Transport properties of nonequilibrium systems under the application of light: Photoinduced quantum Hall insulators without Landau levels

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In this paper, we study transport properties of nonequilibrium systems under the application of light in many-terminal measurements, using the Floquet picture. We propose and demonstrate that the quantum transport properties can be controlled in materials such as graphene and topological insulators, via the application of light. Remarkably, under the application of off-resonant light, topological transport properties can be induced; these systems exhibit quantum Hall effects in the absence of a magnetic field with a near quantization of the Hall conductance, realizing so-called quantum Hall systems without Landau levels first proposed by Haldane.

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I. INTRODUCTION

Application of light is a powerful method to change material properties. For example, light can induce currents through mechanisms such as photovoltaic effect,[1] photothermoelectric effect,[2] and photo-drag effects.[3] Moreover, light can change the response of materials and induce insulator-to-metal transitions[4] or change the characteristics of *p-n* junctions.[5]

In recent years, there has been tremendous development and interest in the induction of quantum phases through light applications. For example, experiments have demonstrated that superconductivity can be induced through infrared pulses in high-temperature cuprate superconductors.[6] Inductions of quantum phases are inherently nonequilibrium phenomena, and thus their understanding is quite challenging. Even some basic questions such as the physical signatures of the induced phases and how such phases can be stabilized in a steady state do not have answers yet. Many of quantum phases manifest themselves through transport and, therefore, understanding of transport properties in nonequilibrium, open systems is crucial for experimental verifications of such induction of quantum phases under the application of light. In this paper, we develop a general formalism for studying nonequilibrium transport under the application of light and, using the formalism, address the possibilities of the induction of topological properties through light.

Motivated by recent rapid development of the understanding in topological phases, the possibility of inducing topological phases such as integer quantum Hall phase and topological insulators through light has been theoretically explored by many different groups.[7–9] Generally speaking, the application of light on electron systems has two important physical effects: (1) photon-dressing of band structures through the mixing of different bands and (2) redistribution of electron occupation numbers through the absorptions/emissions of photons leading to nonequilibrium distributions. Previous works proposed optical induction of band structures with topological properties[7–9] and thus have mostly focused on the analysis of the first effect. On the other hand, most of these works do not address the question of the second effect, the redistribution of electrons in the band structure, and thus its physics is yet poorly understood. Topological properties only appear when certain bands are fully filled, and it is not clear how this band occupation can be achieved and topological properties survive when the system is strongly driven out of equilibrium by the application of light.

In order to answer these questions, we study the physical consequence of the application of light through dc many-terminal transport measurements as in Fig. 1. The coupling of the driven systems with leads, which are in return coupled with equilibrium reservoirs, plays the crucial role to determine the occupations of electrons. Using the formalism for transport properties in periodically driven systems developed by various groups,[10] here we study topological transport phenomena in materials such as graphene and three-dimensional topological insulators.

First of all, we show that nonequilibrium transport properties cannot generally be captured by the photon-dressed, effective band structures. In particular, in addition to the usual transport through such static effective band structures, there are contributions from photon-assisted electron conductions. Thus, the induction of topologically nontrivial band structures does not immediately imply the topological properties of the nonequilibrium systems.

On the other hand, the regime exists in which topological band structures can manifest themselves; we explicitly demonstrate that for *off-resonant* light where electrons cannot directly absorb photons, the transport properties of the nonequilibrium systems attached to the leads are well approximated by the transport properties of the system described by the *static* effective Hamiltonian that incorporates the virtual photon absorption processes. In particular, the occupations of the electrons under this situation are close to the filling of the photon-dressed bands. As examples, we show that the transport properties under the application of off-resonant light is given by the photon-dressed Hamiltonian corresponding to a quantum Hall insulator without Landau levels[11] in the case of graphene and to a gapped insulator with anomalous quantum Hall effects and magnetoelectric response described by axion electrodynamics[12] at the surfaces of three-dimensional topological insulators. In these systems, the measurements in

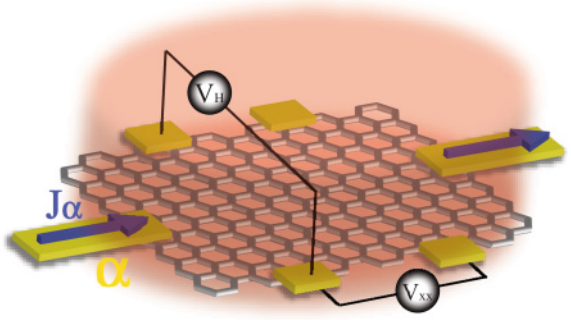

FIG. 1. (Color online) The many-terminal measurements of dc current for graphene under the application of light. Graphene is attached to multiple leads labeled by $\{\alpha\}$, and the leads are connected to reservoirs at chemical potentials, $\{\mu_\alpha\}$. Off-resonant, circularly polarized light is applied to the graphene. In the absence of impurities and interactions, electrons coming from leads coherently propagate under the application of light, asborb or emit photons, and leak out into leads. Current measurements between each lead determine longitudinal and Hall conductances.

six-terminal configurations in Fig. 1 lead to the near quantization of Hall conductance. Thus, the application of off-resonant circularly polarized light leads to intriguing "Hall" effects *without applying a static magnetic field*.

This paper is organized as follows. In Sec. II, we describe the summary of the results, focusing on the analysis of graphene and three-dimensional topological insulators under the application of off-resonant light. Here we provide the physical and intuitive explanations of the phenomenon of light-induced quantum Hall effects and refer to later sections for many important details. In Sec. III, we develop the formalism for studying the nonequilibrium transport properties under periodical drives in many-terminal measurements. Our formalism is based on the extension of the multiprobe Büttiker-Landauer formula[13] to periodically driven systems, a "Floquet Landauer formula." We provide two distinct ways to calculate the transmission amplitudes in the driven systems. The first method expresses the results in terms of the Floquet states and it illuminates the physical origin of the photon-assisted transport. The second method takes advantage of the "Floquet Dyson's equation" to give an elegant solution which is more convenient for a numerical solution.[14] By taking the off-resonant limit of these solutions, the equivalence of transport properties under the application of light and those with effective photon-dressed Hamiltonian is established.

Most of the analysis in this paper assumes the absence of interactions among electrons as well as electron-phonon interactions. We argue in Sec. IV that, in the case of graphene and topological insulators under the off-resonant light, the results given in Sec. II are robust against these interaction effects at low temperatures. While the measurements of transport properties require the attachment of leads, the probe of the effective gap induced by light is plausible even in an isolated system. We propose in Sec. V such measurements through the adiabatic preparation of nonequilibrium systems combined with the transmission of probe laser with small frequencies. The essential ingredients in the arguments of Secs. IV and V are the extensions of adiabatic theorem and Fermi golden rule to periodically driven systems and Floquet states, dubbed the "Floquet adiabatic theorem" and the "Floquet Fermi golden rule." We give a detailed proof of these important statements in the Appendix. In Sec. VI, we conclude with possible extensions of this work.

II. SUMMARY OF RESULTS

A. Garphene effective Hamiltonian

Here we consider graphene as an example of a semimetal and study the change in the transport properties under the application of light. We model graphene using a hexagonal tight-binding model with two π bands, where we first neglect the electron-electron as well as electron-phonon interactions. In Sec. IV, we discuss the effects of these interactions and argue that they do not change the qualitative results of the analysis. We consider the application of circularly polarized light perpendicular to the plane of graphene. For concreteness, here we represent the rotating electric field due to light as a time-dependent vector potential $\mathbf{A}(t) = A(\pm\sin(\Omega t), \cos(\Omega t))$ with $\mathbf{E}(t) = \partial\mathbf{A}(t)/\partial t$, where Ω is the frequency of light. The plus sign is for right circulation of light and minus sign for left circulation. The light intensity is characterized by the dimensionless number $\mathcal{A} = eAa/\hbar$, where e is the electron charge and $a \approx 2.46$ Å is the lattice constant of graphene. For intensity of lasers and pulses available in the frequency regime of our interests $\sim$1000 THz, $\mathcal{A}$ is typically less than 1. In this gauge, electrons accumulate phases as they hop in the lattice;

$$H(t) = -J\sum_{\langle ij\rangle,s} e^{iA_{ij}(t)}c^\dagger_{i,s}c_{j,s}, \tag{1}$$

where $A_{ij}(t) = e/\hbar(\mathbf{r}_j - \mathbf{r}_i)\cdot\mathbf{A}(t)$, with $\mathbf{r}_i$ being the coordinates of the lattice site i, J the the hopping amplitude of electrons, and $s = \uparrow, \downarrow$ the spins of electrons. For simplicity, we only consider the orbital effect of electromagnetic fields on electrons and disregard the small Zeeman effect. The inclusion of the Zeeman effect is straightforward. In this limit, spins trivially double the Hilbert space and thus we suppress the spin indices in the following.

When the light frequency is off-resonant for any electron transitions, light does not directly excite electrons and instead effectively modifies the electron band structures through virtual photon absorption processes. Such off-resonant condition is satisfied for the frequency $\Omega \gg J$ in our model with π bands. A more general case of on-resonant light can be analyzed through the formalism developed in section Sec. III. The influence of such off-resonant light is captured in the static effective Hamiltonian $H_{\rm eff}$ (Ref. 9) defined through the evolution operator U of the system after one period $T = 2\pi/\Omega$ as

$$H_{\rm eff} = \frac{i}{T}\log(U), \tag{2}$$

where $U = \mathcal{T}\exp(-i\int_0^T H(t)dt)$ and $\mathcal{T}$ is the time-ordering operator. Intuitively, $H_{\rm eff}$ describes the dynamics of the system on time scales much longer than T. In the limit of $\mathcal{A}^2 \ll 1$,

$H_{\rm eff}$ is particularly simple near the Dirac points:

$$H_{\rm eff} \approx H_0 + \frac{[H_{-1}, H_1]}{\Omega} + O(\mathcal{A}^4)$$
$$\approx v_G(\sigma_y k_x - \sigma_x k_y \tau_z) \pm \frac{v_G^2 \mathcal{A}^2}{\Omega} \sigma_z \tau_z + O(\mathcal{A}^4) \quad \text{(for infinite system)}, \quad (3)$$

where H_n is the discrete Fourier component of Hamiltonian, that is, $H_n = \frac{1}{T}\int_0^T H(t) e^{it\Omega n} dt$. In the second line, $v_G = 3J/2$ is the velocity of Dirac electrons, k_x and k_y are momenta measured from the Dirac points, and σ_i and τ_i are Pauli matrices representing sublattice and valley degrees of freedom, respectively.

