Anomalous Hall effect in superconductors with spin-orbit interaction

P. D. Sacramento,[1] M. A. N. Araújo,[1,2] V. R. Vieira,[1] V. K. Dugaev,[1,3,4] and J. Barnaś[5]
[1]*CFIF, Instituto Superior Técnico, TU Lisbon, Avenida Rovisco Pais, P-1049-001 Lisboa, Portugal*
[2]*Departamento de Física, Universidade de Évora, P-7000-671 Évora, Portugal*
[3]*Department of Physics, Rzeszów University of Technology, Aleja Powstańców Warszawy 6, PL-35-959 Rzeszów, Poland*
[4]*Institut für Physik, Martin-Luther-Universität Halle-Wittenberg, Heinrich-Damerow-Strasse 4, D-06120 Halle, Germany*
[5]*Department of Physics, Adam Mickiewicz University, Umultowska 85, PL-61-614 Poznań, Poland*
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We calculate the anomalous Hall conductance of superconductors with spin-orbit (SO) interaction and with either uniform or local magnetization. In the first case, we consider a uniform ferromagnetic ordering in a spin-triplet superconductor, whereas, in the second case, we consider a conventional s-wave spin-singlet superconductor with a magnetic impurity (or a diluted set of magnetic impurities). In the latter case, we show that the anomalous Hall conductance can be used to track the quantum-phase transition that occurs when the spin coupling between the impurity and the electronic spin density exceeds a certain critical value. In both cases, we find that, for large SO coupling, the superconductivity is destroyed and the Hall conductance oscillates strongly.

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I. INTRODUCTION

The anomalous Hall effect (AHE) has been observed in metallic ferromagnets as a Hall current generated by an electric field in the absence of an external magnetic field.[1,2] Since then, several physical mechanisms of the AHE have been proposed, related to the side jump and skew scattering from impurities,[3–5] inhomogeneous internal magnetization,[6,7] internal spin-orbit (SO) interaction,[8] and topology of electron-energy bands.[9,10] Recently, the theory of AHE has attracted much attention[3,11–13] because it reveals some very unusual properties of solids, such as the existence of monopoles in the momentum space or the generation of topological gauge fields.

One of the most intriguing models of the AHE is the one based on an intrinsic mechanism[9,10] related to the nontrivial topology of electron-energy bands. In the frame of this mechanism, the main contribution to the AHE is due to electron states well below the Fermi energy.[14] The simplest model in which this mechanism of the AHE can be realized is the model of a magnetized two-dimensional (2D) electron gas with Rashba SO interaction.[15] Unfortunately, it turns out that, if the system is in the metallic state, i.e., if there is no gap at the Fermi surface, then the contribution of electron states at the Fermi surface totally can compensate the other contributions so that the resulting off-diagonal conductivity is zero.[16] In the opposite case, when the chemical potential lies in the gap, the anomalous Hall conductivity σ_{xy} is nonzero and is quantized in units of e^2/h. The theory of quantized AHE is quite similar to the theory of integer quantum Hall effect, where the gap is due to the Landau quantization in a strong magnetic field.[17,18]

In this paper, we consider a 2D electron gas with nonzero magnetization and Rashba SO interaction. Such a model was used earlier for a description of an intrinsic AHE.[16] However, we calculate the AHE in the case when the electron system additionally is superconducting. The superconductivity produces a gap at the Fermi level, suppressing the contribution to the AHE from the Fermi surface. Thus, one can expect that only the filled electronic states below the gap contribute to the AHE. The possibility of the AHE in superconductors has already been considered in the case of ferromagnet-superconductor double-tunnel junctions[19] where side jump and/or skew scattering from impurities have been assumed as possible physical mechanisms responsible for the effect. This is, however, essentially different from our model where we consider the intrinsic mechanism of the AHE. Since, in a superconductor, the charge is not conserved due to the particle-hole mixture, we do not expect any quantization of the anomalous Hall conductance. This already was shown for the usual Hall conductance in conventional superconductors in very high-magnetic fields where the Landau-level description is appropriate.[20]

Various materials are known to show the coexistence of ferromagnetism and superconductivity[21–28] and, in particular, the presence of the SO interaction due to the lack of spatial inversion symmetry.[29–32] We note that, a long time ago, the possibility of magnetoelectric effects in noncentrosymmetric superconductors already was predicted[33] where it was shown that a supercurrent should induce a spin polarization, and, reversely, a Zeeman-like term should induce a supercurrent[34] as a result of strong SO interaction. Other effects due to the interplay of ferromagnetism and superconductivity also have been considered.[35–37] Recently, the interplay among superconductivity, magnetism, and SO interaction (or topological insulators[38,39]) has received additional attention due to the possibility of the Majorana edge states in a finite system or inside superconducting vortices[40–42] with its possible applications in topological quantum computation. Moreover, the coexistence of magnetism and superconductivity turned out to be interesting also from the point of view of possible applications in spintronics.[43,44]

In this paper, in Sec. II, we consider a spin-triplet superconductor, whereas, in Sec. III, we consider a conventional superconductor with a magnetic impurity.[45] In both cases, we analyze the influence of the Rashba SO interaction. In the first case, the magnetization is due to a ferromagnetic order, whereas, in the second case, the system is polarized locally by a magnetic impurity. The latter situation also may be achieved when considering a superconducting film with a magnetic dot juxtaposed. It has been shown before that, if the coupling between the magnetic impurity and the spin density of conduction electrons is strong enough, the system becomes magnetized through a first-order quantum-phase transition[46,47] that leads to discontinuities in various physical quantities.[48] In both cases, we calculate the anomalous Hall conductance. We show that the Hall conductance of a superconductor with a magnetic impurity can be used to reveal the quantum-phase transition. We consider the case of a system close to the equilibrium with the chemical potential within the gap. As we show, the energy spectrum in the state with both magnetic and triplet superconducting orderings is generally gapped. Therefore, possible isolated gapless points at the Fermi surface only can give negligibly small contributions related to the side jump or skew-scattering mechanisms from possible impurities. Finally, we conclude with Sec. IV.

