Majorana zero modes in one-dimensional quantum wires without long-ranged superconducting order

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We show that long-ranged superconducting order is not necessary to guarantee the existence of Majorana fermion zero modes at the ends of a quantum wire. We formulate a concrete model, which applies, for instance, to a semiconducting quantum wire with strong spin-orbit coupling and Zeeman splitting coupled to a wire with algebraically decaying superconducting fluctuations. We solve this model by bosonization and show that it supports Majorana fermion zero modes. We show that electron backscattering in the superconducting wire, which is caused by potential variations at the Fermi wave vector, generates quantum phase slips that cause a splitting of the topological degeneracy, which decays as a power law of the length of the superconducting wire. The power is proportional to the number of channels in the superconducting wire. Other perturbations give contributions to the splitting that decay exponentially with the length of either the superconducting or semiconducting wires. We argue that our results are generic and apply to a large class of models. We discuss the implications for experiments on spin-orbit coupled nanowires coated with superconducting film and for $LaAlO_3/SrTiO_3$ interfaces.

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I. INTRODUCTION

Kitaev[1] showed that a class of superconducting quantum wires supports a pair of Majorana fermion zero modes, one at each end. Lutchyn *et al.*[2] and Oreg *et al.*[3] discovered that, in the presence of a parallel magnetic field, semiconducting wires with strong spin-orbit coupling fall in this class if superconductivity is induced by proximity to a bulk three-dimensional (3D) superconductor (see Fig. 1). As a result of the Majorana zero modes, the ground state is doubly degenerate. The two states differ by fermion parity, which is not locally measurable; therefore, they form a protected qubit. Networks of such semiconducting wires have been proposed for topological quantum information processing.[4–7]

Long-ranged superconducting order is an essential feature of these analyses. Although such order is sufficient, it does not seem necessary. Protected Majorana zero modes also exist in models of the 5/2 fractional quantum Hall state[8–13] and in Kitaev's honeycomb lattice spin model,[14] and neither of these systems has long-ranged superconducting order. Therefore, one might expect that a quantum wire with strong superconducting fluctuations but no long-ranged order could also support Majorana fermion zero modes. Consider, on the other hand, a spinless one-dimensional Luttinger liquid, which has algebraic order, i.e., the two-point correlation function of the superconducting order parameter decays to zero as a power of the separation rather than approaching a constant. Such a system has gapless bulk fermionic excitations, so if Majorana fermion zero modes were found at the ends of such a model, there would be nothing protecting them against small perturbations. Furthermore, in the absence of superconducting order, the two states of a pair of Majorana fermion zero modes would have different electric charges, and not merely fermion parity. Simply changing the electrostatic potential should cause an energy splitting between states with different electric charges. Therefore, one might, instead, conclude that long-ranged superconducting order is necessary to protect Majorana fermion zero modes in quantum wires.

In this paper, we show that this is not the case. We construct a model of a spin-orbit coupled semiconducting wire in a magnetic field, which is coupled to an s-wave superconducting wire with power-law order. A schematic picture of such a heterostructure is depicted in Fig. 2. We show that this model supports Majorana fermion zero modes at the ends of the wire. However, a single wire does not support a qubit; at least two wires are needed. The basic idea is simple. Consider Kitaev's[1] model of a superconducting quantum wire of spinless fermions:

$$H = -t\sum_i (c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}) + |\Delta|\sum_i (e^{i\phi} c_i c_{i+1} - e^{-i\phi} c_i^\dagger c_{i+1}^\dagger). \tag{1}$$

Here, ϕ is the phase of the superconducting order parameter. Let us assume, for the moment, that ϕ is a constant, as in Kitaev's original paper[1] and in Refs.[2] and[3]. If we rotate the fermion operators to the local value of the phase of the order parameter $c_i \to e^{-i\phi/2}\tilde{c}_i$, then the Hamiltonian takes the form

$$H = -t\sum_i (\tilde{c}_{i+1}^\dagger \tilde{c}_i + \tilde{c}_i^\dagger \tilde{c}_{i+1}) + |\Delta|\sum_i (\tilde{c}_i \tilde{c}_{i+1} - \tilde{c}_i^\dagger \tilde{c}_{i+1}^\dagger). \tag{2}$$

At the special point $t = |\Delta|$, this Hamiltonian can be diagonalized by introducing the Majorana fermion operators $\gamma_{2i-1} = \tilde{c}_i + \tilde{c}_i^\dagger$, $\gamma_{2i} = (\tilde{c}_i - \tilde{c}_i^\dagger)/i$:

$$H = i|\Delta|\sum_i \gamma_{2i}\gamma_{2i+1}. \tag{3}$$

These operators satisfy $\gamma_i = \gamma_i^\dagger$ and $\{\gamma_i,\gamma_j\} = 2\delta_{ij}$. Note that γ_1 and γ_{2N} do not appear in the Hamiltonian. Therefore, the ground state is doubly degenerate: $i\gamma_1\gamma_{2N}$ can be either ± 1, while $i\gamma_{2i}\gamma_{2i+1} = -1$ for $1 \leqslant i \leqslant N-1$. Apart from the degeneracy of the ground state, there is a gap $2|\Delta|$ to excitations. The operators γ_1 and γ_{2N} are Majorana fermion zero modes, and the qubit that they form, $i\gamma_1\gamma_{2N} = \pm 1$, is

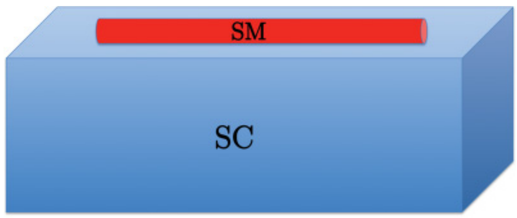

FIG. 1. (Color online) A semiconductor nanowire in contact with a bulk 3D superconductor.

protected since the two states are distinguished only by fermion parity, which can not be measured by a local operation. Only an operator that acts on both sites 1 and N can affect it. Away from the special point $t = |\Delta|$, the physics is very similar: there is a gap in the bulk above two nearly degenerate ground states, which have an energy splitting $\sim e^{-Na/\xi}$, where ξ is inversely proportional to the bulk gap and a is the lattice spacing. This phase persists to the more physical $|\Delta| \ll t$ limit. Electron-electron interactions in the wire determine the region of the phase diagram occupied by this phase.[15–18]

Now suppose that ϕ is a fluctuating dynamical field. We can still perform a change of variables similar to the one that we made in going from Eq. (1) to Eq. (2). This will remove the phase of the order parameter from the second term in Eq. (1), the pairing term. However, it will introduce a coupling between the fermions and gradients of the order parameter. If these terms can be neglected, then we will have mapped a model with fluctuating order parameter to one with fixed order parameter that is decoupled from the fluctuations of ϕ; therefore, it will have Majorana fermion zero modes. However, there are some subtleties involved in the change of variables from c_i to $\tilde{c}_i$ when ϕ fluctuates. These are most easily handled using a bosonized formulation of the electronic degrees of freedom in the wire. We find a special point in Secs. III and IV at which the bosonized formulation simplifies and allows us to completely analyze the model. We then show in Sec. V that our analysis is qualitatively unchanged by perturbations that take the system away from the special point.

The technical subtleties alluded to above have a physical origin related to the conservation of charge. Note that the ground-state energy has the form

$$E(N) = N\mathcal{E} + E_{\text{even,odd}} + O(e^{-aL}) \tag{4}$$

for even and odd electron numbers N, respectively. (See Ref. 19 for the analogous relation for paired quantum Hall states.) The signature of Majorana fermion zero modes at the endpoints of a wire is that $E_{\text{odd}} = E_{\text{even}}$. In a superconducting system without zero modes, we would have $E_{\text{odd}} > E_{\text{even}}$. The difference $E_{\text{odd}} - E_{\text{even}}$ would simply be the energy cost of an unpaired electron. In the presence of zero modes, this cost vanishes. As may be seen from Eq. (4), however, *a single wire does not have degenerate states* unless the electrostatic potential is tuned so that $\mathcal{E} = 0$. Note that, in the presence of unscreened Coulomb interactions, $\mathcal{E}$ is not a constant but also includes a term that is linear in N. One can find a particular N at which $\mathcal{E} = 0$, but this requires fine tuning.

If, however, we consider two such wires, then there are two degenerate states for fixed total electron number without any fine tuning. Suppose that there are $2N$ electrons in the system. Let us denote the energy of the two wires, isolated from each other by $E_1(N)$, $E_2(N)$. They are given by Eq. (4) with $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$ and $E^{(1)}_{\text{even,odd}}$, $E^{(2)}_{\text{even,odd}}$ taking the place of $\mathcal{E}$ and $E_{\text{even,odd}}$. If there are Majorana zero modes at the endpoints of both wires in isolation, then $E^{(1)}_{\text{odd}} = E^{(1)}_{\text{even}}$ and $E^{(2)}_{\text{odd}} = E^{(2)}_{\text{even}}$. Then,

$$E_1(N) + E_2(N) = E_1(N-m) + E_2(N+m) \tag{5}$$

for any m, so long as $\mathcal{E}^{(1)} = \mathcal{E}^{(2)}$. Now, suppose that the two semiconducting wires are coupled to the same (power-law) s-wave superconducting wire (which is assumed to be much longer than either semiconducting wire so that it can be coupled to both while keeping them far apart), so that the electrochemical potential must be the same in the two wires. Then, $\mathcal{E}^{(1)} = \mathcal{E}^{(2)}$. Furthermore, Cooper pairs can tunnel from either semiconducting wire to the superconductor. Therefore, rather than a degenerate ground state for each value of m in (5), there will be two nearly degenerate states, corresponding to an even or odd number of electrons in each wire. (Furthermore, there will only be a charging energy for the whole system, not for each wire separately, which justifies taking $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$ as constants.) Such a protected qubit exists for any fixed electron number. If the electron number were odd, then the two states would correspond, instead, to (a) an even electron number in wire 1, odd in wire 2; and (b) an odd electron number in wire 1, even in wire 2.