The modification of the Hamiltonian with respect to the static component H_0 is the second term in Eq. (3). This term can be easily understood as the sum of two second-order processes as illustrated in Fig. 2(a): one where electron absorbs a photon and then emits a photon $H_1 \frac{1}{\omega-(\omega+\Omega)} H_{-1}$ and another where electrons first emit a photon and then absorb a photon, which leads to $H_{-1} \frac{1}{\omega-(\omega-\Omega)} H_1$, where ω is the energy of the electron before the photon absorption/emission. By summing these two contributions, we obtain the correction due to the second-order process, given in the second term of Eq. (3). In the second line, the plus sign is for right circulation of light polarization and the minus sign is for left circulation. For more rigorous derivation of this result, see Sec. III. We note that the expression of the effective Hamiltonian in Eq. (3) is only valid in the gauge in which light is represented as time-dependent vector potential, and the effective Hamiltonian has different forms for other gauge such as the one in which light represented as time-dependent electric fields.

The effect of virtual photon absorptions at the degenerate Dirac points is to open a gap with magnitude $\Delta = \frac{2v_x^2 \mathcal{A}^2}{\Omega}$. In Fig. 3, we illustrate the opening of the gap near one of the Dirac points upon the application of light for both infinite and finite systems. This Hamiltonian $H_{\rm eff}$ with two bands separated by a band gap corresponds to a quantum Hall insulator, where each band is characterized by a nonzero Chern number.[11,15] In Fig. 3(b), we have plotted the spectrum of $H_{\rm eff}$ where the system is infinite in the x direction and 150 sites in the y direction with armchair edges. Here we have chosen the

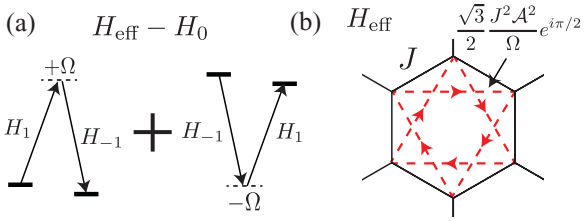

FIG. 2. (Color online) (a) The modification of the Hamiltonian due to the virtual photon process can be intuitively understood as the sum of two second-order processes where electrons absorb and then emit a photon and electrons first emit and absorb a photon. (b) The illustration of the structure of $H_{\rm eff}$ in real space for graphene under the application of right circularly polarized light in Eq. (3). The commutator $[H_1, H_1]$ is the second-neighbor hopping with phase $\varphi = \pi/2$. Thus, the tight-binding model under the application of light realizes Haldane model proposed in Ref. 11.

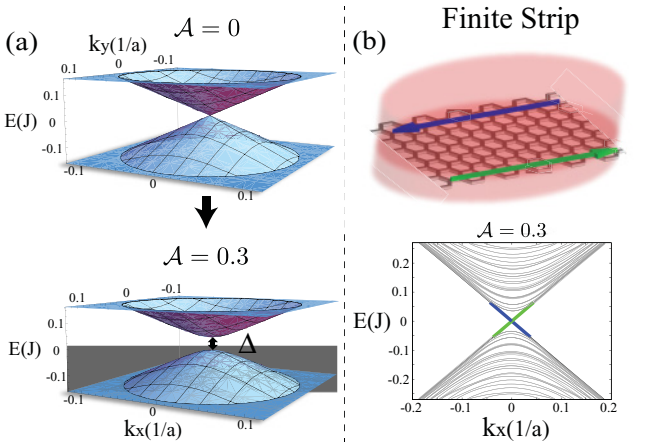

FIG. 3. (Color online) (a) The spectrum of graphene for a single spin near one of the Dirac points for an infinite system. Here we plot the energy spectrum of a static system corresponding to $\mathcal{A} = 0$ (top figure) and the spectrum of $H_{\rm eff}$ for the system under the application of light with light intensity $\mathcal{A} = 0.3$ (bottom figure). A finite gap Δ opens at the Dirac point. (b) The spectrum of $H_{\rm eff}$ for a single spin near one of the Dirac points for infinite system in the x direction and 150 sites in the y direction for $\mathcal{A} = 0.3$. The frequency of light is chosen to be off-resonant with $\Omega = 7.5J$. Here we chose the armchair edges, and in this case the Dirac points are at $k_x = 0$. $H_{\rm eff}$ shows the existence of gapless chiral edge states for each spin originating from nonzero Chern numbers of the bands, which are colored as blue and green, corresponding to the edge states in the upper and lower edges, respectively. The propagation of chiral edge states for a single spin is illustrated in the top figure.

intensity and frequency of light to be $\mathcal{A} = 0.3$ and $\Omega = 7.5J$, respectively. As a result of nonzero Chern number of the bands, $H_{\rm eff}$ shows the existence of gapless chiral edge states, colored as blue and green, corresponding to the edge state in the upper and lower edge, respectively.

Further intuition can be obtained by writing the effective Hamiltonian in Eq. (3) in real space. In the lowest order in $\mathcal{A}$, H_1 and H_{-1} are the hopping between nearest neighbors with phase accumulations that depend on the direction of the hopping. Their commutators contain the second-neighbor hopping with amplitudes $\frac{\sqrt{3}}{2}\frac{J^2\mathcal{A}^2}{\Omega} e^{i\pi/2}$ as illustrated in Fig. 2(b). Thus, the effective Hamiltonian is nothing but the Hamiltonian proposed by Haldane[11] with sublattice potential $M = 0$ and second-neighbor hopping strength $t_2 = \frac{\sqrt{3}}{2}\frac{J^2\mathcal{A}^2}{\Omega}$ with the flux $\varphi = \pi/2$.

Our results above predict the induction of nonzero Chern number for the tight-binding model without second-neighbor hoppings and thus differ from the results obtained in Inoue and Tanaka.[16] In their work, the effect of circularly polarized light on the Haldane model[11] has been considered. They focused on the zero-photon sector of the Hamiltonian and concluded that the Chern number is zero whenever the second-neighbor hopping t_2 is zero or the staggered magnetic flux φ is zero, as is presented in Eq. (7) of their paper. Their work showed no transition from topologically trivial band insulators to topological nontrivial bands with Chern numbers. Here we considered a simple tight-binding model without second-neighbor hopping or staggered-magnetic fields, corresponding

to $t_2 = 0$ and $\varphi = 0$ of the Haldane model, and demonstrated that the Chern number can be induced in this system. In contrast with the result of Ref. 16, we show above (also see Sec. III B 2) that the virtual photon absorption and emission process represented by the second term of Eq. (3) induces a gap at the Dirac point and leads to a nonzero Chern number. Such photon dressing, which was neglected in the study of Ref. 16, has dramatic effects at the degenerate Dirac points and should be taken into account.

There are a few different ways to probe the gap Δ in the effective Hamiltonian. For example, the gap opening might be confirmed through the observation of the transmission of low-frequency probe lasers. In Sec. V, we propose the possibility to probe this dynamically opened gap in an isolated system through adiabatic preparation of Floquet states. The main focus of this paper, however, is the study of the manifestations of the gapped Hamiltonian H_{eff} in many-terminal transport measurements depicted in Fig. 1, which we now describe.

B. Floquet Landauer formalism and Hall current

One of the central results of this paper is the demonstration that measurements of dc current of the nonequilibrium system under the application of off-resonant light is determined by the static, photon-dressed Hamiltonian H_{eff}. To this end, we consider the many-terminal measurements of dc current under the application of light as in Fig. 1.[13] The nonequilibrium transport properties of mesoscopic, periodically driven systems in this configuration have been studied previously, using Floquet theory[17] combined with the Keldysh formalism.[10] We express the general result obtained in these works as the extension of the multiprobe Büttiker-Landauer formula[13] to periodically driven systems, a "Floquet Landauer formula":

$$J_\alpha^{\text{dc}} = J_\alpha^{\text{pump}} + J_\alpha^{\text{res}}, \tag{4}$$

$$J_\alpha^{\text{res}} = \sum_\beta \left(\sum_n T_{\alpha\beta}(n) \right) (\mu_\beta - \mu_\alpha). \tag{5}$$

Here J_α^{dc} is the dc component of the current at lead α, $J_\alpha^{\text{dc}} = \frac{1}{T}\int_0^T J_\alpha(t)dt$. Here we assumed that the reservoirs are at zero temperature, and their chemical potentials $\{\mu_\alpha\}$ are near the Dirac points, that is, $\mu_\alpha \approx 0$. The dc current of a periodically driven system in Eq. (4) consists of two physically distinct contributions; the pump current,[18] J_α^{pump}, which can be present even when all the reservoirs have the same chemical potential μ_α, and the response current, J_α^{res}, which arises from the response of the driven system to the chemical potential differences of the reservoirs. For the application of light we consider in this paper, the pump current J_α^{pump} is zero for inversion symmetric geometries, so we focus on the properties of the response current J_α^{res} in the following. The transmission coefficients $T_{\alpha\beta}(n)$ in Eq. (5) represent the transmission of electrons with energy $\mu_\beta \approx 0$ from lead β to lead α during which electrons absorb (emit) n photons as illustrated in Fig. 4.