II. AHE IN A TRIPLET SUPERCONDUCTOR

First, we consider a superconductor with a uniform magnetization. Since magnetism and superconductivity compete, a spin-singlet superconductor is not stable due to Cooper pair breaking. Therefore, we consider a spin-triplet superconductor where magnetism and superconductivity can coexist. The system is described by the tight-binding model in 2D, to which we add a superconducting-pairing term with the appropriate symmetry. Additionally, we include the Rashba SO term,[15] which generally is allowed in noncentrosymmetric materials. Due to the SO term, a spin-singlet component Δ_s generally is induced, and therefore, there is a pairing mixture in the system.[49]

We write the electron operators $\psi_{\vec{k},\sigma}$ in terms of the Bogoliubov operators $\gamma_{n,\vec{k}}$ as

$$\psi_{\vec{k},\sigma} = \sum_n (u_n(\vec{k},\sigma)\gamma_{n,\vec{k}} - \sigma v_n(\vec{k},\sigma)^* \gamma^\dagger_{n,-\vec{k}}), \tag{1}$$

where $\vec{k},n$ label the eigenstates of the system. The wave functions and energy eigenvalues satisfy the Bogoliubov–de Gennes equations,[50] which can be written as

$$\begin{pmatrix} \epsilon_{\vec{k}} - h_z & \alpha(\sin k_y + i\sin k_x) & -d_x + i d_y & d_z + \Delta_s \\ \alpha(\sin k_y - i\sin k_x) & \epsilon_{\vec{k}} + h_z & d_z - \Delta_s & d_x + i d_y \\ -d_x - i d_y & d_z - \Delta_s & -\epsilon_{\vec{k}} + h_z & \alpha(\sin k_y - i \sin k_x) \\ d_z + \Delta_s & d_x - i d_y & \alpha(\sin k_y + i\sin k_x) & -\epsilon_{\vec{k}} - h_z \end{pmatrix} \begin{pmatrix} u_n(\vec{k},\uparrow) \\ u_n(\vec{k},\downarrow) \\ v_n(-\vec{k},\uparrow) \\ v_n(-\vec{k},\downarrow) \end{pmatrix} = \epsilon_{\vec{k},n} \begin{pmatrix} u_n(\vec{k},\uparrow) \\ u_n(\vec{k},\downarrow) \\ v_n(-\vec{k},\uparrow) \\ v_n(-\vec{k},\downarrow) \end{pmatrix}. \tag{2}$$

Here, $\epsilon_{\vec{k}} = -2t(\cos k_x + \cos k_y) - \epsilon_F$ is the kinetic part, where t denotes the hopping parameter set in the following as the energy scale $t = 1$, ϵ_F is the chemical potential, chosen in the following as $\epsilon_F = -1$, $\vec{k}$ is a wave vector in the xy plane, and we have taken the lattice constant to be unity $a = 1$. Furthermore, h_z in Eq. (2) is the magnetization, in energy units, along the z direction, while the vector $\vec{d} = (d_x,d_y,d_z)$ is the vector representation of the superconducting pairing (p wave). Finally, the Rashba SO term is written as $H_R = \vec{s}\cdot\vec{\sigma} = \alpha(\sin k_y\sigma_x - \sin k_x\sigma_y)$, where α is measured in the energy units, and σ_x,σ_y are the Pauli matrices.

The pairing matrix can be written as[51]

$$\Delta = \begin{pmatrix} \Delta_{\uparrow,\uparrow} & \Delta_{\uparrow,\downarrow} \\ \Delta_{\downarrow,\uparrow} & \Delta_{\downarrow,\downarrow} \end{pmatrix} = \begin{pmatrix} -d_x + i d_y & d_z \\ d_z & d_x + i d_y \end{pmatrix}. \tag{3}$$

Thus, we can write $d_x = (\Delta_{\downarrow,\downarrow} - \Delta_{\uparrow,\uparrow})/2$, $d_y = -i(\Delta_{\downarrow,\downarrow} + \Delta_{\uparrow,\uparrow})/2$, and $d_z = \Delta_{\uparrow,\downarrow}$, whereas, the vector $\vec{q} = i\vec{d}\times\vec{d}^*$ is given by $q_x = \mathrm{Re}[(\Delta_{\downarrow,\downarrow} + \Delta_{\uparrow,\uparrow})\Delta^*_{\uparrow,\downarrow}]$, $q_y = \mathrm{Im}[(\Delta_{\downarrow,\downarrow} - \Delta_{\uparrow,\uparrow})\Delta^*_{\uparrow,\downarrow}]$, and $q_z = \frac{1}{2}[|\Delta_{\uparrow,\uparrow}|^2 - |\Delta_{\downarrow,\downarrow}|^2]$. When this vector vanishes, the pairing is called unitary. We have verified that considering the s-wave component generally has a very small effect on our results, and therefore, we assume $\Delta_s = 0$ in the following.

The energy eigenvalues of Eq. (2) can be written (for $\Delta_s = 0$) as

$$\epsilon_{\vec{k},\alpha_1,\alpha_2} = \alpha_1\sqrt{z_1 + \alpha_2 2\sqrt{z_2}}, \tag{4}$$

where

$$z_1 = \vec{d}\cdot\vec{d} + \vec{s}\cdot\vec{s} + \epsilon_{\vec{k}}^2 + h_z^2,$$
$$z_2 = (\vec{d}\cdot\vec{s})^2 + (\epsilon_{\vec{k}}^2 + d_z^2)(\vec{s}\cdot\vec{s} + h_z^2), \tag{5}$$

and $\alpha_1,\alpha_2 = \pm$.

In the normal phase ($\vec{d} = 0$), the SO coupling lifts the spin degeneracy of the energy bands in the tight-binding model, except at $\vec{k} = (0,0)$, (π,π), and $(0,\pi)$ (and equivalent points). These remaining degeneracies are lifted when including the magnetization. This is shown in Fig. 1 where the two energy bands are shown as a function of momentum for $\lambda_{so} = \alpha/2 = 2$ and various values of h_z. As can be seen from Eq. (4), the lowest band is gapless at the points where

$$(\vec{s}\cdot\vec{s} + h_z^2) + \epsilon_{\vec{k}}^2 = 2\sqrt{(\vec{s}\cdot\vec{s} + h_z^2)\epsilon_{\vec{k}}^2}. \tag{6}$$

In a general case ($\vec{d} \neq 0$), the lowest band has gapless points that are solutions of the equation $z_1 = 2\sqrt{z_2}$, which yields

$$\vec{d}\cdot\vec{d} + \vec{s}\cdot\vec{s} + \epsilon_{\vec{k}}^2 + h_z^2$$
$$= 2\sqrt{(\vec{d}\cdot\vec{s})^2 + (\epsilon_{\vec{k}}^2 + d_z^2)(\vec{s}\cdot\vec{s} + h_z^2)}. \tag{7}$$

Thus, in the superconducting phase, the system generally is gapped. In particular, without the SO interaction, the gapless

