Fermionic Hopf solitons and Berry phase in topological surface superconductors

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An interesting phenomenon in many-body physics is that quantum statistics may be an emergent property. This was first noted in the Skyrme model of nuclear matter, where a theory of a bosonic order parameter field contains fermionic excitations. These excitations are smooth field textures and are believed to describe neutrons and protons. We argue that a similar phenomenon occurs in topological insulators when superconductivity gaps out their surface states. Here, a smooth texture is naturally described by a three-component vector. Two components describe superconductivity, while the third captures the band topology. Such a vector field can assume a "knotted" configuration in three-dimensional space—the Hopf texture—that cannot smoothly be unwound. Here we show that the Hopf texture is a fermion. To describe the resulting state, the regular Landau-Ginzburg theory of superconductivity must be augmented by a topological Berry phase term. When the Hopf texture is the cheapest fermionic excitation, unusual consequences for tunneling experiments on mesoscopic samples are predicted. This framework directly generalizes the phenomenon of period doubling of Josephson effect to three-dimensional topological insulators with surface superconductivity.

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There has been much recent activity relating to topological insulators (TIs)—a new phase of matter with protected surface states.[1,2] Particularly rich phenomena are predicted to arise when this phase is combined with conventional orders such as magnetism,[3,4] crystalline order,[5] and superconductivity. The latter is particularly interesting. Superconductivity induced on the surface of a TI was predicted to have vortices harboring Majorana zero modes.[6] These are of interest to quantum information processing, since they are intrinsically robust against errors. Recently, superconductivity was discovered in a doped TI,[7] which could be used to induce surface superconductivity. Below we discuss a new theoretical approach to studying this remarkable superconducting phase, which provides different insights and directions for experiments.

We focus on smooth configurations where the energy gap never vanishes. In this case, the low-energy description of the system is entirely in terms of bosonic coordinates (order parameter), much as the Landau Ginzburg order parameter theory describes superconductors at energies below the gap. Can fermions ever emerge in such a theory? While it is easy to imagine obtaining bosons from a fermionic theory, the reverse is harder to imagine. However, it has been shown in principle that bosonic theories that contain additional Berry phase (or Wess-Zumino-Witten) terms, can accomplish this transmutation of statistics. We show that this indeed occurs in the superconductor-TI (Sc-TI) system; the order parameter theory contains a Berry phase term which implies that a particular configuration of fields—the Hopf soliton (or Hopfion)—carries fermionic statistics. While such statistics transmutation is common in one dimension,[8] it is a rare phenomenon in higher dimensions. In the condensed matter context, a physically realizable example exists in two dimensions: solitons of quantum Hall ferromagnets (skyrmions) are fermionic and charged and have been observed.[9–11] However, the superconductor-TI system is, to our knowledge, the first explicit condensed matter realization of this phenomenon in three dimensions.

The organization of this paper is as follows: First, we introduce our simplified model of a topological insulator with surface superconductivity and review properties of the Hopf texture. We then discuss evidence from numerical calculations on a lattice model that demonstrate that Hopfions are fermions. We also discuss the connection between fermionic Hopfions and three-dimensional (3D) non-Abelian statistics of Ref. 12. A simplified two-dimensional example, where skyrmions are fermionic, is also discussed. Next, we provide a field theoretical derivation of the same result in the continuum and introduce the necessary theoretical tools to compute a topological term that leads to the fermionic statistics. Finally, we mention the physical consequences for tunneling experiments as well as for Josephson junctions.

I. MODEL AND HOPF TEXTURE

The essential properties of a topological insulator are captured by a simplified low-energy theory with a three-dimensional Dirac dispersion (a microscopic realization is described later):

$$H_D = \psi^\dagger[v_F\alpha\cdot\mathbf{p} + m\beta_0]\psi, \tag{1}$$

where $(\alpha_1,\alpha_2,\alpha_3,\beta_0)$ are 4×4 anticommuting matrices which square to the identity and involve both spin and sublattice degrees of freedom. The dispersion then is $\epsilon(p) = \pm\sqrt{v_F^2p^2+m^2}$. An insulator is obtained for $m\neq 0$. Changing the sign of m results in going from a trivial to a topological insulator. Which sign of m is topological is set by the band structure away from the node—we assume $m<0$ is topological. Consider now adding (onsite) superconducting pairing, which may be proximity induced by an s-wave superconductor. Then

$$H_{\rm pair} = \Delta\psi^\dagger\beta_5\psi^\dagger + \text{H.c.}, \tag{2}$$

where $\beta_5 = \alpha_1\alpha_2\alpha_3\beta_0$. The matrices α_i can be taken to be symmetric, while β_0, β_5 are antisymmetric. We can write the total Hamiltonian $H_f = H_D + H_{\rm pair}$ then as

$$H_f = [\,\psi^\dagger\ \psi\,]\mathcal{H}_f\begin{bmatrix}\psi\\ \psi^\dagger\end{bmatrix},$$
$$\mathcal{H}_f = \begin{bmatrix}-iv_F\alpha\cdot\partial + m\beta_0 & \Delta\beta_5\\ \Delta^*\beta_5 & -iv_F\alpha\cdot\partial + m\beta_0\end{bmatrix}. \quad (3)$$

The spectrum now is $\epsilon(p) = \pm\sqrt{v_F^2p^2 + M^2}$ where $M^2 = m^2 + |\Delta|^2$. The single particle Hamiltonian can be written as $\mathcal{H}_f = [-iv_F\alpha\cdot\partial + m\beta_0]\mathbf{1} + \beta_5(\Delta\nu^+ + \Delta^*\nu^-)$ where the ν^a are Pauli matrices acting on the particle-hole space. It is readily verified that this Hamiltonian enjoys particle-hole symmetry: $C\mathcal{H}_f^*C = -\mathcal{H}_f$, where $C = \nu^x$ and we have used that fact that the α_i are symmetric and $\beta_{0,5}$ are antisymmetric.

It is convenient to define a three-vector $\vec{n} = (\mathrm{Re}\Delta, \mathrm{Im}\Delta, m)$, such that $|\vec{n}| = M$. A singular configuration is one where all three components of this vector go to zero. Since the vacuum can be taken to be a trivial insulator, m changes sign at the topological insulator surface. The components of the pairing Δ vanish in the vortex core. This can also be viewed as a hedgehog[12] configuration in $\vec{n}$. It has been pointed out that a TI-surface vortex with odd winding number will give rise to an unpaired Majorana zero mode.[6] Note that, at the core of these singular configurations, the gap closes, allowing for the possibility of localized bound states at zero energy. In this work we will only consider smooth textures of the $\vec{n}$ field, where the single-particle gap is nonzero everywhere. An effective theory of slow fluctuations of the "order parameter" field $\vec{n}(r,t)$, occurring over spatial (time) scales much larger than $\xi = \hbar v_F/M$ ($\tau = \xi/v_F$), can be obtained by integrating out the gapped fermions. An analogous procedure is well know in the context of the BCS theory of superconductivity, where it leads to the Landau-Ginzburg action. Here, we will find that an extra topological term arises that transmutes statistics and leads to fermionic solitons.

