Approximate method for controlling solid elastic waves by transformation media

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By idealizing a general mapping as a series of local affine ones, we derive approximately transformed material parameters necessary to control solid elastic waves within classical elasticity theory. The transformed elastic moduli are symmetric, and can be used with Navier's equation to manipulate elastic waves. It is shown numerically that the method can provide a powerful tool to control elastic waves in solids in case of high frequency or small material gradient. Potential applications can be anticipated in nondestructive testing, structure impact protection, petroleum exploration, and seismology.

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The transformation method[1–3] provides an efficient tool to find necessary material distribution when a wave pattern is prescribed in a space. Many interesting devices have been proposed, especially with the help of electromagnetic or acoustic metamaterials.[4–8] The method is also extended to control heat conduction[9] and matter waves.[10] These exotic properties are bestowed by form invariance of Maxwell or Helmholtz equations under a general spatial transformation. A comprehensive theory for the transformation method of generalized acoustics was developed by Norris,[11] and an acoustic cloak has been recently demonstrated experimentally by Zhang *et al.*[12] However, for elastic waves in solid materials, Milton *et al.*[13] show that Navier's equation is transformed to Willis' equation for a general spatial transformation, therefore, the machinery developed for electromagnetic or acoustic waves cannot be directly applied to solid elastic waves. Due to the complicated nature of Willis' equation and its unusual constitutive equation, efforts are continually made to find approximate methods to control the elastic wave in some degrees, since elastic waves are involved in broad engineering applications. For example, in the quasistatic limit, Zhou *et al.*[14] proposed an elastic cloak with an impedance-matched method; Brun *et al.*[15] reported a cylindrical cloak for an in-plane elastic wave with an asymmetric elasticity tensor; Farhat *et al.*[16] suggested a cloak for shielding a thin plate bending wave in a long wavelength limit. In this Rapid Communication, we will idealize a general space transformation by local affine ones point by point, and propose an approximate method to control elastic waves. The approximation of the method will also be clarified in the context of elastic ray theory.

When an initial space Ω is transformed to Ω' by a mapping $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, a physical process F prescribed on the initial space with field quantity $\mathbf{u}$ and material parameter $\mathbf{C}$ is also transformed to the new space Ω' with a new field quantity $\mathbf{u}'$ and material parameter $\mathbf{C}'$. The transformed relation between $\mathbf{u}'$ and $\mathbf{u}$, and $\mathbf{C}'$ and $\mathbf{C}$ can be derived if the governing equation of the physical process is globally form invariant. For elastic waves governed by Navier's equation, we will idealize a general transformation by a series of local affine ones point by point. Since Navier's equation locally keeps its form under an affine transformation,[17] we will derive the transformed material parameters in this circumstance. To this end, we establish at any point $\mathbf{x}$ of Ω a local Cartesian frame $\mathbf{e}_i$ and at the point $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$ of Ω' another local Cartesian frame

$\mathbf{e}'_i$, uniquely determined by the mapping. With the assumption of a local affine transformation, the governing equation will retain locally its form in the two spaces with respect to the two local Cartesian systems, i.e.,

$$F(\mathbf{x},\mathbf{u},\mathbf{C},t) = 0, \ \ \mathbf{x} \in \Omega, \tag{1}$$

$$F(\mathbf{x}',\mathbf{u}',\mathbf{C}',t) = 0, \ \ \mathbf{x}' \in \Omega'. \tag{2}$$