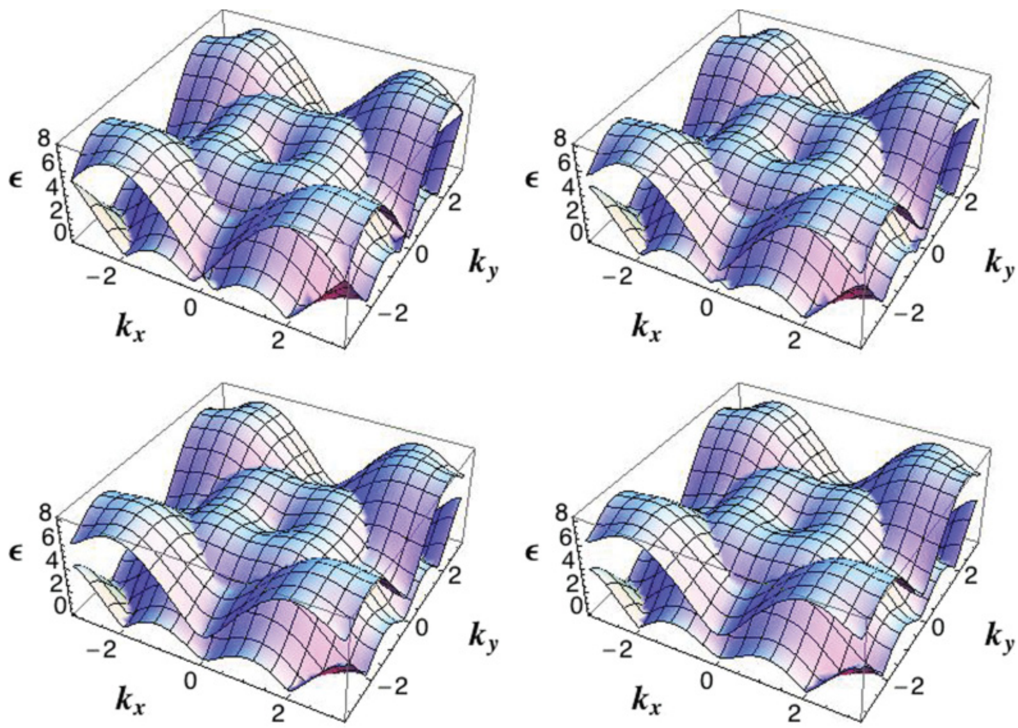

FIG. 1. (Color online) Energy bands in units of the hopping t as a function of momenta k_x,k_y in the normal phase for $\lambda_{\mathrm{so}} = 2$ and various values of the magnetization: From left to right, $h_z = 0$ and $h_z = 0.5$ (top); $h_z = 1$ and $h_z = 1.2$ (bottom).

points are obtained by $\vec{d} \cdot \vec{d} + \epsilon_{\vec{k}}^2 = 0$, which implies particular values for the chemical potential.

The charge current along a link in the lattice can be obtained by adding a vector potential to the kinetic and SO terms and taking a functional derivative of the Hamiltonian with respect to the vector potential,[50,52] or through its definition in the charge continuity equation.[53] The zero-momentum charge current in the $\mu = x,y$ direction can be written as

$$j_\mu = \sum_{\vec{k}} \bar{\psi}_{\vec{k}}^\dagger V_{\vec{k}}^\mu \bar{\psi}_{\vec{k}}, \tag{8}$$

where $\bar{\psi}_{\vec{k}} = (\psi_{\vec{k},\uparrow}\ \psi_{\vec{k},\downarrow})^T$ and

$$\begin{aligned} V^x &= \frac{2e}{\hbar}\left(-t\eta_{\vec{k},-}^x I + \lambda_{\mathrm{so}}\eta_{\vec{k},+}^x \sigma_y\right), \\ V^y &= \frac{2e}{\hbar}\left(-t\eta_{\vec{k},-}^y I - \lambda_{\mathrm{so}}\eta_{\vec{k},+}^y \sigma_x\right) \end{aligned} \tag{9}$$

is a velocity matrix operator.[54] Here, $\eta_{\vec{k},+}^\mu = \cos(\vec{k} \cdot \vec{\delta}_\mu)$ and $\eta_{\vec{k},-}^\mu = \sin(\vec{k} \cdot \vec{\delta}_\mu)$, where $\vec{\delta}_\mu$ is a vector displacement (in units of the lattice constant) between nearest neighbors along the μ direction. In turn, I is the 2×2 unit matrix.

The Hall conductance now can be calculated using a Kubo-like formula,[55] which in the limit of uniform and stationary currents $\vec{q} \to 0$ and $\omega \to 0$, is given by

$$\begin{aligned} \mathrm{Re}(\sigma_{xy}) = -i\frac{\hbar}{N}\sum_{\vec{k}}\sum_{\alpha,\beta}\sum_{\gamma,\delta}\sum_{n,m} \frac{f_{n,\vec{k}} - f_{m,\vec{k}}}{(\epsilon_{n,\vec{k}} - \epsilon_{m,\vec{k}} + i0^+)^2}\big[&V_{\vec{k};\alpha,\beta}^x V_{\vec{k};\gamma,\delta}^y u_n^*(\vec{k},\alpha)u_n(\vec{k},\delta)u_m(\vec{k},\beta)u_m^*(\vec{k},\gamma) \\ &-V_{\vec{k};\alpha,\beta}^x V_{-\vec{k};\gamma,\delta}^y \gamma\delta u_n^*(\vec{k},\alpha)v_n(-\vec{k},\gamma)u_m(\vec{k},\beta)v_m^*(-\vec{k},\delta)\big]. \end{aligned} \tag{10}$$

where N is the number of sites and $f_{n,\vec{k}}$ is the Fermi function for the state described by n and $\vec{k}$. In the normal phase, the wave functions u and v are decoupled. The presence of superconducting pairing mixes the particle and hole characters and, as already mentioned above, the charge no longer is a good quantum number. Then, the results for the Hall conductance depend on the choice of the pairing matrix.[51,56]

Let us now assume that the pairing amplitude is a free parameter. This describes the situations where superconductivity is induced by proximity, and therefore, no self-consistent solution is implied. This also applies to a situation where σ_{xy} is measured on a normal sample in which superconductivity pairing exists due to the proximity effect in the presence of a nearby triplet superconductor. We consider both unitary and nonunitary cases. Then, we consider the case where the pairing amplitude is determined by solving the Bogoliubov–de Gennes equations self-consistently. In the latter case, we consider a nonunitary situation, for which the

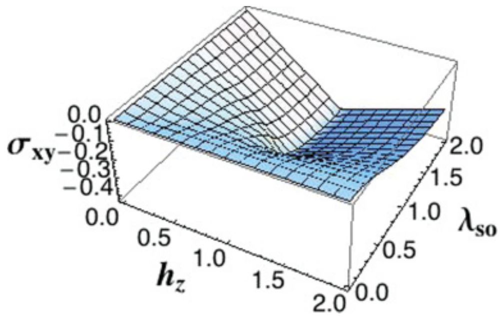

FIG. 2. (Color online) Anomalous Hall conductance in units of e^2/h in the normal phase as a function of h_z and λ_{so}.

amplitudes $\Delta_{\uparrow,\uparrow}$ and $\Delta_{\downarrow,\downarrow}$ are real, to simplify. This, in turn, implies that d_y is imaginary. In all cases, we take $\Delta_{\uparrow,\downarrow} = 0$ ($d_z = 0$), which means that only the q_z component may be nonvanishing.

