

Complete classification of one-dimensional gapped quantum phases in interacting spin systemsXie Chen,¹ Zheng-Cheng Gu,² and Xiao-Gang Wen^{1,3}¹*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*²*Kavli Institute for Theoretical Physics, University of California, Santa Barbara, California 93106, USA*³*Institute for Advanced Study, Tsinghua University, Beijing, 100084, P.R. China*

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Quantum phases with different orders exist with or without breaking the symmetry of the system. Recently, a classification of gapped quantum phases which do not break time reversal, parity, or on-site unitary symmetry has been given for 1D spin systems by X. Chen, Z.-C. Gu, and X.-G. Wen [*Phys. Rev. B* **83**, 035107 (2011)]. It was found that such symmetry-protected topological (SPT) phases are labeled by the projective representations of the symmetry group which can be viewed as a symmetry fractionalization. In this paper, we extend the classification of 1D gapped phases by considering SPT phases with combined time reversal, parity, and/or on-site unitary symmetries and also the possibility of symmetry breaking. We clarify how symmetry fractionalizes with combined symmetries and also how symmetry fractionalization coexists with symmetry breaking. In this way, we obtain a complete classification of gapped quantum phases in 1D spin systems. We find that in general, symmetry fractionalization, symmetry breaking, and long-range entanglement (present in 2 or higher dimensions) represent three main mechanisms to generate a very rich set of gapped quantum phases. As an application of our classification, we study the possible SPT phases in 1D fermionic systems, which can be mapped to spin systems by Jordan-Wigner transformation.

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I. INTRODUCTION

Quantum phases of matter with exotic types of order have continued to emerge over the past decades. Examples include fractional quantum Hall states,^{1,2} the 1D Haldane phase,³ chiral spin liquids,^{4,5} Z_2 spin liquids,⁶⁻⁸ non-Abelian fractional quantum Hall states,⁹⁻¹² quantum orders characterized by projective symmetry group (PSG),¹³ topological insulators,¹⁴⁻¹⁹ etc. Why are there different orders? What is a general framework to understand all these seemingly very different phases? How to classify all possible phases and identify new ones? Much effort has been devoted to these questions, yet the picture is not complete.

First, we want to emphasize that quantum phase is a property of a class of Hamiltonians, not of a single Hamiltonian. We call such a class of Hamiltonians an H class. For an H class of a certain dimension and with possible symmetry constraints, we ask whether the Hamiltonians in it are separated into different groups by phase transition and hence form different phases. Two Hamiltonians in an H class are in the same/different phase if they can/cannot be connected *within the H class* without going through phase transition. We see that without identifying the class of Hamiltonians under consideration, it is not meaningful to ask which phase a Hamiltonian belongs to. Two Hamiltonians can belong to the same/different phases if we embed them in different H classes.

For an H class with certain symmetry constraints, one mechanism leading to distinct phases is symmetry breaking.^{20,21} Starting from Hamiltonians with the same symmetry, the ground states of them can have different symmetries, hence resulting in different phases. This symmetry-breaking mechanism for phases and phase transitions is well understood.^{20,22}

However, it has been realized that systems can be in different phases even without breaking any symmetry. Such phases are often said to be “topological” or “exotic.” However,

the term “topological” in literature actually refers to two different types of quantum order.

The first type has “intrinsic” topological order. This type of order is defined for the class of systems without any symmetry constraint, which corresponds to the original definition of “topological order.”^{23,24} That is, it refers to quantum phases in an H class which includes all local Hamiltonians (of a certain dimension). If we believed that Landau symmetry-breaking theory describes all possible phases, this whole H class would belong to the same phase as there is no symmetry to break. However, in two and three dimensions, there are actually distinct phases even in the H class that has no symmetries. These phases have universal properties stable against any small local perturbation to the Hamiltonian. To change these universal properties, the system has to go through a phase transition. Phases in this class include quantum Hall systems,²⁵ chiral spin liquids,^{4,5} Z_2 spin liquids,⁶⁻⁸ the quantum double model,²⁶ and the string-net model.²⁷ Ground states of these systems have “long-range entanglement” as discussed in Ref. 28.

The “topological” quantum order of the second type is “symmetry protected.” The class of systems under consideration have certain symmetry and the ground states have only short-range entanglement,²⁸ like in the symmetry-breaking case. However, unlike in the symmetry-breaking phases, the ground states have the same symmetry as the Hamiltonian and, even so, the ground states can be in different phases. This quantum order is protected by symmetry; as according to the discussion in Ref. 28, if the symmetry constraint on the class of systems is removed, all short-range entangled states belong to the same phase. Only when symmetry is enforced can short-range entangled states with the same symmetry belong to different phases. Examples of this type include the Haldane phase³ and topological insulators.¹⁴⁻¹⁹

Phases in these two classes share some similarities. For example, they both are beyond Landau symmetry-breaking theory. Also quantum Hall systems and topological insulators both have stable gapless edge states.^{29–31} However, the latter requires symmetry protection while the former do not.

Despite the similarities, these two classes of topological phases are fundamentally different, as we can see from quantities that are sensitive to long-range entanglement. For example, “intrinsic” topological order has a robust ground-state degeneracy that depends on the topology of the space.^{23,24} The ground states with “intrinsic” topological order also have nonzero topological entanglement entropy,^{32,33} while ground states in “symmetry protected” topological phases are short-range entangled and therefore have zero topological entanglement entropy. Also, the low-energy excitations in “symmetry protected” topological phases do not have nontrivial anyon statistics, unlike in “intrinsic” topological phases.^{5,34,35}

In the following discussion, we will use “topological phase” to refer only to the first type of phases (i.e. “intrinsic” topological phases). For the second type, we will call them “symmetry-protected topological” (SPT) phases, as in Ref. 36. Similar to the quantum orders characterized by PSG,¹³ different SPT phases are also characterized by the projective representations of the symmetry group of the Hamiltonian.³⁷ The PSG and projective representations of a symmetry group can be viewed as a “fractionalization” of the symmetry. Thus, we may say that different SPT phases are caused by “symmetry fractionalization.”

Long-rang entanglement, symmetry fractionalization, and symmetry breaking represent three different mechanisms to separate phases and can be combined to generate a very rich quantum-phase diagram. Figure 1 shows a phase diagram with possible phases generated by these three mechanisms. In order to identify the kind of quantum order in a system, we first need to know whether topological orders exist, that is, whether the ground state has long-range entanglement. Next, we need to identify the symmetry of the system (of the Hamiltonian and allowed perturbation). Then we can find out whether all or part of the symmetry is spontaneously broken in the ground state. If only part is broken, what is the SPT order due to the fractionalization of the unbroken symmetry? Combining these data together gives a general description of a quantum

phase. Most of the phases studied before involve only one of the three mechanisms. Examples where two of them coexist can be found in Refs. 5 and 38 which combine long-range entanglement (the intrinsic topological order) and symmetry breaking and in Refs. 13,39–41 which combine long-range entanglement and symmetry fractionalization. In fact, the PSG provides a quite comprehensive framework for symmetry fractionalization in topologically ordered states.^{13,39,40}

Based on this general understanding of quantum phases, we address the following question in this paper: What quantum phases exist in one-dimensional gapped spin systems? The systems we consider can have any finite-strength finite-range interactions among the spins.

In Ref. 37, we gave a partial answer to this question. We first showed that one-dimensional gapped spin systems do not have nontrivial topological order. So to understand possible 1D gapped phases, we just need to understand symmetry fractionalization and symmetry breaking in short-range entangled states. In other words, quantum phases are only different because of symmetry breaking and symmetry fractionalization.

In Ref. 37, we then considered symmetry fractionalization, and gave a classification of possible SPT phases with time reversal, parity, and on-site unitary symmetry, respectively. In this paper, we complete the classification by considering SPT phases of combined time reversal, parity, and/or on-site unitary symmetry and finally incorporate the possibility of symmetry breaking. We find that in 1D gapped spin systems with on-site symmetry of group G , the quantum phases are basically labeled by

- (1) the unbroken symmetry subgroup G' ,
- (2) the projective representation of the unbroken part of on-site unitary and antiunitary symmetry, respectively, and
- (3) the “projective” commutation relation between representations of unbroken symmetries.

