



Scaling of the anomalous Hall effect in the insulating regime

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We develop the theory of the scaling of the anomalous Hall effect (AHE) in the insulating regime, which is observed in experiments to relate the anomalous Hall and diagonal conductivities by $\sigma_{xy}^{\text{AH}} \propto \sigma_{xx}^{1.40 \sim 1.75}$ for a large range of materials. This scaling is qualitatively different from that observed in metals. Basing our theory on the phonon-assisted hopping mechanism and percolation theory in random networks, we derive a general formula for the anomalous Hall conductivity, in which the percolation theory averaging of the random-linked triad clusters is a key aspect that captures the correct observed physics. We show that it scales with the longitudinal conductivity as $\sigma_{xy}^{\text{AH}} \sim \sigma_{xx}^{\gamma}$ with γ predicted to be $1.33 \leq \gamma \leq 1.76$, quantitatively in agreement with the experimental observations. Our theory predicts that this scaling remains similar regardless of whether the hopping process is of long-range (variable-range hopping) or short-range type (activation hopping), or is influenced by interactions, i.e., the Efros-Shklovskii regime. Our theory completes the understanding of the AHE phase diagram in the insulating regime.

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I. INTRODUCTION

The anomalous Hall effect (AHE) is a central topic in the study of ferromagnetic materials.¹ It exhibits the empirical relation $\rho_{xy} = R_0 B_z + R_S M_z$ between the total Hall resistivity and the magnetization M_z and external magnetic field B_z . Here, R_0 and R_S are, respectively, the ordinary and anomalous Hall coefficients. In experiment, the anomalous Hall resistivity (AHR) is usually observed to follow a power-law form versus the longitudinal resistivity $\rho_{xy}^{\text{AH}} \sim \rho_{xx}^{\beta}$, with ρ_{xx} varied by changing the temperature T , disorder scattering, or density of states (DOS) around the Fermi surface. When transformed to the anomalous Hall conductivity (AHC) σ_{xy}^{AH} , the scaling relation takes the form $\sigma_{xy}^{\text{AH}} \approx \rho_{xy}^{\text{AH}} / \sigma_{xx}^2 \sim \sigma_{xx}^{2-\beta}$. Three regimes are observed with respect to their dependence on the diagonal conductivity σ_{xx} .¹ In the metallic regime, the AHC σ_{xy}^{AH} is observed to be linearly proportional to σ_{xx} for the highest metallic systems ($\sigma_{xx} > 10^6 \Omega^{-1} \text{cm}^{-1}$) and roughly constant for the rest of the metallic regime. This dependence indicates the different dominant mechanisms in ferromagnetic metals. These are understood to be the skew scattering, side jump scattering, and intrinsic deflection mechanisms. The intrinsic contribution is induced by a momentum-space Berry phase linked to the electronic structure of the multiband spin-orbit (SO) coupled system.^{1,2} The side jump scattering mechanism gives the same scaling relation as the intrinsic contribution, i.e., $\sigma_{xy}^{\text{AH-sj}} \propto \sigma_{xx}^0$, and the skew scattering is linear in the longitudinal conductivity $\sigma_{xy}^{\text{AH-sk}} \propto \sigma_{xx}$. While these mechanisms are now better understood, the maximum scaling exponent of the AHC can not exceed unity in the metallic regime.¹

On the other hand, experiments in the insulating regime exhibit an unexpected scaling relation of the AHC: $\sigma_{xy}^{\text{AH}} \propto \sigma_{xx}^{1.40 \sim 1.75}$ with the scaling exponent generically larger than unity.³⁻¹⁴ Earlier experiments on AHE in this regime were done in magnetite Fe_3O_4 ,³ and the recent experimental observations of this scaling are reported in a large range of materials including granular Fe/SiO_2 films, magnetite epitaxial thin

films, dilute magnetic semiconductor (DMS) $\text{Ga}_{1-x}\text{Mn}_x\text{As}$, and ferromagnetic semiconductor anatase $\text{Ti}_{1-x}\text{Co}_x\text{O}_{2-\delta}$. The observed scaling in the insulating regime has remained unexplained and a major challenge in understanding fully the phase diagram of the AHE.

The hopping transport regime prevails when a system is in the disordered insulating regime, with the impurity on-site energies randomly distributed. At low temperature ($T \ll E_0/k_B$ with E_0 the ionization energy of the bound states), the charge transport in such a system will be dominated by the phonon-assisted hopping of electrons and holes between impurity sites.^{16,17}

To capture the Hall effect, one requires the hopping process between impurity sites (Fig. 1) to break the time-reversal (TR) symmetry. The two-site direct hopping preserves TR symmetry, and contributes only to the longitudinal charge transport. The hopping through triads (three sites) is the minimum requirement to model theoretically the Hall effect.¹⁸ The total hopping amplitude is obtained by adding the direct and indirect (through the intermediate k site) hopping terms from i to j sites. The two hopping paths give rise to an interference term for the transition rate that breaks TR symmetry and is responsible for the Hall current in the hopping regime. For the ordinary Hall effect (OHE), the interference is a reflection of the Aharonov-Bohm phase, and for the AHE, it reflects the Berry phase due to SO coupling.

While the hopping through triads reveals the minimum element contributing to the AHE in the hopping conduction regime, *the crucial step to understand the insulating AHE and the observed scaling relation is hidden in the evaluation of the AHC within percolation theory.* In the hopping conduction regime, the charge transport is *not* dominated by the whole impurity system but by specific percolation clusters, which span the whole material, but cover only part of the impurity sites. To correctly evaluate the AHC, one must address a highly nontrivial issue: *how to average the AHC over the percolation clusters with triads.*

The few previous studies of the AHE in this regime have been focused on manganites and $\text{Ga}_{1-x}\text{Mn}_x\text{As}$ by employing

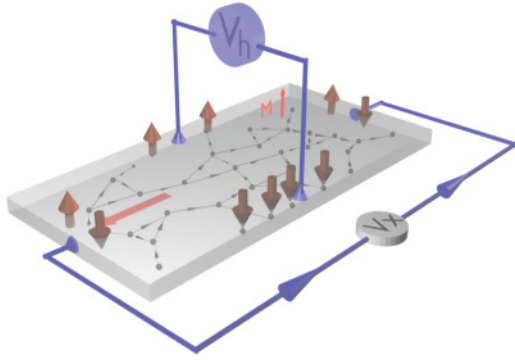


FIG. 1. (Color online) AHE in the insulating regime. In this regime, charge transport occurs via hopping between impurity sites.