These arguments are supported by explicit calculations in Secs. III and IV. First, we show in Sec. II how the topological degeneracy is manifested when a semiconducting nanowire is coupled to a bulk 3D superconductor. Pair tunneling between the wire and the 3D superconductor is represented by a term in the bosonized effective Hamiltonian of the form

$$H_{\text{pair tun.}} \propto \sin 2\theta, \tag{6}$$

where θ is the bosonic field satisfying $\rho = \frac{1}{\pi}\partial_t\theta$, where ρ is the charge density. The two ground states of the system correspond to the two minima of $\sin 2\theta$ as a function of θ. As we discuss in Sec. II, these two states differ in fermion parity, as expected for a pair of Majorana zero modes. Furthermore, if the two ends of the wire are connected to form a ring, then the ground-state degeneracy disappears because only the equal amplitude superposition of the two minima is allowed for periodic boundary conditions of the electrons (while the orthogonal superposition occurs for antiperiodic electronic boundary conditions). When we turn in Secs. III and IV to the case in which the superconductor is also one dimensional

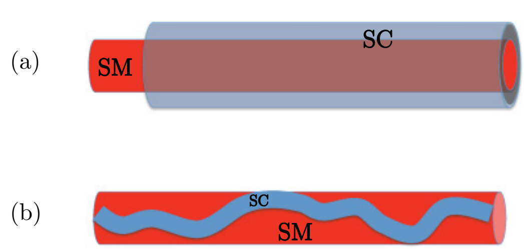

FIG. 2. (Color online) A semiconductor nanowire in contact with a 1D superconducting wire. The superconducting wire could be a coating that (a) completely covers the semiconductor or (b) only covers part of it.

and, therefore, does not have long-ranged order, our analysis will depend on a careful treatment of the target space of the bosonic fields. The periodicity conditions satisfied by these fields encode the quantization of charge, and the ground-state degeneracy can not be counted properly without accounting for them. The use of bosonization techniques also requires a careful treatment of locality: putative Majorana modes in a transformed system may simply be a reflection of a spontaneously broken global $\mathbb{Z}_2$ in the original variables (cf. the duality between the transverse-field Ising model and a Majorana wire). We wish to stress the topological nature of the Majorana degeneracy in our model: no local observable can distinguish the two states. A key feature of these models is that there is a single-fermion gap even though there are gapless superconducting phase fluctuations, as is already apparent in Eq. (2) if the second line is benign (as we show it to be). This may be viewed as a form of the "spin-gap proximity effect."[20,21] This gap protects the Majorana fermion zero modes. However, as we show below, in addition to fermion tunneling events, which lift the topological degeneracy even in models with long-range superconducting order, there is another error-causing process involving quantum phase slips that will have a vanishingly small probability of occurring in a bulk 3D superconductor. The effect of a quantum phase slip in the middle of a superconducting wire can be understood as that of a vortex encircling a pair of Majorana zero modes. Such a process results in reading out the fermionic parity via the Aharonov-Casher effect and effectively leads to a splitting of the degeneracy. We show that backscattering from impurities generates quantum phase slips in the middle of the wire and causes a splitting of the topological degeneracy that decays algebraically with the size of the system rather than exponentially. However, the exponent is proportional to the number of channels in the superconducting wire. Thus, by making a superconducting wire with sufficiently many channels, we can make the splitting decay as a high power of the length.

When this is the case, it is sufficient for the wires and wire networks of Refs. 2–4 to be in proximity to systems with power-law superconducting order; long-ranged order is not necessary. Consequently, it may be possible to sputter superconducting grains onto the semiconducting wire or to coat it with superconducting film of a finite thickness. This is important because it may be difficult to tune a semiconducting wire between topological and nontopological phases by applying a gate voltage if it is in contact with a bulk superconductor, which will presumably fix its chemical potential.

Recently, it has been shown that quasi-1D wires can be "written" on $LaAlO_3/SrTiO_3$ (LAO/STO) interfaces,[22] which have substantial Rashba spin-orbit coupling.[23] These wires show strong superconducting fluctuations. As we will discuss in detail elsewhere,[24] a possible model for this system is a spin-orbit coupled quantum wire in contact with local superconducting regions, which fail to percolate across insulating $SrTiO_3$, but can induce still superconducting fluctuations in quantum wires at the LAO/STO interface. Our results imply that these superconducting fluctuations may be sufficient to support Majorana zero modes at the ends of such wires if a parallel magnetic field is applied.

II. A SEMICONDUCTOR NANOWIRE COUPLED TO A BULK 3D SUPERCONDUCTOR: BOSONIZED FORMULATION

Before introducing our model, we briefly review the proposal for realizing Majorana quantum wires in semiconductor-superconductor heterostructures[2,3] and recast it in bosonic form. Its basic ingredient is a semiconductor nanowire with strong spin-orbit interactions. Superconductivity is induced via the proximity effect. The Hamiltonian for the nanowire is ($\hbar = 1$)

$$H_{\rm NW} = \int_{-L/2}^{L/2} dx\, \psi_\sigma^\dagger(x)\left(-\frac{\partial_x^2}{2m^*} - \mu + i\alpha\sigma_y\partial_x + V_x\sigma_x\right)_{\sigma\sigma'}\psi_{\sigma'}(x),$$
$$H_P = \int_{-L/2}^{L/2} dx[\Delta_0\psi_\uparrow\psi_\downarrow + \text{H.c.}], \tag{7}$$

where m^*, μ, and α are the effective mass, chemical potential, and strength of spin-orbit Rashba interaction, respectively. An in-plane magnetic field B_x leads to a spin splitting $V_x = g_{\rm NW}\mu_B B_x/2$, where $g_{\rm NW}$ and μ_B are the g factor in the semiconducting nanowire and the Bohr magneton, respectively. In the simplest model for the nanowire, we assume that the semiconductor nanowire (NW) is in tunneling contact with a bulk 3D superconductor (SC), as depicted in Fig. 1. Then, electron tunneling between the NW and the SC leads to the proximity effect described by the Hamiltonian H_P. The superconducting pairing potential Δ_0 is assumed to be a static classical field, and quantum fluctuations of the superconducting phase are neglected.

The nanowire described by the Hamiltonian $H_T = H_{\rm NW} + H_P$ can be driven into a nontrivial topological state by adjusting the chemical potential so that it lies in the gap $|\mu| < \sqrt{V_x^2 - \Delta_0^2}$. Under these conditions, the Hamiltonian can be projected to the lower band of the two bands that form as a result of the combined effect of the spin-orbit coupling and magnetic field. The low-energy limit of this Hamiltonian then takes the same form as Eq. (2) for low energies $E \ll t$, assuming $|\Delta| \ll t^2$. Therefore, the topological superconducting phase described by H_T harbors Majorana fermion operators γ_L and γ_R, which are zero modes, up to exponential corrections, localized about the two endpoints:

$$\gamma_a = \gamma_a^\dagger, \quad \{\gamma_a,\gamma_b\} = 2\delta_{ab}, \tag{8}$$

$$[H_T,\gamma_a] = 0 + O(e^{-L/\xi}), \tag{9}$$

$$\{\gamma_L,\psi_\sigma(x)\} \sim e^{-|x+L/2|/\xi}, \quad \{\gamma_R,\psi_\sigma(x)\} \sim e^{-|x-L/2|/\xi}. \tag{10}$$

Here, ξ is the effective coherence length. The presence of these zero modes leads to topological degeneracy up to an exponential splitting energy $\delta E \propto e^{-L/\xi}$. The two nearly degenerate states correspond to the two eigenvalues of $i\gamma_1\gamma_2$ and have even and odd fermion parity,[1] respectively, which can be exploited for topological quantum computation.[25]

These results were obtained[1–3] using the properties of the free-fermion band structure embodied by H_T. We now rederive them using a bosonic representation. In later sections, we will use this representation to analyze the case when there is no long-ranged superconducting order, unlike in H_T. First, we bosonize the semiconductor Hamiltonian (7). In the helical regime corresponding to a large Zeeman gap, $H_{\rm NW}$ can be approximated by projecting the system to the lowest subband and writing the field operator $\Psi(x) \equiv (\psi_\uparrow(x),\psi_\downarrow(x))$ as

$$\Psi(x) \approx \Phi_-(p_F)e^{ip_Fx}c_R(x) + \Phi_-(-p_F)e^{-ip_Fx}c_L(x), \quad (11)$$

where the spinor $\Phi_-(p_F) = \frac{1}{\sqrt{2}}(-e^{i\kappa(p_F)},1)$ and $\kappa(p_F) = \tan^{-1}(\alpha p_F/V_x)$. Substituting Eq. (11) into $H_{\rm NW}$, the Hamiltonian can be written in terms of the spinless right- and left-moving fermions $c_R(x)$ and $c_L(x)$ and eventually bosonized using $c_{R/L} = \frac{1}{\sqrt{2\pi a}}e^{-i(\pm\phi-\theta)}$:

$$H_{\rm NW} \approx v\int_{-L/2}^{L/2} dx[ic_L^\dagger(x)\partial_x c_L(x) - ic_R^\dagger(x)\partial_x c_R(x)] \quad (12)$$

$$\approx \frac{v}{2\pi}\int_{-L/2}^{L/2} dx[K(\partial_x\theta)^2 + K^{-1}(\partial_x\phi)^2]. \quad (13)$$

Here, v is the fermion velocity $v = p_F(\frac{1}{m^*} - \frac{\alpha^2}{\sqrt{V_x^2+\alpha^2p_F^2}})$ and K is the Luttinger parameter for the nanowire. The fields ϕ and θ satisfy the canonical commutation relation

$$[\partial_x\phi(x),\theta(x')] = i\pi\delta(x-x'). \quad (14)$$

The charge density and current near wave vector zero are given by $\rho = \frac{1}{\pi}\partial_x\phi = \frac{1}{\pi}\partial_t\theta$ and $j = -\frac{1}{\pi}\partial_t\phi = \frac{1}{\pi}\partial_x\theta$. The fields ϕ and θ can be interpreted as the phase of the density at wave vector $2k_F$ and the pair field, respectively:

$$\rho_{2k_F}(x) = e^{-2i\phi(x)},$$
$$\Psi_{\rm pair}(x) \equiv \psi_\uparrow(x)\psi_\downarrow(x) = e^{2i\theta(x)}. \quad (15)$$

For the Hamiltonian $H_{\rm NW}$, in which electron-electron interactions in the semiconductor have been neglected, $K = 1$ (the free-fermion value). However, the bosonic representation accommodates short-ranged interactions in the nanowire such as

$$H_{\rm NW\ int.} = u\int_{-L/2}^{L/2} dx\ \psi_\sigma^\dagger(x)\psi_\sigma(x)\,\psi_{\sigma'}^\dagger(x)\psi_{\sigma'}(x) \quad (16)$$

simply by shifting the value of K and rescaling v. Here, $K > 1$ for repulsive interactions and $K > 1$ for attractive interactions. As we shall see below, the Majorana degeneracy persists for a whole range of K, which includes the free Fermi point $K = 1$. The bosonic form for H_P in Eq. (7) is

$$H_P = \frac{\Delta_P}{(2\pi a)}\int_{-L/2}^{L/2} dx\ \sin(2\theta). \quad (17)$$

Therefore, H_T can be written in the bosonic form

$$H_T = \int_{-L/2}^{L/2} dx\bigg(\frac{v}{2\pi}[K(\partial_x\theta)^2 + K^{-1}(\partial_x\phi)^2] + \frac{\Delta_P}{(2\pi a)}\sin(2\theta)\bigg). \quad (18)$$

This interaction term H_P is relevant unless there are very strong repulsive interactions in the nanowire. To be more precise, the lowest-order renormalization-group (RG) equation for the dimensionless coupling $y = 2\Delta_P a/v$ is

$$\frac{dy}{dl} = (2 - K^{-1})\,y. \quad (19)$$

For noninteracting electrons, $K = 1$, and even for repulsive interactions up until $K = 1/2$, this is a relevant perturbation. If y is initially small at short distances, then we can use Eq. (19) to conclude that $y(l) \sim 1$ at the length scale $l = \ln(\xi/a_0)$, where the effective coherence length ξ in the semiconducting nanowire is given by $\xi \sim a_0(v/2\Delta_P a_0)^{K/(2K-1)}$. Here, a_0 is the short-distance cutoff, which is the shortest length scale at which the effective description (18) is valid. We can take it to be the coherence length or the Josephson length of the bulk 3D superconductor, but, at any rate, it must be larger than the Fermi wavelength in the semiconducting wire.