Here the projective representation and the “projective” commutation relation represent the symmetry fractionalization. Parity is not an on-site symmetry and its SPT phases are not characterized by projective representations. The classification involving parity does not fall into the general framework above, but proceeds in a very similar way, as we will show in Sec. III. Actually, (2) and (3) combined give the projective representation of G' and if parity is present, it should be treated as an antiunitary Z_2 element. Our result is consistent with that obtained by Schuch *et al.*⁴²

Our discussion is based on the matrix product state representation^{43,44} of gapped 1D ground states. The matrix product formalism allows us to directly deal with interacting systems and its entangled ground state. In particular, the SPT order of a system can be identified directly from the way matrices in the representation transform under symmetries.⁴⁵

Moreover, symmetry breaking in entangled systems can be represented in a nice way using matrix product states.^{43,44,46} The traditional understanding of symmetry breaking in quantum systems actually comes from intuition about classical systems. For example, in the ferromagnetic phase of the classical Ising model, the spins have to choose between two possible states: either all pointing up or all pointing down. Both states break the spin-flip symmetry of the system. However, for quantum system, the definition of symmetry breaking becomes

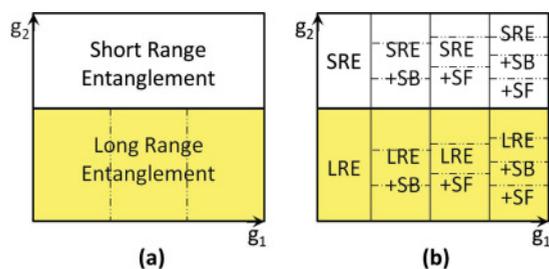


FIG. 1. (Color online) (a) The possible phases for class of Hamiltonians without any symmetry. (b) The possible phases for class of Hamiltonians with some symmetries. Each phase is labeled by the phase-separating mechanisms involved. The shaded regions in (a) and (b) represent the phases with long-range entanglement. SRE stands for short-range entanglement, LRE for long-range entanglement, SB for symmetry breaking, SF for symmetry fractionalization.

a little tricky. In the ferromagnetic phase of the quantum Ising model, the ground space is twofold degenerate with basis states $|\uparrow\uparrow\dots\uparrow\rangle, |\downarrow\downarrow\dots\downarrow\rangle$. Each basis state breaks the spin-flip symmetry of the system. However, quantum systems can exist in any superposition of the basis states and in fact the superposition $\frac{1}{\sqrt{2}}|\uparrow\uparrow\dots\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\dots\downarrow\rangle$ is symmetric under spin flip. What is meant when a *quantum* system is said to be in a symmetry-breaking phase? Can we understand symmetry breaking in a quantum system without relying on its classical picture?

In matrix product representation, the symmetry-breaking pattern can be seen directly from the matrices representing the ground state. If we choose the symmetric ground state in the ground space and write it in matrix product form, the matrices can be reduced to a block diagonal “canonical form.” The canonical form contains more than one block if the system is in a symmetry-breaking phase. If symmetry is not broken, it contains only one block.^{43,44,46} Hence, the canonical form of the matrices gives a nice illustration of the symmetry breaking pattern of the system. This relation will be discussed in more detail in Sec. IV.

Our classification is focused on 1D interacting spin systems; however it also applies to 1D interacting fermion systems as they are related by Jordan-Wigner transformation. As an application of our classification result, we study quantum phases (especially SPT phases) in gapped 1D fermion systems. Our result is consistent with previous studies.^{47,48}

The paper is organized as follows: In Sec. II, we review the previous classification results of SPT phases with time reversal, parity, and on-site unitary symmetry, respectively. We also introduce notations for matrix product representation; in Sec. III, we present classification results of SPT phases with combined time reversal, parity, and/or on-site unitary symmetry; in Sec. IV, we incorporate the possibility of symmetry breaking; in Sec. V, we apply classification results of spins to the study of phases in 1D fermion systems; and finally we conclude in Sec. VI.

II. REVIEW: MATRIX PRODUCT STATES AND SPT CLASSIFICATION

In Ref. 37, we considered the classification of SPT phases with time reversal, parity, and on-site unitary symmetry, respectively. Instead of starting from Hamiltonians, we classified 1D gapped ground states which do not break the symmetry of the system. The set of states under consideration can be represented as short-range correlated matrix product states and we used the local unitary equivalence between gapped ground states, which was established in Ref. 28, to classify phases. Here we introduce the matrix product representation and give a brief review of previous classification result and how it was achieved.

Matrix product states give an efficient representation of 1D gapped spin states^{49,50} and hence provide a useful tool to deal with strongly interacting systems with many-body entangled ground states. A matrix product state (MPS) is expressed as

$$|\phi\rangle = \sum_{i_1, i_2, \dots, i_N} \text{Tr}(A_{i_1} A_{i_2} \dots A_{i_N}) |i_1 i_2 \dots i_N\rangle, \quad (1)$$

where $i_k = 1 \dots d$ with d being the physical dimension of a spin at each site; the A_{i_k} 's are $D \times D$ matrices on site k with D being the inner dimension of the MPS. In our previous studies³⁷ and also in this paper, we consider states which can be represented with a finite inner dimension D and assume that they represent all possible phases in 1D gapped systems. In our following discussion, we will focus on states represented with site-independent matrices A_i and discuss classification of phases with or without translational symmetry. Non-translational-invariant systems have in general ground states represented by site-dependent matrices. However, site dependence of matrices does not lead to extra features in the phase classification and their discussion involves complicated notation. Therefore, we will not present the analysis based on site-dependent MPS. A detailed discussion of site-dependent MPS can be found in Ref. 37 and all results in this paper can be obtained using similar methods.

A mathematical construction that will be useful is the double tensor

$$\mathbb{E}_{\alpha\gamma, \beta\chi} = \sum_i A_{i, \alpha\beta} \times (A_{i, \gamma\chi})^* \quad (2)$$

of the MPS. \mathbb{E} is useful because it uniquely determines the matrix product state up to a local change of basis on each site.^{44,51} Therefore, all correlation and entanglement information of the state is contained in \mathbb{E} and can be extracted.

First we identify the set of matrix product states that need to be considered for the classification of SPT phases. The ground state of SPT phases does not break any symmetry of the system and hence is nondegenerate. The unique ground state must be short-range correlated due to the existence of the gap, which requires that \mathbb{E} has a nondegenerate largest eigenvalue (set to be 1).^{43,44} This is equivalent to an “injectivity” condition on the matrices A_i . That is, for large enough n , the set of matrices corresponding to physical states on n consecutive sites $\mathcal{A}_I = A_{i_1} \dots A_{i_n} (I \equiv i_1 \dots i_n)$ span the space of $D \times D$ matrices.⁴⁴ On the other hand, all MPSs satisfying this injectivity condition are the unique gapped ground states of a local Hamiltonian which has the same symmetry.^{43,44} Therefore, we only need to consider states in this set.

The symmetry of the system and hence of the ground state sets a nontrivial transformation condition on the matrices A_i . With on-site unitary symmetry of group G , for example, the A_i 's transform as^{45,52}

$$\sum_j u(g)_{ij} A_j = \alpha(g) R^{-1}(g) A_i R(g), \quad (3)$$

where $u(g)$ is a linear representation of G on the physical space, and $\alpha(g)$ is a one-dimensional representation of G . One important realization from this equation is that in order to satisfy this equation, $R(g)$ only has to satisfy the multiplication rule of group G up to a phase factor.⁴⁵ That is, $R(g_1 g_2) = \omega(g_1, g_2) R(g_1) R(g_2)$, $|\omega(g_1, g_2)| = 1$. $\omega(g_1, g_2)$ is called the factor system. $R(g)$ is hence a projective representation of group G and belongs to different equivalence classes labeled by the elements in the second cohomology group of G , $\{\omega | \omega \in H^2(G, \mathbb{C})\}$.

We showed in Ref. 37 that two matrix product states symmetric under G are in the same SPT phase if and only if they are related to $R(g)$ in the same equivalence class ω . For

matrices A_i under the two symmetries separately in terms of $\alpha(g)$, $R(g)$, $\alpha(P)$, N . $\alpha(g)$ labels the 1D representation the state forms under G , $R(g) \in \omega$ is the projective representation of G on the boundary spin, $\alpha(P)$ labels parity even or odd, and $N^T = \beta(P)N = \pm N$ corresponds to parity even/odd between sites.

Moreover, parity and on-site $u(g)$ commute. First, it is easy to see that P_1 and $u(g) \otimes u(g) \dots \otimes u(g)$ commute. Therefore, $P_2 = w \otimes w \dots \otimes w$ must also commute with $u(g) \otimes u(g) \dots \otimes u(g)$. Without loss of generality, we will consider the case where w and $u(g)$ commute, $wu(g) = u(g)w$. This leads to certain commutation relation between N and $R(g)$ as shown below.

If we act parity first and on-site symmetry next, the matrices are transformed as

$$\begin{aligned} A_i &\xrightarrow{P} A'_i = \sum_j w_{ij} A_j^T = \alpha(P) N^{-1} A_i N, \\ A'_i &\xrightarrow{G} A''_i = \sum_j u_{ij}(g) A'_j = \alpha(P) N^{-1} \left(\sum_j u_{ij} A_j \right) N \\ &= \alpha(g) \alpha(P) N^{-1} R^{-1}(g) A_i R(g) N. \end{aligned} \quad (6)$$

Combining the two steps together we find that

$$\sum_j \sum_k u_{ij}(g) w_{jk} A_k^T = \alpha(g) \alpha(P) N^{-1} R^{-1}(g) A_i R(g) N. \quad (7)$$

If on-site symmetry is acted first and then parity follows, the matrices are transformed as

$$\begin{aligned} A_i &\xrightarrow{G} A'_i = \sum_j u_{ij}(g) A_j = \alpha(g) R^{-1}(g) A_i R(g), \\ A'_i &\xrightarrow{P} A''_i = \sum_j w_{ij} (A'_j)^T \\ &= \alpha(g) R^T(g) \left(\sum_j w_{ij} A_j^T \right) (R^T)^{-1}(g) \\ &= \alpha(P) \alpha(g) R^T(g) N^{-1} A_i N R^*(g). \end{aligned} \quad (8)$$