Consider a smooth configuration of $\vec{n}(r)$, which can be normalized to give a unit vector $\hat{n}(r)$ at each point. This defines a mapping from each point of three-dimensional space, to a unit three-vector, which describes the surface of a sphere S^2. We require that the mapping approaches a constant at infinity: $\hat{n}(|r|\to\infty) = $ const. (e.g., the vacuum). Can all such mappings be smoothly distorted into one another? A surprising result due to Hopf[13] in 1931 is that there are topologically distinct mappings, which can be labeled by distinct integers h (the Hopf index). No smooth deformation can connect configurations with different Hopf indices. Mathematically, Hopf showed that the homotopy group $\Pi_3[S^2] = Z$. A straightforward way to establish the index is to consider the set of points in space that map to a particular orientation of $\hat{n}$. In general this is a curve. If we consider two such orientations $(\hat{n}_1,\hat{n}_2)$, we get a pair of curves. The linking number of the curves is the Hopf index. A configuration with unit Hopf index can be constructed by picking a reference vector $\hat{n} = \hat{z}$, say, and rotating it: $\hat{n}(\vec{r})\cdot\vec{\sigma} = U(\vec{r})\sigma_z U^\dagger(\vec{r})$ by the spatially varying rotation $U(\vec{r}) = \exp[i\frac{\theta(r)}{2}\hat{r}\cdot\vec{\sigma}]$. Here the angle of rotation varies as we move in the radial direction, from $\theta = 0$ at the origin to $\theta = 2\pi$ at radial infinity, and the axis of rotation is the radial direction $\hat{r}$. By studying which spatial points map to (e.g., $\hat{n} = \pm\hat{z}$), as in Fig. 1, one can conclude that this texture has unit Hopf index. What is the physical interpretation of this Hopf texture in the context of TIs? A torus of TI (Fig. 2) has superconductivity induced on its surface. There is vacuum far away and through the hole of the torus, which counts as a trivial insulator, $\hat{n} = \hat{z}$. The center of the strong topological insulator corresponds to $\hat{n} = -\hat{z}$. On the topological insulator surface $n_z = 0$, and the superconducting phase varies such that

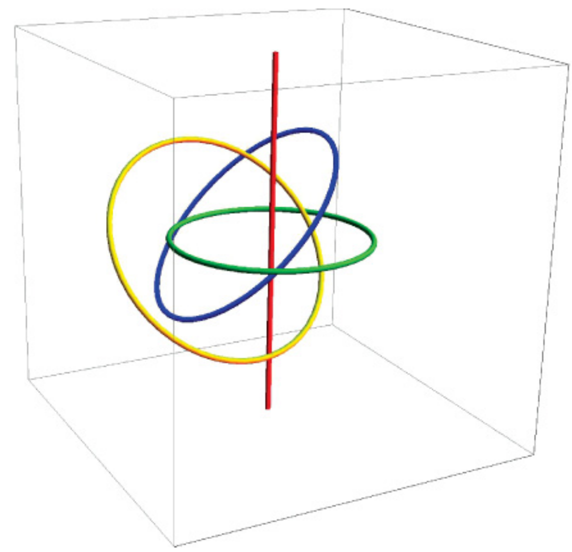

FIG. 1. (Color online) Hopf map: $f_H : \vec{r} \to \hat{n}$ is shown by displaying contours of equal $\hat{n}$. Points at infinity are all mapped to the same point on the sphere $f_H(\infty) = \hat{z}$. In red is $f_H^{-1}[\hat{n} = \hat{z}]$, in green $f_H^{-1}[-\hat{z}]$, in blue $f_H^{-1}[\hat{x}]$, and in yellow $f_H^{-1}[\hat{y}]$. Note the unit linking of any pair of curves, which can be used to define the Hopf texture.

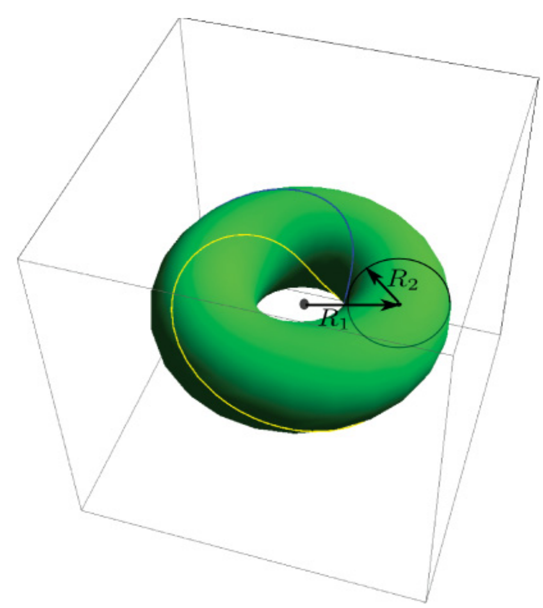

FIG. 2. (Color online) Torus of strong topological insulator (green) in vacuum, whose surface is superconducting. There is unit superconductor phase winding about each cycle of the torus. We plot the equal phase contours on the surface whose pairing phase is 0 (blue) and π (yellow). The unit linking of these curves indicates that this is the Hopf mapping.

there is a unit-vortex trapped in each cycle of the torus. We now argue that such a texture is a fermion.

II. HOPF SOLITONS ARE FERMIONS

The mean-field Hamiltonian (3) for a general texture has no physical symmetry (charge conservation is broken by the superconducting pairing and time reversal is broken by the nontrivial phases associated with the texture). However, a fermion Hamiltonian always has the symmetry of changing the sign of the fermion field (since a term with odd number of fermions does not appear in a local Hamiltonian). The conserved quantity associated with this symmetry is fermion parity. So the only quantum number that can be assigned to the ground state is the parity of the total number of fermions $(-1)^F$. Superconducting pairing only changes the number of fermions by an even number, hence one can assign this fermion parity quantum number to any eigenstate. We now argue that the fermion parity of a smooth texture is simply the parity of its Hopf index h:

$$(-1)^F = (-1)^h. \tag{4}$$

First, we argue that the ground state with a topologically trivial texture has an even number of fermions. As a representative configuration, consider a configuration where the superconductor pairing amplitude is real. This is a time-reversal-invariant Hamiltonian. If the ground state had an odd number of fermions, it must be at least doubly degenerate by the Kramers theorem. However, the ground state of any smooth texture is fully gapped and hence unique. Thus, this configuration must have an even fermion parity. Now, any other texture in the same topological class can be reached by a continuous deformation, during which the gap stays open. The fermion parity stays fixed during this process. Note, this argument cannot be applied to configurations with nonzero Hopf index, since these necessarily break time-reversal symmetry. For example, the configuration shown in Fig. 1 contains phase windings.

To find the fermion parity of the nontrivial Hopf configurations, we consider evolving the Hamiltonian between the trivial and $h = 1$ configuration. In this process we must have $\vec{n} = 0$ at some point, which will allow the gap to close and a transfer of fermion parity to potentially occur. Indeed, as shown below in separate calculations, a change in fermion parity is induced when the Hopf index changes by one.