At longer length scales, the field θ is pinned to the minimum of $\sin(2\theta)$. Since there are two minima, $\theta = -\pi/4,\ 3\pi/4$, there are two degenerate ground states in the $L \to \infty$ limit. These two ground states are related to each other by the global $\mathbb{Z}_2$ symmetry of the model $\theta \to \theta + \pi$. To understand this symmetry better, it is helpful to note that the fermion parity $(-1)^{N_F}$ can be written in the form

$$(-1)^{N_F} = e^{i[\phi(L/2)-\phi(-L/2)]}. \quad (20)$$

Therefore, using the commutation relation (14), we see that the fermion parity $(-1)^{N_F}$ generates the symmetry transformation $\theta \to \theta + \pi$. Since the two degenerate ground states corresponding to $\theta = -\pi/4,\ 3\pi/4$ are transformed into each other by fermion parity, the following quantum superpositions are fermion-parity eigenstates:

$$|\text{even,odd}\rangle = \frac{1}{\sqrt{2}}(|-\pi/4\rangle \pm |3\pi/4\rangle). \quad (21)$$

The ends of the wire are crucial for this qubit. If we were to connect the two ends of the wire to form a ring of circumference L, then we would expect only a single ground state, not a degenerate pair. To see that this is indeed the case, consider the fermion annihilation operators

$$c_{R,L}(x) = \frac{1}{\sqrt{2\pi a}}\,e^{-i(\pm\phi-\theta)}. \quad (22)$$

Since $\rho = \frac{1}{\pi}\partial_x\phi$, the ring will have even fermion parity if the boundary conditions on ϕ are

$$\phi(x+L) = \phi(x) + 2n\pi$$

for integer n. If the fermions have periodic boundary conditions $c_{R,L}(x+L) = c_{R,L}(x)$, then the boundary condition on θ must be

$$\theta(x+L) = \theta(x) + 2n'\pi$$

for integer n'. Since constant solutions are allowed for this boundary condition on θ, the ground state $|\text{even}\rangle = \frac{1}{\sqrt{2}}(|-\pi/4\rangle + |3\pi/4\rangle)$, which is a linear superposition of constant solutions, is allowed in this case. This state has even fermion parity (20), so it is consistent with the boundary conditions on ϕ. If the ring has odd fermion parity, however, then $\phi(x+L) = \phi(x) + (2n+1)\pi$. Consequently, if the fermions have periodic boundary conditions, the boundary condition on θ must be $\theta(x+L) = \theta(x) + (2n'+1)\pi$. This precludes a constant solution. Therefore, the state $|\text{odd}\rangle = \frac{1}{\sqrt{2}}(|-\pi/4\rangle - |3\pi/4\rangle)$, which is odd under fermion parity (20), is not an allowed state if the fermions have periodic boundary conditions. As expected, we conclude that there is only a single ground state for a ring, in contrast with a line segment, which has a doubly degenerate ground state.

The Majorana fermion zero modes of this system are manifested on a ring by the presence of a corresponding state for antiperiodic boundary conditions on the fermions. If $c_{R,L}(x+L) = -c_{R,L}(x)$, then for odd fermion parity $\phi(x+L) = \phi(x) + (2n+1)\pi$, the boundary condition on θ must be $\theta(x+L) = \theta(x) + 2n'\pi$ for integer n'. This boundary condition allows constant solutions, so the ground state is $|\text{odd}\rangle = \frac{1}{\sqrt{2}}(|-\pi/4\rangle - |3\pi/4\rangle)$. Therefore, the ground state with periodic boundary conditions and the ground state with antiperiodic boundary conditions have the same energy density and opposite fermion parities. This can already be seen in the Kitaev chain. On a line segment, the operators γ_1 and γ_{2N} do not appear in the Hamiltonian, as we saw in the Introduction. On a ring with periodic boundary conditions, there is a term $it\gamma_{2N}\gamma_1$. If the boundary conditions are antiperiodic, the term is instead $-it\gamma_{2N}\gamma_1$. The ground-state energy is the same in both cases, but the ground states differ in fermion parity, $i\gamma_{2N}\gamma_1 = \pm 1$.

Returning now to the case of open boundary conditions, we observe that, for finite L, these two states are split in energy because there are instantons that tunnel between the two minima. The Euclidean action in the strong-coupling limit is

$$S = \frac{v}{2\pi}\int dx\, d\tau \left[(\partial_x\theta)^2 + v^{-2}(\partial_\tau\theta)^2 + \frac{y}{\xi^2}\sin(2\theta)\right]. \quad (23)$$

The splitting is then given by $\delta E \propto N_f e^{-S_0}$, where S_0 is the action of the Euclidean instanton $\theta_0(x,\tau)$ satisfying $\theta_0(x,-\infty) = -\frac{\pi}{4}$, $\theta_0(x,\infty) = \frac{3\pi}{4}$ and N_f is a prefactor that comes from fluctuations. Clearly, the lowest action instanton is translationally invariant, at least away from $x = -L/2, L/2$, so the problem reduces to a $(0+1)$-dimensional problem, with action

$$S_{\text{QM}} = \frac{L}{\pi}\int dz \left[\frac{1}{2}(\partial_z\theta)^2 + V(\theta)\right], \quad (24)$$

where $V(\theta) = \frac{y}{\xi^2}\sin(2\theta)$ and $z = v\tau$. Following Ref. 26,

$$S_0 = \frac{L}{\pi}\int_{-\pi/4}^{3\pi/4} d\theta \sqrt{2[V(\theta) - E]} = \frac{4\sqrt{y}}{\pi}\frac{L}{\xi}, \quad (25)$$

where $E = -y/\xi^2$ is the energy of the minimum of the potential. The splitting then scales like $\delta E \propto \exp(-\frac{4\sqrt{y}}{\pi}\frac{L}{\xi})$, as expected.

Since 2θ changes by 2π while the phase of the bulk superconductor is unchanged, such an instanton can be interpreted roughly as the motion of a vortex between the NW and the bulk superconductor. (We say "roughly" because our instanton is a spatially uniform phase slip rather than a spatially localized vortex.) Since it causes a transition between the states $|-\pi/4\rangle$ and $|3\pi/4\rangle$, it splits the states $|\text{even}\rangle$ and $|\text{odd}\rangle$. Thus, it can also be interpreted as a Majorana fermion tunneling between the two ends of the wire.

III. A SINGLE SEMICONDUCTING NANOWIRE COUPLED TO AN ALGEBRAICALLY ORDERED SUPERCONDUCTING WIRE

We now include the effect of quantum fluctuations by replacing the bulk superconductor in the above proposal with an s-wave superconducting wire with power-law order. This model preserves the overall U(1) charge symmetry [there is no spontaneous U(1) breaking] and allows for the study of the topological superconducting phase in the particle number-conserving setting. For the sake of concreteness and simplicity, we will take the Hamiltonian for the superconducting wire to be the attractive U Hubbard model. However, our results hold for any spin-gapped system with s-wave superconducting fluctuations.[27]

We use the standard bosonization procedure for spinful fermions, with the convention[28] that

$$\psi_{r,\sigma} = \frac{1}{\sqrt{2\pi a}} e^{-\frac{i}{\sqrt{2}}[(r\phi_\rho - \theta_\rho) + \sigma(r\phi_\sigma - \theta_\sigma)]}, \quad (26)$$

where $r = \pm$ and $\sigma = \pm$ for right- and left-moving fermions with up and down spin, and a the lattice cutoff. The fields $\phi_{\rho,\sigma}$ and $\theta_{\rho,\sigma}$ satisfy the same commutation relations (14). In terms of these fields, the Hamiltonian for the superconducting wire can be written as

$$H_{\text{SC}} = H_{\text{SC}}^{(\rho)} + H_{\text{SC}}^{(\sigma)}, \quad (27)$$

$$H_{\text{SC}}^{(\rho)} = \frac{v_F}{2\pi}\int_{-L/2}^{L/2} dx \left[K_\rho(\partial_x\theta_\rho)^2 + K_\rho^{-1}(\partial_x\phi_\rho)^2\right], \quad (28)$$

$$H_{\text{SC}}^{(\sigma)} = \frac{v_F}{2\pi}\int_{-L/2}^{L/2} dx \left[K_\sigma(\partial_x\theta_\sigma)^2 + K_\sigma^{-1}(\partial_x\phi_\sigma)^2\right] - \frac{2|U|}{(2\pi a)^2}\int_{-L/2}^{L/2} dx \cos(2\sqrt{2}\phi_\sigma), \quad (29)$$

where v_F, a, and U are the Fermi velocity, the effective cutoff length, and the interparticle interaction potential, respectively.

Tunneling between the superconducting wire and the semiconducting wire can be described using a simple model Hamiltonian

$$H_t = t\sum_\sigma \int_{-L/2}^{L/2} dx (\psi_\sigma^\dagger \eta_\sigma + \eta_\sigma^\dagger \psi_\sigma), \quad (30)$$

where t is the tunneling amplitude and ψ_σ and η_σ represent fermion annihilation operators in the semiconducting and superconducting systems, respectively. Given that single-electron tunneling into the superconducting wire is suppressed due to the presence of the spin gap E_g (see below), the dominant contribution to the action comes from pair hopping. The perturbative expansion in t to second order leads to the following imaginary-time action:

$$S_{\text{PH}} = -t^2 \sum_\sigma \int dx\, d\tau\, dx' d\tau' \times [\psi_\sigma^\dagger(x,\tau)\psi_{-\sigma}^\dagger(x',\tau')\eta_\sigma(x,\tau)\eta_{-\sigma}(x',\tau') + \text{H.c.}]. \quad (31)$$

We now analyze the bosonized action. First, the spin field ϕ_σ orders as a result of the last term in Eq. (29), opening a spin gap E_g in the superconducting wire. The dual field