Thus, the response conduction in the presence of periodic drive can be seen as an extension of the static conduction, where the transmission can now happen with the absorptions/emissions of photons. We note that the expression of dc current in Eq. (5) is valid for arbitrary strength and frequency of the drives. The transmission probabilities $T_{\alpha\beta}(n)$ can be efficiently computed by dressing the propagators with photon absorptions/emissions[14] and are described in Sec. III. We emphasize that the response current is the sum of the contributions from n photon absorption/emission processes [see Eq. (5)] and thus its transport property generally cannot be described by effective static Hamiltonians. The off-resonant case described below is an exceptionally simple case in this respect.

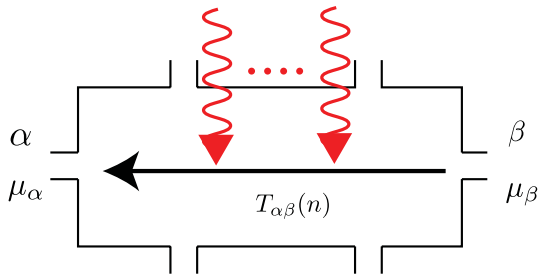

FIG. 4. (Color online) Illustration of the response conduction current in the Floquet Landauer formula given in Eq. (5). The transmission of the electrons can now happen with n photon absorption/emissions. The total conductance is simply given by the sum of these contributions for each n.

We employ this Floquet Landauer formalism to study the off-resonant, large-frequency regime $J \ll \Omega$ with weak intensity of light, that is, $\mathcal{A}^2 \ll 1$. In this regime, absorptions or emissions of photons are suppressed by $\mathcal{A}^2$, and the transmission coefficients $T_{\alpha\beta}(n)$ with $n \neq 0$ is small and of the order of $O(\mathcal{A}^{2n})$. On the other hand, the zero-photon absorption/emission transmission coefficient $T_{\alpha\beta}(0)$ is modified due to virtual photon processes. Such modifications are included in H_{eff} and the transmission probability is given by $T_{\alpha\beta}(0) = T_{\alpha\beta}^{\text{eff}} + O(\mathcal{A}^2)$, where $T_{\alpha\beta}^{\text{eff}}$ is the transmission probability of the static system described by H_{eff}. These results are rigorously established in Sec. III.

This correspondence demonstrates, under our assumptions, that graphene under the application of off-resonant light behaves as an insulator with gap $\Delta = \frac{2v_*^2\mathcal{A}^2}{\Omega}$ with Hall conductance quantized at $2e^2/h$ with possible corrections up to the order of $O(\mathcal{A}^2)$. Here the factor of 2 comes from spin degrees of freedom. While we established the results in the perturbation theory on $\mathcal{A}$, it is possible to analytically confirm the insulating behavior for all orders in $\mathcal{A}$ for weak contact couplings with leads (see Sec. III). We emphasize that although the effective Hamiltonian is perturbative in $\mathcal{A}$, the Hall conductance at zero temperature is nonperturbative: An infinitesimal gap Δ is sufficient to yield a topological band with nonzero Chern number.

A distinct feature of this light-induced Hall effect above is that the Hall conductance switches its sign under the change of circulations of light polarization. This can be easily checked for the geometry of the system which is symmetric under $x \to -x$, under which the circulation of light reverses. Such reversal of Hall current can be used in the experiments to distinguish this light-polarization-*dependent* current from light-polarization-*independent* current, which could originate from mechanisms we did not consider in this paper.

We briefly describe the requirements to observe the proposed phenomena with off-resonant light in graphene. The bandwidth of graphene in the π orbital is given by $6J$, where $J \approx 2.4$ eV, placing the required frequency of off-resonant light to be soft x-ray regime with $\Omega = 3500$ THz. For this frequency of light, the gap of the system Δ can reach $\Delta \approx 300$ K for the strong light intensity $I \approx 3 \times 10^{12}$ W/cm^2 (Ref. 19), which gives $\mathcal{A} \approx 0.09$, where we expect the Hall conductance to be quantized with possible correction of 1% of $2e^2/h$. In reality, even such high frequency of light is expected to be absorbed in graphene. Such direct electron excitations lead to reconfiguration of electron occupation numbers, which modifies the Hall conductance from its quantized values.

C. Three-dimensional topological insulators

The analysis of graphene above can be directly extended to three-dimensional topological insulators such as Bi_2Se_3. The low-energy description of electrons on surfaces of Bi_2Se_3 is given by two-dimensional Dirac fermions[20] and is described by the Hamiltonian $H^{\rm surf} = v_{\rm TI}(k_x\sigma_y - k_y\sigma_x)$, where $v_{\rm TI}$ is the velocity of the Dirac fermion, and σ_i are Pauli matrices corresponding to two bands near the Dirac point. As before, we assume the application of weak, off-resonant, circularly polarized light. The orbital effect of the light is taken into account through the replacement $\mathbf{k} \to \mathbf{k} - \mathbf{A}(t)$. At the Dirac cone, the virtual photon process again opens a gap and the effective Hamiltonian is [see Eq. (3)]

$$H_{\rm eff}^{\rm surf} = v_{\rm TI}(k_x\sigma_y - k_y\sigma_x) \pm \frac{\mathcal{A}^2 v_{\rm TI}^2}{\Omega}\sigma_z, \tag{6}$$

where $+(-)$ corresponds to the gap due to right (left) circularly polarized light. The consequences of the gap coming from the third term in Eq. (6) are extensively investigated in Refs. 12 and 21. Just as in the case of graphene, the induced insulator is topologically nontrivial and expected to result in an anomalous quantum Hall effect with Hall conductance $\pm\frac{e^2}{2h}$ with possible corrections up to the order of $O(\mathcal{A}^2)$. Here we propose to probe the unique magnetoelectric response of the gapped topological insulator through pump-probe-type measurements, where circularly polarized light is used to open a gap at the Dirac point and linearly polarized light with small frequency within the gap is used to probe the Faraday/Kerr rotations,[22] as illustrated in Fig. 5(a) (also see Sec. V). Unlike other schemes proposed previously with ferromagnetic layers, here the Faraday/Kerr rotations can only result from the topological insulators and they give an unambiguous signature of magnetoelectric effects. In a similar fashion, the existence of magnetic monopoles can be probed by placing an electric charge near the surface of the topological insulator in the presence of circularly polarized light [see Fig. 5(b)].

D. Discussion

In the analysis of graphene and topological insulators above, we assumed the off-resonance of light for entire bands, but the gap in the effective Hamiltonian opens whenever the light is off-resonant near the Dirac points, which requires much less stringent condition on the light frequency. However, when interband electronic transitions occur due to photon absorptions, subsequent relaxation processes are expected to change the electron occupation numbers in the steady state and modify Hall conductance away from quantized values. Thus, in the case of on-resonant light, the system is expected to display nonquantized Hall effects without magnetic fields. Moreover, as we show in Sec. III, the application of on-resonant light leads to the photoassisted conductance and the resulting nonequilibrium transport property can no longer simply be described by the static effective Hamiltonian. The transport under the on-resonant light contains rich physics in itself, and it will be studied in the future works. On the other hand, it is possible to achieve the off-resonance with small frequency of light in, for example, the gapped systems such as boron-nitride by applying the subgap frequency of light. Many of these possible extensions can be studied through the formalism developed in the following sections. The understanding and formalism obtained in this paper is likely to guide future searches for the optimal systems to study photodriven quantum Hall effects without magnetic field.

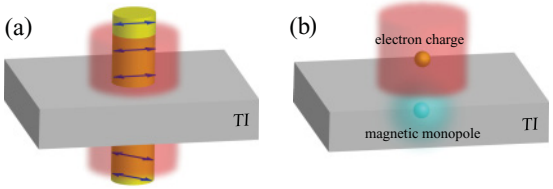

FIG. 5. (Color online) (a) Measurements of Faraday/Kerr rotations in three dimensional topological insulator. Circularly polarized light with a large frequency is used to open a gap at the Dirac point and linearly polarized light with small frequency within the gap is used to measure the Faraday/Kerr rotations. The polarization angle of the light is denoted by blue arrows in the figure. (b) The induction of magnetic monopole inside a three dimensional topological insulator through the application of light. Electron charge is placed near the surface, and circularly polarized light is applied on the surface to break the time-reversal symmetry and open the effective gap. The magnetic monopole is induced as a mirror image of the electron charge.

III. NONEQUILIBRIUM TRANSPORT: FORMALISM

A. Floquet Landauer formula

In this paper, we study the transport properties of systems under the application of light in the Landauer-type configuration, where the systems are attached to the leads as in Fig. 1. Previous works[10] obtained the dc current in periodically driven systems in terms of the Floquet Green's functions $\hat{\mathcal{G}}(\omega,n)$ which represent the Fourier transform of retarded Green's function, and is a propagator with frequency ω which absorbs (emits) n photons. Here we express the general results in physically transparent form:

$$\begin{aligned} J_\alpha^{\rm dc} &= J_\alpha^{\rm pump} + J_\alpha^{\rm res}, \\ J_\alpha^{\rm pump} &= \sum_n \sum_\beta \int \frac{d\omega}{2\pi} T_{\alpha,\beta}(n,\omega)[f_\alpha(\omega) - f_\alpha(\omega + n\Omega)], \\ J_\alpha^{\rm res} &= \sum_n \sum_\beta \int \frac{d\omega}{2\pi} T_{\alpha,\beta}(n,\omega)[f_\beta(\omega) - f_\alpha(\omega)], \end{aligned} \tag{7}$$

$$T_{\alpha,\beta}(n,\omega) = \Gamma_\beta(\omega)\Gamma_\alpha(\omega + n\Omega)|\mathcal{G}_{j_\alpha j_\beta}(n,\omega)|^2, \qquad (8)$$

where j_α are the sites in graphene that are connected with leads α, and $\Gamma_\alpha(\omega)$ represents the coupling strength with leads $\Gamma_\alpha(\omega) = t_\alpha^2\rho_\alpha(\omega)$, where t_α is the hopping strength from graphene to lead α and $\rho_\alpha(\omega)$ is the density of states in lead α at energy ω. Also, $f_\alpha(\omega)$ is the Fermi function at lead α, $f_\alpha(\omega) = \frac{1}{e^{\beta_\alpha(\omega-\mu_\alpha)}+1}$, where $\beta_\alpha = 1/k_B T_\alpha$ is the inverse temperature, and μ_α is the chemical potential of the reservoir connected to lead α. If we take the zero-temperature limit and assume that differences of chemical potentials at each lead are small, the expression in Eq. (7) is reduced to the simpler Floquet Landauer formula given in Eq. (5).