In Fig. 2, the anomalous Hall conductance in the normal phase (zero-pairing amplitude) is plotted as a function of the magnetization h_z and SO coupling λ_{so}. The Hall conductance vanishes if either the magnetization or the SO coupling vanishes. Then, the absolute value of the Hall conductance increases as either parameter increases. Dependence on h_z is more complex, as the conductance reaches a minimum around $h_z = 1 \sim -\epsilon_F$, which shifts if we change the chemical potential. The minimum in the Hall conductance as a function of the magnetization h_z (keeping the SO constant, for instance, $\lambda_{so} = 2$) is associated with the gaplessness of the spectrum at the point $(0,\pi)$ and the equivalent points (see also Fig. 1).

Now, we consider the superconducting phase. Since the SO coupling renders the type of pairing undefined (with the mixture of spin-triplet and spin-singlet pairings), the strength of the triplet pairing is expected to be weakened in comparison to the same superconductor with a vanishing SO coupling. However, it was shown before[57] that the amplitude of the triplet pairing was not affected by the SO term when vector $\vec{d}$ was parallel to the SO vector $\vec{s}$. We have found that this pairing choice leads to results for the anomalous Hall conductance, which are very similar to those for the Hall conductance in the normal phase. This indicates that, for this particular case, the superconducting order does not change the Hall conductance significantly, and therefore, we do not show the corresponding results. We also have considered other choices of pairing, for which vector $\vec{d}$ is not parallel to the SO vector $\vec{s}$. We have considered both unitary and nonunitary cases. It is already known for a unitary case[57] that, even though the amplitude of the triplet coupling is somewhat weakened with respect to the case of vanishing SO term, it is still finite.

In Fig. 3, we show the anomalous Hall conductance in the superconducting phase for the two choices of the triplet pairing. We consider a unitary choice given by

$$\begin{aligned} &\Delta_{\uparrow,\uparrow} = d(-\sin k_y + i \sin k_x), \Delta_{\uparrow,\downarrow} = \Delta_{\downarrow,\uparrow} = 0, \\ &\Delta_{\downarrow,\downarrow} = d(\sin k_y + i \sin k_x) \\ &\quad q_x = 0, \quad q_y = 0, \quad q_z = 0. \end{aligned} \tag{11}$$

and a nonunitary choice given by

$$\begin{aligned} &\Delta_{\uparrow,\uparrow} = d \, \sin k_x, \quad \Delta_{\uparrow,\downarrow} = \Delta_{\downarrow,\uparrow} = 0, \quad \Delta_{\downarrow,\downarrow} = 0, \\ &q_x = 0, \quad q_y = 0, \quad q_z = \frac{d^2}{2} \sin^2 k_x. \end{aligned} \tag{12}$$

In the case of unitary coupling (top panels of Fig. 3), $\sigma_{xy} = 0$ if either $\lambda_{so} = 0$ or $h_z = 0$. However, in the case of a nonunitary coupling (bottom panels of Fig. 3), $\sigma_{xy} = 0$ if $\lambda_{so} = 0$, but for a nonzero SO coupling, there is a finite Hall conductance even if $h_z = 0$. In this nonunitary case, there is a magnetization induced by the pairing, which leads to a finite σ_{xy} in a similar way as in ^{3}He.

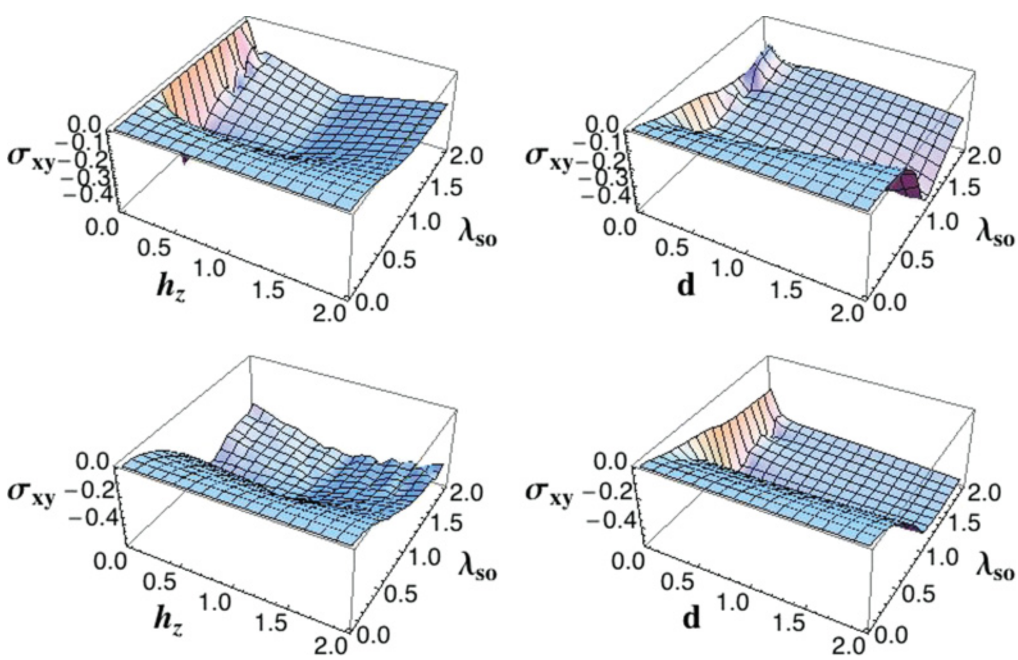

FIG. 3. (Color online) Anomalous Hall conductance for a spin-triplet superconductor. Left panels present a Hall conductance as a function of h_z and λ_{so} for $d = 1$, whereas, right panels present it as a function of d and λ_{so} for $h_z = 0.5$. Top figures correspond to the unitary case, see Eq. (11), while bottom figures correspond to the nonunitary case, Eq. (12).