θ_σ is disordered, and its correlation function decays exponentially $\langle e^{-\frac{i}{\sqrt{2}}\theta_\sigma(x,\tau)} e^{\frac{i}{\sqrt{2}}\theta_\sigma(0,0)}\rangle_\sigma \sim a/\sqrt{x^2+(v_F\tau)^2}\exp[-E_g\sqrt{\tau^2+x^2/v_F^2}]$. This allows us to simplify the action (31) and make a local approximation

$$S_{\rm PH} \approx -\frac{\Delta_P}{(2\pi a)}\int d\tau \int_{-L/2}^{L/2} dx \sin(\sqrt{2}\theta_\rho - 2\theta) \qquad (32)$$

valid in the long-time limit $|\tau-\tau'| \gg E_g^{-1}$. Here, the Cooper-pair-hopping amplitude Δ_P is given by $\Delta_P \sim \frac{t^2}{E_g}\frac{\alpha p_F}{\sqrt{(\alpha p_F)^2+V_x^2}}$ similarly to the proximity-induced gap in the perturbative tunneling limit $t \ll E_g$. If the field θ_ρ were pinned (i.e., $\theta_\rho = 0$), we would recover the model considered in Refs. 2 and 3. In the present case, however, overall U(1) symmetry is not broken due to the presence of fluctuating field θ_ρ. Henceforth, we thus analyze the following effective low-energy model:

$$\begin{aligned} H_M = &\frac{v}{2\pi}\int_{-L/2}^{L/2} dx[K(\partial_x\theta)^2 + K^{-1}(\partial_x\phi)^2] \\ &+ \frac{v_F}{2\pi}\int_{-L/2}^{L/2} dx\left[K_\rho(\partial_x\theta_\rho)^2 + K_\rho^{-1}(\partial_x\phi_\rho)^2\right] \\ &- \frac{\Delta_P}{(2\pi a)}\int_{-L/2}^{L/2} dx \sin(\sqrt{2}\theta_\rho - 2\theta), \end{aligned} \qquad (33)$$

and study the effect of quantum fluctuations of θ_ρ on the stability of the topological superconducting phase. This model is quadratic, except for the interaction Δ_P. The dimensionless coupling $y = 2\Delta_P a/v$ has the RG equation

$$\frac{dy}{dl} = \left(2 - \frac{1}{2}K_\rho^{-1} - K^{-1}\right) y. \qquad (34)$$

For $\frac{1}{2}K_\rho^{-1} + K^{-1} > 2$, this interaction is irrelevant, and we can ignore Cooper-pair tunneling between the wires. However, interwire pair tunneling is relevant for $\frac{1}{2}K_\rho^{-1} + K^{-1} < 2$, which includes the case of weakly attractive interactions in the superconducting wire $K_\rho \lesssim 1$, and weakly repulsive interactions in the semiconducting wire $K \gtrsim 1$.

This model simplifies significantly at the special point $v_F = v$ and $2K_\rho = K$. At this point, one can diagonalize the Hamiltonian (33) by introducing new variables $\theta_+ = \theta_\rho/\sqrt{2} + \theta$ and $\theta_- = \theta_\rho/\sqrt{2} - \theta$:

$$\begin{aligned} H = &\frac{v}{2\pi}\int_{-L/2}^{L/2} dx\left[K_\rho(\partial_x\theta_+)^2 + K_\rho^{-1}(\partial_x\phi_+)^2\right] \\ &+ \frac{v}{2\pi}\int_{-L/2}^{L/2} dx\left[K_\rho(\partial_x\theta_-)^2 + K_\rho^{-1}(\partial_x\phi_-)^2\right] \\ &- \frac{\Delta_P}{(2\pi a)}\int_{-L/2}^{L/2} dx \sin(2\theta_-). \end{aligned} \qquad (35)$$

The first line of this Hamiltonian describes gapless superconducting phase fluctuations. The second and third lines, which are decoupled from these gapless fluctuations, are identical to the Hamiltonian (18) for the proximity effect from a bulk 3D superconductor with long-ranged superconducting order parameter. At this point, the dimensionless coupling $y = 2\Delta_P a/v$ has the RG equation

$$\frac{dy}{dl} = (2 - K_\rho^{-1})y. \qquad (36)$$

Therefore, a fermionic gap $\Delta_F \sim \frac{v}{a_0}(\Delta_P a_0/v)^{1/(2-K_\rho^{-1})}$ opens up as a result of the coupling between the wires.

The single wire model, however, does not exhibit Majorana degeneracy without fine tuning of the electrostatic potential. Semiclassically, this is because the moduli space of low-energy field configurations (i.e., those where θ_- is pinned) has only one connected component. Naively, the $\Delta_P \sin(\sqrt{2}\theta_\rho - 2\theta)$ term might lead one to expect two connected components, corresponding to the two minima $\theta = \frac{\theta_\rho}{\sqrt{2}} - \frac{3\pi}{4}, \frac{\theta_\rho}{\sqrt{2}} - \frac{7\pi}{4}$. However, these are in fact connected in the θ_ρ,θ moduli space (see Fig. 3). One can interpolate from one to the other by winding $\sqrt{2}\theta_\rho \to \sqrt{2}\theta_\rho + 2\pi$ and simultaneously winding θ half as fast, so that $\theta \to \theta + \pi$. θ_- remains pinned throughout the interpolation, but the two vacua are exchanged. Therefore, there is no potential barrier; the field $\theta_\rho/\sqrt{2} + \theta$ is free to fluctuate along a flat direction of the potential between these two points. Consequently, there is just a single vacuum, not two degenerate states. This reflects the conservation of charge: when $\theta_\rho/\sqrt{2} + \theta$ has large fluctuations, the total charge is fixed. Given that the two states $|\text{even/odd}\rangle$ correspond to different fermion-parity states and, thus, satisfy different boundary conditions for the field ϕ_+, there is a degeneracy splitting determined by the charging energy of the system $E_c = v_\rho/2LK_\rho$.

Note that, in the argument above, the two minima were exchanged if we could identify $\sqrt{2}\theta_\rho \equiv \sqrt{2}\theta_\rho + 2\pi$. Naively, these two field values are not equivalent since a shift of $\sqrt{2}\theta_\rho$ by 2π changes the sign of the fermion according to Eq. (26). However, $(\sqrt{2}\theta_\rho, \sqrt{2}\theta_\sigma) \equiv (\sqrt{2}\theta_\rho + 2\pi, \sqrt{2}\theta_\sigma + 2\pi)$. Since ϕ_σ is fixed, θ_σ is disordered, so there is no energy cost for shifting θ_σ. Thus, we can treat $\sqrt{2}\theta_\rho$ as 2π periodic, rather than

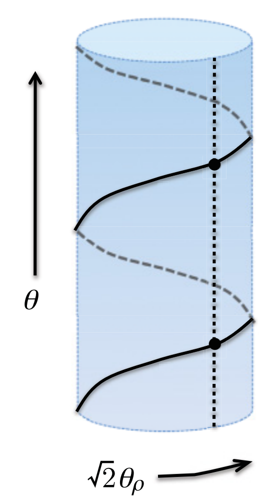

FIG. 3. (Color online) The moduli space of semiclassical vacua is the torus, which is here depicted as a cylinder with the top and bottom edges identified. For any fixed θ_ρ, there are two different semiclassical ground states, depicted by the intersection points between the dotted vertical line and the line of fixed $\sqrt{2}\theta_\rho - 2\theta$, which winds twice around the cylinder. If θ_ρ is, indeed, fixed, as in Sec. II, then there is a tunneling barrier between these two ground states. However, if the total charge mode $\frac{1}{\sqrt{2}}\theta_\rho + \theta$ can fluctuate, as in Sec. III, then the entire line of fixed $\sqrt{2}\theta_\rho - 2\theta$ is the same quantum ground state, and there is no degeneracy.

4π periodic, and the flat direction of the potential connects the two putative minima.

IV. TWO MAJORANA WIRES

As discussed in the Introduction and as we saw in the previous section, if the electron number is fixed, then states with different electron numbers will not be degenerate without fine tuning. However, if we have two semiconducting wires of length ℓ coupled to the same superconducting wire of length L, then there will be two degenerate states of the system for any fixed total charge. These states correspond to even or odd electron numbers in each semiconducting wire, with a constraint that the sum of the parities of the two wires must equal the parity of the total electron number. There need not be literally two separate wires. We could instead have a single wire similar to the spin-orbit coupled semiconducting wire of Eq. (7). In the regions $-L/2 < x < -L/2 + \ell$ and $L/2 - \ell < x < L/2$, we would need to adjust the chemical potential so that $|\mu| < \sqrt{V_x^2 - \Delta_0^2}$, and in the region $-L/2 + \ell < x < L/2 - \ell$, we would need $|\mu| > \sqrt{V_x^2 - \Delta_0^2}$. Then, the system would be in a topological (power-law) superconducting phase for $L/2 - \ell < |x| < L/2$ and in a nontopological (power-law) superconducting phase for $-L/2 + \ell < x < L/2 - \ell$. While we will sometimes call the region $-L/2 + \ell < x < L/2 - \ell$ the "nontopological region," we will usually simply treat the system as if there were no wire there since the section of nontopological wire in this region has a qualitatively similar effect to the absence of a wire.

Let us analyze this setup in more detail. The Hamiltonian for such a system takes the form

$$\begin{aligned} H_{2\,\text{wires}} = &\int_{-L/2}^{-L/2+\ell} dx \left(\frac{v_1}{2\pi}\left[K_1(\partial_x\theta_1)^2 + K_1^{-1}(\partial_x\phi_1)^2\right] - \frac{\Delta_{P1}}{(2\pi a)} \sin(\sqrt{2}\theta_\rho - 2\theta_1)\right) \\ &+ \int_{L/2-\ell}^{L/2} dx \left(\frac{v_2}{2\pi}\left[K_2(\partial_x\theta_1)^2 + K_2^{-1}(\partial_x\phi_1)^2\right] - \frac{\Delta_{P2}}{(2\pi a)} \sin(\sqrt{2}\theta_\rho - 2\theta_2)\right) \\ &+ \int_{-L/2}^{L/2} dx\, \frac{v_\rho}{2\pi}\left[K_\rho(\partial_x\theta_\rho)^2 + K_\rho^{-1}(\partial_x\phi_\rho)^2\right]. \end{aligned} \tag{37}$$

The first two lines are the Hamiltonian for the first semiconducting wire, of length $\ell \ll L$, and its Josephson coupling to the superconducting wire of length L. The third and fourth lines are the analogous terms for the second semiconducting wire. The final line reflects the charge degrees of freedom of the Hamiltonian for a wire with power-law superconducting fluctuations. The gapped spin degrees of freedom have been integrated out. In Eq. (37), we have neglected exponentially small corrections $\propto \exp[-E_g(L-2l)/v_F]$ due to tunneling between the wires (see Sec. V for details).