A. Numerical calculation

We study numerically the microscopic topological insulator model defined in Ref. 14 with a pair of orbitals ($\tau_z = \pm 1$) on each site of a cubic lattice. The tight-binding Hamiltonian

$$H = \sum_k [\psi_k^\dagger \mathcal{H}_k \psi_k + H_{\mathrm{pair}}(k)] \tag{5}$$

is written in momentum space using a four-component fermion operator ψ_k with two orbital and two spin components. Then,

$$\mathcal{H}_k = -2t \sum_{a=1}^{3} \alpha_a \sin k_a - m\beta_0 \left[\lambda + \sum_{a=1}^{3} (\cos k_a - 1)\right], \tag{6}$$

where $(\vec{\alpha},\beta_0) = (\tau_x\sigma_x, \tau_x\sigma_y, -\tau_y, \tau_z)$. For $t,m > 0$ a strong topological insulator is obtained when $\lambda \in (0,2)$. In addition we introduce onsite singlet pairing:

$$H_{\mathrm{pair}}(k) = \Delta[\psi_k^\dagger]^{\mathrm{T}} \sigma_y \psi_{-k}^\dagger + \mathrm{H.c.} \tag{7}$$

Note that, when $\lambda \approx 0$, $\vec{k} \approx (0,0,0)$, Eq. (3) is recovered as the low-energy theory.

The energy spectrum is studied as we interpolate between a topologically trivial texture ($h = 0$) and the Hopf texture ($h = 1$). We choose to define a torus-shaped strong topological insulator with trivial insulator (vacuum) on the outside, as in Fig. 2. The surface is gapped by superconducting pairing Δ, which in the trivial texture is taken to be real Δ_0. (Note that the energy gap in the bulk arises from the insulating gap, while on the surface the gap arises from pairing. There is a smooth evolution between these limits over a scale associated with the surface states.) In the Hopf texture, the superconducting pairing Δ_1 has a phase that winds around the surface, with a unit winding about both cycles of the torus. This can be interpreted as "vortices" inside the holes of the torus. Note, the vortex cores are deep inside the insulators, so there is a finite gap in the Hopf texture. We interpolate between these two fully gapped phases by defining $\Delta(\lambda) = \lambda\Delta_1 + (1-\lambda)\Delta_0$ and changing $\lambda = 0 \to 1$. On the way, the gap must close since the two textures differ in topology. We study the evolution of eigenvalues, as shown in Fig. 3. We find that exactly one pair of $\pm E$ eigenvalues are pumped through zero energy. As argued below, this signals a change in the fermion parity of the ground state on the two sides. Since the trivial texture has even fermion parity from time-reversal symmetry, the Hopf texture must carry odd fermion number.

To see why the crossing of a $\pm E$ conjugate pair of levels corresponds to a change in fermion number, consider a single-site model $H = E_0(c^\dagger c - cc^\dagger)$. This has a pair of single particle levels at $\pm E_0$, which will cross if we tune E_0 from, say, positive to negative values. However, writing this Hamiltonian in terms of the number operators $H = E_0(2\hat{n} - 1)$ shows that the ground-state fermion number changes from $n = 0$ to $n = 1$ in this process. Thus the ground-state fermion parity is changed whenever a pair of conjugate levels cross zero energy. We mention that it is possible to confirm the numerical results analytically by solving for the low-energy modes in the vicinity of the vortex core, where the insulating mass term is set to be near zero. The linking of vortices in the Hopf texture plays a crucial role in deriving this result.

1. The Pfaffian

Previously, the ground-state fermion parity was found by interpolating between two topological sectors. Can one directly calculate the fermion parity for a given Hamiltonian's ground state? We show that this is achieved by calculating the Pfaffian of the Hamiltonian in the Majorana basis. The Pfaffian of an antisymmetric matrix is the square root of the determinant—but with a fixed sign. It is convenient to recast the Hamiltonian in terms of Majorana or real fermions defined via $\psi_a = (\chi_{1a} + i\chi_{2a})/2$. Since a pair of Majorana fermions

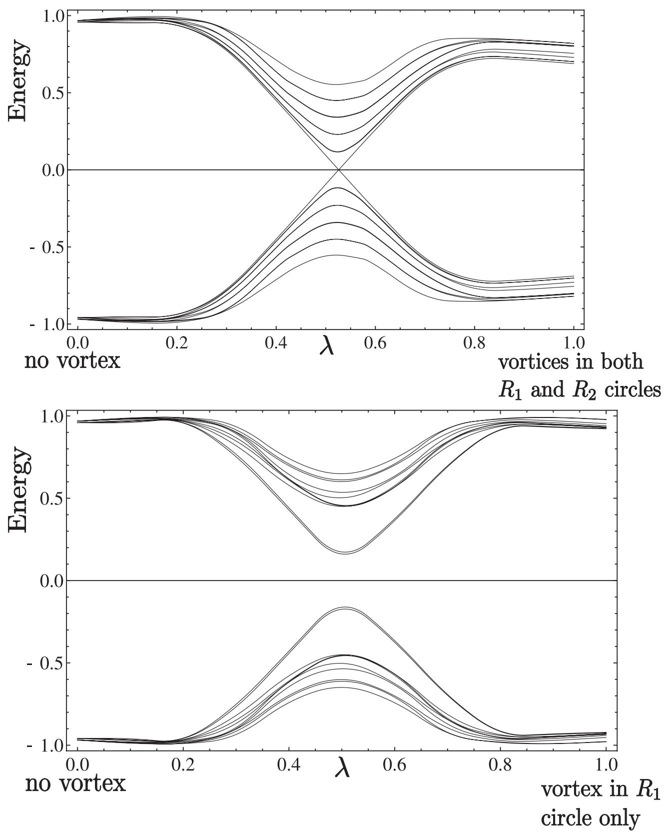

FIG. 3. Spectral flow of the lowest 20 eigenvalues when the pairing on the surface of the topological insulator in Fig. 2 is linearly interpolated between two limits: $\Delta(x,\lambda) = (1-\lambda)\Delta_0(x) + \lambda\Delta_1(x)$. $\Delta_0(x)$ is constant over the whole surface. Top panel shows that $\Delta_1(x)$ has a unit phase winding (vortex) in both the R_1 and R_2 cycles of the torus (i.e., the Hopf texture). Bottom panel shows that $\Delta_1(x)$ has a unit phase winding in only the R_1 cycle. It is clear that there is a single level crossing in the top-panel case, meaning the ground-state fermion parity is changed in the process. The initial state has even fermion parity, so the final state (i.e., the Hopf texture) must have odd fermion parity. The calculation is on the cubic lattice model defined in the text, with $R_1 = 13$ and $R_2 = 5$ lattice units, and only points within the torus are retained. Parameters used are $t = M = \lambda = 1$ and pairing $|\Delta| = 1$.

anticommute $\{\chi_i,\chi_j\} = \delta_{ij}$, the Hamiltonian written in these variables will take the form

$$H = -i\sum_{ij} \mathbf{h}_{ij}\chi_i\chi_j, \tag{8}$$

where $\mathbf{h}_{ij}$ is an even-dimensional antisymmetric matrix, with real entries and the Majorana fields appear as a vector $\chi = (\ldots,\chi_{1a},\chi_{2a},\ldots)$, where a refers to site, orbital, and spin indices. The $\pm E$ symmetry of the spectrum is an obvious consequence of h being an antisymmetric matrix. The ground-state fermion parity in this basis is determined via

$$(-1)^F = \mathrm{sgn}\,[\mathrm{Pfaffian}(\mathbf{h})]\,. \tag{9}$$

We numerically calculated the Pfaffian of a Hamiltonian with a single Hopf texture for small systems and confirmed it has a negative sign. In contrast, the trivial Hopf texture Hamiltonian has positive Pfaffian in the same basis.