Holstein's theory in the anomalous Hall system. In manganites, a nonuniform magnetic system, the AHC is determined only by optimal triads and, therefore, this material does not exhibit the scaling observed in the typical disordered insulators.^{19,20} On the other hand, the studies on insulating $\text{Ga}_{1-x}\text{Mn}_x\text{As}$ did not appreciate the central aspect of the hopping conduction that the AHC should be averaged over percolation cluster, and thus failed in explaining the observed scaling.^{21,22} A numerical study of the AHE using metallic theory observed a scaling in a disordered metallic regime, but both the AHC and σ_{xx} remained metallic in this study, while the scaling was only present for a particular sign of the impurities.^{1,15} The AHE theory in the metallic regime is generally based on the perturbation expansion in terms of small parameter $1/(k_F l)$. Here, k_F is the magnitude of the Fermi wave vector and l is the length of mean-free path. Hence, the available microscopic theories of metals fail in the insulating regime since the condition $k_F l \gg 1$ is no longer satisfied for disordered insulators by its own construction.^{1,15}

In this paper, we develop a theoretical approach to study the scaling of the AHE in the insulating strongly disordered amorphous regime. The AHC in this regime is correctly formulated and exactly calculated by averaging over the percolation cluster. In light of the fact that the charge transport is dominated by the percolation cluster, we derive rigorously a new configuration averaging formula for the AHC, with the key physics that the Hall currents are averaged over percolation cluster containing triads completely considered. With our formalism, we calculate the upper and lower limits of the AHC, which correspond to different extreme situations for the triad spatial distribution, and show that they scale with $\sigma_{xy}^{\text{AH}} \sim \sigma_{xx}^\gamma$, where γ is predicted to be $1.33 \leq \gamma \leq 1.76$ with only a slightly quantitative dependence on the specific hopping types. Namely, the scaling remains similar in the Mott variable-range hopping (VRH), Efros-Shkolovskii (E-S) regime, and in activation E_3 hopping regime. This matches the experimental observation that the scaling is seen in many types of insulators with different hopping types dominating.

II. THE MODEL

Our theory is based on a minimal tight-binding Hamiltonian. With the particle-phonon coupling considered, the total

Hamiltonian $H = H_p + H_c + H_{ph}$, with

$$H_p = \sum_{i\alpha} \epsilon_i \hat{c}_{i\alpha}^\dagger \hat{c}_{i\alpha} - \sum_{i\alpha, j\beta} t_{i\alpha, j\beta} \hat{c}_{i\alpha}^\dagger \hat{c}_{j\beta} + \sum_{i\alpha\beta} \mathbf{M} \cdot \tau_{\alpha\beta} \hat{c}_{i\alpha}^\dagger \hat{c}_{i\beta},$$

$$H_c = i\eta \sum_{i\alpha\lambda} (\vec{q}_\lambda \cdot \vec{e}_\lambda) \omega_\lambda^{-1/2} (b_\lambda e^{i\vec{q}_\lambda \cdot \vec{r}} - b_\lambda^\dagger e^{-i\vec{q}_\lambda \cdot \vec{r}}) \hat{c}_{i\alpha}^\dagger \hat{c}_{i\alpha},$$

$$H_{ph} = \sum_\lambda \omega_\lambda b_\lambda^\dagger b_\lambda.$$

Here, H_p describes localized states, H_c gives the particle-phonon coupling with η the coupling constant, H_{ph} is the phonon Hamiltonian, α is the local on-site total angular momentum index, and ϵ_i is the energy measured from the Fermi level. Here, we consider that the magnetization is saturated and thus assume $\mathbf{M} = M\hat{e}_z$. We rewrite the Hamiltonian H_p in the diagonal basis of the exchange term and obtain

$$H_p = \sum_\alpha \epsilon_{i\alpha} \hat{c}_{i\alpha}^\dagger \hat{c}_{i\alpha} - \sum_{i\alpha, j\beta} t_{i\alpha, j\beta} \hat{c}_{i\alpha}^\dagger \hat{c}_{j\beta}, \quad (1)$$

where $\epsilon_{i\alpha} = \epsilon_i + M\tau_{\alpha\alpha}$. The hopping matrix t_{ij} is generally off diagonal due to SO coupling. For example, for the dilute $\text{Ga}_{1-x}\text{Mn}_x\text{As}$, the matrix $t_{i\alpha, j\beta}$ describes the hopping of the holes localized on the Mn impurities. Under the spherical approximation, $t_{i\alpha, j\beta}$ can be obtained based on a unitary rotation $U(\mathbf{R}_{ij})$ from the \hat{e}_z direction to the hopping direction $i \rightarrow j$.²³ We thus have $t_{i\alpha, j\beta} = [U^\dagger(\mathbf{R}_{ij}) t_{\text{diag}} U(\mathbf{R}_{ij})]_{\alpha\beta}$ with $t_{\text{diag}} = \text{diag}[t_{3/2}, t_{1/2}, t_{-1/2}, t_{-3/2}]$ representing the situation that the hopping direction is along the z axis. Another example is for the localized s -orbital electrons. In this case, the hopping matrix is given by $t_{ij} = U^\dagger(\mathbf{R}_{ij}) [\tilde{t}_{ij} (1 + i\vec{v}_{ij} \cdot \vec{\sigma})] U(\mathbf{R}_{ij})$.²⁰ Here, $\tilde{t}_{ij} = \text{diag}[t_{1/2}, t_{-1/2}]$ and $\vec{v}_{ij} = \frac{\alpha}{\hbar} \int_{\mathbf{r}_i}^{\mathbf{r}_j} [\nabla V(\mathbf{r}) \times d\vec{r}']$ with $V(\mathbf{r})$ including the ion and external potentials, the SO coupling coefficient $\alpha = \hbar/(4m^2 c^2)$, and m the effective mass of the electron. The localization regime has the condition $|t_{i\alpha, j\beta}| \ll |\epsilon_i - \epsilon_j|$ in average. The specific forms of the relevant parameters (t_{ij} , M , spin operator $\tau_{\alpha\beta}$) are material dependent and do not affect the scaling relation between σ_{xy}^{AH} and σ_{xx} .

Considering the dominant contributions to the longitudinal and Hall transports, which include the one- and two-phonon processes through triads (Fig. 2),¹⁸ we obtain the

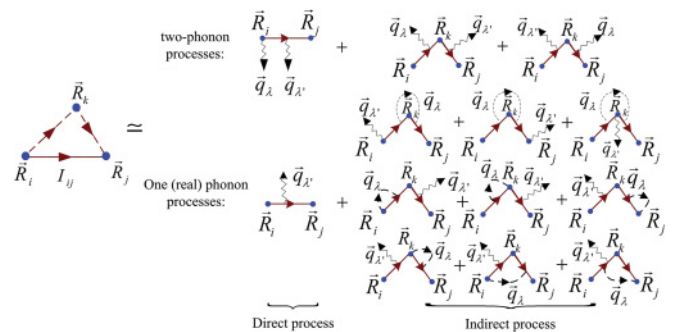


FIG. 2. (Color online) The hopping processes through triads with up to two real phonons absorbed or emitted. (Top) Typical diagrams of the two-phonon direct and indirect hopping processes. (Bottom) One-phonon direct process and typical three-phonon (one real phonon) indirect hopping processes.

