Topological classification of adiabatic processes

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Certain band insulators allow for the adiabatic pumping of quantized charge or spin for special time dependences of the Hamiltonian. These "topological pumps" are closely related to two-dimensional topological insulating phases of matter upon rolling the insulator up to a cylinder and threading it with a time-dependent flux. In this paper we extend the classification of topological pumps to the Wigner-Dyson and chiral classes, coupled to multichannel leads. The topological index distinguishing different topological classes is formulated in terms of the scattering matrix of the system. We argue that similar to topologically nontrivial insulators, topological pumps are characterized by the appearance of protected gapless end states during the course of a pumping cycle. We show that this property allows for the pumping of quantized charge or spin in the weak-coupling limit. Our results may also be applied to two-dimensional topological insulators, where they give a physically transparent interpretation of the topologically nontrivial phases in terms of scattering matrices.

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I. INTRODUCTION

Topological insulating states of matter differ from regular band insulators by the fact that they support protected gapless surface states. Holding promise for numerous applications, this observation has considerably motivated the search for materials that realize such topological phases. The first experimental observation of a topologically nontrivial insulator dates back to the discovery of the quantum Hall effect 30 years ago.[1,2] The recent discovery of the quantum spin Hall effect[3–5] has lead to a full classification of insulators with topological order based on their underlying symmetries and spatial dimensions.[6]

The study of the quantum Hall effect has instigated numerous theoretical works dedicated to the understanding of its nontrivial topological nature. One particular appealing argument was introduced by Laughlin,[7] who considered a pump formed by placing the two-dimensional system on a cylinder and threading it with a time-dependent magnetic flux. As the flux is varied periodically in time, an integer number of charges are transferred across the pump upon completing one cycle. This charge quantization is directly related to the quantized Hall conductance of the underlying Hall insulator.[8–14]

In this paper we extend Laughlin's considerations to pumps formed by two-dimensional insulators belonging to the Wigner-Dyson and the chiral classes. Based on their underlying symmetries, applying the above construction imposes a symmetry constraint on the pumping cycle. This allows for the classification of topological pumps in terms of invariants of their scattering matrix and gives rise to a physically transparent interpretation of the topologically nontrivial phases in terms of quantized pumping properties. Similarly to topologically nontrivial two-dimensional insulators, topologically nontrivial pumps are characterized by the appearance of gapless end states during the course of a pumping cycle.[7,15,16] We show that in the weak-coupling limit the nontrivial pumps allow for noiseless pumping of quantized charge or spin. This paper extends previous work by the authors on topological pumps with time-reversal restriction connected to single-channel leads[17] and on topological pumps where the gap is induced by electron-electron interactions,[18] to the case of other symmetry relations and multichannel leads.

Consider a pump formed by placing the two-dimensional insulator on a cylinder and threading it with a magnetic flux which is varied in time[8,9] (see Fig. 1). The cylinder is connected to ballistic leads with N (or, in presence of spin degeneracy, $2N$) transverse channels, and scattering off the pump can be described in terms of a $2N \times 2N$ (or, in the presence of spin degeneracy, $4N \times 4N$) scattering matrix. Assuming the pump's extension exceeds the attenuation length associated with the bulk energy gap, the scattering matrix reduces into two unitary $N \times N$ ($2N \times 2N$) blocks describing the reflection of electrons incident from the left- and right-hand leads. In the following we consider one of these blocks only, and refer to it as the "scattering matrix" S of the two-dimensional insulator.

The underlying symmetries $\hat{\mathcal{S}}$ of the two-dimensional Hamiltonian $\hat{\mathcal{S}}\hat{H}\hat{\mathcal{S}}^{-1} = \pm\hat{H}$ impose a similar restriction on the pumping cycle $\hat{\mathcal{S}}\hat{H}(t)\hat{\mathcal{S}}^{-1} = \pm\hat{H}[\mathcal{S}^{-1}(t)]$. In this paper we study the topological classification of pumps formed of insulators belonging to the Wigner-Dyson and chiral classes, characterized by the presence or absence of time-reversal $\hat{\mathcal{S}} = \Theta$, $\Theta H \Theta^{-1} = H$, and sublattice symmetries $\hat{\mathcal{S}} = \mathcal{C}$, $CHC^{-1} = -H$, as well as the presence or absence of strong spin-orbit interactions.

