eq. (1), takes on the quantized value

$$\sigma_{xy}^{(\mathrm{spin})} \Big/ \frac{(\hbar/2)^2}{h} = \frac{n}{2} \tag{2}$$

[n-times half the "spin-conductance quantum" $\frac{(\hbar/2)^2}{h}$] as the time-reversal symmetry-breaking perturbation is reduced to zero.[8,29,45]

Thermal surface response. As we show in Sec. III B of this paper, an analogous effect occurs for the thermal response at the surface of the time-reversal-invariant topological superconductor in symmetry class DIII in $d = 3$ spatial dimensions: subjecting its surface, as above, to a weak time-reversal symmetry-breaking perturbation (in the low-temperature limit), a temperature gradient within the surface leads to a heat (energy) current in the perpendicular direction in the surface. The so-defined surface thermal Hall conductance σ_{xy}^T (when divided by temperature) tends, similar to Eqs. (1) and (2), in the zero-temperature limit to a quantized value

$$\left(\sigma_{xy}^T/T\right) \Big/ \frac{(\pi k_B)^2}{3h} = \pm c/2, \quad \text{where} \quad c = n/2 \tag{3}$$

as the symmetry-breaking perturbation is reduced to zero.[8,22,45] [$c \times \frac{(\pi k_B)^2}{3h}$ is the thermal conductance for a Majorana fermion when $c = 1/2$ (its central charge).]

If we start out with a noninteracting topological insulator, one can explicitly compute the theory describing various space-time-dependent responses. [For the thermal responses of the DIII topological superconductor in $d = 3$ spatial dimension, this is done in Sec. III B of this paper. For the SU(2) spin responses of the topological singlet superconductor in symmetry class CI this was done in Ref. 29. For a significant part of the list of all topological insulators (superconductors), this is done more generally in Sec. V of this paper for all dimensionalities.] Due to the fact that the underlying insulators are topological, the field theories for the responses turn out to be described by what are called anomalies. The anomalies turn out to provide[63] an alternative characterization of topological insulators (superconductors) [except in certain one-dimensional cases[53]]. The charge, spin, and thermal surface responses discussed above are consequences of such anomalies.[30] Anomalies are known to be insensitive to the presence or absence of interactions. They are thus independent of the strength of the interactions and can only change when a bulk quantum phase transition is crossed (at which the bulk gap closes).

While these surface responses are quantized and theoretically useful in that they permit one to understand the stability of the topological insulator (superconductor) phases to interactions (for the cases discussed above, and in Sec. V for general dimensionalities), they may not all be directly accessible experimentally. Therefore, we discuss below also the various *bulk* responses.

The *bulk* responses that we find are of three types: *charge* response, previously shown to lead to a quantized $\mathbf{E} \cdot \mathbf{B}$ term in the ordinary $\mathbb{Z}_2$ topological insulator ("axion electrodynamics");[18–20] *gravitational* response, when energy flows lead to an analog of this term for gravitational fields, leading to a Lense-Thirring frame-dragging effect[31] when a temperature gradient is applied; and *magnetic dipole* response, when a magnetic dipole current induced by an applied perturbation leads to an electrical field. A single phase may show more than one of these effects; for example, a phase with a conserved SU(2) spin current can show a non-Abelian response of this type in the presence of an SU(2) gauge field coupling to this current, but will also show a magnetic dipole response via its coupling to ordinary U(1) electromagnetism. We obtain these possible responses for each of the five symmetry classes in three dimensions supporting topological phases.[8,9] As in the classification in Ref. 8, the approach we take is based upon the surfaces of these topological phases; these surfaces carry currents leading to new terms in the effective action of gravitational and electromagnetic fields. Our results for the various symmetry classes with topological invariants in three dimensions are summarized in Table I.

These bulk responses are "topological" to varying degrees. The charge response is topological both in its spatial dependence and as a term of the effective action: quantization of the response is tied to quantization of the electrical charge and the Dirac quantization condition. The gravitational response is topological in terms of the spatial dependence, but its coefficient is related to the mass or energy of the underlying particles and hence not quantized to the same degree as the charge response. The magnetic dipole response is not topological in the sense of being metric-independent, but it does arise from sample boundaries in the same way as the other responses.

This paper is organized as follows: We begin in Sec. II by reviewing the axion electromagnetism for the three-dimensional topological insulators in the spin-orbit symmetry class (symmetry class AII). In Sec. III, the thermal response of three-dimensional time-reversal invariant topological superconductors (such as the B-phase of ^{3}He) is discussed by exploiting a close analogy of electromagnetism and gravity in Newtonian approximation. In Sec. IV, the dipole response is discussed for three-dimensional topological phases when at least one component of spin is conserved. All these responses will be discussed from a much broader perspective in Sec. V in terms of anomalies of various kinds (chiral anomaly, gauge anomaly, gravitational anomaly), and the descent relation pertaining to these anomalies. We conclude in Sec. VI.

II. CHARGE RESPONSES

For an explicit example, consider a cylinder of a topological insulator with surface Hall conductance $\pm e^2/(2h)$, defined with reference to the outward normal (see Fig. 1). (Below, we choose a plus sign for the surface Hall conductance by subjecting the surface to a weak external time-reversal symmetry source.) The motivation for considering this example in some detail is that it will lead to a direct interpretation of the corresponding gravitational response below. The current response to an applied electrical field along the cylinder axis is (see Fig. 1)

$$\mathbf{j} = j_\theta\, \hat{\boldsymbol{\theta}}, \quad \text{where} \quad j_\theta = \frac{e^2}{2h} E_z. \tag{4}$$

Now the magnetic field induced by this current follows from one of Maxwell's equations,

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}, \tag{5}$$

which leads to the magnetostatic equation

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c} \int \mathbf{j}(\mathbf{x}') \times \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3x'. \tag{6}$$

The result for a thin cylinder is that the magnetic field at the cylinder axis, well away from the cylinder ends, is given by $\mathbf{B} = B_z \hat{\mathbf{z}}$ with

$$B_z = \frac{1}{c} \int_{-\infty}^{\infty} \frac{r(2\pi r) j_\theta}{(r^2 + a^2)^{3/2}}\, da = \frac{4\pi}{c} j_\theta = \frac{2\pi e^2 E_z}{hc}. \tag{7}$$

This magnitude follows from minimizing the magnetic energy,

$$H_B = \frac{B^2}{8\pi} - \frac{e^2}{2hc} \mathbf{E} \cdot \mathbf{B}, \tag{8}$$

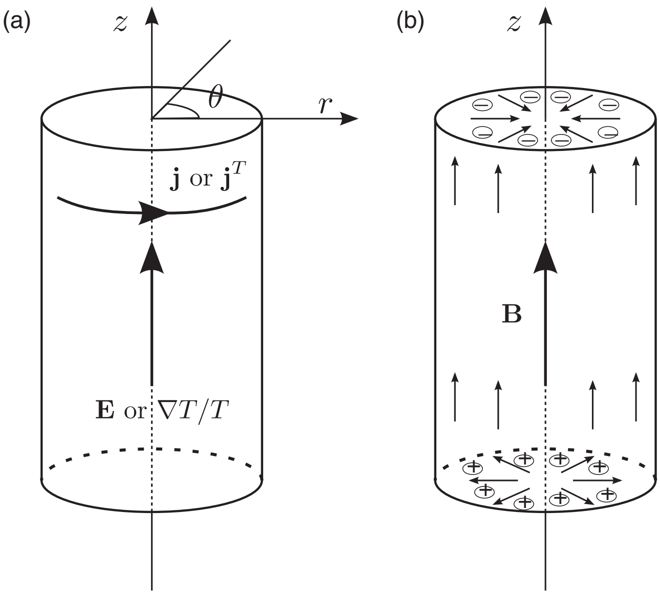

FIG. 1. Electric and thermal response of topological insulators, and thermal response of topological triplet superconductors, in a cylindrical geometry. (a) Electric (**j**) or thermal ($\mathbf{j}^T$) current driven by applied electric field (**E**) or thermal gradient ($\nabla T/T$). (b) A response dual to (a) where an applied magnetic field in the z direction induces charge polarization.

which follows from the Maxwell Lagrangian supplemented with the θ term (axion term)

$$\mathcal{L}_\theta = \frac{\theta e^2}{2\pi hc}\mathbf{E}\cdot\mathbf{B} = \frac{\theta e^2}{16\pi hc}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda} \tag{9}$$

for the coupling $\theta = -\pi$. (The negative sign in this equation is picked out by the choice of the direction of the current flow around the cylinder.)

To understand the dual response (see Fig. 1), which is an electrical field induced by an applied magnetic field, one needs to include the ends of the cylinder. Applying a magnetic field normal to a Hall layer increases or decreases the charge density depending on the direction of the field, as is required for the charge continuity equation to follow from Maxwell's equation

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = \mathbf{0}. \tag{10}$$

Hence an applied magnetic field induces an electrical polarization along the interior of the cylinder. We now turn to a gravitational version of the above physics, generated by energy flows from surface thermal Hall layers.