The combined operation is then

$$\sum_j \sum_k w_{ij} u_{jk}(g) A_k^T = \alpha(P) \alpha(g) R^T(g) N^{-1} A_i N R^*(g). \quad (9)$$

Because w and $u(g)$ commute, $\sum_j u_{ij}(g) w_{jk} = \sum_j w_{ij} u_{jk}(g)$. Therefore, the combined operation in Eqs. (7) and (9) should be equivalent:

$$N^{-1} R^{-1}(g) A_i R(g) N = R^T(g) N^{-1} A_i N R^*(g). \quad (10)$$

This condition is derived for matrices on each site, $i = 1 \dots d$. However, it is easy to verify that it also holds if n consecutive sites are combined together with representing matrices $\mathcal{A}_I = A_{i_1} A_{i_2} \dots A_{i_n}$:

$$N^{-1} R^{-1}(g) \mathcal{A}_I R(g) N = R^T(g) N^{-1} \mathcal{A}_I N R^*(g). \quad (11)$$

As \mathcal{A}_I is injective (spans the whole space of $D \times D$ matrices), we find $R(g) N R^T(g) N^{-1} \propto I$. That is,

$$N^{-1} R(g) N = e^{i\theta(g)} (R^T)^{-1}(g) = e^{i\theta(g)} R^*(g). \quad (12)$$

Different $e^{i\theta(g)}$ corresponds to different ‘‘projective’’ commutation relations between parity and on-site unitary. It must satisfy certain conditions. As

$$\begin{aligned} N^{-1} R(g_1 g_2) N &= e^{i\theta(g_1 g_2)} R^*(g_1 g_2) \\ &= e^{i\theta(g_1 g_2)} \omega^{-1}(g_1, g_2) R^*(g_1) R^*(g_2), \\ N^{-1} R(g_1 g_2) N &= \omega(g_1, g_2) N^{-1} R(g_1) N N^{-1} R(g_2) N \\ &= \omega(g_1, g_2) e^{i\theta(g_1)} R^*(g_1) e^{i\theta(g_2)} R^*(g_2). \end{aligned} \quad (13)$$

Therefore

$$e^{i\theta(g_1 g_2)} e^{-i\theta(g_1)} e^{-i\theta(g_2)} = \omega^2(g_1, g_2). \quad (14)$$

Hence, ω^2 must be trivial. Without loss of generality, assume that the factor systems we have chosen (as discussed in Sec. II) satisfy $\omega^2 = 1$ and $e^{i\theta(g)}$ forms a linear representation of G , denoted by $\gamma(g)$.

Let us interpret this result. First we see that the combination of parity with on-site G restricts the projective representation that can be realized on the boundary spin to those ω that square to identity. This can be clearly seen from the structure of the representative state as in Fig. 2. Because of on-site symmetry G , the left and right black spins in a pair form projective representation in class ω and ω^* , respectively. If the chain has further reflection symmetry, $\omega^* = \omega$ and therefore $\omega^2 = 1$. However, if $\omega^2 \neq 1$, then $\omega^* \neq \omega$. The chain has a direction and cannot have reflection symmetry.

Different $\gamma(g)$ corresponds to different ‘‘projective’’ commutation relation between parity and on-site G on the boundary spin. For example, suppose $G = Z_2$. Consider the state as in Fig. 2 where each black pair consists of two qubits. Suppose that parity on the black pair is defined as exchange of the two qubits (P_1) and unitary operation $P_2 = Z \otimes Z$ on the two qubits. Z_2 symmetry on the pair can be defined as $Z \otimes Z$ or $X \otimes X$. They both commute with parity. However, if we only look at one end of pair, Z_2 either commutes or anticommutes with P_2 . Hence these two cases correspond to two different $\gamma(Z_2)$. Because of this, the two phases cannot be connected without breaking the symmetry group generated by parity and Z_2 .

However, $\gamma(g)$ can be changed by changing the phase of $R(g)$. Remember that the phase of $R(g)$ is only determined (by fixing ω) up to a 1D representation, $\tilde{\alpha}(g)$. From Eq. (12), we can see that if the phase of $R(g)$ is changed by $\tilde{\alpha}(g)$, $\gamma(g)$ is changed to $\gamma(g)/\tilde{\alpha}^2(g)$. Therefore, $\gamma(g)$ and $\gamma'(g)$ which differ by the square of another 1D representation $\tilde{\alpha}(g)$ are equivalent. On the other hand, for a fixed ω ($\omega^2 = 1$) any $\gamma(g)$ can be realized as the ‘‘projective’’ commutation relation, as we will show in Appendix B.

Therefore, with commuting parity and on-site unitary symmetry, SPT phases in a 1D spin chain can be classified by the following data:

- (1) $\alpha(P)$, parity even/odd;
- (2) $\beta(P)$, parity even/odd between sites;
- (3) $\alpha(g)$, 1D representation of G ;
- (4) ω , projective representation of G on boundary spin, $\omega^2 = 1$;
- (5) $\gamma(g) \in \mathcal{G}/\mathcal{G}_2$, 1D representation of G related to commutation relation between parity and on-site G , where \mathcal{G} is

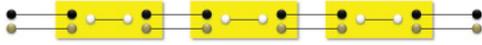


FIG. 3. (Color online) Representative states for different SPT phases. Each box represents one site, containing six spins. Every two connected spins form an entangled pair.

the group of 1D representation of G , \mathcal{G}_2 is the group of 1D representation squared of G .

Following the method used in Appendix G of Ref. 37, we can show that states symmetric under parity and on-site G are in the same SPT phase if and only if they are labeled by the same set of data as given above. We will not repeat the proof here.

A representative state for each phase labeled by $\alpha(P)$, $\beta(P)$, $\alpha(g)$, ω , and $\gamma(g)$ can be constructed as in Fig. 3. We will describe the state of each pair and how it transforms under symmetry operations. The parameters describing the state will then be related to the phase labels.

Each entangled pair is invariant under P and on-site G . First, the on-site white pair forms 1D representations $\eta(P)$ and $\eta(g)$ of parity and G . For projective representation class ω that satisfies $\omega^2 = 1$ and any 1D representation $\lambda(g)$, we show in Appendices A and B that there always exist projective representation $R(g) \in \omega$ and symmetric matrix $N(N^T = N)$ such that $N^{-1}R(g)N = \lambda(g)R^*(g)$. Suppose that $R(g)$ and N are D -dimensional matrices; choose the intersite black pair to be composed of two D -dimensional spins. Define parity on this pair to be exchange of two spins and define on-site symmetry to be $R(g) \otimes R(g)$. If the state of the black pair is chosen to be $(N \otimes I) \sum_i |i\rangle \otimes |i\rangle$, $i = 1, \dots, D$, it is easy to check that it is parity even, forms 1D representation $\lambda(g)$ for on-site G , and contains projective representation ω at each end. Finally, define the state of the intersite gray pair to be $|0\rangle \otimes |1\rangle + \rho(P)|1\rangle \otimes |0\rangle$, $\rho(P) = \pm 1$. Parity acts on it as exchange of spins. On-site G acts trivially on it. The 1D spin state constructed in this way is symmetric under parity and on-site unitary G and belongs to the SPT phase labeled by $\alpha(P) = \eta(P)\rho(P)$, $\beta(P) = \rho(P)$, $\alpha(g) = \eta(g)\lambda(g)$, ω , and $\gamma(g) = \lambda(g)$.

Finally, we consider some specific cases:

- (1) For translation + parity + $\text{SO}(3)$, there are $2 \times 2 \times 1 \times 2 \times 1 = 8$ types of phases.
- (2) For translation + parity + D_2 , there are $2 \times 2 \times 4 \times 2 \times 4 = 128$ types of phases.

B. Time reversal + on-site G

Now consider a 1D spin system symmetric under both time reversal $T = v \otimes v \dots \otimes vK$ and on-site unitary $u(g) \otimes u(g) \dots \otimes u(g)$. On each site, $T^2 = vv^* = I$ and $u(g)$ forms a linear representation of G . Equations (3) and (4) are satisfied due to the two symmetries separately with some choice of $\alpha(g)$, $R(g)$, and M . $\alpha(g)$ labels the 1D representation the state forms under G , $R(g) \in \omega$ is the projective representation of G on the boundary spin, and $\hat{T} = MK$ is the time reversal operator on the boundary spin which squares to $\beta(T)I = \pm I$. Note that while $\hat{T}^2 = \pm I$ on the boundary spin, T always squares to I on the physical spin at each site.