The scattering matrix S of the pump is related to the two-dimensional Hamiltonian through the general formula[19]

$$S = 1 + 2i\pi W^\dagger (H - i\pi W W^\dagger)^{-1} W, \tag{1}$$

where W describes the coupling to the leads. Here we assume the leads couple equally to both spin orientations at the edges of the insulator and that the coupling itself does not break time-reversal or sublattice symmetry. This relation allows to obtain the symmetry restrictions on the reflection matrices, summarized in Table I.

Similarly to topologically nontrivial two-dimensional insulators, topologically nontrivial pumps are characterized by the appearance of gapless end states during the course of a pumping cycle. These emerge as resonances of the scattering matrix that introduce a π phase shift of the incoming scattering states. [The presence of a bound state at the edge of the sample follows from Eq. (1) or from a Bohr-Sommerfeld quantization

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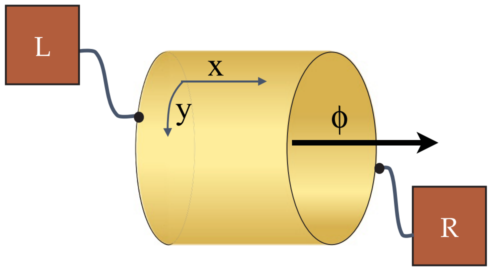

FIG. 1. (Color online) A pump formed by rolling a two-dimensional insulator on a cylinder and threading it with a time-dependent flux ϕ.

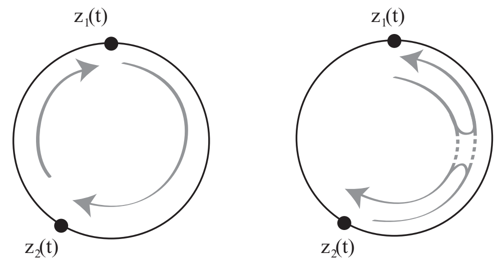

FIG. 2. A pumping cycle can be visualized as a set of trajectories formed by the ring coordinates, shown for $N = 2$. The left-hand side portrays a nontrivial winding of the center-of-mass coordinate, while the right-hand side shows avoided level crossing corresponding to a nontrivial winding of the center of mass.

argument, when the scattering states acquire a π phase shift at resonance.] In a topologically nontrivial pump, the appearance of resonances *during a pumping cycle* is topologically protected and is independent of smooth deformation of the Hamiltonian, although the specific moment of its appearance may vary. As we discuss below, these resonances are manifested in the form of topologically protected pumping properties.

In order to relate the appearance of resonances during a pumping cycle to topological properties of the scattering matrix, we note that the eigenvalues of the unitary scattering matrix are restricted to the unit circle $\{z_1 = e^{i\phi_1},\ldots,z_N = e^{i\phi_N}\}$. The pumping cycle can be visualized as the set of trajectories formed by these coordinates $\{z_1(t),\ldots,z_N(t)\}$ on the unit circle, such that the original set of eigenvalues is recovered after a cycle is completed. From Eq. (1), the appearance of an edge state, i.e., a scattering resonance, corresponds to an eigenvalue $z_i = -1$ in the resonant channel i. As slight deformations of the Hamiltonian may shift the eigenvalues and lead to detuning away from the resonance, topologically protected resonances can only arise if the trajectory of the eigenvalue forms a noncontractable loop around the unit circle. Moreover, in the absence of any symmetries, any crossings of energy levels (and similarly of eigenvalues z_i) are generally avoided, or can be avoided in the presence of small perturbations of the Hamiltonian (see Fig. 2). Hence, in the absence of any symmetries, topologically protected resonances can only arise if the center of mass of the ring coordinates $\Phi = \sum_{i=1}^{N}\phi_i$ forms a noncontractable loop within a pumping cycle. The sum of the eigenvalue phases is related to the determinant $\det S = \Pi_i z_i = e^{i\Phi}$, and the winding of the phase during the course of a pumping cycle, gives rise to an integer index n:[14]

$$n = \oint_0^T \frac{dt}{2\pi}\,\dot{\Phi} = \oint_0^T \frac{dt}{2\pi}\frac{d}{dt}\ln\det S. \qquad (2)$$