III. GRAVITATIONAL RESPONSES

A. Gravitoelectromagnetism

Our approach will be to start from the energy flow at surfaces of a topological phase, which is the microscopic source of the gravitational response. The importance of this response is that it is the only one that exists in the important symmetry class DIII, which includes superfluid ^{3}He. We use this phase as an explicit example in the following. The surface Majorana mode that exists in this phase does not carry charge, but it does carry heat, leading to a thermal Hall effect. Hence a temperature gradient applied to a cylinder leads to an energy flow perpendicular to the applied gradient,

$$j_\theta^T = \sigma_{xy}^T(-\partial_z T) = c^{-2}T\sigma_{xy}^T E_{g,z}, \tag{11}$$

where for future use we have treated temperature as a scalar potential generating a field $\mathbf{E}_g = -c^2(\nabla T)/T$ with units of acceleration. The physical meaning of this scalar potential was worked out by Luttinger in his derivation of the thermal transport coefficients:[32] in a near-equilibrium system, the effect of a thermal gradient is equivalent to that obtained from a gravitational potential ψ such that

$$\nabla\psi = \frac{\nabla T}{T}, \tag{12}$$

where ψ is the gravitational potential energy per mass, divided by c^2.

This rotational energy flow couples to the gravitational field at the first post-Newtonian approximation (i.e., the coupling is down by a factor v/c compared to the static gravitational effect present in the absence of the applied gradient). Because temperature couples to the local energy density in the same way as an applied gravitational potential, as used by Luttinger in his derivation of the thermal Kubo formula,[32] we can view this effect similarly to the charge response above, as a gravitational "magnetic" field resulting from the energy flow that was induced by a gravitational "electric" field reflecting the temperature gradient.

This analogy can be made precise in the near-Newtonian limit using the gravitoelectromagnetic equations[33] that apply to a near-Minkowski metric. The relevant equation is that a mass current induces a gravitomagnetic field B_g, defined more precisely below, via the equation

$$\nabla\times\mathbf{B}_g = \frac{-4\pi G\mathbf{j}_m}{c}. \tag{13}$$

Here $\mathbf{j}_m$ is the (three-dimensional) mass current density, satisfying $\mathbf{j}_m = \mathbf{j}^T/c^2$, and G is the effective Newton constant of the material. The negative sign in this equation compared to the corresponding Maxwell's equation is physically significant and results from the difference that equal masses attract, while equal charges repel. The field E_g, like B_g, has units of acceleration, and the gravitational force on a test particle of small mass m_{test} is

$$\mathbf{F} = m_{\text{test}}\left(\mathbf{E}_g + 2\frac{\mathbf{v}}{c}\times\mathbf{B}_g\right), \tag{14}$$

where **v** is the particle velocity. The factor of 2 here results from the spin-2 nature of the gravitational field.

Now, by the same steps as above, there is an induced field along the cylinder axis,

$$B_g = \frac{4\pi G\mathbf{j}_\theta^T}{c^3} = \frac{4\pi G}{c^3}\frac{T\sigma_{xy}^T E_{g_z}}{c^2}. \tag{15}$$

Since σ_{xy}^T has the units $k_B^2 T/h$ of a two-dimensional thermal conductivity, the ratio between B_g and E_g is of the form $G(\text{energy}^2)/(hc^5)$, which is dimensionless (the gravitational analog of the fine-structure constant that appears in the charge case).

The gravitomagnetic field then has exactly the same spatial dependence as the magnetic field in the axion case computed above. In particular, it is topological (e.g., the field at the cylinder axis does not fall off as the cylinder radius becomes larger) and scales with the energy flow, which in turn scales quadratically with the mass of the underlying particles.

B. Gravitational instanton term

We now discuss the gravitational response in topological insulators and superconductors from a more formal point of view. When discussing electromagnetic responses in topological insulators, we can couple electrons to an external (background) U(1) gauge field. The θ term in the effective action for the gauge field then results by integrating over the gapped electrons. To discuss gravitational and thermal responses, we can take a similar approach: we can introduce an external gravitational field that couples to fermions (electrons for topological insulators, and fermionic Bogoliubov quasiparticles for topological superconductors). By integrating over the gapped fermions, we obtain an effective gravitational action. The derivation of the effective action proceeds in a way quite parallel to that of the U(1) case: Indeed, both of them are related to a chiral anomaly, as we will see below.

For topological insulators or superconductors defined on a lattice, it is not obvious how to couple fermions to gravity in a way fully invariant under general coordinate transformations. Also, there is of course no Lorentz symmetry on a lattice. Yet, energy and momentum are conserved, and one can think of introducing an external field that couples to these conserved quantities. The gravitoelectromagnetic approach discussed in the previous subsection is based on a particular background (flat Minkowski metric), and is an approximation of the full Einstein gravity in the limit where the mass flows are small in some particular reference frame defined by the system with no thermal perturbation.

However, all topological insulators (superconductors) are known[22] to possess a representative in the same topological phase, which is described by a Dirac Hamiltonian. Fermions whose dynamics is described by a Dirac Hamiltonian can naturally be coupled to a gravitational background field. (The theory is fully Lorentz invariant, and the coupling to gravity is fully invariant under general coordinate transformations, and can be described in terms of the spin connection.) For this reason, we provide (below) a derivation of the effective action in terms of the Dirac representative of the topological phases. The topological features of the effective action for the gravitational responses are expected to be independent of the choice of representative in the topological class, and thus to have a much more general applicability. Physically, such gravitational responses describe thermal response functions.[32]

We thus consider the following single 4×4 continuum Dirac model:

$$H = \int d^3x\, \psi^\dagger (-i\boldsymbol{\partial} \cdot \boldsymbol{\alpha} + m\beta)\, \psi, \tag{16}$$

where $\psi^\dagger$ and ψ represent creation and annihilation operator of complex fermions, respectively, and $\boldsymbol{\alpha} = \sigma_1 \otimes \boldsymbol{\sigma}$ and $\beta = \sigma_3 \otimes \sigma_0$ are the Dirac matrices ($\sigma_{0,1,2,3}$ are standard Pauli matrices). (In this subsection, we use natural units, $c = \hbar = 1$, and set the Fermi velocity to be 1 for simplicity.) For topological superconductors, we need to use real (Majorana) fermions instead of complex fermions.

We assume the Dirac model is in a topologically nontrivial phase for $m > 0$ while it is in a trivial phase for $m < 0$: While this does not look apparent from the action in the continuum limit, when the Dirac model is derived from an appropriate lattice model, the sign of the mass does determine the nature of the phase. In the presence of a gravitational background, the fermionic action is given by[34]

$$\begin{aligned} S[m,\bar{\psi},\psi,e] &= \int d^4x\, \sqrt{g}\mathcal{L}, \\ \mathcal{L} &= \bar{\psi} e_a{}^{\mu} i\gamma^a \left(\partial_\mu - \frac{i}{2}\omega_\mu{}^{ab}\Sigma_{ab}\right)\psi - m\bar{\psi}\psi, \end{aligned} \tag{17}$$

where $\mu,\nu,\ldots = 0,1,2,3$ is the space-time index, and $a,b,\ldots = 0,1,2,3$ is the flat index; $e_a{}^{\mu}$ is the vielbein, and $\omega_\mu{}^{ab}$ is the spin connection; $\Sigma_{ab} = [\gamma_a,\gamma_b]/(4i)$. (See Ref. 35 for our conventions of metric, vielbein, spin connection, etc.) The effective gravitational action $W_{\text{eff}}[m,e]$ for the gravitational field is then obtained from the fermionic path integral

$$e^{iW_{\text{eff}}[m,e]} = \int \mathcal{D}[\bar{\psi},\psi] e^{iS[m,\bar{\psi},\psi,e]}. \tag{18}$$

A key observation is that the continuum Hamiltonian H enjoys a continuous chiral symmetry: we can flip the sign of mass, in a continuous fashion, by the following chiral rotation:

$$\psi \to \psi = e^{i\phi\gamma_5/2}\psi', \quad \psi^\dagger \to \psi^\dagger = \psi^{\dagger\prime} e^{-i\phi\gamma_5/2}, \tag{19}$$

under which

$$\begin{aligned} \bar{\psi}(i\partial_\mu\gamma_\mu - m)\psi &= \bar{\psi}'[i\partial_\mu\gamma_\mu - m'(\phi)]\psi', \\ m'(\phi) &= me^{i\phi\gamma_5} = m[\cos\phi + i\gamma_5 \sin\phi], \end{aligned} \tag{20}$$

so that $m'(\phi = 0) = m$ and $m'(\phi = \pi) = -m$. Since m can continuously be rotated into $-m$, one would think, naively, $W_{\text{eff}}[m,e] = W_{\text{eff}}[-m,e]$. This naive expectation is, however, not true because of chiral anomaly. The chiral transformation that rotates m continuously costs the Jacobian $\mathcal{J}$ of the path integral measure,

$$\mathcal{D}[\bar{\psi},\psi] = \mathcal{J}\mathcal{D}[\bar{\psi}',\psi']. \tag{21}$$