Moreover, the two symmetries commute; i.e., $u(g)v = vu^*(g)$. This leads to nontrivial relations between $R(g)$ in

Eq. (3) and M in Eq. (4). In particular, suppose we act G first followed by T ; the matrices A_i transform as

$$\begin{aligned} A_i &\xrightarrow{G} A'_i = \sum_j u_{ij}(g)A_j = \alpha(g)R^{-1}(g)A_iR(g), \\ A'_i &\xrightarrow{T} A''_i = \sum_j v_{ij}(A'_j)^* \\ &= \alpha^*(g)(R^*)^{-1}(g) \left(\sum_j v_{ij}A_j^* \right) R^*(g) \\ &= \alpha^*(g)(R^*)^{-1}(g)M^{-1}A_iMR^*(g). \end{aligned} \quad (15)$$

Acting T first followed by G gives

$$\begin{aligned} A_i &\xrightarrow{T} A'_i = \sum_j v_{ij}A_j^* = M^{-1}A_iM, \\ A'_i &\xrightarrow{G} A''_i = \sum_j u_{ij}(g)A'_j = M^{-1} \left(\sum_j u_{ij}(g)A_j \right) M \\ &= \alpha(g)M^{-1}R^{-1}(g)A_iR(g)M. \end{aligned} \quad (16)$$

Because $u(g)v = vu^*(g)$, the previous two transformations should be equivalent. That is,

$$\begin{aligned} \alpha^*(g)(R^*)^{-1}(g)M^{-1}A_iMR^*(g) \\ = \alpha(g)M^{-1}R^{-1}(g)A_iR(g)M. \end{aligned} \quad (17)$$

Denote $Q = MR^*M^{-1}R^{-1}$. It follows that $A_i = \alpha^2(g)QA_iQ^{-1}$. Suppose that the MPS is injective with blocks larger than n sites, hence $\mathcal{A}_I = A_{i_1}A_{i_2}\dots A_{i_n}$ satisfies $\mathcal{A}_I = \alpha^{2n}(g)Q\mathcal{A}_IQ^{-1}$. But \mathcal{A}_I spans the whole space of $D \times D$ matrices, therefore $Q \propto I$. That is

$$M^{-1}R(g)M = e^{i\theta(g)}R^*(g). \quad (18)$$

Moreover, it follows from Eq. (17) that $\alpha^2(g) = 1$. That is, the 1D representation of G must have order 2.

Similar to the parity + on-site G case, we find, $\omega^2 = 1$, and $e^{i\theta(g)}$ forms a 1D representation, denoted by $\gamma(g)$. Two $\gamma(g)$'s that differ by the square of a third 1D representation are equivalent.

Therefore, different phases with translation, time reversal, and on-site G symmetries are labeled by

- (1) $\beta(T)$, $\hat{T}^2 = \pm I$ on the boundary spin;
- (2) $\alpha(g)$, 1D representation of G , $\alpha^2(g) = 1$;
- (3) ω , projective representation of G on boundary spin, $\omega^2 = 1$;
- (4) $\gamma(g) \in \mathcal{G}/\mathcal{G}_2$, 1D representation of G related to commutation relation between time reversal and on-site G , where \mathcal{G} is the group of 1D representation of G , \mathcal{G}_2 is the group of 1D representation squared of G .

Representative states are again given by Fig. 3. The on-site white pair forms 1D representation $\eta(g)$ for G and is invariant under T . Similar to the parity + on-site symmetry case, we can show that for any $\omega(\omega^2 = 1)$ and 1D representation $\lambda(g)$ there exist D -dimensional projective representation $R(g) \in \omega$ and matrix M such that $MM^* = I$ and $M^{-1}R(g)M = \lambda(g)R^*(g)$. Choose the intersite black pair to be composed of two D -dimensional spins. Define time reversal on this pair to be $(M \otimes M)K$ and define on-site symmetry to be $R(g) \otimes R(g)$. If the

state of the black pair is chosen to be $(M \otimes I) \sum_i |i\rangle \otimes |i\rangle$, $i = 1, \dots, D$, it is easy to check that it is invariant under time reversal and forms 1D representation $\lambda(g)$ for G and contains projective representation ω at each end. Finally, define the state of the intersite gray pair to be $|0\rangle \otimes |1\rangle + \rho(T)|1\rangle \otimes |0\rangle$, $\rho(T) = \pm 1$. Time reversal acts on it as $(|0\rangle\langle 1| + \rho(T)|1\rangle\langle 0|) \otimes (|0\rangle\langle 1| + \rho(T)|1\rangle\langle 0|)K$. On-site G acts trivially on it. The 1D spin state constructed as this is symmetric under time reversal and on-site unitary G and belongs to the SPT phase labeled by $\beta(T) = \rho(T)$, $\alpha(g) = \eta(g)\lambda(g)$, ω , and $\gamma(g) = \lambda(g)$.

Applying the general classification result to specific cases we find the following:

(1) For translation $+T + \text{SO}(3)$, there are $2 \times 1 \times 2 \times 1 = 4$ types of phases.

(2) For translation $+T + D_2$, there are $2 \times 4 \times 2 \times 4 = 64$ types of phases.

If translation symmetry is not required, different phases with time reversal and on-site G symmetries are labeled by

(1) $\beta(T)$, time reversal even/odd on the boundary spin;
 (2) ω , projective representation of G on boundary spin, $\omega^2 = 1$;

(3) $\gamma(g) \in \mathcal{G}/\mathcal{G}_2$, 1D representation of G related to commutation relation between time reversal and on-site G , where \mathcal{G} is the group of 1D representation of G , \mathcal{G}_2 is the group of 1D representation squared of G .

$\alpha(g)$ can no longer be used to distinguish different phases. We find the following:

(1) For $T + \text{SO}(3)$, there are $2 \times 2 \times 1 = 4$ types of phases.
 (2) For $T + D_2$, there are $2 \times 2 \times 4 = 16$ types of phases.

C. Parity + time reversal

When parity is combined with time reversal, what SPT phases exist in 1D gapped spin systems? First we realize that due to parity and time reversal separately, different SPT phases exist labeled by different $\alpha(P)$, $\beta(P)$, $\beta(T)$ as defined in Eqs. (4) and (5). $\alpha(P)$ labels parity even or odd, $\beta(P)$ labels parity even/odd between sites, and $\hat{T}^2 = \beta(T)I$ on the boundary spin.

Does the commutation relation between parity and time reversal give more phases? The combined operation of parity first and time reversal next gives

$$\begin{aligned} A_i \xrightarrow{P} A'_i &= \sum_j w_{ij} A_j^T = \alpha(P) N^{-1} A_i N, \\ A'_i \xrightarrow{T} A''_i &= \sum_j v_{ij} (A'_j)^* = \alpha(P) (N^{-1})^* \left(\sum_j v_{ij} A_j^T \right) N^* \\ &= \alpha(P) (N^{-1})^* M^{-1} A_i M N^*, \end{aligned} \quad (19)$$

and the operation with time reversal first and parity next gives

$$\begin{aligned} A_i \xrightarrow{T} A'_i &= \sum_j v_{ij} A_j^* = M^{-1} A_i M, \\ A'_i \xrightarrow{P} A''_i &= \sum_j w_{ij} (A'_j)^T = M^T \left(\sum_j w_{ij} A_j^T \right) (M^T)^{-1} \\ &= \alpha(P) M^T N^{-1} A_i N (M^T)^{-1}. \end{aligned} \quad (20)$$

As parity and time reversal commute, $wv = vw^*$. The above two operations should be equivalent:

$$(N^{-1})^* M^{-1} A_i M N^* = M^T N^{-1} A_i N (M^T)^{-1}. \quad (21)$$

As A_i is injective, $M N^* M^T N^{-1} \propto I$. That is,

$$M N^\dagger M N^\dagger = e^{i\theta} I. \quad (22)$$

But $e^{i\theta}$ can be set to be 1 by changing the phase of M or N ; therefore, the commutation relation does not lead to more distinct phases.

There are hence eight SPT phases with both parity and time reversal symmetry, labeled by

- (1) $\alpha(P)$, parity even/odd;
- (2) $\beta(P)$, parity even/odd between sites;
- (3) $\beta(T)$, $\hat{T}^2 = \pm I$ on boundary spins.

The representative states of each phase can be given as in Fig. 3. Each pair of spins forms a 1D representation of parity and time reversal. The on-site white pair is in the state $|0\rangle \otimes |1\rangle + \eta(P)|1\rangle \otimes |0\rangle$ with $\eta(P) = \pm 1$. Parity on this pair is defined as exchange of spins and time reversal as K . Therefore, this pair has parity $\eta(P)$ and is invariant under T . The intersite black pair is in the state $|0\rangle \otimes |1\rangle + \lambda|1\rangle \otimes |0\rangle$ with $\lambda = \pm 1$. Parity acts on it as exchange of spins and time reversal as $(|0\rangle\langle 1| + \lambda|1\rangle\langle 0|) \otimes (|0\rangle\langle 1| + \lambda|1\rangle\langle 0|)K$. This pair therefore has parity $\lambda(P) = \lambda$ and is invariant under time reversal. Time reversal on one of the spins squares to $\lambda(T)I = \lambda I$. Finally, the intersite gray pair is in state $|0\rangle \otimes |1\rangle + \rho(P)|1\rangle \otimes |0\rangle$ with $\rho(P) = \pm 1$. Parity on this pair is defined as exchange of spins and time reversal as K . Therefore this pair has parity $\rho(P)$ and is invariant under T . Time reversal on one of the spins squares to I . This state is in the SPT phase labeled by $\alpha(P) = \eta(P)\lambda(P)\rho(P)$, $\beta(P) = \lambda(P)\rho(P)$, $\beta(T) = \lambda(T)$.

D. Parity + time reversal + on-site G

Finally we put parity, time reversal, and on-site unitary symmetry together and ask how many SPT phases exist if the ground state does not break any of the symmetries. From Eqs. (3), (4), and (5), we know that due to the three symmetries separately, states with different $\alpha(g)$, ω , $\alpha(P)$, $\beta(P)$, $\beta(T)$ belong to different SPT phases. $\alpha(g)$ labels the 1D representation the state forms under G , ω is the projective representation of G on the boundary spin, $\alpha(P)$ labels parity even or odd, $\beta(P)$ labels parity even/odd between sites, and $\hat{T}^2 = \beta(T)I$ on the boundary spin.