The calculation of conductance given by $J_\alpha^{\rm res}$ reduces to the calculation of the Floquet Green's function $\hat{\mathcal{G}}(\omega,n)$. Here $\hat{\mathcal{G}}_{l,l'}(\omega,n)$ is nothing but a Fourier transform of the retarded Green's function $G^R_{l,l'}(t,t')$. Starting from the usual definition of the retarded Green's function,

$$G^R_{l,l'}(t,t') = -i\theta(t-t')(\langle c_l(t)c^\dagger_{l'}(t')\rangle + \langle c^\dagger_{l'}(t')c_l(t)\rangle), \qquad (9)$$

we take the Fourier transform to obtain

$$G^R_{l,l'}(t,\omega) = \int_{-\infty}^{\infty} dt' G^R_{l,l'}(t,t')e^{i(\omega+i0^+)(t-t')}. \qquad (10)$$

Because we are driving the system at the given frequency Ω, this Green's function, as a function of t, should contain only the discrete Fourier components of Ω. Therefore, we can expand above expression of the Green's function as

$$G^R_{l,l'}(t,\omega) = \sum_{k=-\infty}^{\infty} \mathcal{G}_{l,l'}(n,\omega)e^{-in\Omega t}. \qquad (11)$$

The equation of motion followed by $\hat{\mathcal{G}}(\omega,n)$ can be obtained by writing out the equation of motion for $G^R_{l,l'}(t,t')$ and taking its Fourier transform. The resulting equation can be written in the most compact form in the matrix equation whose elements correspond to different (discrete) frequency components, $n\Omega$. Explicitly, the equation is given by

$$(\omega + \mathbf{\Omega} - \mathbf{H} - i\mathbf{\Gamma}/2)\mathbf{g} = I, \qquad (12)$$

where

$$\mathbf{H} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \hat{H}_0 & \hat{H}_1 & & \\ & \hat{H}_{-1} & \hat{H}_0 & \hat{H}_1 & \\ & & \hat{H}_{-1} & \hat{H}_0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbf{\Omega} = \begin{pmatrix} \ddots & & & & \\ & 1\Omega & & & \\ & & 0\Omega & & \\ & & & -1\Omega & \\ & & & & \ddots \end{pmatrix},$$

$$\mathbf{\Gamma} = \begin{pmatrix} \ddots & & & & \\ & \hat{\Gamma}(\omega+\Omega) & & & \\ & & \hat{\Gamma}(\omega) & & \\ & & & \hat{\Gamma}(\omega-\Omega) & \\ & & & & \ddots \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \mathcal{G}(0,\omega) & \mathcal{G}(1,\omega) & & \\ & \mathcal{G}(-1,\omega) & \mathcal{G}(0,\omega) & \mathcal{G}(1,\omega) & \\ & & \mathcal{G}(-1,\omega) & \mathcal{G}(0,\omega) & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \qquad (13)$$

$\mathbf{H}$ is the matrix of the Hamiltonian, whose elements are the Fourier components of the Hamiltonian given by $\hat{H}_n = \frac{1}{T}\int_0^T \hat{H}(t)e^{it\Omega n}dt$. $\mathbf{\Omega}$ is the diagonal matrix whose element is just the discrete frequency of the component. $\mathbf{\Gamma}$ is a diagonal matrix that represents coupling of the systems with leads. Its element $\hat{\Gamma}(\omega)$ has nonzero value only at site j_α that couples with leads, and with value $\Gamma_{j_\alpha,j_\alpha}(\omega) = \Gamma_\alpha(\omega) = t_\alpha^2\rho_\alpha(\omega)$. Finally, $\mathbf{g}$ is the matrix composed of the Floquet Green's functions.

Equation (12) can be thought of as the extension of the equation of motion for free electrons coupled with leads to periodically driven systems. The rest of this section is devoted to the solution of this equation and to the explanations of its physical significance for the response current given by Eq. (7).

In the following, we give two different solutions of Eq. (12). In Sec. III B, we solve the equation by expressing the Green's functions in terms of Floquet states, the "stationary states" of periodically driven systems after one period of time. This solution illustrates the physical origin of the transport given in Eq. (7). In Sec. III B 2, we derive the equivalence of nonequilibrium transport and the transport given by the effective photon-dressed Hamiltonian $H_{\rm eff}$ claimed in Sec. II by taking the off-resonant and weak intensity limit. In Sec. III C, we give another solution of Eq. (12), which is valid for a certain class of periodic drive including the application of circularly polarized light. This solution is derived by writing the Floquet Dyson's equation and has the advantage of being numerically efficient. Again, by taking the off-resonant, weak intensity limit of this solution, we arrive the result reported in Sec. II.

B. Floquet states and Floquet Green's functions

1. General relation

The "stationary states" of the Schrödinger equation for periodically driven systems are the states which return to themselves after one period of time, $T = 2\pi/\Omega$, with possible phase accumulations. These so-called Floquet states are the eigenstates of the evolution operator over one period and thus also eigenstates of effective Hamiltonian $H_{\rm eff}$ defined in Eq. (2). Green's functions $\hat{\mathcal{G}}(\omega,n)$ that describes the

propagation of particles with possible absorptions/emissions of photons have natural expressions in terms of these Floquet states.

The time evolution of the Floquet states $|\varphi_a(t)\rangle$ can be expanded in the discrete Fourier component of the driving frequency Ω and can be expressed as

$$|\varphi_a(t)\rangle = e^{-iE_a t}\sum_n e^{-in\Omega t}\left|\varphi_a^n\right\rangle, \tag{14}$$

where E_a is the quasienergy of the Floquet state $|\varphi_a\rangle$ for $H_{\rm eff}$ and $|\varphi_a^n\rangle$ is nth Fourier component of the Floquet state. The eigenstates of effective Hamiltonian $H_{\rm eff}$ are given as $|\varphi_a\rangle = \sum_n |\varphi_a^n\rangle$. As one can see from the expression above, the quasienergy E_a is only well defined up to the driving frequency Ω; that is, we can equally define $E_a + m\Omega$ as the quasienergy of $|\varphi_a(t)\rangle$ by redefining $|\varphi_a^n\rangle \to |\varphi_a^{n-m}\rangle$. Physically, this means the quasienergy is only conserved up to the driving frequency Ω because the system can absorb or emit photon energies. Also, this fact can be seen as a natural consequence of the breaking of continuous time-translation invariance through external drivings where the system only possesses discrete time-translation invariance under $t \to t + T$. In the following, we assume that $-\Omega/2 \leqslant E_a \leqslant \Omega/2$ without loss of generality. We take the normalization of Floquet states such that $\sum_n \langle\varphi_a^n|\varphi_b^n\rangle = \delta_{ab}$. The Schrödinger equation for the Fourier components of the Floquet states is time independent,

$$(E_a + n\Omega)\left|\varphi_a^n\right\rangle = \sum_m H_{n-m}\left|\varphi_a^m\right\rangle, \tag{15}$$

where H_n is the discrete Fourier component of Hamiltonian; that is, $H_n = \frac{1}{T}\int_0^T H(t)e^{it\Omega n}dt$. This equation encapsulates the evolution of states that allows the absorptions/emissions of photons; the application of Hamiltonian H_m leads to the absorption of m photons and the state $|\varphi^n\rangle$ is the component of the state with n photons. Here we considered the evolution of the systems in the absence of coupling with leads, but in the presence of the coupling with leads, zero frequency component of the Hamiltonian H_0 contains the imaginary "leaking" term $i\Gamma/2$.

This Schrödinger equation takes, in the matrix form,

$$E_a|\varphi_a\rangle = (-\mathbf{\Omega} + \mathbf{H})|\varphi_a\rangle, \tag{16}$$

where

$$|\varphi_a\rangle = \begin{pmatrix} \vdots \\ \left|\varphi_a^1\right\rangle \\ \left|\varphi_a^0\right\rangle \\ \left|\varphi_a^{-1}\right\rangle \\ \vdots \end{pmatrix}. \tag{17}$$

The Hamiltonian matrix $\mathbf{H}$ and driving frequency matrix $\mathbf{\Omega}$ are given in Eq. (13).

Thus, Floquet state $|\varphi_a\rangle$ is nothing but the eigenstates of the composite Hamiltonian $(-\mathbf{\Omega} + \mathbf{H})$. Notice that "shifted" state

$$\left|\varphi_a^{\rm shift}(n)\right\rangle = \begin{pmatrix} \vdots \\ \left|\varphi_a^{-n+1}\right\rangle \\ \left|\varphi_a^{-n}\right\rangle \\ \left|\varphi_a^{-n-1}\right\rangle \\ \vdots \end{pmatrix} \tag{18}$$

is also an eigenstate with eigenvalue $E_a + n\Omega$.