charge current between i and j sites in a single triad with applied voltages²¹:

$$I_{ij} = G_{ij}V_{ij} + \mathcal{G}_{ijk}(V_{ik} + V_{jk}), \quad (2)$$

where the direct conductance

$$G_{ij} = \frac{2\lambda_0 e^2}{k_B T} \sum_{\alpha\beta} |t_{i\alpha,j\beta}|^2 T_{ij}^{(2)}, \quad (3)$$

with $T_{ij}^{(2)} = |\Delta_{ij}| e^{-\frac{1}{2}\beta(|\epsilon_{i\alpha}| + |\epsilon_{j\beta}| + |\epsilon_{i\alpha} - \epsilon_{j\beta}|)}$, $\Delta_{ij} = \epsilon_{i\alpha} - \epsilon_{j\beta}$, $\epsilon_{i\alpha} = \epsilon_i + M\tau_{\alpha\alpha}$, and the constant $\lambda_0 \propto \eta^2$.¹⁸ The second term in I_{ij} is responsible for the Hall transport and

$$\mathcal{G}_{ijk} = \frac{4\lambda_0^2 e^2}{k_B T} \sum_{\alpha\beta\gamma} [\text{Im}(t_{i\alpha,j\beta} t_{j\beta,k\gamma} t_{k\gamma,i\alpha}) T_{ijk}^{(3)}], \quad (4)$$

where $T_{ijk}^{(3)} = T_{i,jk}^{(3)} + T_{k,ij}^{(3)} + T_{j,ki}^{(3)}$, with $T_{i,jk}^{(3)} = |\Delta_{ij}\Delta_{ik}| e^{-\frac{1}{2}\beta(|\epsilon_{j\beta}| + |\epsilon_{k\gamma}| + |\epsilon_{i\alpha} - \epsilon_{k\gamma}| + |\epsilon_{i\alpha} - \epsilon_{j\beta}|)}$. The function $\text{Im}(t_{i\alpha,j\beta} t_{j\beta,k\gamma} t_{k\gamma,i\alpha})$ gives a geometric phase term corresponding to the closed-path hopping $i \rightarrow j \rightarrow k \rightarrow i$. The formula of I_{ij} gives the microscopic conductance in any single triad. To evaluate the macroscopic AHC, one shall properly average it over all triads in the random system. This is achieved with the aid of percolation theory, a fundamental tool to understand the hopping transport.

III. PERCOLATION THEORY

We first map the random impurity system to a random resistor network by introducing the connectivity between impurity sites with the help of a cutoff $G_c(T)$. When the conductance between two impurity sites satisfies $G_{ij} \geq G_c$, we consider that the i, j sites are connected with a finite resistor $Z_{ij} = 1/G_{ij}$. Otherwise, they are treated as disconnected, i.e., $G_{ij} \rightarrow 0$. The Hall effect will be treated as a perturbation to the obtained resistor network. The cutoff G_c should be properly chosen so that the long-range critical percolation paths and clusters appear and span the whole material and dominate the charge transport in the hopping regime. The macroscopic physical quantities will finally be obtained by averaging over the percolation path and cluster.

The hopping coefficient generally has the form $t_{i\alpha,j\beta} = t_{i\alpha,j\beta}^{(0)} e^{-aR_{ij}}$, with a^{-1} the localization length and $R_{ij} = |\mathbf{R}_i - \mathbf{R}_j|$. The direct conductance holds the form $G_{ij} = G_0(T) e^{-2aR_{ij} - \frac{1}{2}\beta(|\epsilon_{i\alpha}| + |\epsilon_{j\beta}| + |\epsilon_{i\alpha} - \epsilon_{j\beta}|)}$, and then the cutoff can be introduced by $G_c = G_0 e^{-\beta\xi_c(T)}$.²⁴ Here, $\beta\xi_c$ is a decreasing function of T , indicating the material in the insulating regime. The number of impurity sites connected to a specific site i with energy ϵ_i can be calculated by

$$n(\epsilon_i, \xi_c) = \int d\epsilon_j \int d^3 \vec{R}_{ij} \rho(\epsilon_j, \vec{R}_i) \Theta(G_{ij} - G_c). \quad (5)$$

Here, $\Theta(x)$ is the step function and the DOS $\rho(\epsilon, \vec{R}_i) \approx \frac{1}{V} \sum_i \delta(\epsilon - \epsilon_i)$ is approximated to be spatially homogeneous. The number $n(\epsilon_i, \xi_c)$ can also be given by $n(\epsilon_i, \xi_c) = \sum_n P_n(\epsilon_i, \xi_c)$, with $P_n(\epsilon_i, \xi_c)$ being the probability that the n th

smallest resistor connected to the site i has the resistance less than $1/G_c$. The function P_n reads²⁵ as

$$P_n(\epsilon_i, \xi_c) = \frac{1}{(n-1)!} \int_0^{n(\epsilon_i)} e^{-x} x^{n-1} dx, \quad (6)$$

which can be derived according to the Poisson distribution. The percolation path and cluster appear when the average connections per impurity site $\bar{n} = \langle n(\epsilon_i) \rangle_c$ reach the critical value \bar{n}_c , where the definition of $\langle \dots \rangle_c$ is given in Eq. (7). Suppose a physical quantity $F(\epsilon_1, \dots, \epsilon_m; \vec{r}_1, \dots, \vec{r}_m)$ is an m -site function, requiring the i th site to have at least η_i sites connected to it. The averaging of $F(\epsilon; \vec{r})$ reads as

$$\langle F(\epsilon; \vec{r}) \rangle_c = \frac{1}{\mathcal{N}_F} \int d\epsilon_1 \dots \int d\epsilon_m \int d^3 \vec{r}_{12} \dots \int d^3 \vec{r}_{m-1,m} \times \prod_{i=1}^m \mathcal{P}_{\eta_i}(\epsilon_i) F(\epsilon_1, \dots, \epsilon_m; \vec{r}_1, \dots, \vec{r}_m), \quad (7)$$

where \mathcal{N}_F is a normalization factor and the probability function $\mathcal{P}_{\eta_i}(\epsilon_i) = \rho(\epsilon_i) \sum_{k \geq \eta_i} P_k(\epsilon_i)$. The term $\sum_{k \geq \eta_i} P_k(\epsilon_i)$ entering the probability function has an important physical reason. The configuration averaging is not conducted over the whole impurity system, but over the percolation cluster, which covers only a portion of the impurity sites. Therefore, the probability that an impurity site belongs to the percolation cluster must be taken into account for the probability function. Moreover, this probability function also distinguishes the physical origins of the AHC and σ_{xx} . For σ_{xy}^{AH} , one has $\eta_i = 3$, and for σ_{xx} , one has $\eta_i = 2$. This indicates the averaging of σ_{xx} is performed along the one-dimensional (1D) percolation path, while for AHE, which is a two-dimensional (2D) effect, one shall evaluate AHC over all triads connected in the 2D percolation cluster.

IV. CONFIGURATION AVERAGING OF THE ANOMALOUS HALL CONDUCTIVITY

According to the formula (7), the average value of the one-site function $n(\epsilon_i, \xi_c)$ in the percolation cluster is calculated by

$$\bar{n} = \frac{\int d\epsilon_i n(\epsilon_i) \rho(\epsilon_i) n(\epsilon_i)}{\int d\epsilon_i n(\epsilon_i) \rho(\epsilon_i)}. \quad (8)$$

When the DOS $\rho(\epsilon_i) = \rho_0$ is a constant, the number $n(\epsilon_i)$ is given by $n(\epsilon_i) = \frac{2\pi}{3} \frac{\rho_0}{(2ak_B T)^3} (\xi_c - |\epsilon_i|)^2 (\xi_c^2 - |\epsilon_i|^2)$. Then, we have $\bar{n} = 0.406\pi \frac{\rho_0}{(2ak_B T)^3} \xi_c^4$. The hopping conduction occurs when the average value \bar{n} reaches the critical value \bar{n}_c . We obtain then the cutoff value ξ_c by $\xi_c(T) = \left[\frac{(2ak_B T)^3 \bar{n}_c}{0.406\pi \rho_0} \right]^{1/4}$. Thus, it gives

$$\beta\xi_c = \left(\frac{T_0}{T} \right)^{1/4}, \quad T_0 = 16 \frac{a^3 \bar{n}_c}{k_B \rho_0}, \quad (9)$$

which is the Mott law.¹⁷ Accordingly, if we assume the density of states $\rho(\epsilon) \sim \epsilon^2$, we obtain straightforwardly the E-S law $\beta\xi_c = (T_0/T)^{1/2}$.^{26,27}