The chiral anomaly (the chiral Jacobian $\mathcal{J}$) is responsible for the θ term. The Jacobian $\mathcal{J}$ can be computed explicitly by the Fujikawa method,[36] with the result

$$\begin{aligned} W_{\text{eff}}^{\theta} &:= -\ln\mathcal{J} \\ &= \theta\frac{1}{2}\left[\frac{1}{2\times 384\pi^2}\int d^4x\sqrt{g}\epsilon^{cdef}R^a{}_{bcd}R^b{}_{aef}\right] \end{aligned} \tag{22}$$

when $m > 0$ while $W_{\text{eff}}^{\theta} = 0$ when $m < 0$. The expression in square brackets is the so-called Dirac genus (see Sec. V below for details), which is equal,[34] by the Atiyah-Singer index theorem, to the index of the Dirac operator in the curved background. The multiplicative prefactor 1/2 arises because of the Majorana nature of the Bogoliubov quasiparticles. The index in square brackets is in fact an even integer (by Rochlin's theorem[39]). Therefore, (1/2) of that expression, i.e., half the index, is an integer. Thus the gravitational effective action W_{eff}^{θ} in Eq. (22) equals θ times an integer, i.e., it is a so-called θ term.

As we rotate the angle $\theta = \phi$, Eq. (20), from zero to 2π, the partition function winds an integer number of times around the origin in the complex plane. This winding number measures the integer[8,10,22] of the topological insulator (superconductor). See also Ref. 63. [This winding number is ultimately related to a property of the underlying massless theory. See, e.g., Eq. (46) and its generalizations.] Now, since $\theta \to -\theta$ under time reversal, the θ angle is fixed by time-reversal symmetry and periodicity to either $\theta = 0$ or $\theta = \pi$. The former corresponds to a topologically trivial state, and $\theta = \pi$ to the topologically nontrivial state. [For a similar discussion on the derivation of the θ term, i.e., the $\mathbf{E}\cdot\mathbf{B}$ term, for the electromagnetic response, see Ref. 26, and for the non-Abelian SU(2) response, see Ref. 29.] Note that if instead we consider complex (Dirac) fermions in the background gravity field, the theta angle θ is an integer multiple of 2π, but not of π as in the Majorana case.

The part of the effective action that is not related to the Fujikawa Jacobian takes the form of the Einstein-Hilbert action $W_{\rm EH} = (16\pi G)^{-1}\int d^4x\,\sqrt{g}R$, where G is the effective Newton constant in the bulk of the topological insulator (superconductor). The gravitoelectromagnetism equations mentioned above can be derived from the effective action by taking the Newtonian limit (near Minkowski limit).

To make the connection with the existence of topologically protected surface modes, we note that when there are boundaries (say) in the x^3 direction at $x^3 = L_+$ and at $x^3 = L_-$, the gravitational instanton term $W^{\theta}_{\rm eff}$, at the nontrivial time-reversal invariant value $\theta = \pi$ of the angle θ, can be written in terms of the gravitational Chern-Simons terms at the boundaries,

$$W^{\theta}_{\rm eff} = I_{\rm CS}|_{x^3=L_+} - I_{\rm CS}|_{x^3=L_-}, \tag{23}$$

where $(i,j,k = 0,1,2)$

$$I_{\rm CS} = \frac{1}{2}\frac{1}{4\pi}\frac{c}{24}\int d^3x\,\epsilon^{ijk}{\rm tr}\left(\omega_i\partial_j\omega_k + \frac{2}{3}\omega_i\omega_j\omega_k\right) \tag{24}$$

with $c = 1/2$. This kind of relationship between the θ-term and the Chern-Simons type term in one lower dimension is a special case of the so-called descent relation and will be discussed further in Sec. V. This value of the coefficient of the gravitational Chern-Simons term is one-half of the canonical value $(1/4\pi)\times(c/24)$ with $c = 1/2$. As before, for fermions with a reality condition (Majorana fermions), the canonical value of the coefficient of the gravitational Chern-Simons term corresponds to $c = 1/2$, as opposed to $c = 1$ for fermions without a reality condition. As discussed by Volovik[37] and Read and Green[38] in the context of the two-dimensional chiral p-wave superconductor, the coefficient of the gravitational Chern-Simons term is directly related to the thermal Hall conductivity, which in our case is carried by the topologically protected surface modes.[40] [See Eq. (3) of the Introduction.]

IV. DIPOLE RESPONSES

A. Topological singlet superconductor (class CI) and spin chiral topological insulator (class CII)

The last response we consider can be measured in systems with a conserved spin or magnetic dipole current. Among the five symmetry classes that admit a topological phase in three-spatial dimensions, we thus focus on topological singlet superconductors in symmetry class CI (possessing time-reversal and spin rotation invariance), and also on topological insulators in symmetry class CII (possessing time-reversal but without spin rotation invariance) (see Table I).

Simple lattice models of the three-dimensional topological singlet superconductor in symmetry class CI were discussed previously on the diamond lattice[29] and on the cubic lattice,[26] for which, in the presence of a boundary (surface), there is a stable and nonlocalizing Andreev bound state. Similar to the quantized $\mathbf{E}\cdot\mathbf{B}$ term for the charge response in the topological insulator, the response of topological singlet superconductors to a fictitious external SU(2) gauge field (a "spin" gauge field, which couples to conserved spin current) is described by the θ term at $\theta = \pi$ in the $(3+1)$-dimensional SU(2) Yang-Mills theory.[29] The θ term predicts the surface quantum Hall effect for spin transport (the spin quantum Hall effect), as already mentioned in the Introduction (Sec. I).

To detect such a quantum Hall effect for the SU(2) symmetric spin current requires a fictitious external spin gauge field, and hence one would think it cannot be detected experimentally. Nevertheless, we discuss in this section that the electromagnetic response carried by the dipole moment of the spin current can be measurable. (See Ref. 41 for a similar discussion on the dipole response in a ^{3}He-A superfluid thin film or two-dimensional p-wave paired states.)

The topological insulator in symmetry class CII (called a "spin chiral topological insulator" in Ref. 26) is in many ways analogous to the more familiar quantum spin Hall effect in two spatial dimensions, but requires the chiral symmetry in addition to time-reversal symmetry. (For a lattice model of the $\mathbb{Z}_2$ topological insulator in symmetry class CII, see Ref. 26.) Just as an intuitive understanding of the quantum spin Hall effect can be obtained by starting from two decoupled and independent quantum Hall systems with opposite chirality for each spin and then gluing them together, this spin chiral topological insulator can be obtained by considering two independent topological insulators in symmetry class AIII. More general quantum spin Hall states or spin chiral topological insulators can then be obtained by destroying the S_z conservation by mixing spin-up and -down components. The dipole response for class CII topological insulators, which we will describe below, assumes that a U(1) part of the SU(2) spin rotation symmetry is conserved (i.e., one component of spin is conserved). However, even when there is no such symmetry, if mixing between two species is weak, we can still have such a dipole response.

B. Magnetic dipole responses

The spin current response at the surface of such a system to an applied magnetic field $\mathbf{B}$ via the Zeeman effect can be written as

$$j^a_i = \alpha\epsilon_{ijk}(\partial_j\theta)\partial_k B_a, \tag{25}$$

where α is some constant. Here we have introduced a scalar field θ ("axion" field),[29] by analogy with the local electromagnetic polarizability of the (AII, spin-orbit) topological insulator, to describe the spatial location of the dipole current, which as before is a surface property. Here j^a_i represents the ath component of a magnetic dipole current of dipoles in spatial direction i. Such a current can generate two types of *static* electromagnetic responses: a dipole density through the continuity equation

$$\partial_i j_i^a + \partial_t n^a = 0, \tag{26}$$

and an electrical field through the equation

$$(\nabla \times \mathbf{E})_i = \epsilon_{ijk}\partial_j E_k = \frac{\mu}{4\pi}\partial_a j_i^a, \tag{27}$$

where μ is the permeability of the material of interest. (One could alternately have a time-varying magnetic field, just as a current density can produce either a constant magnetic field or a time-varying electrical field.) The second response may be unfamiliar but can be derived from elementary principles; see Ref. 42 for a discussion of how it can be measured experimentally. Start from a dipole field in the laboratory frame. Take one copy with the dipoles pointing along some direction $\hat{\mathbf{n}}$ and boost that along $\mathbf{v}$, and take another copy with the dipoles pointing along $-\hat{\mathbf{n}}$ and boost that along $-\mathbf{v}$. For a dipole density n^a, this leads, in the comoving frame, to the field $B_a = (\mu/4\pi)n^a$, and hence

$$\nabla \cdot \mathbf{B} = \frac{\mu}{4\pi}\partial_a n^a. \tag{28}$$

Using the nonrelativistic Lorentz transformation law

$$\mathbf{E} \to \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{29}$$

with $\gamma \simeq 1$ leads to Eq. (27), with $j_i^a = v_i n^a$.