Equation (2) imposes a constraint condition on $\mathbf{u}'$ and $\mathbf{C}'$. In addition, we assume that each type of energy is conserved at every point during the mapping; this will lead to another constraint condition, and these constraint conditions will determine directly the transformed relations for $\mathbf{u}'$ and $\mathbf{C}'$. During a mapping, the space will experience locally a rigid rotation and a stretch operation, i.e., $\mathbf{A} = \nabla_{\mathbf{x}}\mathbf{x}' = \mathbf{VR}$,[18] where $\mathbf{R}$ is a rigid rotation, $\mathbf{V}$ is a stretch operation, and it can be expressed by its eigenvector $\mathbf{e}'_i$ and eigenvalues λ_i as $\mathbf{V} = \lambda_1\mathbf{e}'_1 \otimes \mathbf{e}'_1 + \lambda_2\mathbf{e}'_2 \otimes \mathbf{e}'_2 + \lambda_3\mathbf{e}'_3 \otimes \mathbf{e}'_3$. The $\mathbf{e}'_i$ establish a local Cartesian frame at each point in the transformed space Ω'. Any physical quantity $\mathbf{q}$ in the initial space will experience with the space element the same rigid rotation, then it will stretched to reach $\mathbf{q}'$ in the transformed space, i.e., $\mathbf{V_q R} : \mathbf{q} \to \mathbf{q}'$. The transformed physical quantities must rescale themselves to satisfy Eq. (2) and the energy conservation condition; this provides a way to determine them. This general idea has been applied to electromagnetic and acoustic waves.[19] In the following, we will apply the above idea to elastodynamics, namely, Navier's equation

$$\nabla \cdot \sigma = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \ \ \sigma = \mathbf{C} : \nabla\mathbf{u}, \tag{3}$$

where $\mathbf{u}$ denotes the displacement vector, σ is the two-order stress tensor, $\mathbf{C}$ is the four-order elasticity tensor, and ρ is the density. Any transformed physical quantity $\mathbf{q}'$ can be written symbolically as

$$\mathbf{V_q R} : \mathbf{q} \to \mathbf{q}', \ \ \mathbf{q} = \sigma,\mathbf{u},\mathbf{C},\rho, \tag{4}$$

where $\mathbf{R}$ establishes a local Cartesian frame $\mathbf{e}'_i$ from $\mathbf{e}_i$ (chosen arbitrary due to isotropy in the initial space) by a rotation, $\mathbf{V_q}$ has a diagonal form expressed in $\mathbf{e}'_i$ for the physical quantity $\mathbf{q}$, i.e.,

$$\mathbf{V}_\sigma = \mathrm{diag}[a_1,a_2,a_3], \ \ \mathbf{V_u} = \mathrm{diag}[b_1,b_2,b_3],$$
$$\mathbf{V_C} = \mathrm{diag}[c_1,c_2,c_3], \ \ \mathbf{V}_\rho = \mathrm{diag}[d_1,d_2,d_3], \tag{5}$$

a_i, b_i, c_i, and d_i are scaling factors, respectively, for stress, displacement, modulus, and density, and they will be determined with the help of the constraint conditions. During a rigid rotation, any attached physical quantity together with the frame $\mathbf{e}_i$ are rotated to $\mathbf{e}'_i$, so the component of the physical quantity in the local frame $\mathbf{e}'_i$ will not be altered by this rigid rotation. In the local Cartesian frame $\mathbf{e}'_i$, $\mathbf{V}_q$ is of diagonal form, so the transformed relations for the physical quantities from frame $\mathbf{e}_i$ to frame $\mathbf{e}'_i$ at each point can be written as[20]

$$\sigma'_{ij} = a_I a_J \sigma_{ij}, \ \ u'_i = b_I u_i, \ \ C'_{ijkl} = c_I c_J c_K c_L C_{ijkl},$$
$$\rho'_{ij} = d_I \delta_{ij} \rho, \tag{6}$$

where δ_{ij} is the Kronecker delta, and the capital letter in the index means the same value as its corresponding lowercase letter but without the summation.