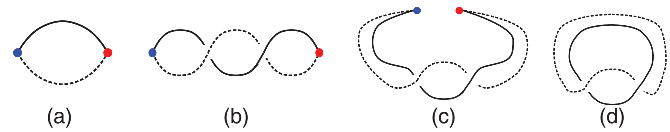

FIG. 4. (Color online) Creating a Hopf soliton by a hedgehog-antihedgehog pair. The blue (red) dot is the hedgehog (antihedgehog). The solid and dotted lines are pre-images of two different points on the 2-sphere. (a) Creating a pair of hedgehog and antihedgehog. (b) Rotating the hedgehog by 2π while leaving the antihedgehog invariant. (c), (d) Annihilating the hedgehog-antihedgehog pair. The final state in (d) clearly shows linking number 1 of the two pre-image loops, which indicates the nontrivial Hopf index.

B. Connections to 3D non-Abelian statistics

It is well known that vortices piercing a superconductor on the topological insulator surface carry Majorana zero modes in their cores.[6,12] In the $\vec{n}$ vector representation of Eq. (3), this corresponds to a hedgehog defect,[12] a singular configuration where the vector points radially outwards from the center. Although in this paper we only work with smooth textures, we discuss below an indirect connection with those works. Note that we can go from a trivial texture to the Hopf texture by creating a hedgehog-antihedgehog pair, rotating one of them by an angle of 2π and annihilating them to recover a smooth texture. This is just the Hopf texture, as can be seen in Fig. 4. However, as pointed out in Ref. 12, in the process of rotation, the Majorana mode changes sign. This signals a change in fermion parity, consistent with our results.

III. A TWO-DIMENSIONAL ANALOG

We briefly mention a two-dimensional analog of the physics described earlier, since it provides a simpler setting to discuss the relevant ideas. Readers interested in the 3D results alone can skip this section.

Note that Eq. (3) with the third component of momentum absent (i.e., $p_z = 0$) describes a quantum spin Hall (QSH) insulator (trivial insulator) when $m > 0$ $(m < 0)$ in the presence of singlet pairing Δ. Again, as before a three-vector characterizes a fully gapped state, and the nontrivial textures are called skyrmions $[\Pi_2(S_2) = Z]$. A unit skyrmion can be realized with a disk of quantum spin Hall insulator with superconductivity on the edge, whose phase winds by 2π on circling the disk. Again, one can show that the skyrmion charge Q determines the fermion parity $(-1)^Q = (-1)^F$. An important distinction from the three-dimensional case is that the low-energy theory here has a conserved charge. If, instead of superconductivity, one gapped out the edge states with a time-reversal-symmetry-breaking perturbation which had a winding, then this charge is the electrical charge. It is readily shown that the charge is locked to the skyrmion charge Q. Hence odd strength skyrmions are fermions. The effective field theory for the $\vec{n}$ vector in this case includes a Hopf term,[15,16] which ensures fermionic skyrmions. This is closely analogous to the quantum Hall ferromagnet, where charged skyrmions also occur.[9,10] Returning to the case with pairing, since that occurs on a one-dimensional edge, it is difficult to draw a clear-cut separation between fermions and collective bosonic coordinates, in contrast to the higher-dimensional version. Hence we focused on the 3D TIs.

A. Topological term for D = 2 + 1 quantum spin Hall case

As a warmup, consider the simpler case of $D = 2 + 1$, briefly discussed above. Here, we start with the Hamiltonian in Eq. (3), but drop the term $p_z\alpha_z$ since we are in $D = 2 + 1$. This describes a quantum spin Hall insulator with singlet superconductivity induced near the edge. The order parameter is still a three-vector, whose smooth textures in this case are skyrmions $[\Pi_2(S_2) = Z]$, and the skyrmion density is given by $\frac{1}{4\pi}\hat{n}\cdot\partial_x\hat{n}\times\partial_y\hat{n}$. On integrating out the fermions one obtains a topological term. Assuming $\hat{n}(\infty) = \text{const.}$, the topological term depends on $\Pi_3(S_2) = Z$, which is nothing but H, the Hopf index of the spacetime configuration of $\hat{n}$. We will show below that $S_{\text{top}} = i\pi H[\hat{n}(x,y,t)]$. Although there is no simple way to express the Hopf invariant directly in terms of the vector field, its physical effect is well known[15]—it modifies the statistics of the skyrmions so that they are now fermionic.

B. Calculating S_{topo} in D = 2 + 1

Although charge is not conserved in this mean-field Hamiltonian, the minimal Dirac model has a U(1) conservation—which can be seen from the fact that the operator $i\psi^\dagger\alpha_z\beta_0\psi$ commutes with Eq. (3). This introduces a simplicity not present on going to $D = 3 + 1$. In particular, the $D = 2 + 1$ problem can be mapped onto the one studied in Ref. 16. It is convenient to introduce the CP^1 representation of the unit vector $\hat{n}$. Introduce a pair of complex fields $z^T = (z_1 z_2)$ that satisfy $z^\dagger z = 1$. Then $\hat{n}\cdot\vec{\sigma} = (2zz^\dagger - 1)$, where σ are the Pauli matrices. In order to calculate the topological term, we would like to expand the order parameter space so as to smoothly go between the different topological configurations. In this way we can calculate the topological term perturbatively. A suitable expansion of the order parameter space from CP^1 to CP^M, $M > 1$, was suggested in Ref. 16. Here, the complex fields $z = (z_1\, z_2\, \ldots\, z_{M+1})$. Since $\Pi_3(CP^M) = 0$ for $M > 1$, different topological sectors can be connected. Following this procedure the topological term can be expressed in terms of the vector potential $a_\mu = \frac{i}{2}z^\dagger\partial_\mu z$:

$$S_{\text{top}} = i\pi H\,[\hat{n}(x,y,t)]\,, \tag{10}$$

$$H = \frac{1}{4\pi^2}\int d^2x\,dt\,\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma, \tag{11}$$

as also noted in Ref. 15.

IV. EFFECTIVE THEORY AND TOPOLOGICAL TERM

We now present a field theoretical calculation of the results of the previous section. The gap to the ψ fermions never vanishes since $|\vec{n}| > 0$, so one can integrate them out to obtain a low-energy theory written solely in terms of the bosonic order parameter $\vec{n}$. How can this field theory describe a fermionic texture? As described below, this is accomplished by a topological Berry phase term which appears in the effective action for the $\hat{n}$ field.

In computing the topological term, it is sufficient to consider a gap whose magnitude is constant $\vec{n}(r,t) = M\hat{n}(r,t)$. Integrating out the fermion fields with action $S_f = \int d^4x[\psi^\dagger\partial_t\psi - H_f]$, where the integral is over space and (Euclidean) time, one obtains the effective action for the bosonic fields:

$$\exp[-S_B(\hat{n}(r,t))] = \int \mathcal{D}\psi\mathcal{D}\psi^\dagger\exp(-S_f[\hat{n},\psi,\psi^\dagger]). \tag{12}$$

This computation may be performed using a gradient expansion (i.e., assuming slow variation of the $\hat{n}$ field over a scale set by the gap). Two terms are obtained $S_b = S_0 + S_{\text{top}}$. The first[17] is a regular term that penalizes spatial variation: $S_0 = \frac{1}{2g}\int(\partial_\mu\hat{n})^2$. The second is a topological term which assigns a different amplitude to topologically distinct spacetime configurations of $\hat{n}$. We first discuss the structure of this topological term and its physical consequences, before describing a calculation to compute it in $D = 3 + 1$ dimensions.