Now we consider these responses for the surface spin current of a three-dimensional topological singlet superconductor. The spin Hall current is always divergence-free by commutation of derivatives,

$$\partial_i j_i^a = \alpha\epsilon_{ijk}\partial_i(\partial_j\theta\partial_k B_a) = 0, \tag{30}$$

since whichever term the ∂_i acts on gives zero. However, the electromagnetic response can be nonzero:

$$\epsilon_{ijk}\partial_j E_k = \frac{\mu}{4\pi}\partial_a j_i^a = \frac{\mu\alpha}{4\pi}\partial_a(\epsilon_{lmn}\partial_m\theta\partial_n B_a). \tag{31}$$

There are two parts to this: one "monopole" part is only nonzero if $\partial_a B^a \neq 0$, and we therefore neglect it. There is also a term

$$\frac{\mu\alpha}{4\pi}\epsilon_{lmn}(\partial_a\partial_m\theta)\partial_n B_a. \tag{32}$$

C. Example

As an example, we compute this response for the case of a surface of a topological singlet superconductor, where the theta angle θ varies as a function of the distance from the surface (Fig. 2). For the response to be nonzero, we need $a = m = z$, so the response is to the z component of the magnetic field. We get, up to a possible sign,

$$(\nabla \times \mathbf{E})_x = -\frac{\alpha\mu}{4\pi}\partial_z^2\theta\partial_y B_z, \quad (\nabla \times \mathbf{E})_y = \frac{\alpha\mu}{4\pi}\partial_z^2\theta\partial_x B_z. \tag{33}$$

For the case in which θ is first constant, then changes linearly in z within a surface surface layer, and is then constant again outside this layer (Fig. 2), this response will occur entirely at the top and bottom surfaces of the region of linear change.

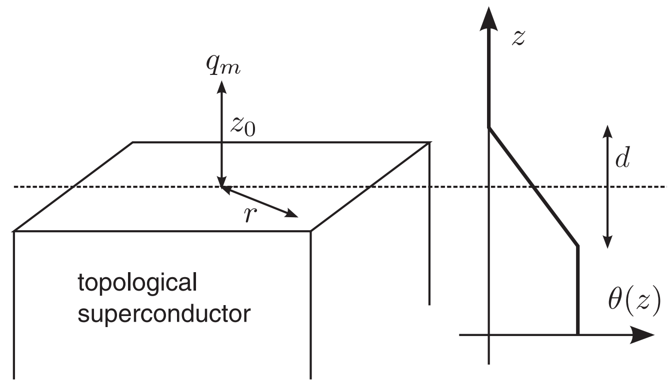

FIG. 2. Surface of a spin chiral topological insulator (class CII) or topological singlet superconductor (class CI).

As an example relevant to possible experiments, we compute this response for the magnetic field produced by a magnetic monopole field of strength q_m (i.e., from one end of a long magnetic dipole), suspended a distance z_0 above a spin Hall surface layer where θ changes linearly across a thickness d. This surface layer gives two surfaces with

$$(\nabla \times \mathbf{E})_x = j_x^m = \mp\beta\partial_y B_z, \quad (\nabla \times \mathbf{E})_y = j_y^m = \pm\beta\partial_x B_z, \tag{34}$$

where $\beta = (\alpha\mu)/(4\pi)\,\pi/d$. At the top layer, the z component of magnetic field is, in cylindrical coordinates,

$$B_z = \frac{q_m z_0}{\left(r^2 + z_0^2\right)^{3/2}}, \tag{35}$$

which leads to a surface magnetic current of magnitude,

$$j_\theta^m = \frac{3\beta q_m z_0 r}{\left(r^2 + z_0^2\right)^{5/2}}, \tag{36}$$

at the top surface. Since

$$\mathbf{E}(\mathbf{r}) = \int d^3\mathbf{r}'\frac{(\mathbf{r} - \mathbf{r}') \times \mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2}, \tag{37}$$

we obtain that the electrical field from the top surface, at a height z_1 above the top surface (and directly above or below the original monopole), is

$$E_z(z_1) = \int_0^\infty (2\pi r)\,dr\,\frac{3\beta q_m z_0 r}{\left(r^2 + z_0{}^2\right)^{5/2}}\frac{r}{r^2 + z_1{}^2}. \tag{38}$$

Evaluating this at the original height z_0 gives

$$E_z(z_0) = (6\pi\beta q_m z_0)\frac{2}{15 z_0{}^4} = \frac{4\pi\beta q_m}{5 z_0{}^3}. \tag{39}$$

Comparing this to the case of an image charge above a metal, we see that the electrical field falls off by one more power of height. From the above, the dipole currents are localized to the top and bottom surfaces of the region where θ changes. The bottom surface contributes with an opposite sign and with $z \to z + d$, so we obtain

$$E_z(z_0) = \frac{4\pi\beta q_m}{5}\left[z_0{}^{-3} - (z_0 + d)^{-3}\right], \tag{40}$$

so that for $d \ll z_0$ the electric field falls off as the fourth power of distance.

We can understand the scaling of the result by noting that q_m divided by length cubed has units of magnetic field per length; multiplying by β converts this to a two-dimensional magnetic charge current density, which has the same units as an electric field. While the dipole response originates in a topological phase, it is not itself "topological" but depends sensitively on the geometry used to probe it.

V. TOPOLOGICAL FIELD THEORIES FOR SPACE-TIME-DEPENDENT RESPONSES IN TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS IN GENERAL DIMENSIONS FROM ANOMALIES

The previous sections of this paper complete the list of the (topological) field theories describing the space-time-dependent responses of all topological insulators and superconductors in three spatial dimensions (3 + 1 space-time dimensions). In this section, we will describe, more generally, the (topological) field theories for such responses in general dimensions. Most importantly, the main result obtained in this section is a general connection between the appearance of such topological terms in the field theories for the responses and the appearance of what are called anomalies[43] for the field theories in those space-time dimensions in which topological insulators (superconductors) appear. In fact, we may ask if the existence of a particular type of anomaly in a given dimension allows us to predict the existence of a topological insulator (superconductor) of the "tenfold" classification in that dimension. The answer to this question is affirmative. As we demonstrate below, a large part of the "tenfold" classification can be derived from the existence of the known anomalies in corresponding quantum field theories in space-time. This can then be thought of as yet another derivation of the "tenfold" classification, in addition to the previously known derivations such as that based on Anderson localization at the sample boundaries,[8] and K-theory[9] (as well as a later point of view based on D-branes[46,47]). Moreover, and most importantly, the appearance of an anomaly is a statement about the respective quantum field theory (of space-time linear responses) independent of the assumption of the absence of interparticle interactions. Thus, anomalies provide a description of topological insulators (superconductors) in the context of interacting systems.

A. Topological insulators (superconductors) in the two complex symmetry classes A and AIII from anomalies in the gauge field action

1. The integer quantum Hall effect (class A)

Let us begin by describing the topological field theories describing the space-time-dependent responses of the two "complex" symmetry classes, classes A and AIII in the Cartan (Altland-Zirnbauer) classification.[8,10,22] This includes the most familiar example, namely the integer quantum Hall insulator (IQH), belonging to symmetry class A. In both symmetry classes, A and AIII, there has to exist a conserved U(1) charge (particle number). This is the electromagnetic charge, since these symmetry classes can be realized as normal electronic systems (as opposed to superconducting quasiparticle systems).[48] Therefore, we can minimally couple these topological insulators to an external U(1) gauge field. The field theory describing the space-time-dependent linear responses of the topological insulator can then be obtained by integrating out the gapped fermions. The fact that the underlying insulator is topological is reflected in the fact that the effective action for the external U(1) gauge field, describing the electromagnetic linear responses, contains a term of "topological origin," such as, e.g., a Chern-Simons or a θ term, or corresponding higher-dimensional analogs of these terms (see below for more details).