Moreover, the commutation relation between parity, time reversal, and on-site G yields further conditions. The commutation relation between parity and on-site G constrains $\omega^2 = 1$ and gives

$$N^{-1} R(g) N = \gamma(g) R^*(g). \quad (23)$$

$\gamma(g)$ is a 1D representation of G . $\gamma_1(g)$ and $\gamma_2(g)$ correspond to different SPT phases if and only if they are not related by the square of a third 1D representation. The commutation relation between time reversal and on-site G constrains $\alpha^2(g) = 1$ and gives,

$$M^{-1} R(g) M = \gamma'(g) R^*(g). \quad (24)$$

$\gamma'(g)$ is a 1D representation of G . $\gamma'_1(g)$ and $\gamma'_2(g)$ correspond to different SPT phases if and only if they are not related by the

square of a third 1D representation. The commutation relation between time reversal and parity gives

$$MN^\dagger MN^\dagger \propto I, \quad (25)$$

which is equivalent to, because $N^T = \pm N$ and $M^T = \pm M$,

$$MN^* \propto NM^*. \quad (26)$$

Therefore, MN^* and NM^* conjugating $R(g)$ should give the same result:

$$\begin{aligned} (MN^*)R(g)(MN^*)^{-1} &= \gamma(g)MR^*(g)M^{-1} \\ &= \gamma(g)/\gamma'(g)R(g). \end{aligned} \quad (27)$$

On the other hand,

$$\begin{aligned} (NM^*)R(g)(NM^*)^{-1} &= \gamma'(g)NR^*(g)N^{-1} \\ &= \gamma'(g)/\gamma(g)R(g). \end{aligned} \quad (28)$$

As $R(g)$ is nonzero, $\gamma'(g) = \pm\gamma(g)$.

$\gamma(g)$ and $\gamma'(g)$ are hence related by a 1D representation $\chi(g)$ which squares to 1. As shown in Appendix C, for fixed ω , if N and $R(g)$ exist that satisfy $N^{-1}R(g)N = \gamma(g)R^*(g)$, then any choice of $\gamma'(g) = \chi(g)\gamma(g)$ ($\chi^2(g) = 1$) can be realized. The freedom in $\chi(g)$ is $\mathcal{G}/\mathcal{G}_2$. Considering the degree of freedom in choosing $\gamma(g)$, the total freedom in $\{\gamma(g), \gamma'(g)\}$ is $(\mathcal{G}/\mathcal{G}_2) \times (\mathcal{G}/\mathcal{G}_2)$.

The SPT phases with parity, time reversal, and on-site unitary symmetries are labeled by

- (1) $\alpha(g)$, 1D representation of G , $\alpha^2(g) = 1$;
- (2) ω , projective representation of G on boundary spin, $\omega^2 = 1$;
- (3) $\alpha(P)$, parity even/odd;
- (4) $\beta(P)$, parity even/odd between sites;
- (5) $\beta(T)$, $\hat{T}^2 = \pm I$ on the boundary spin;
- (6) $\{\gamma(g), \gamma'(g)\} \in (\mathcal{G}/\mathcal{G}_2) \times (\mathcal{G}/\mathcal{G}_2)$, 1D representations of G , related to commutation relation between time reversal, parity, and on-site G .

Representative states can be constructed as in Fig. 3.

Each pair is invariant (up to phase) under G , T , and P .

White pair: forms a 1D representation $\eta(g)$ of G , $\eta(P)$ of P , and is invariant under time reversal.

Black pair: G acts nontrivially on it as $R(g) \otimes R(g)$. $R(g)$ is a D -dimensional projective representation and belongs to class ω . According to Appendices A, B, and C, for any 1D representation $\lambda(g)$ and order 2 1D representation $\chi(g)$, we can find matrices N and M such that $N^T = N$, $N^{-1}R(g)N = \lambda(g)R^*(g)$, $MM^* = I$, $M^{-1}R(g)M = \lambda'(g)R^*(g) = \chi(g)\lambda(g)R^*(g)$, $MN^* = NM^*$. Now set the state of this pair to be $N \sum_i |i\rangle \otimes |i\rangle$, where $i = 1 \dots D$. Define parity as exchange of sites and time reversal as $(M \otimes M)K$. It can be checked that the state forms a 1D representation $\lambda(g)$ for G , has even parity, and is invariant under time reversal. Time reversal squares to I at each end.

Gray pair: G acts trivially on it. The pair is in state $|0\rangle \otimes |1\rangle + \rho(P)|1\rangle \otimes |0\rangle$ with $\rho(P) = \pm 1$. Parity on this pair is defined as exchange of spins and time reversal as $(Y \otimes Y)^{\rho(T)+1/2}K$ with $\rho(T) = \pm 1$. Therefore this pair has parity $\rho(P)$ and is invariant under T . Time reversal on one of the spins squares to $\rho(T)I$.

TABLE I. Numbers of different 1D gapped quantum phases that do not break any symmetry. T stands for time reversal, P stands for parity, and ‘‘Trans.’’ stands for translational symmetry.

Symmetry of Hamiltonian	Number of Different Phases
None	1
SO(3)	2
D_2	2
T	2
SO(3) + T	4
D_2 + T	16
Trans. + $U(1)$	∞
Trans. + SO(3)	2
Trans. + D_2	$4 \times 2 = 8$
Trans. + P	4
Trans. + T	2
Trans. + P + T	8
Trans. + SO(3) + P	8
Trans. + D_2 + P	128
Trans. + SO(3) + T	4
Trans. + D_2 + T	64
Trans. + SO(3) + P + T	16
Trans. + D_2 + P + T	1024

This state is representative of the SPT phase labeled by $\alpha(g) = \eta(g)\lambda(g)$, ω , $\alpha(P) = \eta(P)\rho(P)$, $\beta(P) = \rho(P)$, $\beta(T) = \rho(T)$, $\gamma(g) = \lambda(g)$, $\gamma'(g) = \lambda'(g)$.

When $G = \text{SO}(3)$ or $G = D_2$, the classification result gives the following:

(1) For translation + T + P + SO(3), there are $1 \times 2 \times 2 \times 2 \times 2 \times (1 \times 1) = 16$ types of phases.

(2) For translation + T + P + D_2 , there are $4 \times 2 \times 2 \times 2 \times 2 \times (4 \times 4) = 1024$ types of phases

In Table I, we summarize the results obtained above and in Ref. 37.

E. General classification for SPT phases

Besides the cases discussed above, it is possible to have other types of symmetries in 1D spin systems. For example, there could be systems where time reversal and parity are not preserved individually but the combined action of them together defines a symmetry of the system. The general rule for classifying SPT phases under any symmetry is to classify all the projective representations of the total symmetry group, where on-site unitary symmetries should be represented with unitary matrices, on-site antiunitary symmetries should be represented with antiunitary matrices, and parity should be represented with antiunitary matrices. Moreover, if translational symmetry is present, another independent label for SPT phases exists which corresponds to different 1D representations of the total symmetry group. In calculating this label, the representation of the total symmetry group is slightly different from the one for calculating projective representations. In particular, parity should be represented unitarily, i.e., as a complex number, while on-site unitary/antiunitary symmetries should still be represented unitarily/antiunitarily.

IV. CLASSIFICATION WITH SYMMETRY BREAKING

In Ref. 37 and previous sections we have only considered 1D gapped phases whose ground state does not break any symmetry and hence is nondegenerate. These SPT phases correspond to one section (labeled “SF” in Fig. 1) in the phase diagram for short-range entangled states. Of course apart from SPT phases, there are symmetry-breaking phases. It is also possible to have phases where the symmetry is only partly broken and the nonbroken symmetry protects nontrivial quantum order. In this section, we combine symmetry breaking with symmetry protection and complete the classification for gapped phases in 1D spin systems. We find that 1D gapped phases are labeled by (1) the unbroken symmetry subgroup and (2) SPT order under the unbroken subgroup. This result is the same as that in Ref. 42.

A. Matrix product representation of symmetry breaking

Before we try to classify, we need to identify the class of systems and their gapped ground states that are under consideration. As we briefly discussed in the introduction, while the meaning of symmetry breaking is straightforward in classical system, this concept is more subtle in the quantum setting. A classical system is in a symmetry-breaking phase if each possible ground state has lower symmetry than the total system. For example, the classical Ising model has a spin-flip symmetry between spin up $|\uparrow\rangle$ and spin down $|\downarrow\rangle$ which neither of its ground states $|\uparrow\uparrow\dots\uparrow\rangle$ and $|\downarrow\downarrow\dots\downarrow\rangle$ has. However, in quantum Ising model $H = \sum_{(i,j)} -\sigma_z^i \sigma_z^j$, the ground space contains not only these two states but also any superposition of them, including the state $|\uparrow\uparrow\dots\uparrow\rangle + |\downarrow\downarrow\dots\downarrow\rangle$ which is symmetric under spin flip. This state is called the “cat” state or the GHZ state in quantum-information literature. In fact, if we move away from the exactly solvable point by adding symmetry-preserving perturbations (such as transverse field $B_x \sum_i \sigma_x^i$) and solve for the ground state at finite system size, we will always get a state symmetric under spin flip. Only in the thermodynamic limit does the ground space become two dimensional. How do we tell then whether the ground states of the system spontaneously break the symmetry?