Numerical solutions show that the critical site connectivity is $\bar{n}_c = 2.6 \sim 2.7$ for the appearance of a percolation path and cluster in three-dimensional (3D) materials.^{28,29} This indicates that the triads are sparsely distributed in the percolation cluster, as shown in Fig. 3. The AHC can be derived by examining

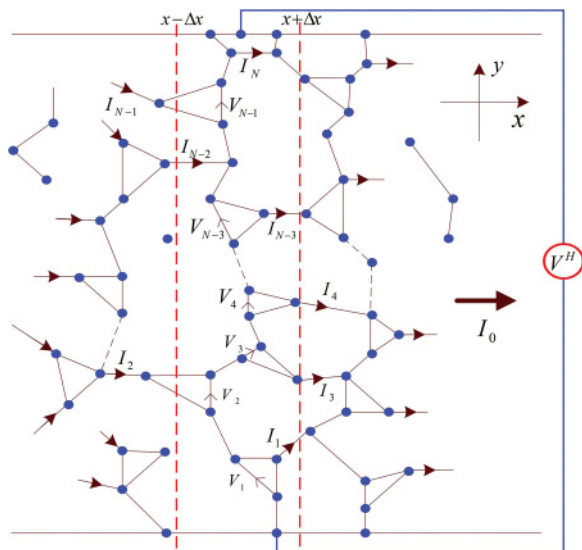


FIG. 3. (Color online) Typical resistor network in the material. The present situation indicates V_{N-2}^H and V_N^H in the region from $x - \Delta x$ to $x + \Delta x$ are zero, where no triads form.

the transverse voltage V_y^H (along the y axis) induced by the applied longitudinal current I_0 . We denote by $N(x)$ the number of triads distributed along the y axis in the region around position x . (Note that \mathbf{M} is along the z axis, hence, we assume the system in this direction to be uniform.) The transverse voltage equals the summation over the voltage drops of the $N(x)$ triads:

$$V_y^H(x) = \sum_{l=1}^{N(x)} V_l^H. \quad (10)$$

For the general situation, we allow some V_i^H 's to be zero (see Fig. 3). In that case, no triad forms for the incoming current I_i under the condition that all direct conductances in a triad must be no less than G_c . To calculate V_i^H , the voltage contributed by the i th triad, we employ perturbation theory to the equation³⁰ $I_{ij} = G_{ij}V_{ij} + \sum_k \mathcal{G}_{ijk}(V_{ik} + V_{jk})$. First, in the zeroth order, we consider only the normal current, namely, the Hall current is zero and, thus, $\sum_j I_{ij} = \sum_j G_{ij}V_{ij}^{(0)} = 0$, with which one can determine the voltage $V_i^{(0)}$ at each site. Then, for the first-order perturbation, we have $\sum_j I_{ij} = \sum_j G_{ij}V_{ij} + \sum_j J_{ij}^{(H)} = 0$, which leads to $J_i^{(H)} = \sum_j J_{ij}^{(H)} = \sum_j \sum_k \mathcal{G}_{ijk}(V_{jk}^{(0)} + V_{ik}^{(0)}) = -\sum_j G_{ij}V_{ij}$. The current $J_i^{(H)}$ can also be written as

$$J_i^{(H)} = \frac{3}{2} \sum_{jk} \mathcal{G}_{ijk} V_{jk}^{(0)}. \quad (11)$$

For the hopping regime, the triads are dilutedly distributed, and the Hall voltages induced by different triads are considered to be uncorrelated. Therefore, we obtain the Hall voltage of the i th triad from the transformation indicated in Fig. 4 that

$$V_i^{(H)} = \frac{3I_i \mathcal{G}_{i_1 i_2 i_3}^{(i)}}{G_{i_1 i_2} G_{i_2 i_3} + G_{i_1 i_3} G_{i_2 i_3} + G_{i_3 i_1} G_{i_1 i_2}}. \quad (12)$$

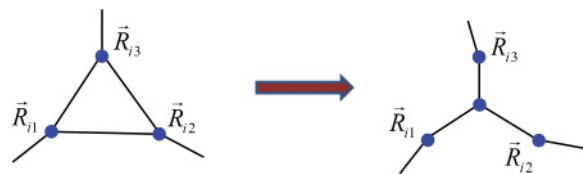


FIG. 4. (Color online) Resistor network transformation.

From the resistor network configuration, one can see $\sum_i^{N(x)} I_i = 2I_0$. For convenience, we denote $I_i = 2I_0 \lambda_i(x)$ with $\sum_i \lambda_i = 1$. For a macroscopic system, one has $N(x) \rightarrow \infty$. Furthermore, we consider that, at the position x for each λ_i , there are $n_i(x)$ triads that have such same current fraction λ_i . Then, the average transverse voltage reads as

$$\bar{V}_y^H = 6I_0 \frac{1}{L_x} \int dx \sum_{\{n_i\}} \lambda_i \times \sum_{j=1}^{n_i \gg 1} \frac{\mathcal{G}_{j_1 j_2 j_3}^{(j)}}{G_{j_1 j_2} G_{j_2 j_3} + G_{j_1 j_3} G_{j_2 j_3} + G_{j_3 j_1} G_{j_1 j_2}}. \quad (13)$$

To simplify this formula, we extend the current distribution $\{\lambda_i\}$ for the region between $x - \Delta x$ and $x + \Delta x$ to the whole space along the x direction, and then we can exchange the order of the integral and the first summation: $\frac{1}{L_x} \int dx \sum_{\{n_i\}} \lambda_i \sum_{j=1}^{n_i \gg 1} \rightarrow \sum_{\{\lambda_i\}} \lambda_i \frac{1}{L_x} \int dx \sum_{j=1}^{n_i(x)}$. In the limit $N(x) \rightarrow \infty$ and when the length L_x is much larger than the typical length L of the triad, the calculation $\frac{1}{L_x} \int dx \sum_{j=1}^{n_i(x)}$ gives the average of all possible configurations of the triads through the percolating cluster. This leads to

$$\bar{V}_y^H = 6I_0 \sum_{\{\lambda_i\}} \bar{n}_i \lambda_i \left\langle \frac{\mathcal{G}_{i_1 i_2 i_3}^{(i)}}{G_{i_1 i_2} G_{i_2 i_3} + G_{i_1 i_3} G_{i_2 i_3} + G_{i_3 i_1} G_{i_1 i_2}} \right\rangle,$$

with $\bar{n}_i = (1/L_x) \int dx n_i(x)$ the average number of triads with ingoing and outgoing current I_i . Note the identity $\sum_i n_i \lambda_i = 1$ is independent of position x and, therefore, we have also $\sum_i \bar{n}_i \lambda_i = 1$. For this, we obtain the AHC

$$\sigma_{xy}^{\text{AH}} = 3L\sigma_{xx}^2 \frac{k_B T}{e^2} \left\langle \frac{\sum_{\alpha\beta\gamma} [\text{Im}(t_{i\alpha, j\beta} t_{j\beta, k\gamma} t_{k\gamma, i\alpha}) T_{ijk}^{(3)}]}{\sum_{i \leftrightarrow j \leftrightarrow k} |t_{ij} t_{jk}|^2 T_{ij}^{(2)} T_{jk}^{(2)}} \right\rangle_c, \quad (14)$$

where L is the correlation length of the network. Note that the configuration integral given by Eq. (7) is derived for the AHC in this paper. This is an essential difference from the former theory by Burkov *et al.*,²¹ where the configuration averaging applies to the whole system rather than to 2D percolation cluster. With our formalism, the key physics that Hall currents are averaged over percolation clusters can be studied, which is a crucial step to understanding the insulating regime of the AHE phase diagram. The above configuration integral can not be solved analytically. In the following, we study the upper and lower limits of the AHC with Eq. (14) by considering different extreme situations for the triad distribution, with which the range of the scaling relation between σ_{xy}^{AH} and σ_{xx} can be determined.