Equation (3) is written in the initial local Cartesian frame $\mathbf{e}_i$ as

$$\frac{\partial \sigma_{ij}}{\partial x_i} = \rho \frac{\partial^2 u_j}{\partial t^2}, \ \ \sigma_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l}. \tag{7}$$

After a local affine transformation, this equation retains its form in the local Cartesian system $\mathbf{e}'_i$ in the transformed space Ω'

$$\frac{\partial \sigma'_{ij}}{\partial x'_i} = \rho'_{ij} \frac{\partial^2 u'_i}{\partial t^2}, \ \ \sigma'_{ij} = C'_{lijkl} \frac{\partial u'_k}{\partial x'_l}. \tag{8}$$

With the help of Eqs. (6) and (8), and the differential relation between the two spaces $\partial/\partial x'_i = \partial/\lambda_I \partial x_i$,[19] we have

$$\frac{a_I a_J}{\lambda_I} \frac{\partial \sigma_{ij}}{\partial x_i} = d_J b_J \rho \frac{\partial^2 u_j}{\partial t^2}, \ \ a_I a_J \sigma_{ij} = c_I c_J c_K c_L \frac{b_K}{\lambda_L} C_{ijkl} \frac{\partial u_k}{\partial x_l}. \tag{9}$$

Here the scaling factors are uniform at the considered point and its neighborhood, respectively, due to the assumption of the local affine transformation. The conservations of strain and kinetic energies during the mapping give the following constraint for the scaling factors:

$$\rho \delta_{ij} \frac{\partial u_i}{\partial t} \frac{\partial u_j}{\partial t} = \lambda_1 \lambda_2 \lambda_3 d_I b_i^2 \rho \left(\frac{\partial u_i}{\partial t} \right)^2, \ \ \sigma_{ij} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$
$$= \lambda_1 \lambda_2 \lambda_3 a_I a_J \sigma_{ij} \left(\frac{b_I}{\lambda_J} \frac{\partial u_i}{\partial x_j} + \frac{b_J}{\lambda_I} \frac{\partial u_j}{\partial x_i} \right). \tag{10}$$

By comparing Eq. (9) directly with (7), together with Eq. (10), the scaling factors are found to satisfy

$$\frac{a_i a_j}{d_J b_J} = \lambda_i, \ \ \frac{a_i a_j}{c_I c_J c_K c_l b_k} = \frac{1}{\lambda_l}, \ \ a_i a_j b_I = \frac{\lambda_j}{\lambda_1 \lambda_2 \lambda_3}. \tag{11}$$

λ_i is known when a mapping is provided, so the scaling factors a_i, b_i, c_i, and d_i can be related to λ_i. Generally, Eq. (11) has a nonunique solution, if we set $\mathbf{u}' = (\mathbf{A}^T)^{-1}\mathbf{u}$ as in Ref. 13, i.e., $b_i = 1/\lambda_i$, then the scaling factors are uniquely determined as

$$a_i = \frac{\lambda_i}{\sqrt{\det \mathbf{A}}}, \ \ b_i = \frac{1}{\lambda_i}, \ \ c_i = \frac{\lambda_i}{\sqrt[4]{\det \mathbf{A}}}, \ \ d_i = \frac{\lambda_i^2}{\det \mathbf{A}}. \tag{12}$$

In the global frame, the transformed relations of σ', $\mathbf{u}'$, and ρ' are given, respectively, by

$$\sigma' = \frac{\mathbf{A}\sigma \mathbf{A}^T}{\det \mathbf{A}}, \ \ \mathbf{u}' = (\mathbf{A}^T)^{-1}\mathbf{u}, \ \ \rho' = \frac{\mathbf{A}\rho \mathbf{A}^T}{\det \mathbf{A}}, \tag{13a}$$

The transformed relation of $\mathbf{C}'$ is expressed in the local Cartesian frame $\mathbf{e}'_i$ as

$$C'_{ijkl} = \frac{\lambda_I \lambda_J \lambda_K \lambda_L}{\det \mathbf{A}} C_{ijkl}. \tag{13b}$$