With matrix product representation, the symmetry-breaking pattern can be easily seen from the matrices.^{43,44,46} Suppose that we solved a system with certain symmetry at finite size and found a unique minimum energy state which has the same symmetry. To see whether the system is in symmetry-breaking phase, we can write this minimum energy state in matrix product representation. The matrices in the representation can be put into a “canonical” form⁴⁴ which is block diagonal:

$$A_i = \begin{bmatrix} A_i^{(0)} & & & \\ & A_i^{(1)} & & \\ & & \ddots & \\ & & & \end{bmatrix}, \quad (29)$$

where the double tensor for each block $\mathbb{E}^{(k)} = \sum_i A_i^{(k)} \otimes (A_i^{(k)})^*$ has a nondegenerate largest eigenvalue λ_i . If in the thermodynamic limit the canonical form contains only one block, this minimum energy state is short-range correlated and the system is in a symmetric phase as discussed in Ref. 37 and

the previous section. However, if the canonical form splits into more than one block with equal largest eigenvalue (set to be 1) when system size goes to infinity, then we say the symmetry of the system is spontaneously broken in the ground states.

The symmetry-breaking interpretation of block diagonalization of the canonical form can be understood as follows. Each block of the canonical form $A_i^{(k)}$ represents a short-range correlated state $|\psi_k\rangle$. Note that here by correlation we always mean connected correlation $\langle O_1 O_2 \rangle - \langle O_1 \rangle \langle O_2 \rangle$. Therefore, the symmetry-breaking states $|\uparrow\uparrow\dots\uparrow\rangle$ and $|\downarrow\downarrow\dots\downarrow\rangle$ both have short-range correlation. Two different short-range correlated states $|\psi_k\rangle$ and $|\psi_{k'}\rangle$ have zero overlap $\langle \psi_{k'} | \psi_k \rangle = 0$ and any local observable has zero matrix elements between them $\langle \psi_{k'} | O | \psi_k \rangle = 0$. The ground state represented by A_i is an equal weight superposition of them $|\psi\rangle = \sum_k |\psi_k\rangle$. Actually the totally mixed state $\rho = \sum_k |\psi_k\rangle \langle \psi_k|$ has the same energy as $|\psi\rangle$ as $\langle \psi_{k'} | H | \psi_k \rangle = 0$ for $k' \neq k$. Therefore, the ground space is spanned by all $|\psi_k\rangle$'s. Consider the operation which permutes $|\psi_k\rangle$'s. This operation keeps ground space invariant and can be a symmetry of the system. However, each short-range correlated ground state is changed under this operation. Therefore, we say that the ground states spontaneously break the symmetry of the system.

This interpretation allows us to study symmetry breaking in 1D gapped systems by studying the block diagonalized canonical form of matrix product states. Actually, it has been shown that for any such state a gapped Hamiltonian can be constructed having the space spanned by all $|\psi_k\rangle$'s as ground space.⁴⁶ Therefore, we will focus on finite dimensional matrix product states in block diagonal canonical form for our classification of gapped phases involving symmetry breaking.

B. Classification with combination of symmetry breaking and symmetry fractionalization

We will consider class of systems with certain symmetry and classify possible phases. For simplicity of notation, we will focus on on-site unitary symmetry. With slight modification, our results also apply to parity and time reversal symmetry and their combination. Suppose that the system has on-site symmetry of group G which acts as $u(g) \otimes u(g) \dots \otimes u(g)$. It is possible that this symmetry is not broken, totally broken, or partly broken in the ground state. In general, suppose that there is a short-range correlated ground state $|\psi_0\rangle$ that is invariant under only a subgroup G' of G . Of course, different G' 's represent different symmetry-breaking patterns and hence lead to different phases. Moreover, $|\psi_0\rangle$ could have different symmetry protected order under G' which also leads to different phases. In the following we are going to show that these two sets of data: (1) the invariant subgroup G' and (2) the SPT order under G' describe all possible 1D gapped phases. Specifically we are going to show that if two systems symmetric under G have short-range correlated ground states $|\psi_0\rangle$ and $|\tilde{\psi}_0\rangle$ which are invariant under the same subgroup G' and $|\psi_0\rangle$ and $|\tilde{\psi}_0\rangle$ have the same symmetry protected order under G' , then the two systems are in the same phase. We are going to construct explicitly a path connecting the ground space of the first system to that of the second system without closing the gap and breaking the symmetry of the system.

Assume that $|\psi_0\rangle$ has a D -dimensional MPS representation $A_i^{(0)}$ which satisfies

$$\sum_{ij} u(g')_{ij} A_j^{(0)} = \alpha(g') M^{-1}(g') A_i^{(0)} M(g'), \quad (30)$$

where $g' \in G'$, $\alpha(g')$ is a 1D representation of G' , and $M(g')$ forms a projective representation of G' .

Suppose that $\{h_0, h_1, \dots, h_m\}$ are representatives from each of the cosets of G' in G , $h_0 \in G'$. Define $|\psi_k\rangle = u(h_k) \otimes u(h_k) \otimes \dots \otimes u(h_k) |\psi_0\rangle$, $k = 1, \dots, m$. $|\psi_k\rangle$ are hence each short-range correlated and orthogonal to each other. The ground space of the system is spanned by $|\psi_k\rangle$. The MPS representation of $|\psi_k\rangle$ is then $A_i^{(k)} = \sum_j u(h_k)_{ij} A_j^{(0)}$, which satisfies similar symmetry conditions

$$\sum_{ij} u(h_k g' h_k^{-1})_{ij} A_j^{(k)} = \alpha(g') M^{-1}(g') A_i^{(k)} M(g'). \quad (31)$$

To represent the whole ground space, put all $A_i^{(k)}$ into a block diagonal form and define

$$A_i = \begin{bmatrix} A_i^{(0)} & & \\ & \ddots & \\ & & A_i^{(m)} \end{bmatrix}. \quad (32)$$

The state represented by A_i is then the superposition of all $|\psi_k\rangle$, which is equivalent to the maximally mixed ground state with respect to any local observable.

Under any symmetry operation $g \in G$, A_i changes as

$$\sum_j u(g)_{ij} A_j = P(g) \Theta(g) Q(g) A_i Q^{-1}(g) P^{-1}(g), \quad (33)$$

where

$$\Theta(g) = \begin{bmatrix} \alpha(g'_{g,0}) & & \\ & \ddots & \\ & & \alpha(g'_{g,m}) \end{bmatrix} \otimes I_n, \quad (34)$$

$$P(g) = p(g) \otimes I_n, \quad (35)$$

with $p(g)$ $m \times m$ permutation matrices and forming a linear representation of G :

$$Q(g) = \begin{bmatrix} M(g'_{g,0}) & & \\ & \ddots & \\ & & M(g'_{g,m}) \end{bmatrix}. \quad (36)$$

To classify phases, we first deform the state into a simpler form by using the double tensor. Define the double tensor for the whole state as

$$\mathbb{E} = \sum_i A_i \otimes A_i^*. \quad (37)$$

As $A_i = \oplus_k A_i^{(k)}$

$$\mathbb{E} = (\oplus_k \mathbb{E}^{(k)}) \oplus (\oplus_{k \neq k'} \mathbb{E}^{(kk')}), \quad (38)$$

where $\mathbb{E}^{(k)} = \sum_i A_i^{(k)} \otimes (A_i^{(k)})^*$, $\mathbb{E}^{(kk')} = \sum_i A_i^{(k)} \otimes (A_i^{(k')})^*$. As $A_i^{(k)}$ for different k only differ by a local unitary on the physical index i , $\mathbb{E}^{(k)}$ all have the same form with a single nondegenerate largest eigenvalue. Without loss of generality, we set it to be 1 and denote the corresponding eigensector as

$\mathbb{E}_0^{(k)}$. On the other hand, $\langle \psi'_k | \psi_k \rangle = \lim_{n \rightarrow \infty} \text{Tr}(\mathbb{E}^{(kk')})^n = 0$; therefore, $\mathbb{E}^{(kk')}$ all have eigenvalues strictly less than 1. Define

$$\mathbb{E}_0 = \oplus_k \mathbb{E}_0^{(k)}. \quad (39)$$

\mathbb{E}_0 is the eigenvalue 1 sector of \mathbb{E} and $\mathbb{E}_1 = \mathbb{E} - \mathbb{E}_0$ has eigenvalues strictly less than 1.

The symmetry condition on A_i can be translated to \mathbb{E} as

$$\mathbb{E} = \bar{P}(g) \bar{\Theta}(g) \bar{Q}(g) \mathbb{E} \bar{Q}^{-1}(g) \bar{P}^{-1}(g), \quad (40)$$

where $\bar{P}(g) = P(g) \otimes P(g)$ [$P(g)$ is real], $\bar{\Theta}(g) = \Theta(g) \otimes \Theta^*(g)$, $\bar{Q}(g) = Q(g) \otimes Q^*(g)$.