Before proceeding further, we would like to present a few remarks on Eq. (14). First of all, this formula is generally valid for the disordered insulating regime, as long as the triads

are sparsely distributed in the percolation cluster. Second, for different types of hopping regimes (Mott, ES, and activation E_3 hopping regimes), the functions of the DOS $\rho(\epsilon)$ and connectivity $n(\epsilon_i)$ in the configuration integral are different.

V. SCALING RELATION BETWEEN ANOMALOUS HALL CONDUCTIVITY AND LONGITUDINAL CONDUCTIVITY

The lower (upper) limit of the AHC can be formulated by keeping only the maximum (minimum) term in the denominator and the minimum (maximum) term in the numerator. In this section, we shall study the scaling relation in the Mott, ES, and activation E_3 hopping regimes, respectively.

A. Mott variable-range hopping regime

In this regime, we first approximate the DOS to be constant although this approximation is relaxed later. Under this condition, one obtains straightforwardly the probability $P_n(\epsilon_i)$ from the number $n(\epsilon_i)$. Note in the hopping conduction mechanism that dependence of the conductivities is dominated by exponential functions. It can then be expected that the scaling relation between σ_{xy}^{AH} and σ_{xx} will be governed by the exponential functions in \mathcal{G}_{ijk} and G_{ij} . To focus on the scaling relation, we first drop off the summation of the spin states. This procedure ignores an important physical consequence that the summation over spin-up and spin-down states contributes oppositely to the AHE (we shall return to this discussion later), but keeps the central result of the scaling relation unchanged between σ_{xy}^{AH} and σ_{xx} . As a result, with further simplification, we find

$$\left\{ \sigma_{xy}^{\text{AH}} \right\}_{\min}^{\max} \simeq 3L\sigma_{xx}^2 \frac{k_B T}{e^2 t_{\max/\min}^{(0)}} \langle R_{ijk}^{\min} \rangle_c \langle \epsilon_{ijk}^{\min} \rangle_c, \quad (15)$$

where $\langle R_{ijk}^{\min} \rangle_c = e^{a(R_{ij}+R_{jk}-R_{ik})} |_{R_{ij}, R_{jk} < R_{ik}}, \langle \epsilon_{ijk}^{\min} \rangle_c = e^{0.5\beta(|\epsilon_i|+|\epsilon_j|+|\epsilon_j-\epsilon_k|+|\epsilon_i-\epsilon_k|)} |_{|\epsilon_i| < |\epsilon_j| < |\epsilon_k|}$, $\langle R_{ijk}^{\max} \rangle_c$ and $\langle \epsilon_{ijk}^{\max} \rangle_c$ hold the same form for the calculation, but the restrictions change to be $R_{ij}, R_{jk} > R_{ik}$, and $|\epsilon_i| > |\epsilon_j| > |\epsilon_k|$, respectively. The coefficient $t_{\max/\min}^{(0)}$ represents the maximum/minimum element in the matrix $t_{ij}^{(0)}$. In obtaining Eq. (15), we have approximated the configuration averaging of the exponential functions to be the configuration averaging of the exponents. This approximation loses the information of the power-law dependence of the AHC on the temperature, and it requires that the dominant temperature dependence of the AHC should be in the exponential form. In the hopping conduction regime, this condition is satisfied.

It is instructive to point out the underlying physics of the two limits. In the hopping regime, charge transport may prefer a short and straight path in the forward direction with larger resistance than a long and meandrous path with somewhat smaller resistance.^{16,25} This picture introduces an additional restriction complementary to the percolation theory for charge transport. The bonds in a triad that play the major role for the current flowing through it are determined by the optimization of the resistance magnitudes and spatial configuration of the three bonds. A quantitative description can be obtained

by phenomenologically introducing an additional probability factor to restrict the charge transport.^{16,25} Here, we only need to adopt this picture to present the two extreme situations corresponding to $\{\sigma_{xy}^{\text{AH}}\}_{\min/\max}$. To get the upper limit, we assume that for each triad of the percolation cluster the two bonds with smaller direct conductance dominate the charge transport, i.e., the product of the two smallest conductances minimize the denominator and take the maximum value for the numerator of Eq. (14). For the opposite limit, the situation that the two bonds with larger conductances in each triad dominate the charge transport corresponds to the lower limit of the AHC.

We study first the lower limit of the AHC. For convenience, we neglect the spin indices. According to Eq. (7), we know the configuration averaging $\langle R_{ij} + R_{jk} - R_{ik} \rangle_c |_{R_{ij}, R_{jk} < R_{ik}}$ is a nine-dimensional (9D) integral over the position $\int d^3 \vec{R}_{ij} \int d^3 \vec{R}_{jk}$ and the on-site energies $\int d\epsilon_i d\epsilon_j d\epsilon_k$. We shall first perform the integral over position. Denote by $\vec{R}_1 = \vec{R}_{ij}, \vec{R}_2 = \vec{R}_{jk}$ for convenience, and then $R_3 = R_{ik} = (R_1^2 + R_2^2 - 2R_1 R_2 \cos \theta)^{1/2}$. We then study the integral $I = \frac{1}{\mathcal{N}_r} \int d^3 \vec{R}_1 \int d^3 \vec{R}_2 (R_1 + R_2 - R_3)$ with $\mathcal{N}_r = \int d^3 \vec{R}_1 \int d^3 \vec{R}_2$. Note that, in the configuration integral, we have the restrictions $R_i \leq R_{i,\max}$ and $R_1, R_2 \leq R_3$, with $R_{i,\max}$ determined through $2aR_{ij}^{\max} + \frac{1}{2}\beta(|\epsilon_i| + |\epsilon_j| + |\epsilon_i - \epsilon_j|) = \beta\xi_c$ (from the condition $G_{ij}^{\min} = G_c$ or $Z_{ij}^{\max} = 1/G_c$). For this, we can show that the integral satisfies

$$\begin{aligned} I \leq & \frac{1}{\mathcal{N}_r} 8\pi^2 \int_0^{R_{2\max}} dR_2 R_2^2 \left[\int_{\pi/2}^{\pi} d\theta \int_0^{R_{1\max}} dR_1 R_1^2 \sin \theta \right. \\ & \times (R_1 + R_2 - \sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \cos \theta}) \\ & + \int_{\pi/3}^{\pi/2} d\theta \int_{2R_2 \cos \theta}^{R_{1\max}} dR_1 R_1^2 \sin \theta \\ & \left. \times (R_1 + R_2 - \sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \cos \theta}) \right], \quad (16) \end{aligned}$$

where $\mathcal{N}_r = \frac{8}{3}\pi^2 R_{2\max}^3 \int_{\pi/2}^{\pi} d\theta \int_0^{R_{1\max}} dR_1 R_1^2 \sin \theta + 8\pi^2 \int_0^{R_{2\max}} dR_2 R_2^2 \int_{\pi/3}^{\pi/2} d\theta \int_{2R_2 \cos \theta}^{R_{1\max}} dR_1 R_1^2 \sin \theta$. By a straightforward calculation, we obtain further

$$I \simeq 0.424\pi^2 R_{\max}^7 / \mathcal{N}_r, \quad (17)$$

with the normalization factor $\mathcal{N}_r = \frac{23}{18}\pi^2 R_{\max}^6$. Here, $R_{\max} = \max\{R_{1\max}, R_{2\max}\}$. We should emphasize that, to this step, we can not cancel the function R_{\max}^7 in the numerator of the Eq. (17) by the normalization factor \mathcal{N}_r . This is because both of them are only part of the original configuration averaging $\langle R_{ij} + R_{jk} - R_{ik} \rangle_c$. The final result needs to further integrate over on-site energies, and gives that

$$\ln \langle R_{ijk}^{\min} \rangle_c \simeq 0.156\beta\xi_c. \quad (18)$$

Now, we evaluate the configuration averaging $\langle \epsilon_{ijk}^{\min} \rangle_c$, which corresponds to a 3D integral over on-site energies.