Now we can relate the Floquet eigenstates given by Eq. (16) and the Floquet Green's function given by Eq. (12). The formal solution of Eq. (12) is obtained in terms of the eigenstates $|\varphi_a\rangle$ of the matrix $\mathbf{\Omega} - \mathbf{H} - i\mathbf{\Gamma}/2$ as

$$\mathbf{g} = \sum_a \frac{|\varphi_a\rangle\langle\tilde{\varphi}_a|}{\omega - \varepsilon_a}. \tag{19}$$

Here $\langle\tilde{\varphi}_a|$ is the state that is determined from $\langle\tilde{\varphi}_a|\varphi_b\rangle = \delta_{ab}$. (Note that in the presence of $i\mathbf{\Gamma}/2$, $\langle\tilde{\varphi}_a|$ is not just a complex conjugate of $|\varphi_a\rangle$.) As is clear from Eq. (16), the eigenstates of $|\varphi_a\rangle$ of the matrix $\mathbf{\Omega} - \mathbf{H} - i\mathbf{\Gamma}/2$ is nothing but the Floquet states in the presence of the coupling with leads, represented by $-i\mathbf{\Gamma}/2$. Notice that the eigenstates $|\varphi_a\rangle$ in Eq. (19) include all the shifted states $|\varphi_a^{\rm shift}(n)\rangle$ in Eq. (18) for all integers n. With this understanding, we obtain the expression of the Floquet Green's function $\hat{\mathcal{G}}(n,\omega)$ as

$$\hat{\mathcal{G}}(n,\omega) = \sum_a\sum_m \frac{\left|\varphi_a^{n-m}\right\rangle\left\langle\tilde{\varphi}_a^{-m}\right|}{\omega - E_a - m\Omega}. \tag{20}$$

We can see that the Floquet Green's function is an intuitive extension of the Green's function for free electrons that allows absorptions and emissions of photons. As is expected, this Green's function transfers the state in $-m$ photon sector $\langle\tilde{\varphi}_a^{-m}|$ to $n - m$ photon sector $|\varphi_a^{n-m}\rangle$ by absorbing n photons.

In the presence of on-resonant light, Floquet states generally contain nonzero amplitudes in $|\varphi_a^n\rangle$ for more than one value of n, and therefore, the contributions to the response current in Eq. (5) from a few photon absorptions/emissions are nonzero. Thus, effective static Hamiltonian or band structures, which are only the description of average of n photon states $\sum_n |\varphi_a^n\rangle$, does not appropriately capture transport properties under the on-resonant light. In this case, it is necessary to compute the full Floquet Green's function given by Eq. (20) and calculate the response current in Eq. (7).

On the other hand, in the case of off-resonant, weak intensity of light, transport properties of nonequilibrium systems can be described by an effective photon-dressed Hamiltonian. In the next section, we provide the proof in the case of semimetals such as graphene and topological insulators.

2. *Effective Hamiltonian description*

As summarized in Sec. II, a rich physics appears when circularly polarized light is applied to graphene and topological insulators. The description of the nonequilibrium transport takes a particularly simple form for the off-resonant light in the limit of small light intensity $\mathcal{A} \ll 1$. Here we apply

the general formalism developed in the previous section to these systems and study the transport property by obtaining the Floquet Green's function in Eq. (20).

From the explicit form of the Hamiltonian in Eq. (1), it is clear that $H_n \sim O(\mathcal{A}^{|n|})$. This simply means the absorptions of photons are suppressed by the factor $O(\mathcal{A}^{|n|})$. Thus, the n photon sectors of the Floquet states $|\varphi_a^n\rangle$ are expected to scale as $|\varphi_a^n\rangle \sim O(\mathcal{A}^{|n|})$ with zeroth order solution being the static part of the Hamiltonian $H_0 + i\Gamma/2$. Here the term $i\Gamma/2$ represents the coupling with leads and we assumed the same strength of the coupling $\Gamma = \Gamma_\alpha$ at each lead α and further assumed that it is independent of frequency. This latter assumption is not important in the off-resonant case because current is essentially conducted only at chemical potential of leads, as we confirm later.

Starting from Eq. (15), we apply a degenerate perturbation theory in the lowest nontrivial order in $\mathcal{A}$ to obtain

$$\left(H_0 + i\Gamma/2 + \frac{[H_{-1}, H_1]}{\Omega}\right)|\varphi_a^0\rangle = E_a|\varphi_a^0\rangle, \tag{21}$$

$$|\varphi_a^n\rangle = \frac{1}{n\Omega} H_n|\varphi_a^0\rangle \quad \text{for } n \neq 0. \tag{22}$$

In the derivation, we assumed $E_\alpha \ll \Omega$, so the expression above is only valid near the Dirac points. Note that since the Hamiltonian H_0 is degenerate at the Dirac points, the mixings of the states due to the perturbations of $\mathcal{A}$ are not small. This result indeed shows that $|\varphi_a^n\rangle \sim O(\mathcal{A}^{|n|})$ and, therefore, the Floquet states $|\varphi_a\rangle$ can be approximated by the zeroth level of the Floquet states $|\varphi_a^0\rangle$, which is given by the eigenstates of the effective Hamiltonian $H_{\text{eff}} = H_0 + \frac{[H_{-1}, H_1]}{\Omega}$, plus the coupling with leads $i\Gamma/2$.

Using the solution of Floquet states above, we can obtain the response current in the lowest order in $\mathcal{A}$. The scaling $|\varphi_a^n\rangle \sim O(\mathcal{A}^n)$ in Eq. (22) directly implies that $|\hat{\mathcal{G}}(n,\omega)|^2 \sim O(\mathcal{A}^{2n})$. Moreover, the Green's function with no photon absorptions or emissions can be approximated as

$$\hat{\mathcal{G}}(0,\omega) = \sum_a \frac{|\varphi_a^0\rangle\langle\tilde{\varphi}_a^0|}{\omega - E_a} + O(\mathcal{A}^2)$$
$$\equiv \hat{\mathcal{G}}^{\text{eff}}(\omega) + O(\mathcal{A}^2),$$

where $|\varphi_a^0\rangle$ is the eigenstate of $H_{\text{eff}} + i\hat{\Gamma}/2$ and, therefore, $\hat{\mathcal{G}}^{\text{eff}}(\omega)$ is the free electron Green's function for the static system with Hamiltonian H_{eff} coupled with leads. Thus, these arguments combined with the expressions of currents in Eqs. (7) and (8) prove that the many-terminal measurements of nonequilibrium systems under the off-resonant light give the same result, as if the system is given by the static effective Hamiltonian H_{eff}. We emphasize that this result immediately implies the following two facts: (1) The nonequilibrium system displays insulating behaviors in the longitudinal conductance, and (2) Hall conductance is nearly quantized with possible correction of $O(\mathcal{A}^2)$, as presented in Sec. II.

3. *Insulating behavior for gapped effective Hamiltonian* H_{eff}

In the analysis of the previous section, we established the insulating behaviors of graphene under the application of off-resonant light through the perturbation theory in $\mathcal{A}$. Such analysis only shows that the longitudinal conductivity is small and of the order of $\mathcal{A}^2$, but does not show, in a strict sense, that the conductivity goes to zero at zero temperature. Using the formalism developed in previous sections, it is possible to show that the nonequilibrium system is an insulator for all orders in $\mathcal{A}$ as long as the chemical potential of leads lies below the effective gap of H_{eff}. The argument does not rely on the off-resonant condition and, in principle, is applicable whenever H_{eff} has a gap.

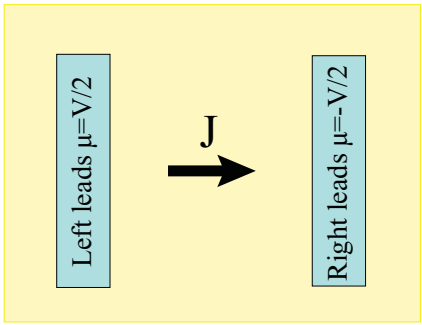

FIG. 6. (Color online) Illustration of the configuration considered in Sec. III B 3 to demonstrate the insulating behavior for gapped effective Hamiltonian H_{eff}. N_L is the number of leads that are attached to the infinite plane of graphene as "left" leads and N_R is the number of leads that are attached as "right" leads. The chemical potentials of the reservoirs connected to left leads are assumed to be at $V/2$ and those of the reservoirs connected to the right leads are at $-V/2$ with $|V| \ll J$

Here we consider an infinite plane of graphene, and we attach N_L number of leads as "left" leads and N_R number of leads as "right" leads, where these leads are separated by a large distance (see Fig. 6). Here we assume that the leads are coupled with the system with equal strength, given by $\Gamma(\omega)$. For clarity, we consider the situation in which the chemical potentials of the reservoirs connected to left leads are at $V/2$ and those of the reservoirs connected to the right leads are at $-V/2$, with $|V| \ll J$.

In the limit of small coupling strength, the Green's function in Eq. (20) can be obtained through the perturbation theory on Γ, and is given by

$$\hat{\mathcal{G}}(n,\omega) \approx \sum_a \sum_m \frac{|\varphi_a^{n-m}\rangle\langle\varphi_a^{-m}|}{\omega - E_a - m\Omega - i\gamma_a(\omega)}, \tag{23}$$

where $\gamma_a(\omega) = \sum_n \langle\varphi_a^n|\hat{\Gamma}(\omega + n\Omega)/2|\varphi_a^n\rangle$, and $|\varphi_a^n\rangle$ and E_a are the Floquet states and (quasi-) energy of the system in the absence of the coupling with leads.

In the limit of small $\gamma_a(\omega)$, the square of the Green's function $\hat{\mathcal{G}}(n,\omega)$ can be approximated by a δ function, so that the transmission probability also becomes a δ function in frequency:

$$T_{L,R}(n,\omega) = \sum_{L_i,R_i} \sum_{a,m} \Gamma(\omega)\Gamma(\omega + n\Omega)$$
$$\times \frac{|\langle j_{L_i}|\varphi_a^{n-m}\rangle|^2 |\langle\varphi_a^{-m}|j_{R_i}\rangle|^2}{2\gamma_a(\omega)} \pi\delta(\omega - E_a - m\Omega),$$

where $j_{L_i(R_i)}$ are the sites of left (right) leads. Now note that the (quasi-) energies E_a are the eigenvalues of H_{eff} in Eq. (2) in the main text. Therefore, if all the chemical

potentials lie within the gap of $H_{\rm eff}$, that is, $|V| \leqslant \Delta$ the δ function gives zero everywhere for $-V/2 \leqslant \omega \leqslant V/2$. Note that we have taken the quasienergies E_a to lie between $-\Omega/2 \leqslant E_a \leqslant \Omega/2$ and thus, by assumption, $\Delta \leqslant \Omega$. The $m\Omega$ term in the δ function of Eq. (24) accounts for the possible transmission of electrons at high/low energies through photon absorption/emission processes. As long as we are interested in the transmission of electrons near the chemical potential which lies within the effective gap, the $m\Omega$ term plays no role in the conduction. Thus, from the expression of the dc current in Eq. (7), it is clear that the current has to be zero when $|V| \leqslant \Delta$.