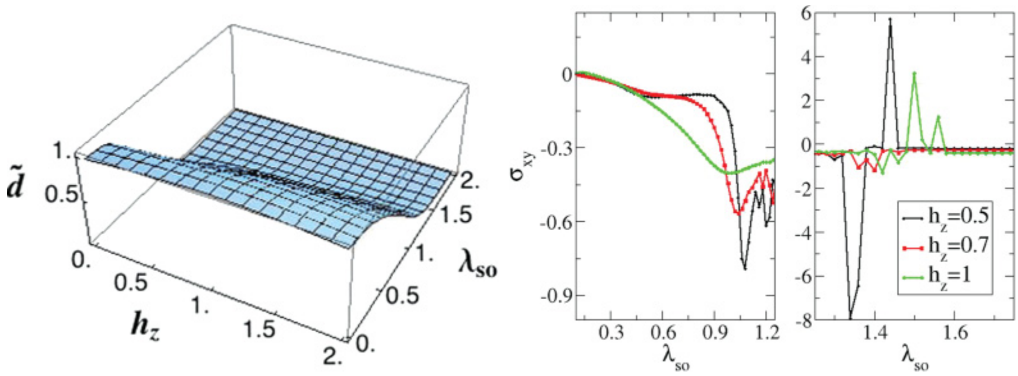

FIG. 4. (Color online) Spin-triplet superconductor calculated self-consistently. Left, $\tilde{d}$ as a function of h_z and λ_{so}. Right, Hall conductance along cuts of constant magnetization as a function of the SO coupling.

In the unitary case, the energy spectrum has a gap at the Fermi energy. This gap decreases as λ_{so} increases. As λ_{so} grows, the gap between the first and the second bands seems to decrease slightly, and then it increases. In general, one can expect that small gaps between the bands will lead to large contributions for the Hall conductance. In the nonunitary case, the energy spectrum also has a gap at the Fermi surface, which is small for small λ_{s0}, increases for slightly larger SO coupling, but vanishes when λ_{so} exceeds $\lambda_{so} \sim 0.7$. As λ_{so} grows further, the gap between the first and the second bands increases.

In the case in which the superconductivity is intrinsic to the material, we have to solve the Bogoliubov–de Gennes equations self-consistently. We look for a situation of the type,

$$\begin{aligned} &\Delta_{\uparrow,\uparrow} = \tilde{d}(-\sin k_x + \sin k_y), \quad \Delta_{\uparrow,\downarrow} = \Delta_{\downarrow,\uparrow} = 0, \\ &\Delta_{\downarrow,\downarrow} = \tilde{d}(\sin k_x + \sin k_y), \quad q_x = 0, \\ &q_y = 0, \quad q_z = \frac{\tilde{d}^2}{2}(-4 \sin k_x \sin k_y), \end{aligned} \tag{13}$$

where amplitude $\tilde{d}$ is determined self-consistently for a given magnetization, taking into account that

$$\tilde{d} = \frac{g}{N}\sum_{\vec{k}}(-\sin k_x + \sin k_y)\langle\psi_{\vec{k}\uparrow}\psi_{-\vec{k}\uparrow}\rangle, \tag{14}$$

where g is the pairing interaction. The corresponding numerical results are shown in Fig. 4. As the left panel shows, the superconductivity is destroyed for a large enough SO coupling. In the right panel, we see that the Hall conductance (as a function of λ_{so}) decreases with increasing λ_{so}, and as the transition to the normal phase appears, there are oscillations of the Hall conductance with relatively large amplitudes.

III. AHE IN A CONVENTIONAL SUPERCONDUCTOR WITH MAGNETIC IMPURITY

Now, consider a classical spin immersed in a 2D s-wave conventional superconductor. We now use a description of the system in real space. In the center of the system $\vec{r} = \vec{l}_c = (x_c, y_c)$, we place a classical spin along the z direction. The kinetic-energy part is described by a tight-binding model with hopping amplitude t, similar to the case of triplet superconductivity. The superconductor pairing is taken as s wave, and the SO interaction[58] is assumed as in the preceding section. The electron operator is written in terms of the Bogoliubov operators,

$$\psi(\vec{r},\sigma) = \sum_n [u_n(\vec{r},\sigma)\gamma_n - \sigma v_n(\vec{r},\sigma)^*\gamma_n^\dagger]. \tag{15}$$

The zero-momentum charge current in the $\mu = x,y$ direction can be written as $j_\mu = \sum_{\vec{r}} \bar{\psi}_{\vec{r}}^\dagger V^\mu \bar{\psi}_{\vec{r}}$, where $\bar{\psi}_{\vec{r}} = (\psi_{\vec{r},\uparrow}\ \psi_{\vec{r},\downarrow})^T$ and the velocity-matrix operators are given by

$$\begin{aligned} V^x &= \frac{e}{\hbar}(it\eta_-^x I + \lambda_{so}\eta_+^x\sigma_y), \\ V^y &= \frac{e}{\hbar}(it\eta_-^y I - \lambda_{so}\eta_+^y\sigma_x). \end{aligned} \tag{16}$$

Here, $f(\vec{r})\eta_+^\mu g(\vec{r}) = f(\vec{r}+\vec{\delta}_\mu)g(\vec{r}) + f(\vec{r})g(\vec{r}+\vec{\delta}_\mu)$ and $f(\vec{r})\eta_-^\mu g(\vec{r}) = f(\vec{r}+\vec{\delta}_\mu)g(\vec{r}) - f(\vec{r})g(\vec{r}+\vec{\delta}_\mu)$, where $\vec{\delta}_\mu$ is a displacement between nearest neighbors along the μ direction, while σ_x,σ_y are Pauli matrices, as above.

The real-space wave functions obey the Bogoliubov–de Gennes equations for the energy excitations ϵ_n,

$$\begin{pmatrix} -h-\epsilon_F - J\delta_{\vec{r},\vec{l}_c} & \Delta_{\vec{r}} & \lambda_{so}(-\eta_x + i\eta_y) & 0 \\ \Delta_{\vec{r}}^* & h+\epsilon_F - J\delta_{\vec{r},\vec{l}_c} & 0 & \lambda_{so}(\eta_x - i\eta_y) \\ \lambda_{so}(\eta_x + i\eta_y) & 0 & -h-\epsilon_F + J\delta_{\vec{r},\vec{l}_c} & \Delta_{\vec{r}} \\ 0 & \lambda_{so}(-\eta_x - i\eta_y) & \Delta_{\vec{r}}^* & h+\epsilon_F + J\delta_{\vec{r},\vec{l}_c} \end{pmatrix} \begin{pmatrix} u_n(\vec{r},\uparrow) \\ v_n(\vec{r},\downarrow) \\ u_n(\vec{r},\downarrow) \\ v_n(\vec{r},\uparrow) \end{pmatrix} = \epsilon_n \begin{pmatrix} u_n(\vec{r},\uparrow) \\ v_n(\vec{r},\downarrow) \\ u_n(\vec{r},\downarrow) \\ v_n(\vec{r},\uparrow) \end{pmatrix}, \tag{17}$$