As in the single-wire case, we introduce the fields

$$\begin{aligned} \theta_+(x) &= \frac{1}{\sqrt{2}}\theta_\rho(x)u_{12}(x) + \theta_\rho(x)[1 - u_{12}(x)] \\ &\quad + \theta_1(x)u_1(x) + \theta_2(x)u_2(x), \\ \theta_-(x) &= \frac{1}{\sqrt{2}}\theta_\rho(x)u_{12}(x) - \theta_1(x)u_1(x) - \theta_2(x)u_2(x), \end{aligned} \tag{38}$$

where $u_1(x) = 1$ for $-L/2 \leqslant x \leqslant -L/2 + \ell$ and $u_1(x) = 0$ otherwise; $u_2(x) = u(-x)$; and $u_{12}(x) = u_1(x) + u_2(x)$. The field $\theta_-(x)$ is only defined for $L/2 - \ell \leqslant |x| \leqslant L/2$. Then, for $v_1 = v_2 = v_\rho = v$ and $K_1 = K_2 = 2K_\rho = K$, the Hamiltonian takes the form

$$\begin{aligned} H_{2\,\text{wires}} = \int_{-L/2}^{L/2} dx \Bigg(&u_{12}(x)\frac{v}{2\pi}\left[K_\rho(\partial_x\theta_-)^2 + K_\rho^{-1}(\partial_x\phi_-)^2\right] \\ &+ \frac{1}{(2\pi a)}[\Delta_{P1}u_1(x) + \Delta_{P2}u_2(x)]\,\sin(2\theta_-) \\ &+ \frac{v}{2\pi}\left[K_\rho(\partial_x\theta_+)^2 + K_\rho^{-1}(\partial_x\phi_+)^2\right]\Bigg). \end{aligned} \tag{39}$$

Naively, this Hamiltonian has four semiclassical ground states: $\theta_-(x) = \vartheta_1 u_1(x) + \vartheta_2 u_2(x)$, with $\vartheta_{1,2} = \frac{3\pi}{4}, \frac{7\pi}{4}$. However, by acting with $(-1)^{N_F^{(1)}}$, $(-1)^{N_F^{(2)}}$, and $(-1)^{N_F^{(1)}+N_F^{(2)}}$ on any one of these states, we can obtain the other three. Thus, we can form two quantum superpositions of these states with $(-1)^{N_F^{(1)}+N_F^{(2)}} = 1$ and two with $(-1)^{N_F^{(1)}+N_F^{(2)}} = -1$. If we fix the total electron number, then one of these two sets will be allowed.

The argument that led us to conclude that a single nanowire has no ground-state degeneracy now shows that the two-wire system (37) at fixed electron number has two (nearly) degenerate ground states. There are two connected components in the moduli space of low-energy field configurations. Here, there is no electrostatic potential breaking the (almost) degeneracy; the leading contributions instead come from instantons, as we shall see below. To see this, note first that θ_1 and θ_2 can be pinned to either $\frac{\theta_\rho}{\sqrt{2}} - \frac{3\pi}{4}$ or $\frac{\theta_\rho}{\sqrt{2}} - \frac{7\pi}{4}$, naively leading to four semiclassical ground states. However, as above, one can wind $\sqrt{2}\theta_\rho \to \sqrt{2}\theta_\rho + 2\pi$, thus connecting the ground state $(\vartheta_1,\vartheta_2) = (\frac{3\pi}{4},\frac{3\pi}{4})$ with $(\frac{7\pi}{4},\frac{7\pi}{4})$, and $(\frac{3\pi}{4},\frac{7\pi}{4})$ with $(\frac{7\pi}{4},\frac{3\pi}{4})$. These two equivalence classes can not be connected to each other, however, since that would require winding $\sqrt{2}\theta_\rho$ by 2π on only one half of the system, leading to an unwanted monopole in the $\sqrt{2}\theta_\rho$ field. This monopole can be removed by a phase slip, leading to degeneracy breaking, as we discuss in Sec. VI. Note that there are no bulk operators local in the fermion variables that can distinguish the two nearly degenerate states. This is because all local terms on, say, wire 1 must be periodic in $2\theta_1$; a term that distinguishes the two ground states must necessarily be odd under $\theta_1 \to \theta_1 + \pi$.

Our two-wire analysis can be generalized in a straightforward manner to N wires in series, coupled to the same superconducting wire, which produce a degeneracy of 2^{N-1}. The ground states correspond to semiclassical vacua $(\frac{3\pi}{4} + n_1\pi, \frac{3\pi}{4} + n_2\pi, \ldots, \frac{3\pi}{4} + n_N\pi)$, where $n_i = 0,1$, subject to the condition that the state $(n_1,n_2,\ldots,n_N) \equiv (n_1 + 1, n_2 + 1, \ldots, n_N + 1)$.

One can formalize this argument for twofold ground-state degeneracy by explicitly constructing an algebra of operators, which commute with the Hamiltonian up to small errors, and satisfy a Pauli algebra, again up to small errors. This algebra then implies an approximate twofold degeneracy in the spectrum, with a small splitting bounded by the size of the errors. To define the algebra, it suffices to construct two approximate symmetry operators A and B, which square to 1 and anticommute. We let $A = (-1)^{N_F^{(1)}}$, the fermionic parity of wire 1. [Given that total fermion parity is fixed, the other operator $(-1)^{N_F^{(2)}}$ is not independent.] B is more subtle: we would like it to act diagonally on the four "phase" eigenstates $(\vartheta_1,\vartheta_2)$ defined above, having eigenvalue 1 on $(\frac{3\pi}{4},\frac{3\pi}{4}),(\frac{7\pi}{4},\frac{7\pi}{4})$ and -1 on $(\frac{3\pi}{4},\frac{7\pi}{4}),(\frac{7\pi}{4},\frac{3\pi}{4})$. Since ϑ_1,ϑ_2 are determined by $\theta_- = \theta_\rho/\sqrt{2} - \theta_{1,2}$, a first guess for B would be

$$B = \cos(\Theta),$$
$$\Theta = \frac{1}{\ell}\int_1 dx \left(\frac{\theta_\rho(x)}{\sqrt{2}} - \theta_1(x)\right) - \frac{1}{\ell}\int_2 dx \left(\frac{\theta_\rho(x)}{\sqrt{2}} - \theta_2(x)\right). \tag{40}$$

However, B is not well defined in Eq. (40) because $\frac{\theta_\rho}{\sqrt{2}}$ is only well defined mod π. To ameliorate the situation, it is useful for convenience to take the limit in which $L \gg \ell$, and treat the topological wires as points (the argument also works away from this limit, but the notation is more cumbersome). Then, the above definition of Θ reduces to $\Theta = [\theta_\rho(0) - \theta_\rho(L)]/\sqrt{2} + \theta_2 - \theta_1$. This is still not well defined but can be made so by replacing the first term with an integral of a total derivative. Thus,

$$A = (-1)^{N_F^{(1)}}, \tag{41}$$

$$B = \cos\left[-\int_0^L \partial_x\theta_\rho(x)/\sqrt{2} + \theta_2 - \theta_1\right] \tag{42}$$

forms our desired set of operators. Clearly, $B^2 = 1$ and $A^2 = 1$ up to errors exponentially small in ℓ because the argument of the cosine in the definition of B is pinned to 0 or π, with fluctuations going like $e^{-\ell/\xi}$. Furthermore, B was constructed to exactly anticommute with A, so we have an approximate Pauli algebra. One can now show that there is topological degeneracy in this system. Consider two different fermion-parity states $|a\rangle \equiv |\vartheta_1 = \frac{3\pi}{4}, \vartheta_2 = \frac{3\pi}{4}\rangle$ and $|b\rangle \equiv |\vartheta_1 = \frac{7\pi}{4}, \vartheta_2 = \frac{3\pi}{4}\rangle$. Since they are eigenstates of B with opposite eigenvalues, and $[H,B] \approx 0$, they must individually be approximate eigenstates of the Hamiltonian, with energies E_a, E_b. Now, using $|a\rangle = A|b\rangle$ and $[H,A] \approx 0$, one can show that the energies E_a and E_b must be equal:

$$E_a|a\rangle = H|a\rangle = HA|b\rangle = AH|b\rangle = AE_b|b\rangle = E_b|a\rangle. \tag{43}$$

Now, what terms in the Hamiltonian could fail to commute with this algebra? In the case of A, the only such term would be an electron tunneling between wire 1 and wire 2, which is exponentially suppressed in $L - \ell$. In the case of B, there are two possibilities: (1) instanton tunneling between two vacua on either wire 1 or 2, as described above, and (2) 2π phase slips in $\sqrt{2}\theta_\rho$ in the middle region between the wires. These processes are both allowed in the Hamiltonian and *anti*commute with B. The first of these is exponentially suppressed with ℓ, but the second, as we shall see later, decays only as a power of L, albeit possibly a large one. Hence, we get a corresponding bound on the degeneracy splitting.

We now compute these various contributions to the degeneracy splitting. First, we consider the instanton contributions and show that they lead to an exponentially suppressed splitting; this is done in the next section. In the following section, we account for impurities and the associated phase slips. These lead to a power-law splitting, which is naively much worse than exponential suppression. However, the exponent in the power law is proportional to the number of channels in the SC, which can be made large, and the prefactor can be made exponentially small in the length scale associated with the smoothness of the disorder.

V. INSTANTON CONTRIBUTIONS

We now discuss the instanton contributions to the degeneracy splitting, starting with the soluble point where, according to (39), θ_+ decouples, leaving a pinned θ_- field on each wire. To tunnel between the two vacua, we need to tunnel from $\theta_- = \frac{3\pi}{4}$ to $\theta_- = \frac{7\pi}{4}$ along a *single* length ℓ wire, say wire 1. The instanton analysis proceeds exactly as that of the proximity-induced case, leading to a splitting $\delta E \sim \exp(-\sqrt{K_\rho}\frac{\ell}{\xi})$. In such a process, $2\theta_1$ winds by $\pm 2\pi$ relative to $\sqrt{2}\theta_\rho$. This can be interpreted as a vortex tunneling between the helical nanowire and SC wire, as depicted in process (a) in Fig. 4. It equivalently can be interpreted as a fermion tunneling from one end of wire 1 to the other.

We now show that the Majorana degeneracy is stable against all possible translationally invariant perturbations around the soluble point. We will consider the effect of impurities

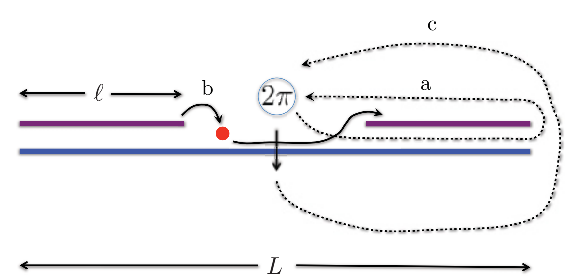

FIG. 4. (Color online) A schematic depiction of the different processes that split the two states of the qubit. Process (a): A vortex can tunnel between a semiconducting wire and the superconducting wire. This causes a splitting that is exponentially small in ℓ. Process (b): An electron can tunnel from one semiconducting wire to another through the superconducting wire. This causes a splitting that is exponentially small in $L - 2\ell$. Process (c): A vortex can tunnel through the superconducting wire between the semiconducting wires as a result of an electron backscattering process. This causes a splitting that decays as a power of L. This model applies also to a situation in which there is a single semiconducting nanowire that extends from $-L/2$ to $L/2$ and is in a topological phase for $L/2 - \ell < |x| < L$ (similar to our two semiconducting nanowires) and is in a nontopological phase for $|x| < L/2 - \ell$ (similar to our superconducting wire).

and phase slips associated with them in the next section. The perturbations come in two varieties. First, there are exponentially small pair-hopping terms that involve electrons in different semiconductor nanowires. Second, there are couplings between the semiconducting and superconducting wires that we have not included in our initial model (37). Third, there are shifts of the parameters that take us away from the point $K_1 = K_2 = 2K_\rho$, $v_1 = v_2 = v_\rho$. As we will see, the second can be accounted for with the third.