Similarly, the formula is given by

$$\ln \langle \epsilon_{ijk}^{\min} \rangle_c = \frac{0.5\beta \int d\epsilon_i d\epsilon_j d\epsilon_k \rho(\epsilon_i) \sum_{l \geq 3} P_l(\epsilon_i) \rho(\epsilon_i) \sum_{l \geq 3} P_l(\epsilon_j) \rho(\epsilon_k) \sum_{l \geq 3} P_l(\epsilon_k) (|\epsilon_i| + |\epsilon_j| + |\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k|)}{\int d\epsilon_i d\epsilon_j d\epsilon_k \rho(\epsilon_i) \sum_{l \geq 3} P_l(\epsilon_i) \rho(\epsilon_i) \sum_{l \geq 3} P_l(\epsilon_j) \rho(\epsilon_k) \sum_{l \geq 3} P_l(\epsilon_k)}. \quad (19)$$

To simplify the above integral, we check $|\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k|$ with the restriction $|\epsilon_i| < |\epsilon_j| < |\epsilon_k|$. For the case (i) $\text{sgn}(\epsilon_i) = \text{sgn}(\epsilon_j) = \text{sgn}(\epsilon_k) = \pm 1$, we have $|\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k| = -|\epsilon_i - \epsilon_j|$; for (ii) $\text{sgn}(\epsilon_i) = \text{sgn}(\epsilon_j) = -\text{sgn}(\epsilon_k) = \pm 1$, we have $|\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k| = -|\epsilon_i - \epsilon_j|$; for (iii) $\text{sgn}(\epsilon_i) = \text{sgn}(\epsilon_k) = -\text{sgn}(\epsilon_j) = \pm 1$, we have $|\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k| = -|\epsilon_i - \epsilon_j|$; and for (iv) $\text{sgn}(\epsilon_j) = \text{sgn}(\epsilon_k) = -\text{sgn}(\epsilon_i) = \pm 1$, we have $|\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k| = |\epsilon_i - \epsilon_j|$. For this, we obtain that $\langle |\epsilon_i| + |\epsilon_j| + |\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k| \rangle_c \simeq \langle |\epsilon_i| + |\epsilon_j| - \frac{1}{2}|\epsilon_i - \epsilon_j| \rangle_c$. Then, by a straightforward calculation, one can verify that

$$\ln \langle \epsilon_{ijk}^{\min} \rangle_c = 0.086\beta\xi_c. \quad (20)$$

Together with the results in Eqs. (18) and (20), we get

$$\langle R_{ijk}^{\min} \rangle_c \langle \epsilon_{ijk}^{\min} \rangle_c \simeq e^{0.242\beta\xi_c}. \quad (21)$$

The lower limit of the AHC is then obtained by

$$\{\sigma_{xy}^{\text{AH}}\}_{\min} = 3L\sigma_{xx}^2 \frac{k_B T}{e^2} \frac{1}{t_{\max}^{(0)}} e^{0.242\beta\xi_c}. \quad (22)$$

Note the longitudinal conductivity σ_{xx} is calculated based on the two-site function of G_{ij} , which should be no less than G_c in a percolation path. The evaluation of σ_{xx} with percolation theory has been well studied in the published literatures.^{16,24,25} It can be shown that the result of σ_{xx} equals G_c divided by the correlation length of the network and takes the form $\sigma_{xx} = \sigma_0(T)e^{-\beta\xi_c}$, where $\sigma_0(T)$ gives at most a power law on T .^{24,25} Comparing this form with the lower limit of the AHC obtained above, we reach that

$$\{\sigma_{xy}^{\text{AH}}\}_{\min} = 3L\sigma_0^{0.242} \frac{k_B T}{e^2} \frac{1}{t_{\max}^{(0)}} \sigma_{xx}^{1.758}. \quad (23)$$

The upper limit can be studied in the same way. Considering the different restrictions, we obtain $\langle R_{ij} + R_{jk} - R_{ik} \rangle_c |_{R_{ij}, R_{jk} < R_{ik}} = 0.483\beta\xi_c/a$ and $\langle |\epsilon_i| + |\epsilon_j| + |\epsilon_j - \epsilon_k| - |\epsilon_i - \epsilon_k| \rangle_c |_{|\epsilon_i| > |\epsilon_j| > |\epsilon_k|} = 0.275\xi_c$. For this, we obtain $\langle R_{ijk}^{\max} \rangle_c \simeq e^{0.483\beta\xi_c}$, $\langle \epsilon_{ijk}^{\max} \rangle_c \simeq e^{0.138\beta\xi_c}$, and the upper limit of the AHC by

$$\{\sigma_{xy}^{\text{AH}}\}_{\max} = 3L\sigma_0^{0.621} \frac{k_B T}{e^2 t_{\min}^{(0)}} \sigma_{xx}^{1.379} \propto \sigma_{xx}^\gamma. \quad (24)$$

Comparing the above results with the AHC, we reach $\{\sigma_{xy}^{\text{AH}}\}_{\min/\max} \sim \sigma_0^{2-\gamma_a/b} \sigma_{xx}^{\gamma_a/b}$ with $\gamma_a = 1.76$ and $\gamma_b = 1.38$. This leads to the scaling relation between σ_{xy}^{AH} and σ_{xx} of the AHE in the Mott VRH regime:

$$\sigma_{xy}^{\text{AH}} \propto \sigma_{xx}^\gamma, \quad 1.38 < \gamma < 1.76. \quad (25)$$

The maximum (minimum) of the AHC corresponds to the smaller (larger) power index γ_b (γ_a). This scaling range can be confirmed with a numerical calculation of Eq. (15).

Furthermore, a direct numerical study for the configuration integral (14) gives the scaling exponent $\gamma \approx 1.62$, which is consistent with the analytical prediction of the lower and upper limits.

It is noteworthy that the configuration averaging over the position $\langle R_{ijk} \rangle_c$ undergoes a relatively large change in magnitude between the upper and lower limits. This result reflects an important property of the (variable-range) hopping conduction regime presented below. In the VRH, the hopping process allows us to go beyond between nearest-neighbor impurity sites to minimize the resistivity. The optimization of the typical hopping length plays a major role in determining the scaling of the conductivities with respect to temperature.¹⁷ The lower and upper limits correspond to the opposite extreme situations of the triad distribution, which have distinct influences on the optimization of the hopping distances for the Hall transport and, thus, lead to very different results for the AHC after spatial averaging. We should emphasize that this remarkable difference between $\langle R_{ijk}^{\max} \rangle_c$ and $\langle R_{ijk}^{\min} \rangle_c$ is obtained in the case of a constant DOS around Fermi energy. One can expect this effect will be suppressed in the E-S hopping regime where the DOS is a parabolic function of the on-site energy and the difference between configuration integrals with respect to energies become more important (refer to the discussion in the next section).