Equation (13) provides the transformed material parameters for controlling solid elastic waves by the transformation method. The transformed moduli possess the necessary symmetry $C'_{ijkl} = C'_{ijlk} = C'_{jikl} = C'_{klij}$, as required in classical elasticity theory. If the material in the initial space is fluid, i.e., $C_{ijkl} = \kappa \delta_{ij} \delta_{kl}$, and $\mathbf{u}' = (\mathbf{A}^T)^{-1}\mathbf{u}$, both ρ' and $\mathbf{C}'$ agree with the results given in Ref. 11 for generalized acoustic waves obtained by a completely different method. If the materials in the initial and transformed spaces are both fluid, i.e., $\sigma_{ij} = P\delta_{ij}$, $\sigma'_{ij} = P'\delta_{ij}$, $C_{ijkl} = \kappa \delta_{ij} \delta_{kl}$, and $C'_{ijkl} = \kappa' \delta_{ij} \delta_{kl}$, the transformed relation given by Refs. 7,8, and 21 for the acoustic wave also can be recovered.

To illustrate the proposed method, we design in the following an elastic rotator with the transformed material parameters given by Eq. (13), and validate them through numerical simulation. We propose the following mapping for the rotator:

$$r' = r, \ \ \theta' = \theta + f(r)\theta_0, \tag{14}$$

where θ_0 is a rotation angle, a and b are the radii of inner and outer boundaries of the rotator, respectively, and $f(r)$ is a function used to satisfy the impedance-matched condition at the boundary. To this end, the quintic polynomial $f(r)$ is used with $f(a) = 1$, and $f(b) = f'(a) = f'(b) = f''(a) = f''(b) = 0$ at the boundaries. Once the transformation is complete, and the material parameters for the elastic rotator are obtained from Eq. (13), the constructed rotator is then simulated with COMSOL Multiphysics. We set $a = 0.1$ m, $b = 0.35$ m, the normalized Lamé constants of the background are $\lambda = 2.3$, $\mu = 1$, and the density is $\rho = 1$ with respect to fused silica.[15] A small circle of radius $r = 0.01$ m, a distance away from the rotator, can emit harmonic P or S waves, respectively, and it in turn impinges on the rotator. The perfectly-matched-layer (PML) conditions are imposed on the boundary of the computed region.[22]

In order to quantify the approximation of the proposed transformed relation, we define a parameter $L = \max_{\Omega', i, j, k, l} |\nabla C'_{ijkl}| \gamma / C'_{ijkl}$, where γ is the wavelength in the initial space; this parameter represents approximately the maximum material variation within a wavelength. It is expected that the proposed method will give better results when L is small, i.e., more approaching the condition necessary for local affine transformation. Figure 1 shows the distributions of total displacement of the designed rotators with different θ_0 and different incident frequencies γ, respectively, and all are with an impinging S wave. It is seen that the wave patterns are all rotated, as expected. The wave pattern is clearer in the case where L is smaller, as shown in Fig. 1. The wave front becomes slightly blurred when L is large, as illustrated in Fig. 1(a). The same effect can also be observed for the P waves.