A. Topological term in D = 3 + 1 dimensions

Assuming $\hat{n}(\infty) = \text{const.}$, it is known that the spacetime configurations of the unit vector $\hat{n}$ are characterized by a Z_2 distinction since $\Pi_4(S_2) = Z_2$.[18] That is, there are two classes of maps: the trivial map, which essentially corresponds to the uniform configuration, and a nontrivial class of maps, which can all be smoothly related to a single representative configuration $\hat{n}_1(r,t)$. If the function $\Gamma[\{\hat{n}(r,t)\}] = 0, 1$ measures the topological class of a spacetime configuration, then the general form of the topological term is $S_{\text{top}} = i\theta\Gamma[\{\hat{n}(r,t)\}]$. The topological angle θ can be argued to take on only two possible values 0 and π since composing a pair of nontrivial maps leads to the trivial map. Via an explicit calculation outlined below, we find $\theta = \pi$.

Let us first examine the consequences of such a term. The nontrivial texture $\hat{n}_1(r,t)$ can be described as a Hopfion-antiHopfion pair being created at time t_1, the Hopfion being rotated slowly by 2π and then being combined back with the antiHopfion at a later time t_2.[15] The topological term assigns a phase of $e^{i\pi}$ to this configuration. This is equivalent to saying the Hopfion is a fermion since it changes sign on a 2π rotation. An intuitive way to understand this is that exchange can be accomplished by a π rotation about the midpoint connecting a pair of identical particles, which is equivalent to a 2π rotation of a single particle, or by invoking the spin-statistics connection. A more pictorial proof for the $D = 2 + 1$ analog, which can be readily generalized to $D = 3 + 1$, is in Ref. 15.

Calculating the topological term requires connecting the pair of topologically distinct configurations. To do this in a smooth way keeping the gap open at all times requires enlarging the order parameter space for this purpose. If $\mathbf{m}(r,t,\lambda)$ is an element of this enlarged state that smoothly interpolates between the trivial configuration $\mathbf{m}(r,t,0) = \text{const.}$ and the nontrivial one $\mathbf{m}(r,t,1) = \hat{n}_1(r,t)$ as we vary λ, then one can analytically calculate the change in the topological term $\partial S_{\text{top}}/\partial\lambda$ and integrate it to get the required result.[16,19–21] The key technical point is finding a suitable enlargement of our order parameter space S_2. Remembering that this can be considered as $S_2 = SU(2)/SO(2)$, we can make a natural generalization $M_3 = SU(3)/SO(3)$. The latter has all the desirable properties of an expanded space (e.g,. there are no nontrivial spacetime configurations), so everything can be smoothly connected [$\Pi_4(M_3) = 0$]. This extension allows us to calculate the topological term (as explained in detail in the section below), which yields $\theta = \pi$.

V. CALCULATION OF TOPOLOGICAL TERM

In this section we discuss how the topological term is calculated. This is the most technical part of the present paper—and the results have already been stated. Therefore, readers interested mainly in physical consequences can skip to the next section.

The calculation of the topological term in $D = 2 + 1$ is relatively straightforward, and we can utilize existing results in the literature.[16] However, the same procedure unfortunately does not work in $D = 3 + 1$, which is technically harder and requires some innovation. We previously discussed the $D = 2 + 1$ dimensional case and utilize some of the general lessons learned there to attack the $D = 3 + 1$ problem. More details can be found in the Appendix.

Note that, although there is no simple way to express the Hopf invariant H directly in terms of the vector field, it may be expressed in terms of the vector potential a as above. Thus, in $D = 2 + 1$, the Hopf index is directly connected to the topological term. However, in $D = 3 + 1$, the Hopf index characterizes the soliton, and we will need to look at a higher-dimensional homotopy group to pin down the topological term. In that case the topological term does not have a simple expression even in terms of the vector potentials a. Also, due to the different form of the Dirac theory, we will not be able to utilize the extension to CP^M to calculate the topological term, and a new construction will be needed.

A. Calculating S_{topo} in $D = 3 + 1$

We begin by rewriting Eq. (3) in terms of an eight-component Majorana fermion field Ψ:

$$H_f = \mathbf{\Psi}^\dagger v_F(-i\partial_i\alpha_i)\mathbf{\Psi} + M\mathbf{\Psi}^\dagger(n_1\beta_0 + n_2\beta_5\eta_x + n_3\beta_5\eta_z)\mathbf{\Psi}, \tag{13}$$

where the αs are symmetric and the βs are antisymmetric matrices, and $\Psi^\dagger = \Psi^T$. For example, we can build them out of three Pauli matrices as below [and as shown in Eq. (3)]:

$$\alpha_1=\sigma_z,\quad \alpha_2=\tau_x\sigma_x,\quad \alpha_3=-\tau_z\sigma_x,\quad \beta_0 = \tau_y\sigma_x,\quad \beta_5 = \sigma_y. \tag{14}$$

Note that, if we think of σ_a as referring to spin indices, then indeed α_i are odd under time reversal (making the Dirac kinetic energy even under time reversal). The β_0 mass term is also time-reversal even, as required for a topological insulator. Finally, the pairing term $\Psi^\dagger\beta_5\Psi^\dagger$ is clearly a spin singlet with the choice above.

We further assume the order parameters n_i are restricted to unit 2-sphere: $\sum_i n_i^2 = 1$ so that $\hat{n} = (n_1,n_2,n_3)$ is a unit vector living on S^2.

We need to show that, starting from this fermionic model Eq. (13) and integrating out the fermions, the obtained S^2 nonlinear sigma model (NLSM) has an imaginary term (topological Berry phase) $i\theta H_{\pi_4(S^2)}(\hat{n}(x,y,z,t))$ with $\theta = \pi$ in the action (from now on we use $H_{\pi_n(M)}$ to denote the homotopy index of a mapping). Because this term is nonperturbative, in order to compute it we need to embed the manifold S^2 into a larger manifold M with $\pi_4(M) = 0$, which allows us to smoothly deform a $H_{\pi_4(S^2)}(\hat{n}(x,y,z,t)) = 1$ mapping to a constant mapping. This means that an $H_{\pi_4(S^2)}(\hat{n}(x,y,z,t)) = 1$ mapping can be smoothly extended over the 5-dimensional disk: $V(x,y,z,t,\rho) : D^5 \to M$ ($\rho \in [0,1]$) such that, on the boundary, $V(x,y,z,t,\rho = 1) = \hat{n}(x,y,z,t)$ and $V(x,y,z,t,\rho = 0) = V_0$ is constant. With an extension V, we can perturbatively keep track of the total change of Berry's phase when going from a constant mapping to a nontrivial mapping.