In turn, the presence of terms of topological origin in the so-obtained effective action for the external U(1) gauge field is closely related to the presence of a so-called anomaly. To see how an anomaly for the theory of the external U(1) gauge field can actually predict the presence of a topological phase, let us consider first, as the simplest example, the IQH insulator in $d = 2$ spatial dimensions—symmetry class A. (The space-time dimension is thus $D = 2 + 1$.) In fact, let us first focus attention on the theory of the sample boundary (the edge state), which has $d = 1$ spatial dimensions. It is known (see below) that the effective theory for the linear responses of the U(1) gauge field in $D = 1 + 1$ space-time dimensions (i.e., of the edge state) can have what is called a "gauge anomaly" since the space-time dimension D is even.[33,36] The presence of this anomaly simply means that U(1) charge conservation is spoiled by quantum mechanics. In the condensed-matter setting of the IQH insulator, the meaning of this anomaly is that the system (i.e., the edge) in $D = 1 + 1$ space-time dimensions, exhibiting the anomaly, does not exist in isolation, but is necessarily realized as the boundary of a topological insulator in one dimension higher. In this case, the breakdown of the conservation law of U(1) charge conservation at the boundary simply means that the current "leaks" into the bulk. Thus, in the condensed-matter setting, the presence of the anomaly in the theory at the boundary is not something abnormal, but it is a physical effect: it is the integer quantum Hall effect. As we will discuss briefly below, the same reasoning applies to all even space-time dimension, $D = 2k$. Consequently, we see that the presence of a U(1) gauge anomaly predicts the presence of a topological insulator in one dimension higher. That is, this predicts the presence of a topological insulator in symmetry class A in $D = 2k + 1$ space-time dimensions, in agreement with the "tenfold" way classification.

2. Three-dimensional insulator (superconductor) in symmetry class AIII

Let us now consider the topological insulator (superconductor) in the other complex symmetry class, class AIII, in $d = 3$ spatial dimensions. Again, the space-time dimension $D = 3 + 1 = 4$ is even. It is known (see below) that in all even space-time dimensions, the effective action for the space-time-dependent U(1) gauge field may also possess a different anomaly [in contrast to the discussion in the preceding subsection], often referred to as the "chiral (or axial) anomaly in a background U(1) gauge field."[34] The meaning of such an anomaly can be explained using Eq. (46) below: the so-called axial (or chiral) U(1) current $J_5^\mu(x)$ is *not* conserved in the presence of a background U(1) gauge field, i.e., $D_\mu J_5^\mu(x) \neq 0$, where D_μ denotes the covariant derivative in the presence of a background gauge field. In the simplest case of a single copy of a massive Dirac fermion (mass m), this covariant derivative of the current is given by Eq. (46) below. As displayed in this equation, there are two sources of the lack of conservation: (i) a finite mass $m \neq 0$ and (ii) the extra "anomaly" term $\mathcal{A}_{2n+2}$ (to be discussed in more detail below), which represents the breaking of the conservation of J_5^μ by quantum effects.[50] Now, as discussed in Ref. 26, the presence of a "chiral (or axial) anomaly in a background U(1) gauge field" implies directly the possibility of having a nonvanishing θ term when deriving the effective action for the external U(1) gauge field.[51] (The θ angle is fixed[22] to $\theta = \pi$ by a discrete symmetry, which is the chiral symmetry for symmetry class AIII.) Thus, the presence of a "chiral (or axial) anomaly in a background U(1) gauge field" in $D = 2k$ space-time dimensions signals the existence of a topological insulator in this space-time dimension through the appearance of a θ term in the (topological) field theory for the linear responses.

3. Anomaly polynomials and descent relation

Observe that above we have used anomalies of *two kinds*, and we used them *in two different ways*:

(i) In case 1. there was an anomaly in the theory of the responses at the *boundary* [which had $D = (d-1)+1$ space-time dimensions]. In this case the anomalous theory (i.e., the one at the boundary) was gapless (critical); we refer to this situation as a gauge anomaly [i.e., nonconservation of the U(1) charge in question]. The presence of this anomaly implied the existence of a topological insulator in one dimension higher, i.e., in $D' = d+1$ space-time dimensions. The responses of this topological insulator are described by an effective Chern-Simons action for the U(1) gauge field in $D' = d+1$ space-time dimensions. [See also Eq. (42).]

(ii) In case 2. there existed an anomaly in the massive *bulk* theory in $D = d+1$ space-time dimensions. This was a chiral anomaly [referring to the violation of the conservation of the global *axial* U(1) current J_5^μ] in the background of a nonvanishing U(1) background gauge field.

There are important relationships between the following different anomalies: (i) the U(1) gauge anomaly in $D = 2n$, (ii) the Chern-Simons term (i.e., parity anomaly) in $D = 2n+1$, and (iii) chiral anomaly in the presence of a background gauge field in $D = 2n+2$, which can be summarized, in terms of the so-called *descent relation* of the "anomaly polynomial."[34] Let us now explain this relation.

As mentioned above, it is known that in even space-time dimensions $D = 2n$, there is a U(1) gauge anomaly. If there is a gauge anomaly, the (Euclidean) effective action $\ln Z[\mathcal{A}]$ in the presence of the gauge field $\mathcal{A}$ is not invariant under a gauge transformation $\mathcal{A} \to \mathcal{A} + v$. Thus we can write

$$\delta_v \ln Z[\mathcal{A}] = 2\pi i \int_{M_{2n}} \Omega_{2n}^{(1)}(v,\mathcal{A},\mathcal{F}), \tag{41}$$

where the variation δ_v is the gauge transformation in question, and $\Omega_{2n}^{(1)}$ is a $2n$-form built from the connection 1-form, $\mathcal{A} = A_\mu dx^\mu$, its field-strength 2-form, $\mathcal{F} = (1/2)F_{\mu\nu}dx^\mu dx^\nu$, and the variation $v = v_\mu dx^\mu$ of the gauge field. [By definition, $\Omega_{2n}^{(1)}$ is linear in v. The integral is taken over the physical $D = 2n$-dimensional (Euclidean) space-time M_{2n}.] Now, the descent relation tells us that $\Omega_{2n}^{(1)}$ can be derived from the so-called anomaly polynomial $\Omega_{2n+2}(\mathcal{F})$, which is a $(2n+2)$-form built from the curvature 2-form $\mathcal{F}$, with the aid of yet another $(2n+1)$-form $\Omega_{2n+1}^{(0)}$, by

$$\Omega_{2n+2} = d\Omega_{2n+1}^{(0)}, \quad \delta_v \Omega_{2n+1}^{(0)} = d\Omega_{2n}^{(1)}. \tag{42}$$

That is, Ω_{2n+2} is closed, and gauge invariant, and hence can be written as a polynomial in $\mathcal{F}$. Here $\Omega_{2n+1}^{(0)}(\mathcal{A},\mathcal{F})$ is its corresponding Chern-Simons form.

There is a simple closed-form expression for the anomaly polynomial Ω_{2n+2} that is given by

$$\Omega_D(\mathcal{F}) = \mathrm{ch}(\mathcal{F})|_D. \tag{43}$$

Let us explain the notation: $\mathrm{ch}(\mathcal{F})$ is the following power series ("characteristic class") constructed from the field-strength 2-form $\mathcal{F}$, and is given by

$$\mathrm{ch}(\mathcal{F}) = r + \frac{i}{2\pi}\mathrm{tr}\,\mathcal{F} - \frac{1}{2(2\pi)^2}\mathrm{tr}\,\mathcal{F}^2 + \cdots. \tag{44}$$

This expression is written for the general case of a gauge field transforming in an r-dimensional irreducible representation of a (possibly non-Abelian) gauge group, where tr denotes the trace in this representation. Observe that $\mathrm{ch}(\mathcal{F})$ consists of a sum of different p-forms with different p where $p =$ even. The notation $\cdots|_D$ in Eq. (43) means we extract a D-form from $\mathrm{ch}(\mathcal{F})$.

While up to this point the differential forms $\Omega_{2n+1}^{(0)}$ and Ω_{2n+2} appear to have been introduced solely to express the $D = 2n$-dimensional gauge anomaly in terms of other objects, they themselves are known to be related to other types of anomalies: the Chern-Simons form $\Omega_{2n+1}^{(0)}$ represents an anomaly in a discrete symmetry (parity or charge-conjugation symmetry, depending on dimensionality) discussed in more detail in Sec. V A 4 below, and Ω_{2n+2} represents[34] the chiral anomaly in the presence of a background gauge field, discussed in Sec. V A 2 above. The integral of Ω_{2n+2} over $D = (2n+2)$-dimensional space-time, on the other hand, represents the θ term (see also Sec. V A 5 below).

4. The Chern-Simons term

The integral of $\Omega_{2n+1}^{(0)}(\mathcal{A},\mathcal{F})$ over $D = (2n+1)$-dimensional space-time is the Chern-Simons-type action for the gauge field $\mathcal{A}$, and represents, as already mentioned, an anomaly in a discrete symmetry: the parity or charge-conjugation anomaly.