We make a further remark here to compare our results with those obtained by Burkov *et al.*²¹ In the final result of the AHC in Ref. 17, the configuration averaging was actually not performed but simply replaced by the maximum value of the integrand in their formalism. This procedure, not surprisingly, can not predict the correct scaling relation. Here, we have performed the exact calculation of the lower and upper limits of the AHC based on the correct configuration averaging formula. With our procedure, the key physics that the Hall currents are averaged over percolation cluster is completely considered and reflected in our evaluation.

So far, in the calculation we have assumed a constant DOS. This approximation is applicable for the ferromagnetic system with strong exchange interaction between local magnetic moments and charge carriers (e.g., oxides, magnetites) and half metals in general. In this case, we do not need to sum over spin-up and spin-down states, which contribute oppositely to the AHE, and the previous results are valid.

However, when the Fermi energy crosses both spin-up and -down impurity states, a symmetric DOS with $\rho(\epsilon) = \rho(-\epsilon)$ leads to zero AHC. This is because under the transformation $\epsilon_{l,\sigma} \rightarrow -\epsilon_{l,-\sigma}$ ($l = i, j, k$), \mathcal{G}_{ijk} changes sign, while G_{ij} is invariant. Thus, the averaging for AHC over all spin states and on-site energies cancels.²¹ We relax the previous simplifying restriction by expanding the DOS by $\rho(\epsilon) = \sum_n \frac{1}{n!} \frac{d^n \rho_0}{d\epsilon^n} \epsilon^n$, where we consider $\rho_0 = \rho(\epsilon_F)$ is finite and has only a

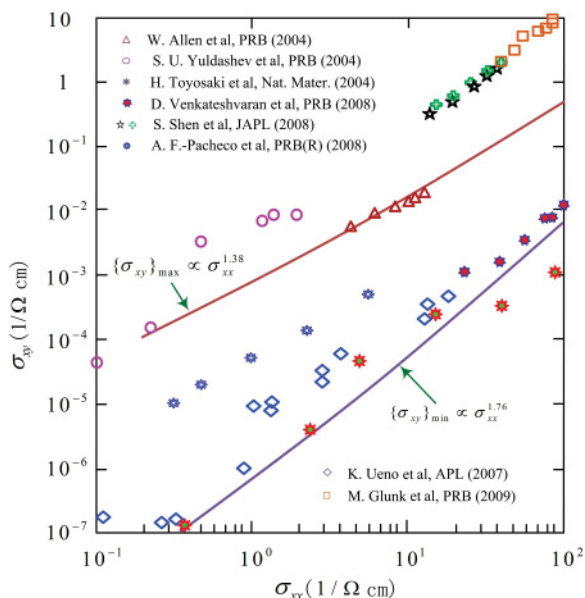


FIG. 5. (Color online) Scaling relation between the AHC and longitudinal conductivity. The theoretical results are compared with the experimental observations.

relatively small variation in the range $|\epsilon| < \xi_c$. Substituting this expansion into Eq. (14) yields $\sigma_{xy}^{\text{AH}} = \sum_{n=0}^{\infty} \sigma_{xy}^{(n)}$, with the first and second nonzero terms, respectively, proportional to $\frac{d\rho_0}{d\epsilon_F}$ and $\frac{d^3\rho_0}{d\epsilon_F^3}$. We can similarly evaluate the lower and upper limits of σ_{xy}^{AH} as before. The first two nonzero terms in the expansion are $\{\sigma_{xy}^{(1)}\}_{\text{min/max}} \sim M \frac{d\rho_0}{d\epsilon_F} \xi_c(T) \sigma_0^{2-\gamma_{a/b}} \sigma_{xx}^{\gamma_{a/b}}$ and $\{\sigma_{xy}^{(2)}\}_{\text{min/max}} \sim 0.002M \frac{d^3\rho_0}{d\epsilon_F^3} \xi_c^3(T) \sigma_0^{2-\gamma_{a/b}} \sigma_{xx}^{\gamma_{a/b}}$. The appearance of M is due to the summation over the spin-up and -down states. We have also employed the result $\langle |\epsilon| \rangle_c = 0.112\xi_c$. Note that $\sigma_{xy}^{(1)}$ and $\sigma_{xy}^{(2)}$ have different physical meanings. The term $\sigma_{xy}^{(1)}$ dominates when the DOS varies monotonically versus ϵ . Furthermore, when the DOS has a local minimum at the Fermi level, which may be obtained due to particle-particle interaction (Coulomb interaction), we have $d\rho/d\epsilon_F = 0$. Then, the term $\sigma_{xy}^{(1)}$ vanishes and $\sigma_{xy}^{(2)}$ dominates the AHE. The result that the AHC σ_{xy}^{AH} is proportional to $d\rho_0/d\epsilon_F$ (when the DOS varies monotonically with respect to energy around Fermi energy) or $d^3\rho_0/d\epsilon_F^3$ (when the DOS has a local minimum at Fermi level) indicates an interesting property that the AHC may change sign when the first- or third-order derivative of DOS with respect to energy changes sign. This result is consistent with the observation by Allen *et al.*⁷

Figure 5 shows that our theoretical prediction is consistent with the experimental observations of the scaling relation in this regime, hence completing the understanding of the phase diagram of the AHE.

B. Efros-Shkolovskii regime

In the case with strong Coulomb interaction, the DOS may be greatly reduced around the Fermi energy in disordered

insulators.^{26,27} In this case, the assumption in the preceding section that the DOS has a small variation relative to ρ_0 is not valid. The limit situation is that both the DOS and the first derivative at Fermi level vanish (i.e., the E-S hopping regime), which corresponds to the appearance of a gap due to Coulomb interaction. In this case, $\rho_0 = 0$ and $d\rho/d\epsilon_F = 0$, and thus

$$\rho(\epsilon) \simeq \frac{1}{2} \frac{d^2\rho_0}{d\epsilon^2} \epsilon^2 + \frac{1}{6} \frac{d^3\rho_0}{d\epsilon^3} \epsilon^3. \quad (26)$$

The E-S hopping regime is different from the cases discussed in the previous section since, around the Fermi energy, DOS is not dominated by a constant but by a parabolic function of on-site energy. This may lead to a quantitative variation of the probability function in the configuration averaging, and finally affect the quantitative but not qualitative result of the scaling relation. The formula of the connectivity $n(\epsilon_i, \xi_c)$ is now given by

$$n(\epsilon_i, \xi_c) = \frac{1}{(2ak_B T)^3} \frac{2\pi}{3} \frac{d^2\rho_0}{d\epsilon^2} \left(\frac{1}{30} \xi_c^6 - \frac{1}{10} \xi_c^5 |\epsilon_i| + \frac{1}{4} \xi_c^4 |\epsilon_i|^2 - \frac{1}{3} |\epsilon_i|^3 \xi_c^3 + \frac{3}{10} |\epsilon_i|^5 \xi_c - \frac{3}{20} |\epsilon_i|^6 \right). \quad (27)$$