where $h = t\hat{s}_{\vec{\delta}}$ with $\hat{s}_{\vec{\delta}} f(\vec{r}) = f(\vec{r}+\vec{\delta})$. Furthermore, $\eta_x = \pm 1$ if the neighbor along x is $i_x + 1$ $(i_x - 1)$ and $\eta_y = \pm 1$ if the neighbor along y is $i_y + 1$ $(i_y - 1)$. Parameter J describes the coupling between the impurity spin and the spin density of the conduction electrons. Note that the solution to this problem requires diagonalization of a $4N \times 4N$ matrix, where N is the number of lattice sites. This is in contrast to the problem of the triplet superconductor described in the previous section where a partial diagonalization was possible due to the translational invariance. Owing to this symmetry, the problem could be reduced to a simple diagonalization of a 4×4 matrix for each momentum value. Since the effect of the magnetic impurity is rather local, a system of 15×15 lattice sites is sufficient to have small finite-size effects as we have shown previously.[47] We solve the problem self-consistently, as in a previous paper (see Ref. 47 for details).

As in the case of triplet superconductivity studied in Sec. II, the Hall conductance can be obtained from a Kubo-like formula, which now reads

$$\mathrm{Re}(\sigma_{xy}) = i\frac{\hbar}{V}\sum_{\vec{r}_1,\vec{r}_2}\sum_{\alpha,\beta}\sum_{\gamma,\delta}\sum_{n,m}\frac{f_n - f_m}{(\epsilon_n - \epsilon_m + i0^+)^2}\big[V^x_{\vec{r}_1;\alpha,\beta}\bar{V}^y_{\vec{r}_2;\gamma,\delta}u_n(\vec{r}_1,\alpha)^* u_n(\vec{r}_2,\delta)u_m(\vec{r}_1,\beta)u_m(\vec{r}_2,\gamma)^* - V^x_{\vec{r}_1;\alpha,\beta}V^y_{\vec{r}_2;\gamma,\delta}\gamma\,\delta u_n(\vec{r}_1,\alpha)^* v_n(\vec{r}_2,\gamma)u_m(\vec{r}_1,\beta)v_m(\vec{r}_2,\delta)^*\big]. \quad (18)$$

In this expression, $\bar{V}$ is the complex conjugate, and $V^{\mu}_{\vec{r}_i}$ means that the operator acts on the coordinate $\vec{r}_i$.

The corresponding numerical results are presented in Fig. 5 where we show the total magnetization, order parameter at the impurity site, and the anomalous Hall conductance as a function of J and λ_{so} for a system of 15×15 lattice sites. The case with no SO coupling ($\lambda_{so} = 0$) was studied before.[45] Note, that the SO interaction shifts the critical value J_c, at which the quantum-phase transition to a magnetic state occurs for higher values. However, if λ_{so} is large enough, the transition is washed out. Note also that the various quantities reveal the quantum-phase transition when fixing λ_{so} and plotting them as a function of J. At the transition point, the impurity captures one electron and breaks a Cooper pair. Note that there still is a transition when we introduce the SO coupling, but one needs larger coupling parameter J as the SO increases. If we increase the SO coupling further, superconductivity is destroyed, and the Hall conductance exhibits strong oscillations as in the case of the triplet superconductor.

In order to emphasize the connection between the behavior of the Hall conductance and the quantum-phase transition, in Fig. 6 (for different SO couplings), we show the Hall conductance, amplitude of the order parameter at the impurity location, and the total magnetization of the conduction electrons as a function of the coupling between the spin density of the conduction electrons and the impurity spin. At the quantum-phase transition, both the amplitude of the order parameter and the total magnetization have discontinuities. At this critical coupling, the Hall conductance has a sharp minimum, which therefore, signals the phase transition.

IV. SUMMARY

We have analyzed the AHE in superconductors, considering only the intrinsic mechanism that results from the interplay of the Rashba SO interaction and magnetization. In the normal phase, the effect appears when both the SO term and the magnetization are nonzero. In a conventional spin-singlet superconductor of s-wave symmetry, an extended magnetization destroys the superconductivity. As we have shown, to have a nonvanishing anomalous Hall conductance in the superconducting phase, it then is sufficient to assume a single magnetic impurity in the presence of SO interaction. However, vanishing coupling between the conduction electrons and the