In turn, the presence of such a Chern-Simons term in the effective (bulk) action for the gauge field $\mathcal{A}$ in $D = (2n+1)$-dimensional space-time signals the presence of a topological phase: when there is a boundary in the system, the integral of the Chern-Simons term is not invariant on its own; rather, upon making use of the descent relation Eq. (42), one obtains

$$\delta_v \int_{M_{2n+1}} \Omega_{2n+1}^{(0)} = \int_{M_{2n+1}} d\Omega_{2n}^{(1)} = \int_{\partial M_{2n+1}} \Omega_{2n}^{(1)}. \tag{45}$$

This is something we are familiar with from the physics of the quantum Hall effect: the presence of the boundary term $\int_{\partial M_{2n+1}} \Omega_{2n}^{(1)}$ appearing on the right-hand side of Eq. (45) signals the presence of an edge mode. In turn, as we have seen in Sec. V A 1, the gauge anomaly in $D=(2n)$-dimensional space-time, which is represented by the integral over $\Omega^{(1)}_{2n}$, itself signals the presence of a topological phase in $D=2n+1$ space-time dimensions, i.e., in one dimension higher.

5. The θ term

The integral of the anomaly polynomial Ω_{2n+2} over $D=(2n+2)$-dimensional space-time is the θ term and represents a chiral anomaly in the presence of a background gauge field (discussed in Sec. V A 2 above). Again, to be more explicit, in the presence of such an axial anomaly, the axial current $J_5^{\mu}(x)$ [which in the present case is an axial U(1) current] is not conserved: $D_\mu J_5^{\mu}(x)\neq 0$, where D_μ is the covariant derivative in the presence of the gauge field. For a single copy of a massive Dirac fermion, it is given by

$$D_\mu J_5^{\mu}(x) = 2im\bar{\psi}\gamma_{2n+1}\psi + 2i\mathcal{A}_{2n+2}(x), \tag{46}$$

where the first term represents the explicit breaking of the chiral symmetry by the mass term, whereas the second term represents the breaking of the chiral symmetry by quantum effects. $\mathcal{A}_{2n+2}$ quantifying the breaking of the axial current conservation by an anomaly is essentially identical to Ω_{2n+2}, and given by removing all dx^{μ} that appear in the differential form Ω_{2n+2}.

Just as was the case for the Chern-Simons term, the presence of such a θ term in the effective action for the gauge field signals the presence of a topological phase. In particular, the descent relation tells us that

$$\int_{M_{2n+2}} \Omega_{2n+2} = \int_{M_{2n+2}} d\Omega^{(0)}_{2n+1} = \int_{\partial M_{2n+2}} \Omega^{(0)}_{2n+1}. \tag{47}$$

This is, again, something we are familiar with from the physics of the three-dimensional topological insulator in class AIII, which is described by the θ term (the axion term). In the presence of a boundary ∂M_{2n+2}, such a topological state supports boundary degrees of freedom, as signaled by the boundary term $\int_{\partial M_{2n+1}} \Omega^{(0)}_{2n+1}$, which is a Chern-Simons term.[52]

Let us summarize: to derive the existence of topological phases in symmetry class A and AIII, we start from the anomaly polynomial Ω_{2n+2}. Then the terms $\int_{M_{2n+2}}\Omega_{2n+2}$ and $\int_{M_{2n+1}}\Omega^{(0)}_{2n+1}$ are the effective actions for the (topological) field theory of the space-time linear responses for the gauge field for the topological phases in class AIII ($D=2n+2$) and A ($D=2n+1$), respectively.

B. Topological insulators (superconductors) in the remaining eight "real" symmetry classes from gravitational and mixed anomalies

1. Gravitational anomaly and axial anomaly in the presence of background gravity

For the remaining eight "real" of the ten symmetry classes, having a conserved U(1) quantity is less trivial. Classes AI, AII, and CII are naturally realized as a normal (as opposed to superconducting) electronic system, and thus for these there is a natural notion of a conserved U(1) quantity (the electrical charge). One realization of the BDI symmetry class, which is only part[53] of the entire symmetry class, can also be considered to have a conserved U(1) quantity, and we consider this realization in this subsection. On the other hand, classes D, DIII, C, and CI are naturally realized as BdG systems. While for classes C and CI, SU(2) spin is conserved [so a conserved U(1) charge exists], for classes D and DIII, there is no conserved U(1) quantity at all.

Since for the latter four of eight real symmetry classes (D, DIII, C, CI) we cannot rely on a conserved U(1) quantity to describe these topological phases, it is not possible to couple these systems minimally to a U(1) gauge field. However, it is natural to consider a coupling of these topological phases to gravity. Let us focus first on topological insulators (superconductors) with an integer topological charge, $\mathbb{Z}$, but not on those with a binary topological charge, $\mathbb{Z}_2$. For now we also do not consider topological insulators or superconductors with a $2\mathbb{Z}$ charge.

An analog of the U(1) gauge anomaly, which we have described in Sec. V A 1 at the boundary (of space-time dimension $D=2n$) of topological phases in symmetry class A, is the gravitational anomaly. It corresponds to the breakdown of energy-momentum conservation, and when it happens, it must be realized in a system that represents the boundary of a topological phase in one dimension higher [in analogy to the case of a U(1) gauge anomaly, Sec. V A 1]. We refer to this anomaly also as a "purely gravitational anomaly." In the following, we will show that one can predict the appearance of the topological phases in symmetry classes D, C, DIII, CI [i.e., those without conserved U(1) charge] from the presence of a purely gravitational anomaly that appears in the field theory for the gravitational (or thermal[32]) responses.

Finally, we will need to discuss the still remaining symmetry classes AI, BDI, AII, and CII. Topological insulators (superconductors) in these symmetry classes can be coupled to both a U(1) gauge field[54] as well as a gravitational background. We will show that the field theories for the space-time-dependent linear responses for these topological insulators possess a so-called mixed anomaly. Indeed, we will show that the appearance of a mixed gravitational and electromagnetic axial anomaly signals the existence of topological phases in these symmetry classes.

2. Topological insulators (superconductors) in symmetry classes D, C, DIII, and CI from the purely gravitational anomaly

As mentioned earlier in this paper, each topological insulator (in any dimension) has a Dirac Hamiltonian representative.[22] We can consider the coupling of this Dirac theory to a space-time-dependent gravitational background. Upon integrating out the massive fermions, we obtain an effective gravitational action in D space-time dimensions. If there is a gravitational anomaly, the (Euclidean) effective action $\ln Z[e,\omega]$ in the presence of the gravitational background is not invariant under a general coordinate transformation $x^{\mu}\to x^{\mu}+\epsilon^{\mu}$, where e is the vielbein and ω is the spin-connection 1-form. That is,

$$\delta_v \ln Z[e,\omega] = 2\pi i \int_{M_D} \Omega^{(1)}_D(v,\omega,\mathcal{R}), \tag{48}$$

where δ_v represents an infinitesimal SO(D) rotation, under which ω, the spin-connection 1-form ω, is transformed as $\omega\to$

$\omega + v$; $\Omega^{(1)}_D(v,\omega,\mathcal{R})$ is a D-form related to the gravitational anomaly. In complete analogy to the case of the gauge anomaly discussed above, $\Omega^{(1)}_D(v,\omega,\mathcal{R})$ can be derived from a corresponding anomaly polynomial $\Omega_{D+2}(\mathcal{R})$ [see Eqs. (54) and (55) below] through its Chern-Simons form $\Omega^{(0)}_{D+1}(\omega,\mathcal{R})$, by using a descent relation that takes a form identical to Eq. (42). Thus, once the existence of the (purely) gravitational anomaly is known for a given dimension D, it predicts the presence of topological phases in $D+1$ and $D+2$ dimensions, using the same logic as in the gauge field case above.

Now, according to Ref. 55, a purely gravitational anomaly can exist in

$$D = 4k + 2 \quad (d = 4k + 1). \tag{49}$$

Thus, breakdown of energy-momentum conservation due to quantum effects can occur in these dimensions. As in the case of symmetry class A, discussed above, we take this as evidence for the existence of a topological bulk in one dimension higher, i.e., in space-time dimensions

$$D = 4k + 3 \quad (d = 4k + 2). \tag{50}$$

This thus predicts the appearance of topological phases in

$$\text{class D } (d = 2), \quad \text{class C } (d = 6), \tag{51}$$

as well as all the other higher-dimensional topological phases that we can obtain from these by Bott periodicity. (These are colored red in Table 2.)

On the other hand, there is an analog of the "axial anomaly in the presence of a background gauge field," which we discussed in Sec. V A 2 in the context of symmetry class AIII in $D = 2n$ space-time dimensions. This analog is the "axial anomaly in the presence of a background gravitational field." If only a background gravitational field is present, this anomaly exists in space-time dimensions

$$D = 4k \quad (d = 4k - 1). \tag{52}$$

This covers symmetry classes

$$\text{class DIII } (d = 3), \quad \text{class CI } (d = 7), \tag{53}$$

as well as all higher-dimensional topological phases that we can obtain from these by Bott periodicity. (These are colored blue in Table II.)