Symmetry constraints on the pumping cycle restrict the “dynamics” of the eigenvalues during the cycle. In particular, a time-reversal restriction relates the time evolution of $S(t)$ during the second part of the cycle $T/2 \leqslant t < T$ to the evolution during the first part $0 \leqslant t < T/2$,

$$S(t) = \hat{O}S^t(T-t)\hat{O}^{-1}. \qquad (3)$$

Here $\hat{O} = e^{i\pi\hat{S}_y/\hbar}$ is the unit matrix for spinless electrons and the Pauli matrix σ_y for spin-full electrons. As a consequence, any winding of the phase Φ during the first part of the cycle is undone in the course of completing the cycle $\Phi(t) = \Phi(T-t)$, resulting in

$$n = \int_0^{T/2}\frac{dt}{2\pi}[\dot{\Phi}(t) + \dot{\Phi}(T/2-t)] = 0,$$

which expresses the fact that pumps with a time-reversal constraint cannot pump charge.

The time-reversal constraint allows, however, for a more subtle classification, when the spin rotational symmetry is broken as a result of strong spin-orbit scattering (e.g., in the symplectic class AII).[15,17] The time-reversal restriction (3), in combination with the periodicity of the pump, ensures the existence of two time-reversal invariant moments (TRIM) $t_i = 0, T/2$ throughout the pumping cycle. At these moments, the scattering matrix is time-reversal symmetric, and its eigenvalues form Kramers degenerate pairs. While the time-reversal restriction (3) inhibits any winding of Φ, topologically protected resonances may still arise if the sum of the phase differences acquired by the Kramer pairs during half a cycle has a topologically nontrivial winding. As we shall see below, the latter is defined up to multiples of 4π, giving rise to a $\mathbb{Z}_2$ index instead of a $\mathbb{Z}$ index (a similar argument is made in Ref. 15 with respect to the “time-reversal polarization”).

At the time-reversal invariant moments, the eigenvalues of the scattering matrix occur in pairs $(e^{i\phi_n}, e^{i\bar{\phi}_n})$ with $n = 1,\ldots,N$. This introduces two possibilities. After half a cycle of the pump is completed, the pairs can either recombine $\phi_n(T/2) = \bar{\phi}_n(T/2)$ or exchange partners $\phi_n(T/2) = \bar{\phi}_{n-1}(T/2)$. In the latter scenario, the sum of all phase differences acquired between former Kramers pairs can pick

TABLE I. Symmetry restrictions on reflection matrices belonging to the Wigner-Dyson (first three rows) and chiral (last three rows) classes. Θ and $\mathcal{C}$ are the time-reversal and sublattice symmetry operations, respectively. Last column: Classes which allow for nontrival topological $\mathbb{Z}$ or $\mathbb{Z}_2$ invariants.

Class	Θ	$\mathcal{C}$	Symmetry restriction	Index
A	0	0	$S(t) \in U(2N)$	$\mathbb{Z}$
AI	1	0	$S(t) = S^T(-t)$	0
AII	−1	0	$S(t) = \sigma_y S(-t)^T \sigma_y$	$\mathbb{Z}_2$
AIII	0	1	$S(t) = S^\dagger(t)$	0
BDI	1	1	$S(t) = S^\dagger(t) \quad S(t) = S^T(-t)$	0
CII	−1	1	$S(t) = S^\dagger(t) \quad S(t) = \sigma_y S(-t)^T \sigma_y$	0

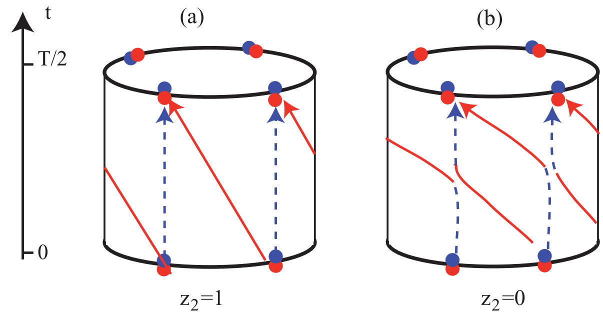

FIG. 3. (Color online) (a) shows a nontrivial winding of the relative phase acquired between a Kramer's pair after half a cycle corresponding to the exchange of partners. The higher winding of the relative phase shown in (b) involves additional crossing points away from the TRIM which are generally avoided, leaving no net phase difference between the Kramers pairs.