How can we find a suitable M? We note the global symmetry of the model Eq. (13) in the massless limit is $U(1)_{\mathrm{chiral}} \times SU(2)_{\mathrm{isospin}}$, whose generators are

$$U(1)_{\mathrm{chiral}} : \gamma_5= -i\beta_0\beta_5,\quad SU(2)_{\mathrm{isospin}} : \eta_y,\gamma_5\{\eta_x,\eta_z\}. \tag{15}$$

In our convention, β_0,β_5,γ_5 are all antisymmetric matrices. Starting from a given mass (for instance, $\mathbf{\Psi}^\dagger\beta_0\mathbf{\Psi}$), one can generate the full order parameter manifold by action of $SU(2)_{\mathrm{isospin}}$: $\mathbf{\Psi}^\dagger W\beta_0 W^\dagger\mathbf{\Psi}$—where $W \in SU(2)_{\mathrm{isospin}}$. $SU(2)_{\mathrm{isospin}}$ is broken down to U(1), which is the invariant group generated by η_y. Thus, the order parameter space is $SU(2)/U(1) = SU(2)/SO(2) = S^2$.

Now we generalize the 8-component fermion $\mathbf{\Psi}$ to the 12-component $\tilde{\mathbf{\Psi}}$. The two-dimensional $\eta_{x,y,z}$ space is enlarged to a three-dimensional space, and we let the eight λ_i matrices ($i = 1,2,\ldots,8$) of the standard SU(3) generators (see, e.g., page 61 in Ref. 22) act within this three-dimensional space, among which $\lambda_2,\lambda_5,\lambda_7$ are antisymmetric while others are symmetric. And $\lambda_{1,2,3}$ are just the old $\eta_{x,y,z}$ matrices. The symmetry of the generalized massless theory of $\tilde{\mathbf{\Psi}}$: $H = \tilde{\mathbf{\Psi}}^\dagger(-i\partial_i\alpha_i)\tilde{\mathbf{\Psi}}$ is $U(1)_{\mathrm{chiral}} \times SU(3)_{\mathrm{isospin}}$, where the generators of the $SU(3)_{\mathrm{isospin}}$ are $\lambda_2,\lambda_5,\lambda_7,\gamma_5\{\lambda_1,\lambda_3,\lambda_4,\lambda_6,\lambda_8\}$.

Starting from a given mass $\tilde{\mathbf{\Psi}}^\dagger\beta_0\tilde{\mathbf{\Psi}}$, we use $SU(3)_{\mathrm{isospin}}$ to generate the order parameter manifold: $\tilde{\mathbf{\Psi}}^\dagger U\beta_0 U^\dagger\tilde{\mathbf{\Psi}} \equiv \tilde{\mathbf{\Psi}}^\dagger V\tilde{\mathbf{\Psi}}$, $U \in SU(3)_{\mathrm{isospin}}$. It is clear that the SO(3) subgroup generated by $\lambda_2,\lambda_5,\lambda_7$ is the invariant group and the order parameter manifold is $M_3 \equiv SU(3)/SO(3)$. We thus embed the original order parameter manifold S^2 into M_3, and it is known that $\pi_4(M_3) = 0$.[23]

The idea is to smoothly extend an $H_{\pi_4(S^2)} = 1$ mapping over D^5 [denoted by $V(x,y,z,t,\rho)$] by embedding S^2 into M_3. In fact, if we can extend an $H_{\pi_4}(SU(2)) = 1$ mapping over D^5 [denoted by $U(x,y,z,t,\rho)$] by embedding $SU(2)_{\mathrm{isospin}}$ into $SU(3)_{\mathrm{isospin}}$, it will generate the extension V by $V = U\beta_0U^\dagger$. This is because an $H_{\pi_4(S^2)} = 1$ mapping can be thought of as a combination of an $H_{\pi_4}(SU(2)) = 1$ mapping and an Hopf $H_{\pi_3(S^2)} = 1$ mapping. Such an extension $U : D^5 \to SU(3)_{\mathrm{isospin}}$ has already been explicitly given by Witten (see Eqs. (9)–(13) in Ref. 24), which we will term Witten's map. Witten's map was introduced to compute an $i\theta H_{\pi_4}(SU(2))$ topological Berry phase. Basically, on the boundary $\partial D^5 = S^4$ ($\rho = 1$), Witten's map is an $H_{\pi_4}(SU(2)) = 1$ mapping defined as rotating an $H_{\pi_3}(SU(2)) = 1$ soliton (by 2π) along the time direction. This $H_{\pi_4}(SU(2)) = 1$ mapping at $\rho = 1$ is smoothly deformed into a trivial mapping at $\rho = 0$ by embedding SU(2) into SU(3).

We use Witten's map to generate V, with which we compute the Berry's phase perturbatively by a large-mass expansion[14,21] of the Lagrangian

$$L = \tilde{\Psi}^\dagger[\partial_\tau + i\partial_i\alpha_i]\tilde{\Psi} + M\tilde{\Psi}^\dagger V\tilde{\Psi} \equiv \tilde{\Psi}^\dagger[D]\tilde{\Psi}, \quad (16)$$

where $D = [\partial_\tau + i\partial_i\alpha_i + MV]$ and the partition function is $Z = \int \mathcal{D}\tilde{\Psi}^\dagger\mathcal{D}\tilde{\Psi}\mathcal{D}V e^{-\int d^4x L}$. After integrating out the fermions, we obtain an NLSM of V: $\tilde{L} = -\frac{1}{2}\mathrm{Tr}\ln[\partial_\tau + i\partial_i\alpha_i + MV]$. Here the factor 1/2 is because we are integrating out Majorana fields. If there is a variation δV, the variation of the imaginary part $\Gamma \equiv \mathrm{Im}(\tilde{L})$ is

$$\delta\Gamma = \frac{-K}{2}\int dx^4 \epsilon^{\alpha\beta\gamma\mu}\mathrm{Tr}\{\gamma_5 V\partial_\alpha V\partial_\beta V\partial_\gamma V\partial_\mu V\delta V\}, \quad (17)$$

where $K = \int \frac{d^4p}{(2\pi)^4}\frac{M^6}{(p^2+M^2)^5} = \frac{1}{192\pi^2}$. Denoting $\partial_\rho = \partial_4$ and after some algebra, the Berry phase can be written in the fully antisymmetric way

$$\Gamma = \frac{-K}{10}\int dx^5 \epsilon^{\alpha\beta\gamma\mu\nu}\mathrm{Tr}\{\gamma_5 V\partial_\alpha V\partial_\beta V\partial_\gamma V\partial_\mu V\partial_\nu V\}. \quad (18)$$

In fact Γ is not fully well defined because the ambiguity of the extension of $\hat{n}(x,y,z,t)$ to the 5-disk: two different extensions $V(x,y,z,t,\rho)$ can differ by a mapping $S^5 \to M_3$. We will soon show that this ambiguity means that Γ is well defined up to mod 2π.