First, we derive an effective action for interwire pair hopping, starting with the following microscopic model:

$$H_t = t_1 \sum_\sigma \int_{-L/2}^{-L/2+\ell} dx_1 [\psi^\dagger_{1\sigma}(x_1)\eta_\sigma(x_1) + \text{H.c.}] + t_2 \sum_\sigma \int_{L/2-\ell}^{L/2} dx_2 [\psi^\dagger_{2\sigma}(x_2)\eta_\sigma(x_2) + \text{H.c.}]. \quad (44)$$

At second order of perturbation theory in t, one obtains cross terms proportional to $t_1 t_2$. These exponentially small terms were neglected in Eq. (37). Now, we take them into account and study their effect on the degeneracy splitting. Consider the term in the Euclidean effective action proportional to $t_1 t_2$:

$$S_t^{(12)} = -2t_1 t_2 \int d\tau_1 \int d\tau_2 \int_{-L/2}^{-L/2+\ell} dx_1 \int_{L/2-\ell}^{L/2} dx_2 \times [\psi^\dagger_{1\uparrow}(1)\psi^\dagger_{2\downarrow}(2)\eta_\downarrow(1)\eta_\downarrow(2) + \text{c.c.}]. \quad (45)$$

Bosonizing the action $S_t^{(12)}$ and integrating out the massive spin fields, one arrives at

$$S_t^{(12)} = \frac{-2t_1 t_2 \alpha p_F}{\sqrt{(\alpha p_F)^2 + V_x^2}} \int d\tau_1 \int d\tau_2 \int_{-L/2}^{-L/2+\ell} dx_1 \int_{L/2-\ell}^{L/2} dx_2 \times \cos\left(\frac{\phi_\rho(1) - \phi_\rho(2)}{\sqrt{2}}\right) \sin\left(\frac{\theta_\rho(1)+\theta_\rho(2)}{\sqrt{2}} - \theta(1) - \theta(2)\right) \times \frac{a}{(2\pi a)^2} \frac{\exp\left[-E_g\sqrt{(\tau_1-\tau_2)^2 + \frac{(x_1-x_2)^2}{v_F^2}}\right]}{\sqrt{(x_1-x_2)^2 + v_F^2(\tau_1-\tau_2)^2}}. \quad (46)$$

The dominant contribution to the integral over $\tau_1 - \tau_2$ comes from short times $[|\tau_1 - \tau_2| \ll (L-2\ell)/v_F]$ and can be approximately carried out. The remaining spatial integral is peaked at $x_{1/2} = \pm(L/2 - \ell)$, and the action can be approximately written as

$$S_t^{(12)} \propto \frac{-t_1 t_2}{E_g} e^{-\frac{E_g}{v_F}(L-2\ell)} \frac{\alpha p_F}{\sqrt{(\alpha p_F)^2 + V_x^2}} \times \int d\tau \cos\left(\frac{\phi_\rho(-x_0,\tau) - \phi_\rho(x_0,\tau)}{\sqrt{2}}\right) \times \sin\left(\frac{\theta_\rho(-x_0,\tau)+\theta_\rho(x_0,\tau)}{\sqrt{2}} - \theta(-x_0,\tau) - \theta(x_0,\tau)\right) \quad (47)$$

with $x_0 = L/2 - \ell$. In fermionic language, the above expression has a very clear physical interpretation: it corresponds to the Josephson coupling between the ends of the two wires. It is due to single-fermion tunneling from one wire to the other, as depicted in process (b) in Fig. 4. The terms on the second and third lines of Eq. (47) are bounded, so such a term causes a splitting, which decays exponentially in $(L-2\ell)$: $\Delta E \sim e^{-\frac{E_g}{v_F}(L-2\ell)}$.

We now consider interactions between the semiconductor and superconductor wires. We assume that because of the Fermi momenta mismatch in these two systems, one can neglect interactions between the charge and spin densities at $2k_F^{\text{SC}}$ in the superconductor and the corresponding densities at $2k_F^{\text{NW}}$ in the semiconducting nanowire since these interactions will be oscillatory. We will now write down all possible operator couplings between the superconductor and the semiconductor and generate all allowed terms preserving U(1) symmetry. For the superconductor, the charge- and spin-density operators are given by

$$O_\rho = \psi^\dagger_\uparrow\psi_\uparrow + \psi^\dagger_\downarrow\psi_\downarrow = -\frac{\sqrt{2}}{\pi}\partial_x\phi_\rho, \quad (48)$$

$$O^z_\sigma = \psi^\dagger_\uparrow\psi_\uparrow - \psi^\dagger_\downarrow\psi_\downarrow = -\frac{\sqrt{2}}{\pi}\partial_x\phi_\sigma, \quad (49)$$

$$O^y_\sigma = -i(\psi^\dagger_\uparrow\psi_\downarrow - \psi^\dagger_\downarrow\psi_\uparrow) = \frac{-2}{\pi a}\sin(\sqrt{2}\theta_\sigma)\cos(\sqrt{2}\phi_\sigma), \quad (50)$$

$$O^x_\sigma = \psi^\dagger_\uparrow\psi_\downarrow + \psi^\dagger_\downarrow\psi_\uparrow = \frac{2}{\pi a}\cos(\sqrt{2}\theta_\sigma)\cos(\sqrt{2}\phi_\sigma), \quad (51)$$

and the singlet and triplet superconducting pairing operators read as

$$O_{\text{SS}} = \psi^\dagger_\uparrow\psi^\dagger_\downarrow - \psi^\dagger_\downarrow\psi^\dagger_\uparrow = \frac{1}{\pi a} e^{-i\sqrt{2}\theta_\rho}\cos(\sqrt{2}\phi_\sigma), \quad (52)$$

$$O^x_{\text{TS}} = \psi^\dagger_\uparrow\psi^\dagger_\uparrow + \psi^\dagger_\downarrow\psi^\dagger_\downarrow = \frac{1}{\pi a} e^{-i\sqrt{2}\theta_\rho}\cos(\sqrt{2}\theta_\sigma), \quad (53)$$

$$O^y_{\text{TS}} = -i(\psi^\dagger_\uparrow\psi^\dagger_\uparrow - \psi^\dagger_\downarrow\psi^\dagger_\downarrow) = \frac{-1}{\pi a} e^{-i\sqrt{2}\theta_\rho}\sin(\sqrt{2}\theta_\sigma), \quad (54)$$

$$O^z_{\text{TS}} = 0, \quad (55)$$

where SS and TS denote triplet and singlet pairing. We now write down these operators for the semiconductor nanowire. Because of the large Zeeman gap, we perform projection to the lowest subband as explained in Sec. II. The charge- and spin-density operators in the semiconductor now become

$$O_\rho = n_R + n_L = -\frac{1}{\pi}\partial_x\phi, \quad (56)$$

$$O^z_\sigma = 0, \quad (57)$$

$$O^y_\sigma = \frac{\alpha p_F}{\sqrt{V_x^2 + \alpha^2 p_F^2}}(n_R - n_L) = \frac{\partial_x\theta}{\pi}\frac{\alpha p_F}{\sqrt{V_x^2 + \alpha^2 p_F^2}}, \quad (58)$$

$$O^x_\sigma = \frac{-V_x}{\sqrt{V_x^2 + \alpha^2 p_F^2}}(n_R + n_L) = \frac{\partial_x\phi}{\pi}\frac{V_x}{\sqrt{V_x^2 + \alpha^2 p_F^2}}, \quad (59)$$

and the superconducting pairing operators read as

$$O_{\text{SS}} = \frac{i\alpha p_F}{\sqrt{V_x^2 + \alpha^2 p_F^2}} c^\dagger_R c^\dagger_L = \frac{i\alpha p_F}{\sqrt{V_x^2 + \alpha^2 p_F^2}} \frac{e^{-2i\theta}}{\pi a}, \quad (60)$$

$$O^x_{\text{TS}} = O^y_{\text{TS}} = O^z_{\text{TS}} = 0. \quad (61)$$

The triplet pairing operators vanish because, in our model, the superconducting wire has a spin gap and, therefore, ϕ_ρ is fixed. Given these operators, one can construct all possible coupling terms between the superconductor and the semiconductor.

In addition to the pair-hopping term, which is essential for our proposal to work and was already included in our model (33), one can have additional couplings that represent various density-density interactions:

$$H_1 = V_{\rho\rho} \int dx\, \partial_x \phi_\rho\, \partial_x \phi, \tag{62}$$

$$H_2 = V_{\sigma\sigma}^{(x)} \int \frac{dx}{a} \sin(\sqrt{2}\theta_\sigma) \cos(\sqrt{2}\phi_\sigma) \partial_x \phi, \tag{63}$$

$$H_3 = V_{\sigma\sigma}^{(y)} \int \frac{dx}{a} \sin(\sqrt{2}\theta_\sigma) \cos(\sqrt{2}\phi_\sigma) \partial_x \theta. \tag{64}$$

The first term above describes the charge density-density interaction between the wires, whereas the Hamiltonians in the second and third lines correspond to spin-spin interactions. The couplings between current fluctuations are similar in form to the density-density interactions and have not been included explicitly because their analysis is so similar. Assuming that $|V_{\sigma\sigma}^{(x,y)}|$ are small compared to $|U|$ in Eq. (29), the terms (63) and (64) can be dropped because the field ϕ_σ orders and θ_σ is disordered. Thus, the only coupling that is relevant in the present setup is H_1 [Eq. (62)]. We show below that this quadratic term does not affect the stability of the Majorana modes.

Therefore, a general perturbation is described by the following Euclidean action:

$$\begin{aligned} S_{2\,\text{wires}}^{(E)} = &\int_{-\frac{L}{2}}^{-\frac{L}{2}+\ell} dx \int d\tau \bigg(\frac{K_1}{2\pi v_1} \big[(\partial_\tau \theta_1)^2 + v_1^2 (\partial_x \theta_1)^2 \big] \\ &+ \frac{\Delta_{P1}}{2\pi\xi} \sin(\sqrt{2}\theta_\rho - 2\theta_1) + V_{\rho\rho}^{(1)} (\partial_\tau \theta_\rho)(\partial_\tau \theta_1) \bigg) \\ &+ \int_{\frac{L}{2}-\ell}^{\frac{L}{2}} dx \int d\tau \bigg(\frac{K_2}{2\pi v_2} \big[(\partial_\tau \theta_2)^2 + v_2^2 (\partial_x \theta_2)^2 \big] \\ &+ \frac{\Delta_{P2}}{2\pi\xi} \sin(\sqrt{2}\theta_\rho - 2\theta_2) + V_{\rho\rho}^{(2)} (\partial_\tau \theta_\rho)(\partial_\tau \theta_2) \bigg) \\ &+ \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \int d\tau \bigg(\frac{K_\rho}{2\pi v_\rho} \big[(\partial_\tau \theta_\rho)^2 + v_\rho^2 (\partial_x \theta_\rho)^2 \big] \bigg). \end{aligned} \tag{65}$$