up a phase of multiple of 2π when evolving into the twofold degenerate configuration at $t = T/2$ [see Fig. 3(a)]. We note that the crossing of the eigenvalues at the TRIM is protected by time-reversal symmetry and therefore cannot be lifted by small deformations of the Hamiltonian. Moreover, higher winding of the relative phase $\delta\varphi = (\varphi - \bar{\varphi})$ (with $\varphi = \sum_{i=1}^{N} \phi_i$ and $\bar{\varphi} = \sum_{i=1}^{N} \bar{\phi}_i$), corresponding to the Kramers pairs recombining with next-to-nearest neighbors eigenvalues, inevitably involve additional crossing points away from the TRIM. As these are not protected by time-reversal symmetry, they are generically avoided or can be removed by small deformations of the Hamiltonian [see Fig. 3(b)]. Hence, an odd number of winding of the relative phase by 2π is topologically protected, while an even number of winding is topologically trivial. The relative phase $\delta\varphi$ acquired during half a cycle can be expressed in terms of the products of eigenvalues $Z = e^{i\varphi}$ and $\bar{Z} = e^{i\bar{\varphi}}$ as

$$\theta = \int_0^{T/2} dt\, \delta\dot{\varphi} = \ln \frac{Z(T/2)\bar{Z}(0)}{Z(0)\bar{Z}(T/2)}.$$

The presence of an even or odd winding number of the relative phase is then expressed in terms of the $\mathbb{Z}_2$ index

$$e^{i\theta/2} = \frac{\sqrt{Z(T/2)\bar{Z}(0)}}{\sqrt{Z(0)\bar{Z}(T/2)}}, \quad (4)$$

which takes the values $e^{i\theta/2} = \pm 1$. Using the relation between Z and the Pfaffian $Z = \mathrm{Pf}(iS\sigma_y)$, this result can be formulated in terms of the scattering matrix

$$e^{i\theta/2} = \frac{\mathrm{Pf}(iS(T/2)\sigma_y)}{\mathrm{Pf}(iS(0)\sigma_y)} \frac{\sqrt{\det S(0)}}{\sqrt{\det S(T/2)}}. \quad (5)$$

where the same branch of the square root is chosen in the numerator and denominator. Equation (5) is the N-channel generalization of the $\mathbb{Z}_2$ index in Ref. 17. The same result appeared in Ref. 20 in the context of a classification of two-dimensional (2D) topological insulators.

Finally, we consider a system with chiral, i.e., sublattice symmetry. Contrary to time-reversal symmetry, the sublattice symmetry is present at every instant in the cycle

$$S(t) = S^{\dagger}(t). \quad (6)$$

As the scattering matrix is Hermitian at all times, its eigenvalues are restricted to the real axis, i.e., $z_i = \pm 1$ throughout the cycle which prevents a nontrivial topology of the trajectory $S(t)$ during a cycle. Consequently all pumps with a chiral symmetry are topologically trivial.

Following the discussion above, we may classify the pumps belonging to the Wigner-Dyson and the chiral classes based on the presence of absence of time-reversal, chiral, and spin rotational symmetry. The result is summarized in the last column of Table I. The classification based on the scattering matrix of the one-dimensional pump reproduces the corresponding table for two-dimensional insulators in the Wigner-Dyson and chiral classes.[21]

The appearance of protected gapless edge states during the course of a pumping cycle alters the pumping properties. In contrast to their trivial counterparts, topological nontrivial pumps allow for the pumping of quantized charge or spin. In the absence of time-reversal or chiral symmetries (class A), the quantization of the charge is evident as the charge pumped through the insulator[22,23] is proportional to the topological index itself[8,9,14,24–29]

$$Q = \frac{e}{2\pi i} \oint dt\, \mathrm{tr}\left(\frac{d\hat{S}}{dt}\hat{S}^{\dagger}\right) = en, \quad (7)$$

where the trace is taken over the N channels.

Imposing a time-reversal restriction on the pumping cycle inhibits the pumping of charge by restricting the winding of the phase of $\det S$. Nonetheless, topological pumps with a time-reversal restriction allow for the pumping of quantized spin even in the presence of spin-orbit scattering. We notice that contrary to topological charge pumps, the spin $\vec{s}$ pumped in a cycle

$$\vec{s} = \frac{\hbar}{2\pi i} \oint dt\, \mathrm{tr}\left(\frac{d\hat{S}_\alpha}{dt}\hat{S}_\alpha^{\dagger}\vec{\sigma}\right) \quad (8)$$

is not directly related to the $\mathbb{Z}_2$ index (5). Hence its quantization is in general not ascertained. Instead, quantization becomes asymptotically exact in the weak-coupling limit when the coupling to the lead becomes arbitrary small.[30]