Because V is generated by U, plugging $V = U\beta_0 U^\dagger$, $U \in \mathrm{SU(3)}_{\mathrm{isospin}}$ in Eq. (18), one can further simplify it by tracing out the $\beta_{0,5}$ space. First note that $\partial_\mu V = U[U^\dagger\partial_\mu U,\beta_0]U^\dagger$ and, because $U^\dagger\partial_\mu U$ is an element of the $\mathrm{SU(3)}_{\mathrm{isospin}}$ Lie algebra spanned by $\lambda_2,\lambda_5,\lambda_7,\gamma_5\{\lambda_1,\lambda_3,\lambda_4,\lambda_6,\lambda_8\}$, $[U^\dagger\partial_\mu U,\beta_0]$ simply picks out the latter five generators. After some algebra,[21] one finds

$$\Gamma = \frac{1}{15\pi^2}\int dx^5 \epsilon^{\alpha\beta\gamma\mu\nu}\mathrm{Tr}\{(g^{-1}\partial_\alpha g)_\perp(g^{-1}\partial_\beta g)_\perp \times (g^{-1}\partial_\gamma g)_\perp(g^{-1}\partial_\mu g)_\perp(g^{-1}\partial_\nu g)_\perp\}, \quad (19)$$

where g is defined as the corresponding 3×3 SU(3) matrix of U: if U is the exponential of $a_1\lambda_2 + a_2\lambda_5 + a_3\lambda_7 + a_4\gamma_5\lambda_1 + a_5\gamma_5\lambda_3 + a_6\gamma_5\lambda_4 + a_7\gamma_5\lambda_6 + a_8\gamma_5\lambda_8$, then g is the exponential of $a_1\lambda_2 + a_2\lambda_5 + a_3\lambda_7 + a_4\lambda_1 + a_5\lambda_3 + a_6\lambda_4 + a_7\lambda_6 + a_8\lambda_8$. g is nothing but the Witten map, and $(g^{-1}\partial_\mu g)_\perp$ denotes the symmetric part: $(g^{-1}\partial_\mu g)_\perp = [(g^{-1}\partial_\mu g) + (g^{-1}\partial_\mu g)^T]/2$.

We simply need to compute Γ by integration. Because it is clear that Γ can only be 0 or π (mod 2π), a numerical integration is enough to determine it unambiguously. We performed the integration of Eq. (19) with Witten's map g by a standard Monte Carlo approach and found

$$\Gamma = (1.000 \pm 0.005)\pi. \quad (20)$$

This proves that the Hopf-skyrmion is a fermion. In addition it confirms that Γ is well defined only up to mod 2π: different extensions of g can differ by a doubled Witten map, which is known to have $H_{\pi_5(\mathrm{SU(3)})} = 1$, and the above calculation indicates that this ambiguity only adds an integer times 2π in Γ.

VI. PHYSICAL CONSEQUENCES

We now discuss two kinds of physical consequences arising from fermionic Hopfions. The first relies on the dynamical nature of the superconducting order parameter, while the second utilizes the Josephson effect to isolate an anomalous response. The first class conceptually parallels experiments used to identify skyrmions in quantum Hall ferromagnets. There, when skyrmions are the cheapest charge excitations, they are detected upon adding electrons to the system.[11]

Consider surface superconductivity on a mesoscopic torus-shaped topological insulator as in Fig. 2. The Hopf texture corresponds to unit phase winding in each cycle of the torus. The energy cost, $E_H = (\rho_s/2)\int(\nabla\phi)^2 d^2x$, is simply proportional to the superfluid density $E_H = A\rho_s$, where A is in general an $O(1)$ constant. If we have $E_H < \Delta$, which is the superconducting gap, then the Hopfion is the lowest-energy fermionic excitation. Tunneling a single electron onto the surface should then spontaneously generate these phase windings in equilibrium. Measuring the corresponding currents ($\rho_s = 1$ K corresponds to a few nano-Ampères of current) can be used to establish the presence of the Hopf texture. A daunting aspect of this scheme is to obtain a fully gapped superconductor with $\rho_s < \Delta$.

A different approach which does not depend on the above inequality but relies on the Josephson effect is illustrated in Fig. 5. A hollow cylinder of topological insulators is partially coated with a superconductor on the top and bottom surfaces, forming a pair of Josephson junctions. The superconducting proximity effect induced through the surface states will ultimately gap out the entire surface. A unit vortex along the C_1 cycle can be induced by enforcing a phase difference of π between the top and bottom surfaces, using the flux $\Phi_1 = \varphi_0/2$ (where $\varphi_0 = h/2e$ is the superconductor flux quantum). This phase difference makes the order parameter wind by π on going from the bottom to the top surface. The lowest-energy

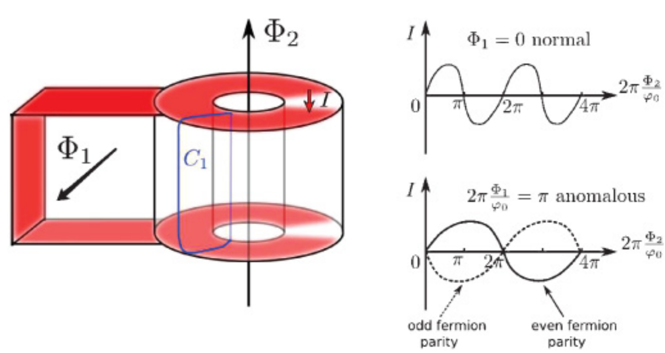

FIG. 5. (Color online) Anomalous Josephson response connected to fermionic Hopfions. An annular cylinder of TI with pairing induced on the top and bottom surfaces via proximity to a superconductor (present in the red regions) is shown. Tuning the flux to $\Phi_1 \approx \varphi_0/2$ induces a vortex in the C_1 cycle. Now, the Josephson effect on tuning Φ_2 will be anomalous, with a part that is not periodic in the flux φ_0. This is directly related to the change in the ground-state fermion parity at $\Phi_2 = 2\pi(\varphi_0/2\pi)$, where the Hopf texture is realized. Adding a fermion inverts this current, indicating that the ground state in this sector is at $\Phi_2 = 2\pi(\varphi_0/2\pi)$. If, however, $\Phi_1 \approx 0$, there is no vorticity in the C_1 cycle, and the Josephson effect is the usual one, which is periodic in the flux quantum $\varphi_0 = h/2e$.

phase configuration that accomplishes this is just the unit winding on going around the circumference, so that the phase is π at the half-way stage.

Now, the vorticity enclosed by the annulus determines the Hopf number and hence the ground state fermion parity. This vorticity can be traded for magnetic flux Φ_2 [parametrized via $f_2 = 2\pi(\Phi_2/\varphi_0)$] threading the cylinder, since only $\nabla\phi - eA$ is gauge invariant. Consider beginning in the ground state with $f_2 = 0$ and then tuning to $f_2 = 2\pi$. One is now in an excited state since the ground state at this point has odd fermion parity. This must be reflected in the Josephson current I. We argue this implies doubling the flux period of the Josephson current. Since $I = \partial E(\Phi_2)/\partial\Phi_2$, the area under the (I,Φ_2) curve, $\int_0^{\varphi_0} d\Phi_2 I[\Phi_2] = E[\varphi_0] - E[0] > 0$, is the excitation energy which does not vanish. Hence, the Josephson current is not periodic in flux φ_0, as in usual Josephson junctions. If we started with an odd fermion number to begin with, then the state of affairs would be reversed—the ground state would be achieved at multiples of $f_2 = 2\pi$. The ground state with a particular fermion parity can be located by studying the slope of the current vs phase curve. Since the energy of the ground state increases on making a phase twist $\partial I/\partial\Phi_2 = \partial^2 E/\partial\Phi_2^2 > 0$, it is associated with a positive I vs Φ_2 slope. This positive slope will be at even (odd) multiples of $f_2 = 2\pi$ for even (odd) fermion parity. If, on the other hand, unit vorticity was not induced in the cycle C_1 (e.g., if $\Phi_1 \sim 0$), there is no Hopf texture, and the Josephson relation would be the usual one (i.e., one that is periodic in $f_2 = 2\pi$). This is summarized in Fig. 5.