The anomaly polynomial related to the gravitational anomalies is known explicitly. It can be written as

$$\Omega_{D=4k} = \hat{A}(\mathcal{R})|_D, \tag{54}$$

where $\hat{A}(\mathcal{R})$ is the so-called Dirac genus given by[36]

$$\hat{A}(\mathcal{R}) = 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\mathrm{tr}\,\mathcal{R}^2 + \frac{1}{(4\pi)^2}\left[\frac{1}{288}(\mathrm{tr}\,\mathcal{R}^2)^2 + \frac{1}{360}\mathrm{tr}\,\mathcal{R}^4\right] + \cdots. \tag{55}$$

Here $\mathcal{R}$ is the $D \times D$ matrix of 2-forms,

$$\mathcal{R}_\mu{}^\nu := \tfrac{1}{2} R_{\alpha\beta\mu}{}^\nu\, dx^\alpha dx^\beta, \tag{56}$$

where $R_{\alpha\beta\mu}{}^\nu$ is the usual Riemann curvature tensor, and the trace refers to the $D \times D$ matrix structure. This defines, by the descent relation [which takes a form identical to Eq. (42)], the differential forms $\Omega^{(0)}_{4k-1}$ and $\Omega^{(1)}_{4k-2}$. As before, the notation $\hat{A}(\mathcal{R})|_D$ extracts a D-form from $\hat{A}(\mathcal{R})$. It is obvious from (55) that the anomaly polynomial exists only for $D = 4k$ because Eq. (55) is a function of $\mathcal{R}^2$. [Note that the descent relation Eq. (42) then implies the existence of a purely gravitational anomaly $\Omega^{(1)}_{4k+2}(\mathcal{R})$ in $D = 4k + 2$ space-time dimensions, in agreement with Ref. 55.]

3. *Topological insulators (superconductors) in symmetry classes AI, BDI, AII, and CII from the mixed anomaly*

Before proceeding, let us briefly summarize the previous subsection: by considering various anomalies related to gravity, we can predict the integer topological phases in the BdG symmetry classes D, DIII, C, and CI. (As mentioned above,

TABLE II. Topological insulators (superconductors) with an integer ($\mathbb{Z}$) classification, (a) in the complex symmetry classes, predicted from the chiral U(1) anomaly, and (b) in the real symmetry classes, predicted from the gravitational anomaly (red), the chiral anomaly in the presence of background gravity (magenta), the mixed anomaly under gauge and coordinate transformations (blue) and the chiral anomaly in the presence of both background gravity and U(1) gauge field (green).

Cartan\d	0	1	2	3	4	5	6	7	8	9	10	11	···
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	···
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	···
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	···
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	···
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	···
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	···
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	···
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	···
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	···
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	···

for the moment we do not consider topological phases with $\mathbb{Z}_2$ or $2\mathbb{Z}$ topological charges.) On the other hand, we have so far not covered the description of topological insulators in symmetry classes AI, BDI, AII, and CII in terms of anomalies.

So far, we have considered for the "real" symmetry classes only those anomalies that involve solely gravity. Since the (gapped) topological insulators in symmetry classes AI, BDI, AII, and CII, also possess a conserved U(1) charge,[54] we can couple those to both a U(1) gauge field as well as a gravitational background. Therefore, it is natural to consider an anomaly that occurs in the presence of both a background gauge and a background gravitational field.

As it turns out, even in the presence of both gauge and gravitational fields, the structure of the anomaly is similar to the one discussed so far: the noninvariance of the effective action under a gauge transformation or coordinate transformation can be expressed as

$$\delta_v \ln Z[\mathcal{A},e,\omega] = 2\pi i \int_{M_D} \Omega_D^{(1)}(v,\mathcal{A},\omega,\mathcal{F},\mathcal{R}), \qquad (57)$$

where $\Omega_D^{(1)}(v,\mathcal{A},\omega,\mathcal{F},\mathcal{R})$ can be derived from an associated anomaly polynomial, which reads[34,36]

$$\Omega_D(\mathcal{R},\mathcal{F}) = \left[\mathrm{ch}(\mathcal{F})\hat{A}(\mathcal{R})\right]|_D. \qquad (58)$$

As the right-hand side is given simply by the product of the anomaly polynomials for a gauge field [Eq. (44)] and gravity [Eq. (55)], by switching off either $\mathcal{R}$ or $\mathcal{F}$, we recover the results discussed in the previous subsections: for all even space-time dimensions $D = d+1 = 2k$ $(k = 1,2,\ldots)$ we obtain a nonvanishing anomaly polynomial $\Omega_D(\mathcal{R}=0,\mathcal{F}) = \Omega_D(\mathcal{F})$, which we have already used to predict topological insulators or superconductors in class A $(D = 2k+1)$ and AIII $(D = 2k)$. For space-time dimensions $D = d+1 = 4k$ $(k = 1,2,\ldots)$ we obtain a nonvanishing anomaly polynomial $\Omega_D(\mathcal{R},\mathcal{F}=0) = \Omega_D(\mathcal{R})$, which we have already used to predict topological insulators or superconductors in class DIII $(D = 4+8k)$ and CI $(D = 8+8k)$.

On the other hand, while the anomaly polynomial $\Omega_D(\mathcal{R},\mathcal{F}=0) = \Omega_D(\mathcal{R})$ vanishes in $D = 4k+2$ dimensions, the one obtained from Eq. (58), namely $\Omega_D(\mathcal{R},\mathcal{F})$, is nonvanishing in these dimensions.

As before, the anomaly polynomial itself is related to a "chiral anomaly in the presence of both gauge field and gravity" of the massive bulk system in $D = 4k+2$ space-time dimensions, $D_\mu J_5^\mu(x) = 2im\bar{\psi}\gamma_{D-1}\psi + 2i\mathcal{A}_D(x)$, where $\mathcal{A}_D(x)$ is given in terms of $\Omega_D(\mathcal{R},\mathcal{F})$. For this reason, one predicts an additional topological insulator (superconductor) in these space-time dimensions (besides the one of Sec. V A 2). Therefore, one predicts the occurrence of topological phases in spatial dimensions $d = 9$ $(d = 1)$ and $d = 5$,

$$\text{class BDI } [d = 9\ (d = 1)], \quad \text{class CII } (d = 5), \qquad (59)$$

as well as of all higher-dimensional topological phases that we can obtain from these by Bott periodicity.[56] (These are colored green in Table II.) Indeed, for classes BDI and CII, we can realize these symmetry classes as a normal (i.e., not superconducting) system, and hence they have a natural U(1) charge.[54] The effective topological field theory for the space-time-dependent linear [electrical and gravitational (thermal)] responses possesses a term of topological origin of the form $\int \Omega_D(\mathcal{R},\mathcal{F})$, where $D = 4k+2$.

Moreover, it turns out that a descent relation that is identical in form to Eq. (42) also holds for the "mixed" anomaly polynomial defined in Eq. (58). Therefore, the space-time integral of the Chern-Simons form $\Omega_{4k+1}^{(0)}$ of Ω_{4k+2}, which is obtained from Ω_{4k+2} by using the descent relation, $d\Omega_{4k+1}^{(0)} = \Omega_{4k+2}$, describes the term of topological origin in the effective action for the linear responses in $D = 4k+1$ space-time dimensions. This corresponds to a "mixed anomaly" $\Omega_{4k}^{(1)}$ in the corresponding boundary theory in $4k$ space-time dimensions. For this reason, one predicts the occurrence of additional topological insulators in spatial dimensionalities $d = 0$ and 4 (besides the ones in Sec. V A 1), for the two symmetry classes

$$\text{class AI } (d = 0), \quad \text{class AII } (d = 4), \qquad (60)$$

as well as for all their higher-dimensional equivalents obtained from the Bott periodicity (These are colored magenta in Table II.)

4. Atiyah-Singer index theorem

For all the symmetry classes with *chiral symmetry*, the Hamiltonian can be brought into block off-diagonal form.[8] Above, we have discussed all symmetry classes of this form that possess topological insulators with a $\mathbb{Z}$ classification (i.e., AIII in $D = 2n$, DIII in $D = 4+8k$, CI in $D = 8+8k$, CII in $D = 6+8k$, BDI in $D = 10+8k$). A Dirac Hamiltonian $\mathcal{H}$ with chiral symmetry possesses an index, and the Atiyah-Singer index theorem[34] relates the integral of the anomaly polynomial discussed above to this index through the formula

$$\mathrm{index}(\mathcal{H}) = \int_{M_D} \Omega_D(\mathcal{R},\mathcal{F}), \qquad (61)$$

where $\Omega_D(\mathcal{R},\mathcal{F})$ is the most general anomaly polynomial, as defined in Eq. (58) above. Here, the Dirac Hamiltonian $\mathcal{H}$ refers to the Hamiltonian in a gravitational background and a background (Abelian or non-Abelian) gauge field. The index $\mathrm{index}(\mathcal{H})$ is by definition an integer. We note that it is because of this theorem that the space-time integral of the anomaly polynomial represents a θ term for the theory of the space-time-dependent linear gauge and gravitational responses, and that the θ terms only occur for symmetry classes possessing a chiral symmetry.