The argument above is general and did not require the condition of off-resonance. In the case of on-resonant light, the effective band structures are given by mixing the static eigenstates whose energies differ by Ω. This "folding" of the band structures generically leads to a large number of states appearing in the effective Hamiltonian near the chemical potential, and subsequently the effective gap in $H_{\rm eff}$ becomes proportional to $O(\mathcal{A}^n)$ with n approximately determined by the ratio of the static bandwidth and the driving frequency Ω. While the insulating behavior should be observable in the small window of the gap, the gap could be small in this case.

C. Floquet Dyson's equation

In this section, we present yet another way to obtain the Floquet Green's function $\hat{\mathcal{G}}(n,\omega)$, which gives an efficient way to numerically evaluate the Floquet Green's function for a certain class of periodical drives.

In this section, we consider the Hamiltonian that depends only on the first harmonics of the driving frequency Ω; namely, the Hamiltonian takes the form

$$H(t) = H_0 + V_1 e^{-i\Omega t} + V_{-1} e^{i\Omega t}. \tag{24}$$

For example, for the application of the circularly polarized light to two-dimensional lattice systems, $V_1 = \sum_j (x_j + iy_j) c_j^\dagger c_j$ and $V_{-1} = V_1^\dagger$ in the gauge in which the light is represented as a circulating potential. However, in this gauge, $V_{\pm 1}$ diverges as $x_j, y_j \to \infty$, and care must be taken to study with this gauge. A conceptually useful gauge is the gauge in which the effect of light is represented as a phase accumulation as in Sec. II. For weak amplitude of light, we can approximate the Hamiltonian in this gauge in the form of Eq. (24) with $V_1 = H_1$ and $V_1 = H_{-1}$. As before, we are interested in the terminal measurements of conductance, and thus we assume that the static part of the Hamiltonian H_0 contains the "leaking" of particles into leads given by $i\Gamma(\omega)/2$.

In order to evaluate Floquet Green's functions, we first rewrite Eq. (12) in the suggestive form of the Floquet Dyson's equation (see Fig. 7):

$$\hat{\mathcal{G}}(n,\omega) = \delta_{n0}\hat{G}^0(n,\Omega) + \hat{G}^0(n,\Omega)\{V_{-1}\hat{\mathcal{G}}(n+1,\omega) + V_1\hat{\mathcal{G}}(n-1,\omega)\}. \tag{25}$$

Here $\hat{G}^0(n,\Omega) = \frac{1}{\omega + n\Omega - H_0}$ represents the bare propagator of a particle with n photons. This equation has the intuitive understanding of the full propagator $\hat{\mathcal{G}}(n,\omega)$ that represents the n photon absorption process as being composed of the full propagation of $\hat{\mathcal{G}}(n \pm 1,\omega)$ followed by the absorption or emission of a photon, followed by the propagation of the bare particle.

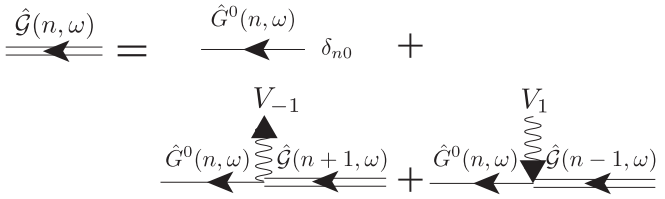

FIG. 7. The Floquet Dyson equation. The propagator goes from right to left. Double lines represent the full propagator $\hat{\mathcal{G}}(n,\omega)$ and single lines are a bare propagator $\hat{\mathcal{G}}(n,\omega) = \frac{1}{\omega + n\Omega - H_0}$ which does not include the effect of photon absorptions.

A particularly elegant solution for $\hat{\mathcal{G}}(n,\omega)$ is provided by continued fraction method.[14] The building block of the solution is the dressed propagator

$$\hat{\mathcal{F}}_+(n,\omega) = \frac{1}{(G^0)^{-1}(n,\omega) - V_{-1}\frac{1}{(G^0)^{-1}(n+1,\omega) - V_{-1}\frac{1}{\ddots}V_1}V_1} \quad \text{for } n > 0,$$

$$\hat{\mathcal{F}}_-(n,\omega) = \frac{1}{(G^0)^{-1}(n,\omega) - V_1\frac{1}{(G^0)^{-1}(n-1,\omega) - V_1\frac{1}{\ddots}V_{-1}}V_{-1}} \quad \text{for } n < 0.$$

The propagator $\hat{\mathcal{F}}_+(n,\omega)$ is dressed only from the higher-photon-number states, and the propagator $\hat{\mathcal{F}}_-(n,\omega)$ is dressed by the lower-photon-number states. The full propagator is then given as

$$\begin{aligned} \hat{\mathcal{G}}(0,\omega) &= (\omega - H_0 - V_{\rm eff})^{-1}, \\ V_{\rm eff} &= V_1\hat{\mathcal{F}}_-(-1,\omega)V_{-1} + V_{-1}\hat{\mathcal{F}}_+(1,\omega)V_1, \end{aligned} \tag{26}$$

$$\begin{aligned} \hat{\mathcal{G}}(n,\omega) &= \hat{\mathcal{F}}_+(n,\omega)V_1\cdots\hat{\mathcal{F}}_+(1,\omega)V_1\hat{\mathcal{G}}(0,\omega) \quad \text{for } n > 0, \\ &= \hat{\mathcal{F}}_-(n,\omega)V_{-1}\cdots\hat{\mathcal{F}}_-(-1,\omega)V_{-1}\hat{\mathcal{G}}(0,\omega) \quad \text{for } n < 0. \end{aligned} \tag{27}$$

This solution is valid for any driving frequency. Remarkably, we see that the zero-photon absorption propagator $\hat{\mathcal{G}}(0,\omega)$ is simply given by the propagator in an effective Hamiltonian $H_{\rm eff} = H_0 + V_{\rm eff}$.

For the gauge in which light is represented as time-dependent vector potential, and weak intensity of light $\mathcal{A} \ll 1$, we can approximate $\hat{\mathcal{F}}_+(1,\omega) = \hat{\mathcal{F}}_-(-1,\omega) = \frac{1}{\Omega}$ in the limit of high frequency. Thus, we reproduce the result we obtained in Sec. III B 2 of the effective Hamiltonian $H_{\rm eff} = H_0 + \frac{[H_{-1},H_1]}{\Omega}$ in this limit.

IV. EFFECT OF ELECTRON-ELECTRON AND ELECTRON-PHONON INTERACTIONS

The nonequilibrium transport properties described in Sec. II are robust against interactions such as electron-electron interactions, interactions between electrons and disorder, and electron-phonon interactions. The electron-electron interactions only renormalize the velocity of Dirac electrons, v_G and $v_{\rm IT}$, and do not change the Dirac nature of the electrons near the Fermi surface.[23] The quantum Hall insulators are insensitive

to disorders due to the topological origin of the phase, as long as the disorder strength is small compared to the gap size, Δ.[24]

The robustness of the phenomena against phonon scatterings originates from the conservation of energy in $H_{\rm eff}$ up to the light frequency Ω. When the chemical potentials of leads lie in the gap of $H_{\rm eff}$, the nonequilibrium current in many-terminal measurements is conducted through electrons in the lower band of $H_{\rm eff}$. Such current can degrade due to electron-phonon interactions if electrons in the lower band can be excited to the higher band. However, such excitations in the bands of effective Hamiltonians require a physical energy greater than the gap Δ as is rigorously established in the Floquet Fermi golden rule in the Appendix. It is, in principle, possible to absorb energies from photons, but because the frequency of photons Ω is assumed to be much larger than bandwidth, the absorption of such large energy requires the excitations of electrons together with many phonons and, therefore, such a process is suppressed. Thus, the transition of an electron from the lower band of the effective Hamiltonian to the higher band is possible only through the absorption of phonon energies. Therefore, at low temperatures, the property of an "insulating" state of the effective Hamiltonian is protected against electron-phonon interactions by the gap.

V. PROBE OF THE INDUCED EFFECTIVE GAP IN AN ISOLATED SYSTEM

In this section, we propose a different way to probe the effective gap induced by off-resonant light through the pump and probe measurements in an isolated system. The essential idea is simple. Given a system under the application of light (called pump laser), suppose that the effective Hamiltonian $H_{\rm eff}$ defined by Eq. (2) has a gap Δ. We prepare the state, in isolation from thermal reservoirs, such that only the lower band of $H_{\rm eff}$ is occupied through a sort of "adiabatic preparation." Here we start from zero-temperature static system and increase the intensity of light gradually to increase the size of the gap, Δ. As we argue below and in the Appendix, the adiabatic theorem in the Floquet picture guarantees that the final state has the electron occupations such that the lower band of photon-dressed Hamiltonian $H_{\rm eff}$ is occupied. Now for this occupation of electrons with a gap to a higher band, it is intuitively clear that the system becomes transparent to the probe light with frequency smaller than Δ. In the case of graphene and topological insulators under the application of light, we expect that the transmitted probe light results in the Faraday rotations.

The pump and probe measurements described above are well understood if the modification of the system from the original Hamiltonian to final Hamiltonian $H_{\rm eff}$ is done through a static field. In this case, the adiabatic preparation is guaranteed by the adiabatic theorem, and the transmission of probe light can be confirmed by looking at the Fermi golden rule, which shows that the photons cannot be absorbed by electrons due to the conservation of energy.