The configuration averaging of AHC can be calculated following a similar procedure. Specifically, for the lower limit, we obtain

$$\langle R_{ijk}^{\text{min}} \rangle_c \simeq e^{0.092\beta\xi_c}, \quad \langle \epsilon_{ijk}^{\text{min}} \rangle_c \simeq e^{0.29\beta\xi_c}. \quad (28)$$

Comparing the above results with those obtained in the Mott VRH regime with a constant DOS, we can see that the magnitude of energy averaging in the E-S hopping regime increases, while the magnitude of position averaging decreases. This is reasonable since the DOS varies as a function of ϵ^2 , which increases the contribution to the Hall effect from the impurity states with energies far away from the Fermi energy and, accordingly, decreases the contribution from hopping between impurity sites with large distances. The lower limit of the AHC is then obtained by

$$\{\sigma_{xy}^{\text{AH}}\}_{\text{min}} \simeq 0.059L\sigma_0^{0.38}(T) \frac{M_0}{e^2} \frac{1}{t_{\text{max}}^{(0)}} \frac{d^3\rho_0}{d\epsilon_F^3} \xi_c^3(T) \sigma_{xx}^{1.62}. \quad (29)$$

For the upper limit of AHC, we get $\langle R_{ijk}^{\text{max}} \rangle_c \simeq e^{0.29\beta\xi_c}$ and $\langle \epsilon_{ijk}^{\text{max}} \rangle_c \simeq e^{0.38\beta\xi_c}$. This leads to the scaling relation in the upper limit

$$\{\sigma_{xy}^{\text{AH}}\}_{\text{max}} \simeq 0.026L\sigma_0^{0.67}(T) \frac{M_0}{e^2} \frac{1}{t_{\text{min}}^{(0)}} \frac{d^3\rho_0}{d\epsilon_F^3} \xi_c^3(T) \sigma_{xx}^{1.33}. \quad (30)$$

Therefore, in the E-S hopping regime, the scaling relation between anomalous Hall and longitudinal conductivities becomes $\sigma_{xy}^{\text{AH}} \propto \sigma_{xx}^\gamma$ with $1.33 \leq \gamma \leq 1.62$, which has a small quantitative shift relative to the scaling obtained in the case with a constant DOS. This result is consistent with the observations in the experiments by Aronzon *et al.*⁴ and by Allen *et al.*,⁷ who found the scaling relation as $1.4 \leq \gamma \leq 1.6$ for the E-S hopping conduction regime. Again, the AHC in the E-S regime σ_{xy}^{AH} is proportional to $d^3\rho_0/d\epsilon_F^3$, and

thus may change sign when the third-order derivative of DOS with respect to energy changes sign. Finally, it can be expected that the general situation with a reduced DOS (not necessarily zero) at Fermi level will be associated with a scaling falling in-between the E-S hopping regime and the case with a constant DOS. This confirms that the scaling relation between σ_{xy}^{AH} and σ_{xx} is insensitive to which types of hopping conduction the material belongs to, and is therefore generic for the disordered insulating regime.

C. Activation E_3 hopping regime

Finally, we present a brief study on the AHE in the activation E_3 hopping regime, which dominates the charge transport in the disordered insulating system when the temperature $T > T_0$. Here, T_0 is given by Eq. (9). In the activation E_3 hopping regime, the hopping between nearest-neighbor impurity sites dominates the charge transport.^{27,31} In this case, the hopping configuration in the position space is not affected by temperature. Thus, the configuration averaging over position space is independent of temperature. The temperature dependence of the conductivities is solely determined by the energy configuration integral. Again, we consider that the impurity sites are homogeneously distributed in position space. Then, connectivity $n(\epsilon_i)$ for a specific impurity site with on-site energy ϵ_i is given by

$$n(\epsilon_i) = \frac{4\pi R_c^3}{3} \int d\epsilon_j \rho(\epsilon_j) \Theta\left(E_3 - \frac{|\epsilon_i| + |\epsilon_j| + |\epsilon_i - \epsilon_j|}{2}\right), \quad (31)$$

where E_3 is the cutoff for on-site energy and R_c represents the typical distance between the neighbor impurity sites. For a constant DOS, one has

$$n(\epsilon_i) = \frac{4}{3}\pi R_c^3 \rho_0 (2E_3 - |\epsilon_i|), \quad (32)$$

with $|\epsilon| \leq E_3$. It can be seen that, for the present regime, $n(E_3) = \frac{4}{3}\pi R_c^3 \rho_0 E_3 > 0$. This is different from the situation in the VRH regime considered in previous sections. By substituting the above formula into Eq. (8), one can calculate the relation between the cutoff E_3 and \bar{n} straightforwardly, with which one can verify that E_3 is a constant independent of temperature and $E_3 \propto 1/(\rho_0 R_c^3)$.³¹ The longitudinal conductivity is then given by

$$\sigma_{xx} = \sigma_0 e^{-E_3/k_B T}. \quad (33)$$

The AHC is still given by Eq. (14) with the function of $n(\epsilon_i)$ given by Eq. (31), but now the configuration integral over position is unrelated to that over on-site energies and,

thus, does not affect the temperature dependence of σ_{xy}^{AH} . For this, we obtain the upper and lower limits of the AHC that

$$\left\{ \sigma_{xy}^{\text{AH}} \right\}_{\min}^{\max} \simeq 3L\sigma_{xx}^2 \frac{k_B T e^{aR_c}}{e^{2I_{\max}^{(0)}}} \left\langle \epsilon_{ijk}^{\min} \right\rangle_c. \quad (34)$$

By a direct numerical evaluation we obtain that $\langle \epsilon_{ijk}^{\max} \rangle_c \approx e^{0.61\beta E_3}$, $\langle \epsilon_{ijk}^{\min} \rangle_c \approx e^{0.34\beta E_3}$, with which we obtain the scaling relation $\sigma_{xy}^{\text{AH}} \propto \sigma_{xx}^\gamma$, where $1.39 \leq \gamma \leq 1.66$. With this result, we conclude that the scaling in the activation E_3 hopping regime has only a quantitative small shift relative to the scaling in the VRH hopping regimes.

VI. CONCLUSIONS

We have developed a theory based on the phonon-assisted hopping mechanism and percolation theory to study the anomalous Hall effect (AHE) in the disordered insulating regime. A general formula for the anomalous Hall conductivity (AHC) has been derived for the hopping conduction regime, with the key physics that the Hall currents are averaged over percolation cluster being completely considered. We calculated the lower and upper limits of the AHC and show that it scales with the longitudinal conductivity as $\sigma_{xy}^{\text{AH}} \sim \sigma_{xx}^\gamma$ with γ predicted to be $1.33 \leq \gamma \leq 1.76$. The predicted scaling only slightly depends on the specific hopping types, and is quantitatively in agreement with the experimental observations.

From our theory, the scaling relation in the insulating AHE is fully determined by the microscopic origin: phonon-assisted hopping conduction mechanism, and by the fact that the AHC is dominated by the percolation clusters. It is clear that these two aspects are generic for the hopping conduction regime of the disordered insulators, and, therefore, the obtained scaling in this regime is qualitatively generic in the disordered insulating AHE. We have shown that this scaling remains similar regardless of whether the hopping process is Mott variable-range hopping, influenced by interactions, or activation E_3 hopping (nearest-neighbor hopping) regime. Our results explain naturally how the scaling between the two quantities remain true even when the diagonal conductivity crosses regimes and why this type of scaling is so prevalent in the insulating regime. Our theory completes the understanding of the AHE phase diagram in the insulating regime.

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