Note that a similar anomalous 4π Josephson periodicity was pointed out in the context of the 2D QSH case with proximate superconductivity in Ref. 25. We interpret this result in terms of the fermionic nature of the solitons there, which lends a unified perspective. In the 3D case, the Hopf texture allows one to tune between the normal and anomalous Josephson effect by tuning the C_1 vorticity via Φ_1.

VII. CONCLUSIONS

The low-energy field theory of the superconductor-TI system was derived and shown to possess a topological Berry phase term, which leads to fermionic Hopf solitons. We note that topological terms are particularly important in the presence of strong quantum fluctuations. In one dimension where fluctuations dominate, the Berry phase term of the spin 1/2 Heisenberg chain[26] leads to an algebraic phase. It would be very interesting to study the destruction of superconductivity on a TI surface driven by quantum fluctuations. The Berry phase or, relatedly, the fact that a conventional insulator must break time reversal on the TI surface will provide an interesting twist to the well-known superconductor-insulator transition studied on conventional substrates.[27]

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APPENDIX: LONG EXACT SEQUENCE IN HOMOTOPY THEORY

A very powerful theorem in algebraic topology is the long exact sequence: given a manifolds E and its submanifold $F \subset E$, the following sequence is exact:

$$\cdots \pi_{n+1}(E,F) \rightarrow \pi_n(F) \xrightarrow{f} \pi_n(E) \xrightarrow{g} \pi_n(E,F) \xrightarrow{h} \pi_{n-1}(F) \cdots. \tag{A1}$$

By "exact" it means that the kernel of one mapping is the same as the image of the previous mapping in the sequence. For example, $\ker(g) = \mathrm{image}(f)$. $\pi_n(F)$ and $\pi_n(E)$ are the n-dimensional homotopy groups of F and E, respectively. $\pi_n(E,F)$ is called the relative homotopy group, which is defined to be all different classes of mappings from an n-dimensional disk D^n to E which map the boundary of $\partial D^n = S^{n-1}$ into F. In particular, when the quotient manifold E/F can be defined, the relative homotopy group $\pi_n(E,F)$ is isomorphic to the homotopy group of the quotient manifold $\pi_n(E/F)$. In this case, the order of the manifolds showing up in each dimension is always the smaller manifold, the larger manifold, their quotient manifold.

Now we specify the definition of the mappings in the long exact sequence. Mapping f is the natural mapping between two homotopy groups induced by inclusion $x \in F \rightarrow x \in E$. Mapping g is also natural. For any mapping from S^n to E, one can choose an arbitrary point $P \in S^n$ and smoothly enlarge that point into a small n-disk U. Then $S^n - U$ is also an n-disk, and the mapping from $S^n - U$ is an element of $\pi_n(E,F)$. When the quotient manifold E/F can be defined, it is easy to see that a mapping in $\pi_n(E,F)$, after modding out the F, will induce a mapping from S^n to E/F. This natural mapping between $\pi_n(E,F)$ and $\pi_n(E/F)$ turns out to be an isomorphism. Finally, the mapping h is induced by restricting an element in $\pi_n(E,F)$ to its boundary S^{n-1}.

As a demonstration, we now use the long exact sequence to study Witten's extension:[24]

$$\begin{array}{ccccccccc}
\pi_5(SU(2)) & \longrightarrow & \pi_5(SU(3)) & \longrightarrow & \pi_5\left(\frac{SU(3)}{SU(2)} = S^5\right) & \longrightarrow & \pi_4(SU(2)) & \longrightarrow & \pi_4(SU(3)) \\
\| & & \| & & \| & & \| & & \| \\
Z_2 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & & & 2k+1 & \longrightarrow & 1 & \searrow & \\
 & & k & \longrightarrow & 2k & \longrightarrow & 0 & \longrightarrow & 0
\end{array} \tag{A2}$$

This means that if we extend an $S^4 \to \mathrm{SU}(2)$ mapping over the 5-disk by embedding SU(2) into SU(3), then the extended mapping $D^5 \to \mathrm{SU}(3)(\partial D^5 \to \mathrm{SU}(2))$ has odd index. In fact this index can be computed by the standard winding number of $\pi_5(S^5)$. Thus, if we glue different mappings $D^5 \to \mathrm{SU}(3)(\partial D^5 \to \mathrm{SU}(2))$ together this index simply adds up. If we glue two identical extensions to form a mapping $S^5 \to \mathrm{SU}(3)$, it must have a odd $\pi_5(\mathrm{SU}(3))$ index $H_{\pi_5(\mathrm{SU}(3))} = 2(2k+1)/2 = 2k+1$. Witten's explicit form of extension (see Eqs. (9)–(13) in Ref. 24) is the element 1 in $\pi_5(\mathrm{SU}(3)/\mathrm{SU}(2))$, and the doubled Witten extension is element $1 \in \pi_5(\mathrm{SU}(3))$.

Finally, we apply the long exact sequence to study the property of $\pi_5(\mathrm{SU}(3)/\mathrm{SO}(3))$, relevant to the calculation of Hopfion statistics. It is known that $\pi_5(\mathrm{SU}(3)/\mathrm{SO}(3)) = Z \oplus Z_2$.

$$
\begin{array}{ccccccccc}
\pi_5(SO(3)) & \longrightarrow & \pi_5(SU(3)) & \longrightarrow & \pi_5\left(\frac{SU(3)}{SO(3)}\right) & \longrightarrow & \pi_4(SO(3)) & \longrightarrow & \pi_4(SU(3)) \\
\| & & \| & & \| & & \| & & \| \\
Z_2 & \longrightarrow & Z & \longrightarrow & Z \oplus Z_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & & & & & & & \\
 & & & & (k,1) & \longrightarrow & 1 & \searrow & \\
 & & k & \longrightarrow & (k,0) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\qquad \text{(A3)}
$$

This means that the Z part of $\pi_5(\mathrm{SU}(3)/\mathrm{SO}(3))$ is completely given by the Z index of $\pi_5(\mathrm{SU}(3))$. For example, if we take a fundamental element of $\pi_5(\mathrm{SU}(3))$ ($H_{\pi_5(\mathrm{SU}(3))} = 1$), for example, a doubled Witten extension, after modding out the SO(3) subgroup, it will be a fundamental element of the Z part of $\pi_5(\mathrm{SU}(3)/\mathrm{SO}(3))$ [$H_{\pi_5(\mathrm{SU}(3)/\mathrm{SO}(3))} = (1,0)$]. Because Berry's Γ for a single Witten extension is found to be π by numerical integration of Eq. (19), we prove that the integration for a doubled Witten extension is 2π. Thus, we prove that Eq. (19), when integrated over S^5, is exactly 2π times the Z index of $\pi_5(\mathrm{SU}(3)/\mathrm{SO}(3)) = Z \oplus Z_2$.

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