Matching the eigenvalue 1 sector on the two side of Eq. (40), it is clear that

$$\mathbb{E}_0 = \bar{P}(g) \bar{\Theta}(g) \bar{Q}(g) \mathbb{E}_0 \bar{Q}^{-1}(g) \bar{P}^{-1}(g). \quad (41)$$

It follows that

$$\mathbb{E}_1 = \bar{P}(g) \bar{\Theta}(g) \bar{Q}(g) \mathbb{E}_1 \bar{Q}^{-1}(g) \bar{P}^{-1}(g). \quad (42)$$

That is, \mathbb{E}_0 and \mathbb{E}_1 satisfy the symmetry condition separately.

Now define the deformation path of the double tensor analogous to Ref. 37 as

$$\mathbb{E}(t) = \mathbb{E}_0 + \left(1 - \frac{t}{T}\right) \mathbb{E}_1. \quad (43)$$

We will show that as t increases from 0 to T , this corresponds to a deformation of the ground space to a fixed point form while the system remains gapped and symmetric under G .

First, it can be checked that for $0 \leq t \leq T$, $\mathbb{E}(t)$ remains a valid double tensor and satisfies symmetry condition Eq. (40). Decomposing $\mathbb{E}(t)$ back into matrices, $A_i(t)$ necessarily contains m blocks each with finite correlation length. The fact that the total state remains gapped is proven by Ref. 46. Because two equivalent double tensors can only differ by a unitary transformation on the physical index, the symmetry condition Eq. (40) for $\mathbb{E}(t)$ gives that there exist unitary transformations $u(g)(t)$ such that $A_i(t)$ transform in the same way as A_i in Eq. (33). The symmetry operation can be defined continuously for all t .

At $t = T$, the state is brought to the fixed point form $\mathbb{E}(T) = \mathbb{E}_0 = \sum_k \mathbb{E}_0^{(k)}$. Each block k represents a dimer state as in Fig. 2 with entangled pairs between neighboring sites supported on dimension $1_k, \dots, D_k$, $|E P_k\rangle = \lambda_{i_k} |i_k i_k\rangle$. For different blocks k and k' , $|i_k\rangle \perp |i_{k'}\rangle$. The total Hilbert space on one site is $(D \times m)^2$ dimensional. The symmetry operation on one site can then be defined as $u(g) = P(g) Q^*(g) \Theta(g) \otimes P(g) Q(g)$. With continuous deformation that does not close the gap or violate symmetry, every gapped ground space is mapped to such a fixed point form. If we can show that two fixed point states with the same unbroken symmetry G' and SPT order under G' are in the same phase, we can complete the classification for combined symmetry breaking and SPT order.

Suppose that $|\psi_0\rangle$ and $|\tilde{\psi}_0\rangle$ are short-range correlated fixed point ground states of two systems symmetric under G . Symmetry operations are defined as $u(g)$ and $\tilde{u}(g)$ respectively. $|\psi_0\rangle$ and $|\tilde{\psi}_0\rangle$ are symmetric under the same subgroup G' and have the same SPT order. As shown in Ref. 37, $|\psi_0\rangle$ and $|\tilde{\psi}_0\rangle$ can be mapped to each other with local unitary transformation W_0 that preserves G' symmetry. The other short-range correlated ground states can be obtained as $|\psi_k\rangle =$

$u(h_k)|\psi_0\rangle$ and $|\tilde{\psi}_k\rangle = \tilde{u}(h_k)|\tilde{\psi}_0\rangle$. At fixed point, $|\psi_k\rangle(|\tilde{\psi}_k\rangle)$ are supported on orthogonal dimensions for different k and $u(h_k)$ and $\tilde{u}(h_k)$ maps between these support spaces. Therefore, we can consistently define local unitary operations mapping between $|\psi_k\rangle$ and $|\tilde{\psi}_k\rangle$ as $W_k = \tilde{u}(h_k)W_0u^\dagger(h_k)$ and the total operation is $W = \bigoplus_k W_k$. W as defined is a local unitary transformation symmetric under G that maps between two fixed point gapped ground states with the same unbroken symmetry G' and SPT order under G' . Combined with the mapping from a general state to its fixed point form, this completes our proof that 1D gapped phases are labeled by unbroken symmetry G' and SPT order under G' .

V. APPLICATION: 1D FERMION SPT PHASES

Although our previous discussions have been focused on spin systems, it actually also applies to fermion systems. Because in 1D, fermion systems and spin systems can be mapped to each other through the Jordan-Wigner transformation, we can classify fermionic phases by classifying corresponding spin phases. Specifically, for a class of fermion systems with certain symmetry we are going to (1) identify the corresponding class of spin systems by mapping the symmetry to spin, (2) classify possible spin phases with this symmetry, including symmetry breaking and symmetry fractionalization, and (3) map the spin phases back to fermions and identify the fermionic order. In the following we are going to apply this strategy to 1D fermion systems in the following four cases respectively: no symmetry (other than fermion parity), time reversal symmetry for spinless fermions, time reversal symmetry for spin half integer fermions, and $U(1)$ symmetry for fermion number conservation. Our classification result is consistent with previous studies in Refs. 47 and 48. One special property of fermionic system is that it always has a fermionic parity symmetry. That is, the Hamiltonian is a sum of terms composed of even numbers of creation and annihilation operators. Therefore, the corresponding spin systems we classify always have an on-site Z_2 symmetry. Note that this approach can only be applied to systems defined on an open chain. For systems with translation symmetry and periodic boundary conditions, the Jordan-Wigner transformation could lead to nonlocal interactions in the spin system.

A. Fermion parity symmetry only

For a 1D fermion system with only fermion parity symmetry, how many gapped phases exist?

To answer this question, first we do a Jordan-Wigner transformation and map the fermion system to a spin chain. The fermion parity operator $P_f = \prod_i (1 - 2a_i^\dagger a_i)$ is mapped to an on-site Z_2 operation. On the other hand, any 1D spin system with an on-site Z_2 symmetry can always be mapped back to a fermion system with fermion parity symmetry (expansion of local Hilbert space may be necessary). As the spin Hamiltonian commutes with the Z_2 symmetry, it can be mapped back to a proper physical fermion Hamiltonian. Therefore, the problem of classifying fermion chains with fermion parity is equivalent to the problem of classifying spin chains with Z_2 symmetry.

There are two possibilities in spin chains with Z_2 symmetry: (1) The ground state is symmetric under Z_2 . As Z_2 does

not have nontrivial projective representation, there is one symmetric phase. (If translational symmetry is required, systems with even numbers of fermions per site are in a different phase from those with odd numbers of fermions per site. This difference is somewhat trivial and we will ignore it.) (2) The ground state breaks the Z_2 symmetry. The ground state will be twofold degenerate. Each short-range correlated ground state has no particular symmetry ($G' = I$) and they are mapped to each other by the Z_2 operation. There is one such symmetry-breaking phase. These are the two different phases in spin chains with Z_2 symmetry.

This tells us that there are two different phases in fermion chains with only fermion parity symmetry. But what are they? First of all, fermion states cannot break the fermion parity symmetry. All fermion states must have a well-defined parity. Does the spin symmetry-breaking phase correspond to a real fermion phase?

The answer is yes and actually the spin symmetry-breaking phase corresponds to a Z_2 symmetric fermion phase. Suppose that the spin system has two short-range correlated ground states $|\psi_0\rangle$ and $|\psi_1\rangle$. All connected correlations between spin operators decay exponentially on these two states. Mapped to fermion systems, $|\psi_0^f\rangle$ and $|\psi_1^f\rangle$ are not legitimate states but $|\tilde{\psi}_0^f\rangle = |\psi_0^f\rangle + |\psi_1^f\rangle$ and $|\tilde{\psi}_1^f\rangle = |\psi_0^f\rangle - |\psi_1^f\rangle$ are. They have even/odd parity respectively. In spin systems, $|\tilde{\psi}_0\rangle$ and $|\tilde{\psi}_1\rangle$ are not short-range correlated states but mapped to fermion systems they are. To see this, note that any correlator between bosonic operators on the $|\tilde{\psi}_0^f\rangle$ and $|\tilde{\psi}_1^f\rangle$ is the same as that on $|\psi_0\rangle$ and $|\psi_1\rangle$ and hence decays exponentially. Any correlator between fermionic operators on the $|\tilde{\psi}_0^f\rangle$ and $|\tilde{\psi}_1^f\rangle$ gets mapped to a string operator on the spin state; for example $a_i^\dagger a_j$ is mapped to $(X - iY)_i Z_{i+1} \dots Z_{j-1} (X - iY)_j$, which also decays with separation between i and j . Therefore, the symmetry-breaking phase in spin chains corresponds to a fermionic phase with symmetric short-range correlated ground states. The degeneracy can be understood as isolated Majorana modes at the two ends of the chain.^{53,54}

On the other hand, the short-range correlated ground state in the spin symmetric phase still correspond to the short-range correlated fermion state after JW transformation. Therefore, the symmetric and symmetry-breaking phases in spin systems both correspond to symmetric phases in fermion systems. The two fermion phases cannot be connected under any physical fermion perturbation.

B. Fermion parity and $T^2 = 1$ time reversal

Now consider the more complicated situation where aside from fermion parity, there is also a time reversal symmetry T . T acts as an antiunitary $T = UK$ on each site. In this section we consider the case where $T^2 = 1$ (spinless fermion).