In the case of periodically varying fields, analogous statements hold. The Floquet adiabatic theorem shows that, under an adiabatic evolution of the periodically varying fields, each Floquet state follows the instantaneous Floquet state given by the instantaneous Hamiltonian. Similarly, the Floquet Fermi golden rule gives the rate in which the transition from one Floquet state to another happens under small perturbations. This result shows that the quasienergies of Floquet states are conserved up to integer multiples of driving frequency Ω. Thus, as we have claimed above, the electrons in the lower band of $H_{\rm eff}$ cannot be excited to higher bands unless the photon energy is larger than the band gap Δ. We give the detailed proof of these theorems in the Appendix.

VI. CONCLUSION

In this paper, we studied the transport properties of nonequilibrium systems under the application of light in many-terminal measurements. Starting from the Floquet Landauer formula, we gave two different solutions of Floquet Green's functions that illustrate the physical origin of transport in this situation. We found that for generic driving frequencies, the transport involves photon-assisted conductance and cannot be described by any static, effective Hamiltonians.

In the case of graphene and topological insulators under the off-resonant light, the nonequilibrium transport does not involve photon absorptions/emissions. Rather, the electron band structures are modified through the virtual photon absorption/emission processes. We established, through the solution of the Floquet Green's function, that such modifications are captured by the static photon-dressed Hamiltonian and that the transport in this system becomes equivalent to that described by the photon-dressed Hamiltonian. Remarkably, the effective Hamiltonian obtained in this way takes the form of a Haldane model[11] with second-neighbor hopping with phase accumulations for graphene under the application of circularly polarized light.

One important aspect of our proposal is the opening of the gap in the photon-dressed Hamiltonian when the original static Hamiltonian is semimetal and gapless. We gave two physical manifestations of such a gap. One is the insulating behavior of the driven system *attached to the leads* (Sec. III B 3). The attachment of leads is crucial to determine the electron occupation numbers. Another is the transmission of low-frequency light in an isolated system (Sec. V) after the adiabatic preparations of states. We argued the possibility of such pump-probe measurements by establishing two important extensions of well-known theorems, the Floquet adiabatic theorem and the Floquet Fermi golden rule (Appendix).

The formalism and intuitive understanding developed in this paper can be used to study the transport properties of a variety of systems under the application of light. It is of interest to analyze, for example, the transport properties of light-induced topological systems proposed in Ref. 8. In addition, our analysis shows that transport under the application of light contains richer physics than static transport. In particular, photon-assisted conductance in which electrons absorb/emit photons during the propagations is the unique feature of driven systems, and it is interesting to analyze how such physical process results in energy conductions. While we focused on the response current $J_{\rm res}$ in this paper, yet another aspect of driven systems is the presence of pump current $J_{\rm pump}$ appearing in Eq. (4). It is of interest to find materials that can pump currents by simply shining light on their surface.

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APPENDIX: FLOQUET FERMI GOLDEN RULE AND FLOQUET ADIABATIC THEOREM

In this section we establish the following two statements about H_{eff} studied in the main text: (1) The result of many-terminal measurements of the systems under the application of light obtained in the main text is robust against electron-phonon interactions, as long as the energy of phonons dictated by the temperature of the systems is smaller than the induced gap Δ; (2) the photoinduced gap Δ can be probed, in a closed system, by the transmission of a laser with frequency $\omega < \Delta$.

We give arguments for the first statement by deriving an analogous theorem as Fermi golden rule in the periodically driven systems. When the chemical potentials of leads lie in the gap of H_{eff}, the nonequilibrium current in many-terminal measurements is conducted through electrons in the lower band of H_{eff}. Such current can degrade due to electron-phonon interactions if electrons in the lower band can be excited to the higher band. By deriving the Floquet Fermi golden rule, we demonstrate that such excitations in the bands of effective Hamiltonians still require a physical energy greater than the gap Δ. It is in principle possible to absorb energies from photons, but because the frequency of photons Ω is assumed to be much larger than the bandwidth, the absorption of such large energy requires the excitations of electrons and many phonons, and therefore such a process is suppressed. Thus, the transition of an electron from the lower band of the effective Hamiltonian to the higher band is possible only through the absorption of phonon energies. Therefore, at low temperatures, the property of an "insulating" state of the effective Hamiltonian is protected against electron-phonon interactions by the gap.

The proof of the second statement requires two steps. If we assume that the closed system with H_{eff} can be prepared in a state such that only the lower band of H_{eff} is occupied, then we can argue from the Floquet Fermi golden rule that the low-frequency laser with $\omega < \Delta$ cannot be absorbed by the electrons. Therefore, such a system is transparent to the light. In order to prepare such a "filled" state of the effective Hamiltonian H_{eff}, we consider an adiabatic preparation. Starting from the half-filled state of original systems whose chemical potential lies at the Dirac points, we adiabatically increase the strength of light. We argue, by deriving the Floquet adiabatic theorem,[25] that such procedure prepares the filled state of H_{eff} except possibly exactly at the Dirac points.

These two statements rely on two general theorems about periodically driven systems, dubbed the Floquet Fermi golden rule and the Floquet adiabatic theorem. In the following, we derive these results, using the elegant approach from "two-time" formalism.[25] We emphasize that these results are general and have wide applications outside of what we discussed in this paper.

1. Two-time Schrödinger equation

In order to study the dynamics of periodically driven systems, it is convenient to separate two time scales: a fast time scale associated with the driving frequency Ω and a slow time scale associated with other dynamics such as those of phonons. We let t denote the former time scale and τ the latter and obtain the Schrödinger equation of the slower dynamics in terms of τ through the replacement $i\partial/\partial t \to i\partial/\partial t + i\partial/\partial\tau$. Then the time evolution of states for slow time scale can be written as

$$i\frac{\partial}{\partial\tau}|\psi(\tau)\rangle = (\mathcal{H} + \hat{V}(\tau))|\psi(\tau)\rangle, \tag{A1}$$

$$\mathcal{H} = H(t) - i\frac{\partial}{\partial t}, \tag{A2}$$

where $H(t)$ corresponds to the Hamiltonian with periodic drives with frequency Ω and $\hat{V}(\tau)$ represents the perturbation of the system with slow frequencies compared to Ω. In the absence of the perturbation $V(\tau)$, the eigenstates of the Schrödinger equation above is given by Floquet states such that

$$E_\alpha|\Phi_\alpha\rangle = \mathcal{H}|\Phi_\alpha\rangle, \tag{A3}$$

where $|\Phi_\alpha(t)\rangle = e^{iE_\alpha t}|\varphi_\alpha(t)\rangle = \sum_n e^{-i\Omega n t}|\varphi_\alpha^n\rangle$ is a state with a periodic structure $|\Phi_\alpha(t)\rangle = |\Phi_\alpha(t+T)\rangle$ and E_α is the quasienergy of the Floquet state, that is, the eigenenergy of H_{eff}. Here $|\varphi_\alpha(t)\rangle$ represents a Floquet state which satisfies the equation $\mathcal{H}|\varphi_\alpha(t)\rangle = 0$. Note that E_α is only defined up to Ω, so that physically the same Floquet state $|\varphi_\alpha(t)\rangle$ in Eq. (14) can be associated with the eigenvalue $E_\alpha + m\Omega$ and the state $|\Phi_\alpha^m(t)\rangle = \sum_n e^{-i\Omega n t}|\varphi_\alpha^{n+m}\rangle$. Here we take the convention that $|\Phi_\alpha^m(t)\rangle$ with $m = 0$ is associated with the quasienergy E_α such that $-\Omega/2 \leqslant E_\alpha \leqslant \Omega/2$. The orthogonality of the eigenstates $|\Phi_\alpha\rangle$ can be recovered by defining the inner product of Floquet states as the average of the usual inner product over one period of time,

$$\langle\langle\chi_\alpha|\chi_\beta\rangle\rangle \equiv \frac{1}{T}\int_0^T \langle\chi_\alpha(t)|\chi_\beta(t)\rangle dt. \tag{A4}$$

Then we have $\langle\langle\Phi_\alpha(t)|\Phi_\beta(t)\rangle\rangle = \delta_{\alpha,\beta}$.

These extensions of the inner products and eigenvalue problem in periodically driven systems can be considered as the extension of Hilbert space to include the fast time variable t as another "spatial" variable. The inner product Eq. (A4) in this Hilbert space integrates over t, and the time variable is now represented by the slow time variable τ. We point out that the inner product Eq. (A4) makes sense only when any dynamics associated with τ occurs in a slower time scale than the period of driving T. In principle, operators and states which depend on τ change during the integration time T of fast time variable t due to the dependence on τ. Since we are treating t and τ as independent variables, the inner product Eq. (A4) ignores such τ dependence. As long as such changes are small, the inner product Eq. (A4) gives a good approximation.

The crucial observation is that the slow time Schrödinger equation in Eq. (A1) has the identical form as the usual Schrödinger equation and, therefore, many results for static systems can be directly extended to periodically driven systems through the extension of the the inner product to Eq. (A4).

2. Floquet Fermi golden rule

The Floquet Fermi golden rule gives the intuition behind the response of periodically driven systems under the influence of perturbations. In particular, the result shows that quasienergy of the effective Hamiltonian $H_{\rm eff}$ is a conserved quantity up to the driving frequency Ω. In the context of our paper, this result implies that electrons in the lower band of $H_{\rm eff}$ cannot be excited to the upper band if the frequencies of the perturbations, such as phonons or probe light, are smaller than the gap Δ. Thus, the transport property of a nonequilibrium system described by $H_{\rm eff}$ is robust against phonon interactions as long as the chemical potentials of leads lie in the gap and phonon energies are smaller than the gap Δ. Moreover, if one can prepare the system in the state with filled lower band of $H_{\rm eff}$, then the gap of $H_{\rm eff}$ can be probed by observing the transmissions of low-frequency lasers.