Following the above discussion, a topological spin pump (class AII) crosses an odd number of resonance pairs during the course of a cycle. As multiple winding of the phase can be removed by small perturbations, a nontrivial pump generically crosses a single resonance pair during a cycle. In the weak-coupling limit, the N channels decouple and only the resonant channel has a time-dependent phase shift. Since only one channel contributes to the pumped spin, the considerations of Ref. 17 for the single-channel case $N = 1$ can be applied: The typical time scale at which the resonance at time t_i is traversed vanishes in the weak-coupling limit, as the broadening of the energy levels goes to zero. Contrarily, the time scale on which the spin quantization axis $\vec{e}_\phi(t)$ changes depends exclusively on microscopic details of the insulator and is independent of the coupling to the leads. Therefore, in the weak-coupling limit, the scattering matrix takes the form

$$S(t) = U^{\dagger}(t_i)\Lambda(t)U(t_i) \quad (9)$$

for t near t_i, where the eigenvector matrix $U(t_i)$ and the eigenphase matrix $\Lambda(t) = \mathrm{diag}\{e^{i\phi_1(t_i)}, \ldots, e^{i\phi_j(t)}, \ldots,$

$e^{i\bar{\phi}_j(t)},\ldots,e^{i\phi_N(t_i)}\}$, can be taken to be time independent, with the exception of a single pair $\phi_j(t),\bar{\phi}_j(t)$ of phases, which abruptly change near $t=t_i$. The spin quantization axis is

$$\vec{e}_\phi(t_i)=\mathrm{tr}[P_{j,\bar{j}}U(t_i)\vec{\sigma}U^\dagger(t_i)], \qquad (10)$$

with $P_{j,\bar{j}}=\mathrm{diag}\{0,\ldots,1,\ldots,1,\ldots,0\}$ the projection on the resonant channel. The quantization in the $\vec{e}_\phi$ direction $\vec{s}=\hbar\vec{e}_\phi(t_i)$ then readily follows from Eq. (8).[17,31]

These results can be extended to systems in which the gap arises due to electron-electron interactions, following the arguments of Refs. 14 and 18. While in the presence of interactions the unitarity of the scattering matrix is in general not guaranteed, there is a unitary scattering matrix under suitable conditions:[14,18] We consider scattering from an insulator with an energy gap Δ, which does not support any low-energy bulk excitations. Moreover, we consider the weak-coupling limit $\Gamma\to 0$ in which one can avoid inelastic scattering from edge states and Kondo-like resonances by operating the pump at rates $1/T$ that satisfy $\beta_{\mathrm{K}}^{-1},\Gamma^2/(\mu^2\nu_0)\ll\hbar/\beta,\hbar/T\ll\Gamma\ll\Delta$—see the full discussion in Ref. 18. Here $1/\beta$ is the temperature, $1/\beta_{\mathrm{K}}$ is the relevant Kondo temperature, μ is the energy of the edge state, and ν_0 is the density of states in the leads. Contrary to a single-particle band gap, however, the many-body nature of an interaction-induced gap can change the Fermi sea of noninteracting electrons into d many-particle ground states. As a consequence, ground states may be interchanged in the course of a pumping cycle, and the pump may operate with an extended period, which is a multiple qT of the period T at which the Hamiltonian is varied.[14,18] In the weak-coupling limit, depending on the symmetry group of the Hamiltonian, each extended periodicity can be characterized by a $\mathbb{Z}$ (class A) or $\mathbb{Z}_2$ (class AII) topological index, corresponding for a transfer of an integer charge n or spin $\hbar$ during the *extended* cycle, respectively. Consequently, a topological pump with an extended pumping cycle qT due to interactions transfers on average a fractional charge en/q or spin $\hbar/q$ during a cycle T.[14,18]

In conclusion, we have extended Laughlin's construction of pumps formed by two-dimensional insulators to the Wigner-Dyson and chiral classes, coupled to multichannel leads. This mapping allows for a pedestrian derivation of the topological classification of insulators in terms of the reflection matrices of the corresponding pumps. We provide a physically transparent interpretation of the topologically nontrivial phases based on their quantized pumping properties.

Note added. Upon finishing this work we became aware of a related work in Ref. 20.

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