5. Global gravitational anomalies

The discussion that we have presented so far for the connection between anomalies and topological insulators and superconductors in "the primary series" (those located in the diagonal of the Periodic Table and characterized by an integer topological invariant) can be extended to some of the "first and second descendants" (the topological insulators and superconductors in the same symmetry class, but in one and two dimensions less than the one with a $\mathbb{Z}$ invariant; these are each characterized by a $\mathbb{Z}_2$ invariant). We propose that for these we need to use so-called global anomalies, instead of the so-called perturbative anomalies that we have made use of in this section. Such anomalies do not affect infinitesimal, but rather large (of order 1) symmetry transformations.

It was found in Ref. 55 that global gravitational anomalies can exist, given certain assumptions are satisfied, (i) in $D = 8k$, (ii) in $D = 8k + 1$, and (iii) in $D = 4k + 2$ space-time dimensions. If so, then following the same reasoning as above, the presence of these anomalies would indicate the existence of a topological insulator in one dimension higher (of which the anomalous system is the boundary). This would then indicate the existence of topological insulators (superconductors) in space-time dimensions (i) $D = 8k + 1$, (ii) $D = 8k + 2$, and (iii) $D = 4k + 3$ [corresponding to spatial dimensions (i) $d = 8k$, (ii) $d = 8k + 1$, and (iii) $d = 4k + 2$]. Indeed, there exist $\mathbb{Z}_2$ topological insulators in these dimensions (Table II). More precisely, there exist *two* $\mathbb{Z}_2$ topological insulators in these dimensions, and at this point we have not yet explored in detail which of the two (or if both) could be related to this global gravitational anomaly. Moreover, we note that there also exist other (i.e., not gravitational) global anomalies, and we propose that the other, as yet not yet covered, $\mathbb{Z}_2$ topological insulators can be obtained from considering these other global anomalies.

We end by mentioning that the notions presented in this section (Sec. V) may also be further supported by the connection with the tenfold classification of D-branes:[46,47] In the D-brane realizations of topological insulators and superconductors, massive fermion spectra arise as open string excitations connecting two D-branes, which are in one-to-one correspondence with the Dirac representative of the tenfold classification of topological insulators and superconductors, and come quite naturally with gauge interactions. The Wess-Zumino term of the D-branes gives rise to a gauge field theory of topological nature, such as those with the Chern-Simons term or the θ term in various dimensions.

VI. CONCLUSIONS

There are various important future research directions in the field of topological insulators and superconductors. Let us mention two here. One is the search for experimental realizations of the topological singlet and triplet superconductors in three spatial dimensions, besides the B phase of the ^{3}He superfluid. Given how fast experimental realizations of the quantum spin Hall effect in two spatial dimensions and the $\mathbb{Z}_2$ topological insulators in three dimensions have been found, one may perhaps anticipate a similar development for these three-dimensional topological superconducting phases. Notably, $Cu_xBi_2Se_3$, which arises from the familiar three-dimensional topological insulators Bi_2Se_3, was found to be superconducting at 3.8 K.[57] Subsequent theoretical work proposed that this superconducting phase should be a topological superconductor.[58] The various linear responses discussed in this paper, as summarized in Table I, may become helpful in the search for, and identification of, such various topological phases.

Another important issue is to complete the study of the effect of interactions for the symmetry classes so far not yet included in the discussion given in Sec. V. (These include, in general dimensionalities, the topological insulators (superconductors) with a $2\mathbb{Z}$ classification, as well as the majority of those with a $\mathbb{Z}_2$ classification.) Moreover, this includes the case of symmetry class BDI in $d = 1$ spatial dimension (recall also Refs. 53 and 56), discussed in the work of Refs. 59 and 61. Further important outstanding questions concern possible topological phases (besides superconductors) which may arise from interactions rather than from band effects. How can one describe "fractional" versions of the topological insulators (superconductors),[23] and how can one classify bosonic systems such as, e.g., spin systems?[62] Clearly, to address any of these interaction-dominated issues, one cannot rely on a topological invariant defined in terms of single-particle Bloch wave functions. Rather, a definition of topological quantum states of matter in terms of responses to physical probes is necessary. In this paper, we have developed a description of this type for all topological insulators in three spatial dimensions, and for a significant part of the topological insulators in general dimensions. From a conceptual point of view, the gravitational responses are the most fundamental ones in that they apply to all topological insulators. Owing to Luttinger's derivation[32] of the thermal Kubo formula, these correspond physically to thermal response functions.

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$$g^{\mu\nu}, \quad g_{\rho\sigma}, \quad \text{with} \quad g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}. \tag{62}$$

The components of the Levi-Civita connection are

$$\Gamma^{\mu}{}_{\nu\rho} = \tfrac{1}{2} g^{\mu\alpha}(\partial_\nu g_{\rho\alpha} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}). \tag{63}$$

The vielbein $e_a{}^{\mu}$ and $e^a{}_{\mu}$ diagonalizes the metric, and is defined by

$$g_{\mu\nu} e_a{}^{\mu} e_b{}^{\nu} = \eta_{ab}, \quad \eta_{ab} e^a{}_{\mu} e^b{}_{\nu} = g_{\mu\nu}. \tag{64}$$

Here, η_{ab} is a flat (Minkowski) metric, and we use Greek indices $\mu,\nu,\ldots$ for coordinates of the manifold, and Roman indices $a,b,\ldots$ for the flat coordinates at some point x_0 of the manifold; they are raised and lowered by $g_{\mu\nu}, g^{\mu\nu}$ and η_{ab}, η^{ab}, respectively. Since the vielbein $e^a{}_{\mu}$ transforms as a covariant vector under general coordinate transformation, it is convenient to introduce a one-form

$$e^a = e^a{}_{\mu} dx^{\mu}. \tag{65}$$

The spin connection is

$$\omega_{\mu}{}^{a}{}_{b} = e^a{}_{\alpha}\Big[\partial_\mu e_b{}^{\alpha} + \Gamma^{\alpha}{}_{\mu\beta} e_b{}^{\beta}\Big]. \tag{66}$$

This can be written in terms of a covariant vector $e_b{}^{\mu}$, which is the bth eigenvector of the metric, by using the covariant derivative with respect to the Levi-Civita connection $\Gamma^{\mu}{}_{\nu\rho}$ as

$$\omega_{\mu}{}^{a}{}_{b} = e^a{}_{\alpha} \nabla_\mu e_b{}^{\alpha}. \tag{67}$$

We define the connection one-form by

$$\omega^a{}_b = \omega_{\mu}{}^{a}{}_{b} dx^{\mu}. \tag{68}$$

The curvature tensor is

$$\begin{aligned} R^{\mu}{}_{\nu\alpha\beta} &= \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\sigma\alpha}\Gamma^{\sigma}{}_{\nu\beta} - \Gamma^{\mu}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\nu\alpha}, \\ R_{\mu\alpha} &= g^{\nu\beta} R_{\mu\nu\alpha\beta}. \end{aligned} \tag{69}$$

The curvature tensor can also be constructed from the spin connection:

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R_{\mu\nu}{}^{c}{}_{b} dx^{\mu} dx^{\nu}, \tag{70}$$

where $R_{\mu\nu}{}^{c}{}_{b} = R_{\mu\nu}{}^{\rho}{}_{\lambda} e^c{}_{\rho} e_b{}^{\lambda}$.
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[51]The argument is essentially the same as that presented in Sec. III B. As explained below, the calculations performed in this subsection amount to a derivation of what we will call below a "chiral $U(1)$ gauge anomaly in the presence of a background gravitational field."

[52]The appearance of this term for the non-Abelian gauge group SU(2) was first pointed out in the context of topological insulators (superconductors) in Ref. 29 for the spin-singlet topological superconductor in symmetry class CI in $d = 3$ spatial dimensions.

[53]For class BDI there exist two distinct physical realizations, one as ("spinless" time-reversal invariant) superconductors and one as normal (nonsuperconducting) electronic systems. Without considering interactions, there is basically no difference between the two, except that the number of species of Majorana fermions is even in the latter case, where a pair of Majorana fermions is thought to be combined into a complex fermion, carrying a U(1) charge, or particle number. The discussion of anomalies, considered in the current section of this paper, is aimed at the discussion of interacting theories (as explained, e.g., in the Introduction). Now, when inclusion of interactions is considered, the two above-mentioned realizations of symmetry class BDI behave very differently. Obviously, in the latter (normal, nonsuperconducting) realization, the interactions are to respect the U(1) symmetry, whereas in the former (superconducting) realization, there is no such constraint on the form of the interactions. In this subsection, we will consider solely the latter realization. In this case, there is thus always a conserved U(1) quantity. The former (superconducting) case was discussed recently in Refs. 59–61. (For similar methods applied to a gapped spin chain, see, e.g., Ref. 62.) At present, we do not have an understanding of that case in terms of anomalies. We hope to be able to address this case in future work.

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