The local affine transformation implies that the material is considered locally to be homogeneous round at a point (piecewise constant); this assumption is commonly used in
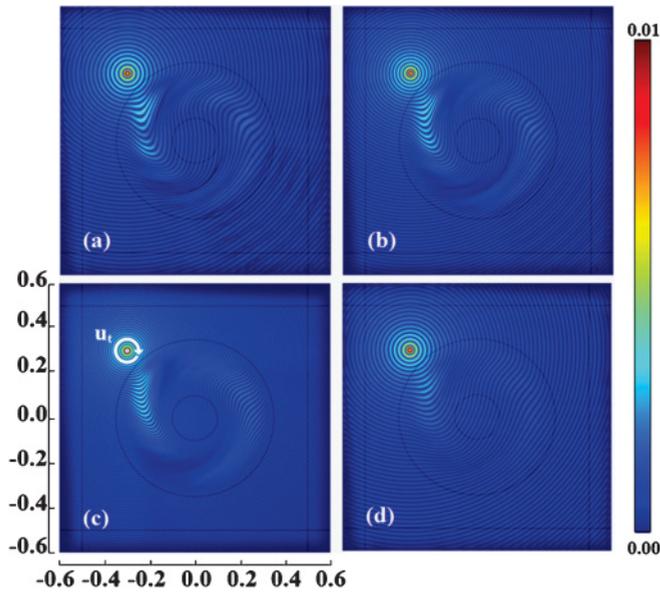
FIG. 1. (Color online) Simulation of rotator for elastic waves, total displacement field ($\sqrt{u_x^2 + u_y^2}$) of the generated S wave: (a) $\theta_0 = \pi/4$, $\gamma = 0.0375$ m, $L \approx 1.748$; (b) $\theta_0 = \pi/4$, $\gamma = 0.03$ m, $L \approx 1.398$; (c) $\theta_0 = \pi/4$, $\gamma = 0.0225$ m, $L \approx 1.049$; (d) $\theta_0 = \pi/8$, $\gamma = 0.0375$ m, $L \approx 0.774$.

elastic ray theory.[23] According to this theory, for a wave of form $\mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x})F(t - T(\mathbf{x}))$, Navier's equation is reduced into the following two equations in a high-frequency approximation (elastic ray theory):

$$C_{ijkl}\frac{\partial T}{\partial x_l}\frac{\partial T}{\partial x_j}U_k - \rho_{ij}U_j = 0, \tag{15a}$$

$$C_{ijkl}\frac{\partial T}{\partial x_j}\frac{\partial U_k}{\partial x_l} + \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial T}{\partial x_l}U_k\right) = 0. \tag{15b}$$

Equation (15a) is called the eikonal equation describing the ray path, and (15b) is the transport equation characterizing the wave amplitude along the ray. It can be shown that the transformed relation (13) verifies the form invariant of

the eikonal equation, but keeps the transport equation (15b) form invariant only if the scaling factors change slowly or their derivatives can be neglected, contrary to transformation optics and acoustic cases. This implies that the transformed material parameters can well control the wave path, but leave the wave amplitude to be controlled approximately. For very high frequency, the eikonal equation dominates; the proposed method will lead to nearly perfect results. Detailed discussions of the transformation method based on the elastic ray theory will be presented elsewhere. As shown by Milton *et al.*,[13] Navier's equation is transformed to Willis' equation, and when the material gradient is small or the frequency is high, Willis' equation can be reduced to Navier's equation, and the transformation relation (13) can also be found from their theory. It must be stressed that the proposed method is different from the transformation methods existing in literature; the transformed relations are derived directly from the constraint conditions, and none of transformed relation is preassumed. As a result, the proposed method can offer more flexibility in material design, such as the impedance-matched condition in isotropic transformed materials.[24]

In summary, we have proposed an approximate method to derive the transformed relation for controlling the solid elastic wave. The method idealizes a general transformation by local affine ones; the local form invariant of Navier's equation and the conservation of energy provide the constraint conditions to derive the transformed relations. It is shown numerically that the obtained transformed relation can be used within the classical elasticity theory to control elastic waves, especially for the regime of high frequency or the small material gradient. It is also shown that the transformed material parameters control well the ray path, but only approximately control the amplitude along the ray. Potential applications of the proposed results can be anticipated in nondestructive testing, structure impact protection, petroleum exploration, and seismology.[23,25]

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[20] A stretch operation $\mathbf{V}_q$ on a physical quantity follows the tensor rule; for example, a stretch operation a_{ij} on the stress σ_{ij} can be written as $\sigma'_{ij} = a_{ik}a_{jl}\sigma_{kl}$. In the frame $\mathbf{e}'_i$, a_{ij} has a diagonal form $\mathbf{a} = \mathrm{diag}[a_1, a_2, a_3]$, so it has $\sigma'_{ij} = a_I a_J \sigma_{ij}$.

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