So now the total symmetry for the fermion system is the Z_2 fermion parity symmetry P_f and $T^2 = 1$ time reversal symmetry. T commutes with P_f . The on-site symmetry group is a $Z_2 \times Z_2$ group and has four elements $G = \{I, T, P_f, T P_f\}$. Mapped to the spin system, the symmetry group structure is kept.

The possible gapped phases for a spin system with on-site symmetry $G = \{I, T, P_f, T P_f\}$ include (1) $G' = G$. Following

the discussion in Sec. III we find that it has four different projective representations. Examples of the four representations are (a) $\{I, K, Z, KZ\}$, (b) $\{I, iYK, Z, iYKZ\}$, (c) $\{I, iYKZ \otimes I, I \otimes Z, iYKZ \otimes Z\}$, (d) $\{I, K, Y, KY\}$. There are hence four different symmetric phases. [If translational symmetry is required, the number is multiplied by 2 due to $\alpha(Z_2)$.] (2) $G' = \{I, P_f\}$ with no nontrivial projective representation; the time reversal symmetry is broken. There is one such phase. (If translational symmetry is required, there are two phases.) (3) $G' = \{I, T\}$, with two different projective representations (time reversal squares to $\pm I$ on boundary spin). The Z_2 fermion parity is broken. There are two phases in this case. (4) $G' = \{I, TP_f\}$, with two different projective representations. The fermion parity symmetry is again broken. There are two different phases. (5) $G' = I$, no projective representation; all symmetries are broken.

Mapped back to fermion systems, fermion parity symmetry is never broken. Instead, the P_f symmetry-breaking spin phases are mapped to fermion phases with Majorana boundary mode on the edge as discussed in the previous section. Therefore the above spin phases correspond in the fermion system to (1) four different symmetric phases; (2) one time reversal symmetry breaking phase; (3) two symmetric phases with Majorana boundary mode; (4) another two symmetric phases with Majorana boundary mode; (5) one time reversal symmetry breaking phase. (1), (3), and (4) contain the eight symmetric phases for time reversal invariant fermion chains with $T^2 = 1$. This is consistent with previous studies in Refs. 47 and 55.

C. Fermion parity and $T^2 \neq I$ time reversal

When $T^2 \neq I$, the situation is different. This happens when we take the fermion spin into consideration and for a single particle, time reversal is defined as $e^{i\pi\sigma_y} K$. With half integer spin, $(e^{i\pi\sigma_y} K)^2 = -I$. Note that for every particle the square of time reversal is $-I$; however, when we write the system in second quantization as creation and annihilation operator on each site, the time reversal operation defined on each site satisfies $T^2 = P_f$. Therefore, the symmetry group on each site is a Z_4 group $G = \{I, T, P_f, TP_f\}$. To classify possible phases, we first map everything to spin.

The corresponding spin system has on-site symmetry $G = \{I, T, P_f, TP_f\}$. $T^2 = P_f$, $P_f^2 = I$. The possible phases are (1) $G' = G$, with two possible projective representations, one with $T^4 = I$, the other with $T^4 = -I$. Examples for the latter include $T = (1/\sqrt{2})(X + Y)K$. Therefore, there are two possible symmetric phases. (If translational symmetry is required, there are four phases.) (2) $G' = \{I, P_f\}$; the time reversal symmetry is broken. There is one phase. (If translational symmetry is required, there are two phases.) (3) $G' = I$; all symmetries are broken. There is one phase.

Therefore, the fermion system has the following phases: (1) two symmetric phases; (2) one time reversal symmetry breaking phase; (3) one time reversal symmetry breaking phase with Majorana boundary mode. (1) contains the time reversal symmetry protected topological phase. Models in this phase can be constructed by first writing out the spin model in the corresponding spin phase and then mapping

it to a fermion system with the Jordan-Wigner transformation.

D. Fermion number conservation

Consider the case of a gapped fermion system with fixed fermion number. This corresponds to an on-site $U(1)$ symmetry, $e^{i\theta N}$. Mapped to spins, the spin chain will have an on-site $U(1)$ symmetry. This symmetry cannot be broken and $U(1)$ does not have a nontrivial projective representation. One thing special about $U(1)$ symmetry, though, is that it has an infinite family of 1D representations. If translational symmetry is required, the fermion number per site is a good quantum number and labels different phases. Therefore, mapped back to fermions, there is an infinite number of phases with different average number of fermions per site.

VI. CONCLUSION

In this paper, we complete the classification of gapped phases in 1D spin systems with various symmetries. Based on our classification of symmetry-protected topological phases with on-site unitary, parity, or time reversal symmetry in Ref. 37, we give explicit results in this paper for the classification of SPT phases with combined on-site unitary, parity, and/or time reversal symmetry. A general rule is also given for the classification of SPT phases with any symmetry group. Moreover, we considered the classification of phases with possible (partial) symmetry breaking. We find that 1D gapped spin phases with symmetry of group G are basically labeled by (1) the unbroken symmetry subgroup G' and (2) projective representations of G' . Note that in calculating projective representations of G' , on-site unitary symmetries are represented unitarily while parity and on-site antiunitary symmetries are represented antiunitarily. We apply this classification result to interacting 1D fermion systems, which can be mapped to spin systems with the Jordan-Wigner transformation, and classify possible gapped phases with no symmetry, time reversal symmetry, and also fermion number conservation.

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APPENDIX A: EXISTENCE OF N SUCH THAT

$$N^{-1}R(g)N = R^*(g)$$

In this section we will show that for any class ω of projective representation of group G which satisfies $\omega^2 = 1$, there is a projective representation $R(g) \in \omega$ and a symmetric matrix $N(N^T = N)$ such that $N^{-1}R(g)N = R^*(g)$.

Suppose that $R_0(g)$ is a d -dimensional projective representation in class ω . Because $\omega^2 = 1$, $R_0^*(g)$ is also a projective representation in this class, and so is

$$R(g) = \begin{bmatrix} R_0(g) & \\ & R_0^*(g) \end{bmatrix}. \quad (\text{A1})$$

Define

$$N = \begin{bmatrix} & I \\ I & \end{bmatrix}, \quad (\text{A2})$$

where I is a d -dimensional identity matrix. It can be checked that $N^T = N$ and $N^{-1}R(g)N = R^*(g)$.

APPENDIX B: FREEDOM IN COMMUTATION RELATION BETWEEN PARITY/TIME REVERSAL AND ON-SITE G

We will show in this section that for a fixed factor system $\omega(\omega^2 = 1)$, if there exists projective representation $R(g)$ and symmetric or antisymmetric matrix $N(N^T = \pm N)$, such that $N^{-1}R(g)N = \gamma(g)R^*(g)$ for one 1D representation $\gamma(g)$, then there are other $R'(g)$ and N' which satisfy the relation for any other $\gamma'(g)$.

Suppose that $\gamma'(g) = \alpha(g)\gamma(g)$.

Define

$$R'(g) = \begin{bmatrix} R(g) & \\ & \alpha(g)R(g) \end{bmatrix}, \quad N' = \begin{bmatrix} N & \\ & N \end{bmatrix} \begin{bmatrix} I & \\ & I \end{bmatrix}. \quad (\text{B1})$$

$R'(g)$ is another projective representation with factor system ω and $N'^T = \pm N'$.

Moreover it can be checked that

$$N'^{-1}R'(g)N' = \alpha(g)\gamma(g)R' = \gamma'(g)R'. \quad (\text{B2})$$

APPENDIX C: FREEDOM IN COMMUTATION RELATION BETWEEN ON-SITE G , TIME REVERSAL, AND PARITY

We will show in this section that for a fixed factor system $\omega(\omega^2 = 1)$, if there exists projective representation $R(g)$ and symmetric or antisymmetric matrix $N(N^T = \pm N)$, such that $N^{-1}R(g)N = \gamma(g)R^*(g)$ for one 1D representation $\gamma(g)$, then there exist $R'(g)$, $N'(N'^T = \pm N')$, $M'(M'^T = \pm M')$, such that $M'N'^* = N'M'^*$, $N'^{-1}R'(g)N' = \gamma(g)R'^*(g)$, $M'^{-1}R'(g)M' = \gamma'(g)R'^*(g)$, for any $\gamma'(g) = \chi(g)\gamma(g)$, $\chi^2(g) = 1$.

Define

$$R'(g) = \begin{bmatrix} R(g) & \\ & \chi(g)R(g) \end{bmatrix}, \quad (\text{C1})$$

$$N' = \begin{bmatrix} N & \\ & N \end{bmatrix}, \quad (\text{C2})$$

$$M' = \begin{bmatrix} & N \\ N & \end{bmatrix}. \quad (\text{C3})$$

It can be checked that $R'(g)$ is a projective representation with factor system ω . $N'^T = \pm N'$, and $M'^T = \pm M'$. Moreover,

$$M'N'^* = \begin{bmatrix} & NN^* \\ NN^* & \end{bmatrix} = N'M'^*, \quad (\text{C4})$$

$$N'^{-1}R'(g)N' = \gamma(g)R'^*(g), \quad (\text{C5})$$

$$M'^{-1}R'(g)M' = \chi(g)\gamma(g)R'^*(g) = \gamma'(g)R'^*(g). \quad (\text{C6})$$

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