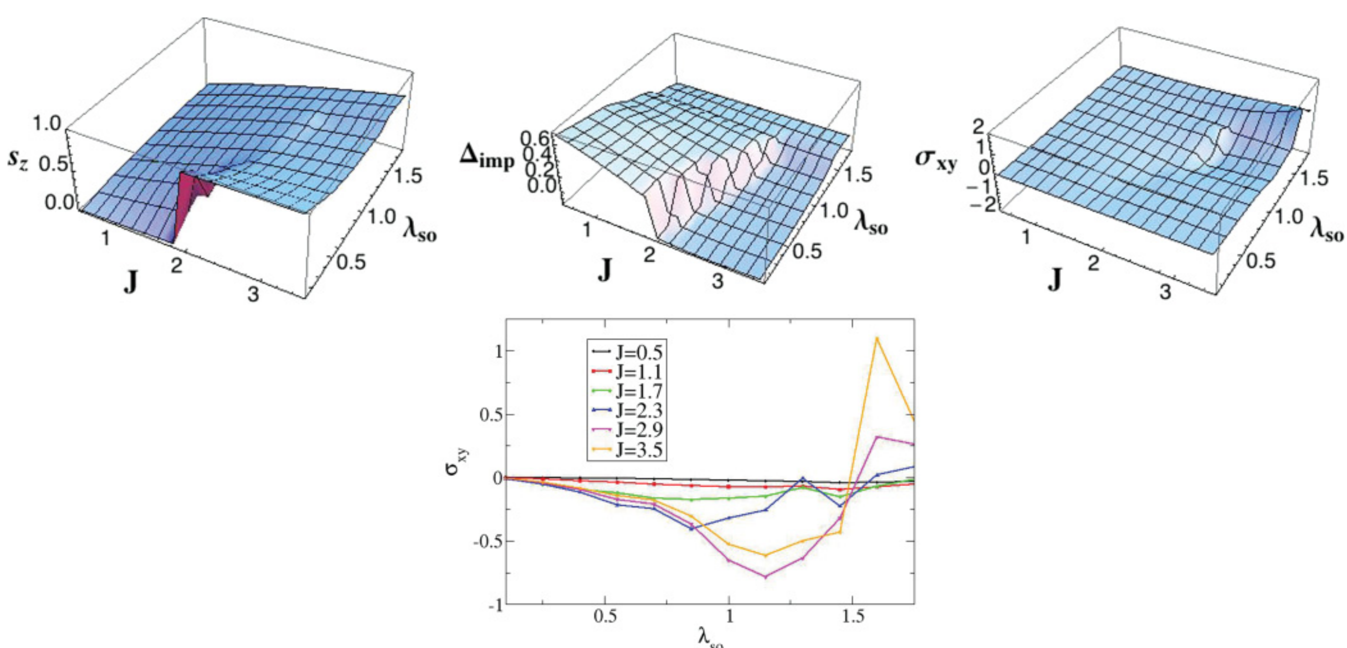

FIG. 5. (Color online) In the top panels from left to right, total magnetization, order parameter at the impurity site, and anomalous Hall conductance for the conventional superconductor with a magnetic impurity, calculated for a finite system including 15×15 lattice points as a function of J and λ_{so}. In the lower panel, we show some cuts of the Hall conductance for J fixed and varying SO coupling.

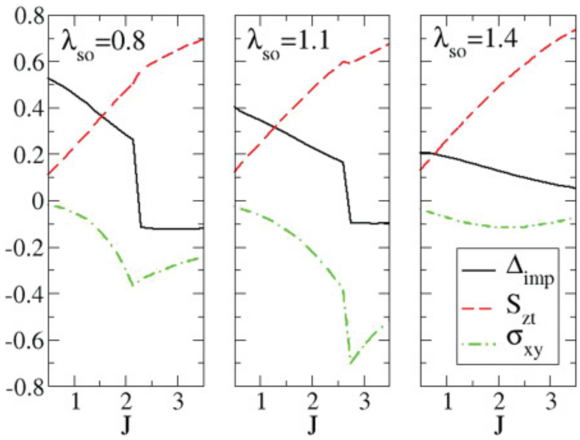

FIG. 6. (Color online) Order parameter at the impurity site, total magnetization, and anomalous Hall conductance for the conventional superconductor as a function of coupling strength for the impurity spin, calculated for the SO coupling corresponding to $\lambda_{so} = 0.8$, $\lambda_{so} = 1.1$, and $\lambda_{so} = 1.4$ as indicated. The first two values cross the quantum-phase transition, and for the highest value, the transition turns into a crossover.

magnetic impurity or vanishing SO coupling, lead to zero Hall conductance.

The case of a spin-triplet superconductor is qualitatively different. An extended magnetization does not destroy the superconducting order. The magnetization, generally, either can be induced by an adjacent ferromagnet owing to the proximity effect (we also may consider the superconducting order as a proximity effect in heterostructures where some metal is coupled to a magnet and a superconductor), or it may be an intrinsic property of the material (described by a self-consistent solution for the pairing amplitude). In the first case, two pairing forms lead to different results. If the pairing is unitary, the results are similar to those for the normal phase, and both magnetization and SO coupling are required for a finite-Hall conductance. The superconducting case also is very similar to the normal phase when $\vec{d}$ is parallel to $\vec{s}$. In the nonunitary case, however, there is a polarization associated with the pairing amplitude, and the Hall conductance is finite as long as the SO coupling is finite (a nonunitary pairing leads to a finite magnetization as in the case of ^{3}He).

Since the SO interaction generates spin flips, its moderate values destroy superconductivity in both the conventional and the triplet superconductors. In the case of s-wave superconductors, critical values of the spin coupling J in the presence of SO coupling are larger than those for zero SO coupling, thus, shifting the point at which the quantum-phase transition appears. In the case of spin-triplet superconductors with the pairing amplitude determined self-consistently, the SO coupling leads to suppression of the superconductivity through a continuous-phase transition. Finally, we have shown that the Hall conductance tracks the quantum-phase transition induced by magnetic impurities in conventional superconductors. This provides transport measurement as a possible tool to detect the transition, related to earlier predictions that transport properties are affected by the presence of magnetic impurities in a superconductor.[59] We note that one of the interests of the AHE is that it can easily be measured.

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[1] E. H. Hall, Philos. Mag. **10**, 301 (1880); **12**, 157 (1881).
[2] *The Hall Effect and Its Applications*, edited by C. L. Chien and C. R. Westgate (Plenum, New York, 1979).
[3] A. Crépieux and P. Bruno, Phys. Rev. B **64**, 014416 (2001).
[4] L. Berger, Phys. Rev. B **2**, 4559 (1970); **5**, 1862 (1972).
[5] J. Smit, Physica (Amsterdam) **24**, 39 (1958).
[6] Y. Taguchi and Y. Tokura, Europhys. Lett. **54**, 401 (2001); Y. Taguchi, Y. Oohara, H. Yoshizawa, N. Nagaosa, and Y. Tokura, Science **291**, 2573 (2001); S. Onoda and N. Nagaosa, Phys. Rev. Lett. **90**, 196602 (2003); Y. Taguchi, T. Sasaki, S. Awaji, Y. Iwasa, T. Tayama, T. Sakakibara, S. Iguchi, T. Ito, and Y. Tokura, *ibid.* **90**, 257202 (2003).
[7] P. Bruno, V. K. Dugaev, and M. Taillefumier, Phys. Rev. Lett. **93**, 096806 (2004).
[8] R. Karplus and J. M. Luttinger, Phys. Rev. **95**, 1154 (1954); J. M. Luttinger, *ibid.* **112**, 739 (1958).
[9] T. Jungwirth, Q. Niu, and A. H. MacDonald, Phys. Rev. Lett. **88**, 207208 (2002); D. Culcer, A. H. MacDonald, and Q. Niu, Phys. Rev. B **68**, 045327 (2003); D. Culcer, J. Sinova, N. A. Sinitsyn, T. Jungwirth, A. H. MacDonald, and Q. Niu, Phys. Rev. Lett. **93**, 046602 (2004).
[10] M. Onoda and N. Nagaosa, J. Phys. Soc. Jpn. **71**, 19 (2002); Phys. Rev. Lett. **90**, 206601 (2003); Z. Fang, N. Nagaosa, K. S. Takahashi, A. Asamitsu, R. Mathieu, T. Ogasawara, H. Yamada, M. Kawasaki, Y. Tokura, and K. Terakura, Science **302**, 92 (2003).
[11] N. Nagaosa, J. Sinova, S. Onoda, A. H. McDonald, and N. P. Ong, Rev. Mod. Phys. **82**, 1539 (2010).
[12] J. Sinova, T. Jungwirth, and J. Cerne, Int. J. Mod. Phys. B **18**, 1083 (2004).
[13] F. D. M. Haldane, Phys. Rev. Lett. **93**, 206602 (2004).
[14] P. Streda, J. Phys. C **15**, L717 (1982).
[15] Y. A. Bychkov and E. I. Rashba, J. Phys. C **17**, 6093 (1984).
[16] T. S. Nunner, N. A. Sinitsyn, M. F. Borunda, V. K. Dugaev, A. A. Kovalev, Ar. Abanov, C. Timm, T. Jungwirth, J.-i. Inoue, A. H. MacDonald, and J. Sinova, Phys. Rev. B **76**, 235312 (2007).
[17] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[18] Q. Niu, D. J. Thouless, and Y. S. Wu, Phys. Rev. B **31**, 3372 (1985).
[19] S. Takahashi and S. Maekawa, Phys. Rev. Lett. **88**, 116601 (2002).
[20] P. D. Sacramento, J. Phys.: Condens. Matter **11**, 4861 (1999).
[21] W. A. Fertig, D. C. Johnston, L. E. DeLong, R. W. McCallum, M. B. Maple, and B. T. Matthias, Phys. Rev. Lett. **38**, 987 (1977).
[22] M. Ishikawa and O. Fischer, Solid State Commun. **23**, 37 (1977).