This Floquet Fermi golden rule can be easily obtained through the two-time formalism described in the previous section. In the following, we consider the perturbations of the system such as phonons with frequency ω much smaller than Ω such that $\omega \ll \Omega$. We take the perturbation in the form $\hat{V}(\tau) = \hat{V} e^{-i\omega\tau}$. The usual derivation of Fermi golden rule can be applied in a straightforward fashion, and we obtain the Floquet Fermi golden rule, which gives the rate $\gamma_{i\to f}$ of exciting the initial Floquet state $|\varphi_i\rangle$ to the final Floquet state $|\varphi_{\rm f}\rangle$ in the presence of the perturbation $\hat{V}(\tau)$:

$$\gamma_{i\to f} = \sum_m \left|\langle \Phi_f^m | \hat{V} | \Phi_i^0 \rangle\right|^2 \times \delta(E_i + \omega - E_f - m\Omega). \tag{A5}$$

Here E_i and E_f are the quasienergies of the initial and final Floquet states, respectively. In order to derive the result above, we represented the Floquet state $|\varphi_i\rangle$ by the specific periodic state $|\Phi_i^0\rangle$. This choice is arbitrary and any other choice gives the same result. Since the physical Floquet state $|\varphi_{\rm f}\rangle$ can be represented as the states $|\Phi_f^m\rangle$ for any integers m, the total transition rate is given by the sum of the rate from the state $|\Phi_i^0\rangle$ to states $|\Phi_f^m\rangle$.

This rate has the same form as the conventional Fermi golden rule, except for the summation over the Floquet energy index m. The δ function in the equation above imposes the conservation of *quasienergy*, which is the eigenenergy of effective Hamiltonian $H_{\rm eff}$, which means the energy is conserved up to the driving frequency Ω. This is a natural consequence of the fact that the system can absorb or emit the energy Ω from the periodic drives.

From this result, it is clear that such conservation of quasienergy prevents the excitations of electrons from lower band to upper band when phonon energy ω is smaller than the gap of the system, and Ω is much larger than the total bandwidth of electrons.

3. Floquet adiabatic theorem

In this section, we show, in analogy with the adiabatic theorem of static systems, that a Floquet state follows an adiabatic change of Hamiltonian and stays in the Floquet state of the instantaneous Hamiltonian. This result indicates that the adiabatic increase of the intensity of light can be used to prepare the state with filled lower band of $H_{\rm eff}$, whose properties can then be probed through low-frequency lasers, as argued above.

Starting from the slow-time Schrödinger equation in Eq. (A1), we can follow the derivation of adiabatic theorem and prove the analogous theorem for periodically driven systems. Here we briefly outline the derivation.

Suppose that the total Hamiltonian $\mathcal{H}(\tau)$ is slowly varying as a function of τ. We are interested in how a Floquet state of $H(0)$ at time $\tau = 0$ evolves under this time evolution. Let $|g\rangle$ be the initial Floquet state and $|G(\tau_0)\rangle$ be the result of evolving $|g\rangle$ under $H(\tau)$ for time τ_0.

We denote the instantaneous eigenstates of $\mathcal{H}(\tau)$ as $|\alpha(\tau)\rangle$ such that $\mathcal{H}(\tau)|\alpha(\tau)\rangle = E_\alpha(\tau)|\alpha(\tau)\rangle$. Then we express the state $|G(\tau)\rangle$ in terms of $|\alpha(\tau)\rangle$ as

$$|G(\tau)\rangle = \exp\left(-i\int_0^\tau E_g(\tau')dt'\right) \times \left(c_g(\tau)|g(\tau)\rangle + \sum_{\alpha\neq g} c_\alpha(\tau)|\alpha(\tau)\rangle\right). \tag{A6}$$

In the absence of degenerate states, we can solve for the coefficients $c_\alpha(\tau)$ in the lowest order for the slow change of Hamiltonian $\mathcal{H}(\tau)$ in the Schrödinger equation of Eq. (A1). The result is given by

$$|G(\tau)\rangle = e^{-i\int_0^\tau E_g(\tau')d\tau'} \exp\left(-i\int_0^\tau i\left\langle \langle g(\tau')|\frac{\partial}{\partial\tau'}|g(\tau')\rangle\right\rangle d\tau'\right) \times \left(|g(\tau)\rangle - i\sum_{\alpha\neq g}|\alpha(\tau)\rangle \frac{\left\langle\langle\alpha(\tau)|\frac{\partial}{\partial\tau}|g(\tau)\rangle\right\rangle}{E_\beta(\tau) - E_g(\tau)}\right). \tag{A7}$$

Thus, to the zeroth order for the slow change of Hamiltonian $\mathcal{H}(\tau)$, $|G(\tau)\rangle$ is the Floquet state of the instantaneous Hamiltonian $\mathcal{H}(\tau)$ with possible accumulations of dynamical and Berry phases. The first-order correction is given by the second term of Eq. (A7).

For static systems of Dirac Fermions studied in this paper, we have shown that a gap proportional to $\mathcal{A}^2$ opens at the Dirac point upon the application of light. If the chemical potential lies at the Dirac point before the application of light, the result above implies that the adiabatic increase of the intensity of light $\mathcal{A}(\tau)$ can be used to prepare the system close to the filled lower band state of $H_{\rm eff}$. At exactly the Dirac points where the spectrum becomes degenerate, the adiabatic theorem above does not apply, but these points represent only a tiny portion of the total states, and thus can be ignored for the calculations of physical quantities. When the initial

system is at finite temperature, such adiabatic increase of $\mathcal{A}(\tau)$ leads to *nonthermal* distributions of electrons in the spectrum of $H_{\rm eff}$, but nonetheless the resulting density matrix can be calculated through the result Eq. (A7) in the adiabatic limit.

This Floquet adiabatic theorem can be used to obtain the Kubo's formula[26] in the noninteracting, periodically driven systems. In Ref. 7, such a result is applied to derive the extension of TKNN formula[24] to periodically driven systems in infinite systems.

[1] A. M. Glass, D. von der Linde, and T. J. Negran, Appl. Phys. Lett. **25**, 233 (1974); E. J. H. Lee *et al.*, Nat. Nano. **3**, 486 (2008).

[2] X. Xu *et al.*, Nano Lett. **10**, 562 (2010).

[3] J. Karch *et al.*, Phys. Rev. Lett. **105**, 227402 (2010); T. Hatano, T. Ishihara, S. G. Tikhodeev, and N. A. Gippius, *ibid.* **103**, 103906 (2009).

[4] K. Miyano, T. Tanaka, Y. Tomioka, and Y. Tokura, Phys. Rev. Lett. **78**, 4257 (1997); M. Fiebig *et al.*, Science **280**, 1925 (1998).

[5] S. V. Syzranov, M. V. Fistul, and K. B. Efetov, Phys. Rev. B **78**, 045407 (2008).

[6] D. Fausti *et al.*, Science **331**, 189 (2011).

[7] T. Oka and H. Aoki, Phys. Rev. B **79**, 081406 (2009)

[8] N. H. Lindner, G. Rafael, and V. Galitski, Nat. Phys. **7**, 490 (2011); W. Yao, A. H. MacDonald, and Q. Niu, Phys. Rev. Lett. **99**, 047401 (2007).

[9] T. Kitagawa, M. S. Rudner, E. Berg, and E. Demler, Phys. Rev. A **82**, 0033429 (2010); T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B **82**, 235114 (2010); L. Jiang, T. Kitagawa, J. Alicea, A. R. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Phys. Rev. Lett. **106**, 220402 (2011).

[10] A. P. Jauho, N. S. Wingreen, and Y. Meir, Phys. Rev. B **50**, 5528 (1994); S. Kohler, J. Lehmann, and P. Hanggi, Phys. Rep. **406**, 379 (2005); M. Moskalets and M. Büttiker, Phys. Rev. B **66**, 205320 (2002); L. Arrachea and M. Moskalets, *ibid.* **74**, 245322 (2006); Hernan L. Calvo *et al.*, Appl. Phys. Lett. **98**, 232103 (2011); D. Martinez, R. Molina, and B. Hu, Phys. Rev. B **78**, 045428 (2008).

[11] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).

[12] X.-L. Qi, T. L. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008); X.-L. Qi, R. Li, J. Zang, and S.-C. Zhang, Science **323**, (2009).

[13] M. Büttiker, Phys. Rev. Lett. **57**, 1761 (1986).

[14] D. F. Martinez, J. Phys. A: Math. Gen. **36**, 9827 (2003); T. Brandes and J. Robinson, Phys. Status Solidi B **234**, 378 (2002); D. F. Martinez and R. A. Molina, Eur. Phys. J. B **52**, 281 (2006).

[15] D. Thouless, M. Kohmoto, M. P. Nightingale, and M. denNijs, Phys. Rev. Lett. **49**, 405 (1982).

[16] J. I. Inoue and A. Tanaka, Phys. Rev. Lett. **105**, 017401 (2010).

[17] H. Sambe, Phys. Rev. A **7**, 2203 (1973); M. Torres and A. Kunold, Phys. Rev. B **71**, 115313 (2005).

[18] M. Switkes *et al.*, Science **283**, 1905 (1999); L. E. F. FoaTorres, Phys. Rev. B **72**, 245339 (2005).

[19] R. W. Schoenlein *et al.*, Science **287**, 2237 (2000).

[20] H. Zhang *et al.*, Nat. Phys. **5**, 438 (2009); Y. Xia *et al.*, *ibid.* **5**, 398 (2009).

[21] L. Fu and C. L. Kane, Phys. Rev. B **76**, 045302 (2007).

[22] T. Oka and H. Aoki, e-print arXiv:1007.5399 (to be published).

[23] K. S. Kovoselov *et al.*, Nature (London) **438**, 197 (2005).

[24] D. Thouless *et al.*, Phys. Rev. Lett. **49**, 405 (1982).

[25] H. Breuer, Phys. Lett. A **140**, 507 (1989); S. C. Althorpe, D. J. Kouri, D. K. Hoffman, and N. Moiseyev, Chem. Phys. **217**, 289 (1997).

[26] Y. Hatsugai, J. Phys. Condens. Matter **9**, 2507 (1997).