We rewrite this action in terms of the new fields θ_+, θ_- defined in Eq. (38). Up to local terms proportional to $\delta[x \pm (\frac{L}{2} - \ell)]$, which we drop because they will contribute negligibly to the bulk instanton action, we obtain

$$\begin{aligned} S_{2\,\text{wires}}^{(E)} = &\int_{-\frac{L}{2}}^{-\frac{L}{2}+\ell} dx \int d\tau \bigg[A_\tau^{(1)} (\partial_\tau \theta_+)^2 + A_x^{(1)} (\partial_x \theta_+)^2 \\ &+ B_\tau^{(1)} (\partial_\tau \theta_-)^2 + A_x^{(1)} (\partial_x \theta_-)^2 + \frac{\Delta_{P1}}{2\pi\xi} \sin(2\theta_-) \\ &+ C_\tau^{(1)} (\partial_\tau \theta_+)(\partial_\tau \theta_-) + C_x^{(1)} (\partial_x \theta_+)(\partial_x \theta_-) \bigg] \\ &+ \int_{\frac{L}{2}-\ell}^{\frac{L}{2}} dx \int d\tau [1 \to 2] \int_{-\frac{L}{2}+\ell}^{\frac{L}{2}-\ell} \\ &\times dx \int d\tau \bigg[\frac{K_\rho}{2\pi v_\rho} (\partial_\tau \theta_+)^2 + \frac{K_\rho v_\rho}{2\pi} (\partial_x \theta_+)^2 \bigg], \end{aligned} \tag{66}$$

where $A_\tau^{(1)} = \frac{K_1}{8\pi v_1} + \frac{K_\rho}{4\pi v_\rho} + \frac{V_{\rho\rho}^{(1)}}{2\sqrt{2}}$, $A_x^{(1)} = \frac{K_1 v_1}{8\pi} + \frac{K_\rho v_\rho}{4\pi}$, $B_\tau^{(1)} = \frac{K_1}{8\pi v_1} + \frac{K_\rho}{4\pi v_\rho} - \frac{V_{\rho\rho}^{(1)}}{2\sqrt{2}}$, $C_\tau^{(1)} = \frac{K_\rho}{2\pi v_\rho} - \frac{K_1}{4\pi v_1}$, $C_x^{(1)} = \frac{K_\rho v_\rho}{2\pi} - \frac{K_1 v_1}{4\pi}$, and similarly with 1 replaced by 2.

We see that, in the general case, θ_+ does not decouple. However, its action is still quadratic, so we can integrate it out exactly. We generate the following terms. On wire 1, we have

$$\begin{aligned} \delta S^{(1)} = &\int d\omega\, dk \frac{2\big(C_\tau^{(1)}\big)^2 \omega^4}{A_\tau^{(1)} \omega^2 + A_x^{(1)} k^2} \theta_-^2 \\ &+ \int d\omega\, dk \frac{2\big(C_x^{(1)}\big)^2 k^4}{A_\tau^{(1)} \omega^2 + A_x^{(1)} k^2} \theta_-^2. \end{aligned} \tag{67}$$

We obtain an analogous expression for $\delta S^{(2)}$. We also obtain the following bilinear, which couples wires 1 and 2:

$$\begin{aligned} \delta S^{(12)} = &\int d\tau_1\, d\tau_2\, dx_1\, dx_2 \\ &\times \big(C_\tau^{(1)} C_\tau^{(2)} [\partial_{\tau_1} \partial_{\tau_2} \langle \theta_+(1)\theta_+(2) \rangle][\partial_\tau \theta_-(1)][\partial_\tau \theta_-(2)] \\ &+ C_x^{(1)} C_x^{(2)} [\partial_{x_1} \partial_{x_2} \langle \theta_+(1)\theta_+(2) \rangle][\partial_x \theta_-(1)][\partial_x \theta_-(2)] \big). \end{aligned} \tag{68}$$

Here, $\langle \theta_+(1)\theta_+(2) \rangle$ is the θ_+ two-point function between wires 1 and 2.

Suppose now that the coupling $\delta S^{(12)}$ were absent. Then, as in the instanton analysis of the proximity-induced case, we could conclude that the lowest action instanton is translationally invariant on 1 and 2 separately; taking $k = 0$ in (67) just gives a renormalization of the kinetic term of the $\theta_-^{(1)}$ center of mass mode, and similarly for $\delta S^{(2)}$. We are interested in an instanton that tunnels from $\theta_- = \frac{3\pi}{4}$ to $\theta_- = \frac{7\pi}{4}$ on wire 1, and remains in the same vacuum on wire 2. In the present context, with $\delta S^{(12)}$ absent, such an instanton has the same form as that obtained in the proximity-induced case on wire 1, and is simply constant on wire 2. According to that analysis, it leads to a splitting $\delta E \propto \exp(-c\frac{\ell}{\xi})$.

Now, put back $\delta S^{(12)}$. We will show that the change in the instanton (and the change in its action) is of order $\frac{\xi}{L-2\ell}$, and thus negligibly small when $L - 2\ell \gg \xi$. To declutter the following argument, we set all velocities equal to 1, set all dimensionless constants equal to 1, and let $\vec{r} = (x,\tau)$. The action is then

$$\begin{aligned} S = &\int_{1\,\text{and}\,2} d\tau\, dx \left[(\nabla\theta_-)^2 + \frac{1}{\xi^2} \sin(2\theta_-) \right] + \delta S^{(1)} + \delta S^{(2)} \\ &+ \int d\vec{r}_1 d\vec{r}_2 \big(f(\vec{r}_2 - \vec{r}_1)[\partial_\tau \theta_-(\vec{r}_1)][\partial_\tau \theta_-(\vec{r}_2)] \\ &+ g(\vec{r}_2 - \vec{r}_1)[\partial_x \theta_-(\vec{r}_1)][\partial_\tau \theta_-(\vec{r}_2)] \big), \end{aligned} \tag{69}$$

where

$$\begin{aligned} f(\vec{r}_2 - \vec{r}_1) &= \partial_{\tau_1} \partial_{\tau_2} \langle \theta_+(\vec{r}_1)\theta_+(\vec{r}_2) \rangle, \\ g(\vec{r}_2 - \vec{r}_1) &= \partial_{x_1} \partial_{x_2} \langle \theta_+(\vec{r}_1)\theta_+(\vec{r}_2) \rangle. \end{aligned} \tag{70}$$

We do not need the precise forms of f and g; rather, all we use is the fact that $|\nabla f(\vec{r}_2 - \vec{r}_1)| < \frac{c'}{(L-2\ell)^3}$ for some constant c', whenever $x_2 - x_1 > L - 2\ell$, a condition that is always satisfied in (69), and a similar condition for g. Let us start with the instanton solution discussed above, i.e., the one that minimizes the action with $\delta S^{(12)}$ absent and tunnels between the two vacua only on wire 1, while staying constant in one of the vacua on wire 2. We plug it into (69) and vary with respect to $\theta_-(\vec{r}_2)$ to obtain

$$\nabla^2\theta_-(\vec{r}_2) = \frac{2}{\xi^2}\cos[2\theta_-(\vec{r}_2)] + \frac{\delta S^{(\vec{r}_2)}}{\delta\theta_-(\vec{r}_2)} + h(\vec{r}_2), \quad (71)$$

where

$$h(\vec{r}_2) = \pi \int d\vec{r}_1\, \delta(\tau_1)\partial_\tau f(\vec{r}_2 - \vec{r}_1) \quad (72)$$

is sourced by the instanton on 1, which, because it varies on a time scale ξ^{-1}, can be taken to be $\pi\delta(\tau_1)$ for the purposes of this calculation. From (72) and the previous bound on $|\nabla f|$, we see that $|h(\vec{r}_2)| < \frac{c'\ell}{(L-2\ell)^3}$. The key point now is that the dimensionful quantity $h(\vec{r}_2)$ is smaller than $\frac{1}{\xi^2}$ by a factor of $\epsilon^2 = \frac{c'\ell\xi^2}{(L-2\ell)^3} \ll 1$. Thus, the inclusion of $h(\vec{r}_2)$ in (71) causes $\theta_-(\vec{r}_2)$ to deviate from its zeroth-order solution only by an amount order ϵ. This is the first step in a perturbative expansion in ϵ, which shows that the inclusion of $\delta S^{(12)}$ causes only a small change, of order ϵ, in the instanton and its action.

Our analysis did not require $2K_\rho - K$ to be small since we were able to integrate out θ_+ exactly, regardless of their values. Therefore, so long as $\frac{1}{2}K_\rho^{-1} + K^{-1} < 2$, which implies that Δ_P is relevant and generates a coherence length ξ, the instanton argument is still valid and leads to a splitting $\delta E \propto \exp(-c\frac{\ell}{\xi})$. Thus, the Majorana degeneracy is stable over this entire region of the phase diagram, which includes more physically interesting values than the soluble point.

VI. ELECTRON BACKSCATTERING AND PHASE SLIPS

We now study the effect of processes in the superconducting wire, which backscatter a right-moving electron into a left-moving one or vice versa. We can include the effect of an electrostatic potential in the superconducting wire by adding a term to the action

$$\begin{aligned}
H_{\text{pot}} &= \int dx\; V(x)\,\psi_\sigma^\dagger(x)\psi_\sigma(x)\\
&= \int dx\; V(x)\,[\psi_{R\sigma}^\dagger(x)\psi_{R\sigma}(x) + \psi_{L\sigma}^\dagger(x)\psi_{L\sigma}(x)\\
&\quad + e^{-2ik_Fx}\psi_{R\sigma}^\dagger(x)\psi_{L\sigma}(x) + e^{2ik_Fx}\psi_{L\sigma}^\dagger(x)\psi_{R\sigma}(x)]\\
&= \int dx\; V(x)\left[\frac{\sqrt{2}}{\pi}\partial_x\phi_\rho + 2\cos(\sqrt{2}\phi_\rho + 2k_Fx)\cos\sqrt{2}\phi_\sigma\right]\\
&= \int dx\; V(x)\left[\frac{\sqrt{2}}{\pi}\partial_x\phi_\rho + 2\cos(\sqrt{2}\phi_\rho + 2k_Fx)\right]. \quad (73)
\end{aligned}$$

In going from the penultimate line to the final one, we have used the fact that there is a spin gap in the SC wire that pins the value of ϕ_σ. The first term in the final line is harmless and can be absorbed by shifting ϕ_ρ, which corresponds to a shift of the chemical potential. Therefore, we will ignore this term from now on. The second term in the final line causes 2π phase slips in the order parameter in the superconducting wire $e^{i\sqrt{2}\theta_\rho}$ since

$$[\sqrt{2}\phi_\rho(x), \partial_x(\sqrt{2}\theta_\rho(x'))] = -2\pi i\,\delta(x - x'). \quad (74)$$

This equation expresses the fact that, when an electron in a 1D system is backscattered, a 2π phase slip occurs.

These phase slips cause transitions between the two states of the qubit (or, in the fermion-parity basis, they cause a splitting between the two states). At a technical level, this occurs because a phase slip at the origin causes $\sqrt{2}\theta_\rho$ to wind by 2π on half of the system. Then, θ_1 can wind by π (while remaining at the minimum of the cosine potential), and the system will make a transition from the state $(\frac{3\pi}{4},\frac{3\pi}{4}) \equiv (\frac{7\pi}{4},\frac{7\pi}{4})$ to $(\frac{3\pi}{4},\frac{7\pi}{4}) \equiv (\frac{7\pi}{4},\frac{3\pi}{4})$. At a more physical level, when a phase slip occurs, a vortex tunnels across the wire quantum mechanically. Since there is no barrier for a vortex to move outside the wire, a vortex that tunnels through the midpoint of the SC wire can then encircle half of the SC wire, along with the NW, which is in contact with that half of the SC wire. The vortex thereby measures the fermion parity of that NW by the Aharonov-Casher effect, as is depicted schematically in process (c) in Fig. 4.