[23] I. Felner, U. Asaf, Y. Levi, and O. Millo, Phys. Rev. B **55**, R3374 (1997).
[24] C. Bernhard, J. L. Tallon, C. Niedermayer, T. Blasius, A. Golnik, E. Brucher, R. K. Kremer, D. R. Noakes, C. E. Stronach, and E. J. Ansaldo, Phys. Rev. B **59**, 14099 (1999).
[25] S. S. Saxena *et al.*, Nature (London) **406**, 587 (2000).
[26] D. Aoki *et al.*, Nature (London) **413**, 613 (2000).
[27] N. T. Huy, A. Gasparini, D. E. de Nijs, Y. Huang, J. C. P. Klaasse, T. Gortenmulder, A. de Visser, A. Hamann, T. Görlach, and H. v. Löhneysen, Phys. Rev. Lett. **99**, 067006 (2007).
[28] G. Cao, S. Xu, Z. Ren, S. Jiang, C. Feng, and Z. Xu, e-print arXiv:1105.3255.
[29] E. Bauer, G. Hilscher, H. Michor, Ch. Paul, E. W. Scheidt, A. Gribanov, Yu. Seropegin, H. Noël, M. Sigrist, and P. Rogl, Phys. Rev. Lett. **92**, 027003 (2004).
[30] K. V. Samokhin, E. S. Zijlstra, and S. K. Bose, Phys. Rev. B **69**, 094514 (2004).
[31] T. Akazawa, H. Hidaka, H. Kotegawa, T. Kobayashi, S. Fukushima, E. Yamamoto, Y. Haga, H. Settai, and Y. Onuki, Physica B **378–380**, 355 (2006).
[32] N. Kimura, Y. Muro, and H. Aoki, J. Phys. Soc. Jpn. **76**, 051010 (2007).
[33] V. M. Edelstein, Phys. Rev. Lett. **75**, 2004 (1995).
[34] S. K. Yip, Phys. Rev. B **65**, 144508 (2002).
[35] V. M. Edelstein, Phys. Rev. B **67**, 020505(R) (2003).
[36] K. V. Samokhin, Phys. Rev. B **70**, 104521 (2004).
[37] S. Fujimoto, Phys. Rev. B **72**, 024515 (2005).
[38] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[39] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057(2011).
[40] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
[41] J. D. Sau, R. M. Lutchyn, S. Tewari, and S. Das Sarma, Phys. Rev. Lett. **104**, 040502 (2010).
[42] M. Sato and S. Fujimoto, Phys. Rev. B **79**, 094504 (2009).
[43] S. Takahashi, S. Hikino, M. Mori, J. Martinek, and S. Maekawa, Phys. Rev. Lett. **99**, 057003 (2007).
[44] T. Yokoyama, Phys. Rev. B **84**, 132504 (2011).
[45] A. Sakurai, Prog. Theor. Phys. **44**, 1471 (1970); A. V. Balatsky, I. Vekhter, and J.-X. Zhu, Rev. Mod. Phys. **78**, 373 (2006).
[46] M. I. Salkola, A. V. Balatsky, and J. R. Schrieffer, Phys. Rev. B **55**, 12648 (1997); D. K. Morr and N. A. Stavropoulos, *ibid.* **67**, 020502(R) (2003).
[47] P. D. Sacramento, V. K. Dugaev, and V. R. Vieira, Phys. Rev. B **76**, 014512 (2007).
[48] P. D. Sacramento, P. Nogueira, V. R. Vieira, and V. K. Dugaev, Phys. Rev. B **76**, 184517 (2007); N. Paunković, P. D. Sacramento, P. Nogueira, V. R. Vieira, and V. K. Dugaev, Phys. Rev. A **77**, 052302 (2008).
[49] L. P. Gor'kov and E. I. Rashba, Phys. Rev. Lett. **87**, 037004 (2001).
[50] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Addison-Wesley, Reading, MA, 1989).
[51] M. Sigrist and K. Ueda, Rev. Mod. Phys. **63**, 239 (1991).
[52] A. C. Durst and P. A. Lee, Phys. Rev. B **62**, 1270 (2000).
[53] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev B **25**, 4515 (1982).
[54] O. Vafek, A. Melikyan, and Z. Tesanovic, Phys. Rev. B **64**, 224508 (2001).
[55] G. D. Mahan, *Many-Particle Physics* (Kluwer, New York, 2000).
[56] G. E. Volovik and L. P. Gor'kov, Sov. Phys. JETP **61**, 843 (1985).
[57] P. A. Frigeri, D. F. Agterberg, A. Koga, and M. Sigrist, Phys. Rev. Lett. **92**, 097001 (2004).
[58] T. P. Pareek and P. Bruno, Phys. Rev. B **65**, 241305(R) (2002).
[59] P. D. Sacramento, V. K. Dugaev, V. R. Vieira, and M. A. N. Araújo, J. Phys.: Condens. Matter **22**, 025701 (2010); P. D. Sacramento and M. A. N. Araújo, Eur. Phys. J. B **76**, 251 (2010); P. D. Sacramento, L. C. Fernandes Silva, G. S. Nunes, M. A. N. Araújo, and V. R. Vieira, Phys. Rev. B **83**, 054403 (2011).