Note that if the phase slip occurs between $-L + \ell$ and $L - \ell$, where the semiconducting wire is nontopological or is absent, there will only be gradient energy in θ_ρ (or its dual equivalent, fluctuation energy in ϕ_ρ). However, if the phase slip occurs at a point x satisfying $|L - \ell| < |x| < |L|$, where there is a topological region of wire, then it will put a kink in $\sqrt{2}\theta_\rho$ in a region where it is locked by the potential $\sin(\sqrt{2}\theta_\rho - 2\theta_1)$. Due to the energy cost of a kink, this will leave the system in a higher energy state. The kink is simply a fermion excited above the gap. In order to return to a ground state, another instanton or an antiinstanton must occur. However, this double process does not mix or split ground states.

We will consider three different types of potentials $V(x)$, which can backscatter electrons. First, we consider a single impurity. For simplicity, we will focus on the case of a δ-function impurity at the origin $V(x) = \frac{v}{2}\,\delta(x)$, but the physics will be the same for any potential that is nonzero only in a region of length much less than $L - 2\ell$ near the middle of the SC wire. Then, the Hamiltonian (73) takes the form

$$H_{1-\text{imp}} = v\cos[\sqrt{2}\phi_\rho(0)]. \quad (75)$$

The RG equation for v follows from the scaling dimension for $\cos(\sqrt{2}\phi_\rho)$:

$$\frac{dv}{dl} = \left(1 - \frac{1}{2}K_\rho\right)v. \quad (76)$$

For $K_\rho > 2$, this is irrelevant; in the large-L, low-temperature limit, the superconductor heals itself and the backscattering amplitude goes asymptotically to zero. However, for $K_\rho < 2$, the SC wire is effectively broken in two by the impurity. The qubit is then lost. Therefore, it is necessary to have sufficiently strong attractive interactions in the SC wire that $K_\rho > 2$. Even when this is satisfied, the backscattering amplitude vanishes as a power law in the system size, not exponentially. Since backscattering and phase-slip processes cause transitions between the two different states of the qubit in the phase basis, they cause an energy splitting between states of different fermion parity:

$$\Delta E \propto \langle v\cos(\sqrt{2}\phi_\rho)\rangle \propto \frac{|v|}{L^{K_\rho/2}}. \quad (77)$$

Since ϕ_ρ is fixed at the ends of the SC wire (since no current flows off the ends), the one-point function for $\cos\sqrt{2}\phi_\rho$ has the L dependence shown above.

Now, suppose, instead, that there is a random distribution of impurities so that

$$\overline{V(x)V(x')} = W\delta(x-x'). \tag{78}$$

Then, we replicate the action by introducing an additional index α on the field ϕ_ρ^α with $\alpha = 1,2,\ldots,N$. We will take $N \to 0$ at the end of the calculation in order to take the quenched average over all realizations of the disorder. The disorder-averaged effective action takes the form

$$S_{\text{random}} = \int d\tau\, d\tau' dx\, W \cos\left\{\sqrt{2}\left[\phi_\rho^\alpha(x,\tau) - \phi_\rho^\beta(x,\tau')\right]\right\}. \tag{79}$$

The RG equation for W is

$$\frac{dW}{dl} = (3-K_\rho)W. \tag{80}$$

Thus, we need a larger K_ρ for the superconductivity to survive a random distribution of impurities, and if $K_\rho > 3$ is satisfied, then there will be an energy splitting:

$$\Delta E \propto \frac{W}{L^{K_\rho - 2}}. \tag{81}$$

Thus far, we have focused on backscattering by impurities, which effectively create weak spots in the wire where a vortex can tunnel through. However, even in a completely clean system, there is some amplitude for backscattering. For instance, let us suppose that $V(x)$ is constant near the middle of the wire and goes to zero smoothly near the ends. To make this concrete, let us take $V(x) = V_0$ for $|x| \ll L/2$ and $V(x) = 0$ for $|x| = L/2$. We will assume that $V(x)$ varies smoothly, so that the Fourier transform $\tilde{V}(q) \sim e^{-q^2b^2}$ for $q \gg 1/L$, where $b > \xi$. Then, from the last line of Eq. (73), we expect a splitting

$$\Delta E \propto \int_{-\frac{L}{2}+\ell}^{\frac{L}{2}-\ell} dx\, \frac{V(x)\cos 2k_F x}{\left(\frac{L}{2} - |x|\right)^{\frac{K_\rho}{2}-2}} < \frac{e^{-4k_F^2 b^2}}{\ell^{\frac{K_\rho}{2}-2}}. \tag{82}$$

Therefore, as the potential becomes smoother and smoother, the splitting, which it induces through electron backscattering and phase slips, goes exponentially to zero with the length scale b over which the potential varies. Inhomogeneities enhance backscattering, as we saw in Eqs. (77) and (81).

VII. DISCUSSION

As we saw in Sec. VI, the effects of electron backscattering by impurities can be mitigated by making K_ρ large. In a superconducting wire, $K_\rho = 2\pi\sqrt{A_w\rho_s\kappa} \propto k_F^2 A_w \propto N_{\text{channels}}$ and $v = \sqrt{A_w\rho_s/\kappa}$ with A_w, ρ_s, and κ being the cross-sectional area, superconducting stiffness, and compressibility, respectively.[29] Therefore, if the superconducting wire has enough channels or, equivalently, if the superconducting wire has sufficiently large cross-sectional area and/or sufficiently large superfluid density, we can have K_ρ large. For a typical quasi-one-dimensional superconductor (e.g., aluminum) with the cross-sectional area $A_w \sim 10^4$ nm^2, the Luttinger parameter $K_\rho \sim 10^6$ and velocity $v \sim 10^5$ m/s. Although this is not as good as exponential decay as a matter of principle, it may be just as good as a practical matter. This may be important since it could be very difficult to tune the chemical potential appreciably in the semiconducting wire (which is necessary to move the Majorana zero modes) if it is in contact with a bulk 3D superconductor. Furthermore, coating the semiconducting wires with superconducting material, as depicted in Fig. 2, may be the easiest way to make a complex network of wires (especially a three-dimensional network), which is in contact with a superconductor. [We thank C. Marcus for his colorful culinary metaphor comparing the situation in Fig. 2(b) to mustard on a hot dog.] However, such an architecture will necessarily be, at best, an algebraically ordered superconductor (except, perhaps, at the lowest temperatures, at which the coupling between wires causes a crossover to 3D superconductivity). Therefore, it is significant that our results show that such a network supports Majorana fermion nearly zero modes and that their splitting can be made small (albeit not exponentially so).

We also note that it is only important that K_ρ be large in the regions between the topological semiconducting wire segments. In the topological semiconducting wire segments, the phase is locked so that 2π phase slips can not occur (although harmless 4π phase slips can occur). Therefore, one can imagine a scenario in which the topological segments are coated with a thin superconducting film, while the nontopological segments between them are in contact with essentially bulk 3D superconductors. This would lead to a protected topological qubit, although it would be difficult (if not impossible) to move the Majorana zero modes since that would involve tuning the chemical potential in the nontopological regions (which are in contact with bulk 3D superconductors) to drive them into the topological phase. One may, alternatively, in a system in which a semiconducting wire is coated with a thin layer of superconducting material, use a gate voltage to occupy a large (even) number of subbands of the semiconducting wire in the nontopological regions. This would lead to a large effective K_ρ for the combined superconductor-semiconductor system in the nontopological regions and, therefore, a large power for the decay of the splitting due to phase slips in these regions.

As the previous sentence anticipates, our methods should be generalizable to multichannel *semiconducting* wires.[30–33] They should also apply to a semiconducting wire, which is near a superconducting grain (as in a model of quasi-1D wires in LAO/STO interfaces[24]). If the linear size of the grain r is smaller than the superconducting coherence length ξ, then we can treat the grain as a zero-dimensional system. Suppose that the wire also has length r. Then, the Hamiltonian for the wire coupled to the grain is simply (1) with ϕ independent of position i but dependent on time. There will also be a charging energy $U(N-N_0)^2$, which causes ϕ to fluctuate. There will be no long-ranged order in the superconducting grain, but it can still induce a single-fermion gap in the semiconducting wire. Of course, if the wire has length $L \gg \xi$, then the grain will only change the behavior of a short section of the wire, and the two ends of this section will be relatively close to each other. But, if the wire passes near many such grains, then they can induce a single-fermion gap in the wire. If the coupling between the grains is large compared to their charging energies, then, in the long-wavelength limit, the grains will develop algebraic order. The superconducting grains can be modeled by a superconducting wire, and this situation can be modeled with the Hamiltonian of Sec. III, but with a very small velocity.[34] If there is Ohmic dissipation, then the grains may not even have power-law superconducting order, but may have exponentially decaying superconducting correlations.

In fact, we will have Majorana zero modes in a system with exponentially decaying superconducting correlations if we simply take our model to finite temperature. Then, the θ_+ field in Eq. (39) will have exponentially decaying correlations, with a correlation length inversely proportional to the temperature. The θ_- field will still be pinned to a minimum of the potential, but it will be possible for the system to be thermally excited over the barrier from one minimum to the other. Therefore, if Δ_F is the bulk single-fermion gap, there will be a contribution due to processes (a) and (b) in Fig. 4 to the coherence time for a Majorana qubit of order $\sim e^{-\Delta_F/T}$, just as if there were long-ranged superconducting order. However, there will also be a contribution from quantum phase slips, process (c), which will increase with temperature as $T^{K\rho/2}$ for a single impurity and $T^{K\rho-2}$ for a random distribution of impurities. We similarly expect Majorana fermion zero modes to survive in two-dimensional structures in which a superconducting gap is induced via the proximity effect to stabilize a phase with Ising anyons,[35–40] but long-ranged superconducting order is disordered by quantum or thermal fluctuations. If the single-particle gap remains, then the Majorana fermion zero modes associated with the Ising anyons could survive. However, quantum phase slips are suppressed,[41] and, therefore, the splitting will be exponentially rather than algebraically decaying. Of course, there is nothing surprising about having protected Majorana zero modes in a system with no long-range order or even algebraic order since this is precisely the case with any true topological phase of matter, as in the examples mentioned in the Introduction. However, the particular route that we have found to such a system is new and interesting.

In this paper, we have shown that a gapless system can be nearly as good as a fully gapped one at supporting protected Majorana fermion zero modes. It is an interesting open question as to whether a gapless system might be capable of supporting protected degrees of freedom, which can not occur in fully gapped 1D systems.[42–44]

Note added: After the initial version of this paper appeared on the arXiv, several other papers[45–47] on related topics were submitted to the arXiv.

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