Klein-Gordon equation approach to nonlinear split-ring resonator based metamaterials: One-dimensional systems

Pradipta Giri,[1] Kamal Choudhary,[2] Arnab Sen Gupta,[2,3] A. K. Bandyopadhyay,[4] and Arthur R. McGurn[5,*]
[1]*Dumkal Institute of Engineering and Technology, West Bengal University of Technology, Dumkal, Murshidabad, India*
[2]*Government College of Engineering and Ceramic Technology, West Bengal University of Technology, 73, Abinash Chandra Banerjee Lane, Kolkata-700010, India*
[3]*Department of Materials Science and Engineering, Pennsylvania State University, University Park, Pennsylvania 16802, USA*
[4]*Ideal Institute of Engineering, Kalyani-741235, Dist. Nadia, India*
[5]*Department of Physics, Western Michigan University, Kalamazoo, Michigan 49008-5252, USA*
(Received 3 July 2010; published 17 October 2011)

The electrodynamics of a one-dimensional split-ring resonator (SRR) based nonlinear metamaterial is studied. The metamaterial is a one-dimensional periodic array of weakly coupled SRRs, with each SRR represented by a nonlinear resistor-inductor-capacitor (RLC) equivalent resonator circuit. Nonlinearity is introduced into the system by the addition of Kerr-type dielectric medium within the SRRs or by the introduction into the system of certain other nonlinear elements (e.g. diodes). In the continuum limit of the system, variations of the charge stored within the capacitive slits of the SRRs in both time and space are shown to be described along the array by a nonlinear Klein-Gordon equation. Analytical solutions of the nonlinear Klein-Gordon equation for various dark and bright envelope, breather, and pulse soliton solutions are obtained and studied. A discussion is given of the relationship between the Klein-Gordon solutions and the solutions of the nonlinear Schrödinger equation approximation to the nonlinear Klein-Gordon equation. A comparison is made of the Klein-Gordon solutions with intrinsic localized mode (discrete breather) solutions of the discrete system and their continuum limits. An additional continuum limit differential equation for the breather modes of the system is obtained which is not bound by a weak coupling assumption, and its relation to the Klein-Gordon equation is studied. Analytic forms are given for the effects of dissipation in the system on the various bright and dark envelope, breather, and pulse solitons. Discussions are given of the effects of further than first neighbor couplings in the SRR system.

DOI: 10.1103/PhysRevB.84.155429 PACS number(s): 78.67.Pt, 05.45.Yv

I. INTRODUCTION

The electrodynamics of substances with simultaneous negative values of dielectric permittivity (ε) and magnetic permeability (μ) has been a subject of much study, with current interests in possible technological applications.[1] Substances with both negative ε and μ are predicted to posses a negative refractive index and, consequently, to exhibit a variety of optical properties not found in positive indexed materials. Negative index materials, however, do not occur in nature, and only recently has it been shown that they can be artificially fabricated. The experimental realization of such materials was demonstrated by Smith *et al*.[2] based on theoretical work of Pendry *et al*.[3,4] Smith *et al*. made a type of metamaterial (MM) as an artificial structure with negative refractive properties. The structure consists of metallic wires responsible for the negative permittivity and metallic split-ring resonators (SRR) responsible for the negative permeability. The optical and electrical properties of the MM are modulated by the proper use of SRRs to give ε, $\mu < 0$ within a region of frequencies, and it is the SRRs that are key in setting negative μ within a material with $\varepsilon < 0$. In particular, unlike naturally occurring materials, the designed MMs show a relatively large magnetic response at THz frequencies. This, in combination with its negative permitivity in the THz, is responsible for an effective negative index in this range of frequencies.

From the standpoint of theory, linear and nonlinear SRR have been shown to be described by equivalent resistor-inductor-capacitor (RLC) circuits[5] featuring a self-inductance L from the ring, a ring Ohmic resistance R, and a capacitance C from the split in the ring. Metamaterials with negative refractive properties are then formed as a periodic array of SRR, which are coupled by mutual inductance and arrayed in a material of dielectric constant ε. From the standpoint of experiment, the requirements for the effective electromagnetic application of metamaterials in the THz region introduce the necessity of very high accuracy in the fabrication of SRR-based MMs to produce materials of uniform and consistent properties.

The electrodynamics of MM consisting of large numbers of loosely coupled SRRs has recently been studied for discrete lattices,[5] treating the array as a set of capacitively loaded loops[6] (see references therein). It was shown that the system of capacitively loaded loops support wave propagation. Since the coupling between the SSRs is due to induced voltages, these extended wave solutions are referred to as magneto-inductive waves (MI waves)[7–10] and represent a vast area of active research in the field of (a) artificial delay lines and filters, (b) dielectric Bragg reflectors, (c) slow-wave structures in microwave tubes, and (d) coupled cavities in accelerators, modulators, antenna array applications, etc. The RLC configuration of the SRRs cause the MMs based on them to exhibit a resonant frequency for both linear and nonlinear systems, and MI waves are observed to propagate within a frequency band near the resonant frequency.

In linear MMs, the SRRs are composed of both linear dielectric and magnetic materials so that the MI dispersion relations do not depend on the EM field intensities. Nonlinearity is incorporated into MMs by embedding Kerr-type medium[11,12] in the SRRs or by inserting certain nonlinear elements (e.g. diodes) in the SRR.[13–15] In the presence of nonlinearity, the excitations of the MMs become more complex and include additional excitations than just the MI waves and their modifications. In such systems, the self-modulation of MI wave propagation has recently been investigated by Kourakis *et al.*,[16] based on a nonlinear Schrödinger equation formulation. It was shown that the MI waves exhibit a transition leading to spontaneous energy localization through the generation of localized envelope structures, i.e. so called envelope solitons. In addition to the MI waves, new sets of solutions of the nature of bright and dark discrete breathers or intrinsic localized modes[5] are also found. These excitations are defined on a discrete lattice and are localized within a finite region of space. Unlike MI waves which continue to exist in the absence of nonlinearity in the system, discrete breathers and envelope solitons are no longer present in the linear limit of the system.

In this paper, we will look at the continuum limit of a one-dimensional discrete lattice model similar to that treated previously by Lazarides *et al.*[5] in studies of discrete lattice bright and dark breather (soliton-like) solutions. The continuum limit is valid in cases in which the envelope of the excitations changes slowly compared to the lattice constant of the periodic array of SRR. In this limit, the dynamics of the excitations of the system are found to be described by a nonlinear Klein-Gordon equation in both space and time variables, which is valid for weak coupling between the SRRs. This is convenient as the nonlinear Klein-Gordon equation has been used in a number of studies on different nonlinear systems (i.e. see references in Refs. 17–19), and in addition, it lends itself to a treatment of dissipative effects in the system. It is a well-known equation of mathematical physics, exhibiting a wide variety of interesting properties. As we shall see, the nonlinear Klein-Gordon equation gives a more general treatment of the system than does the nonlinear Schrödinger Equation approach used in Ref. 5. The nonlinear Schrödinger equation approach arises in a multiple-scale, small-amplitude perturbation theory treatment of the modes of the nonlinear Klein-Gordon equation. Consequently, from the mathematical standpoint, the solutions of the nonlinear Klein-Gordon equations are valid for arbitrary mode amplitudes, whereas the nonlinear Schrödinger equations solutions give small amplitude approximations to the modes of the nonlinear Klein-Gordon equations. A brief comparison of the nonlinear Klein-Gordon and Schrödinger equation envelope and breather modes is given. While both equations give envelope and breather soliton modes, the nonlinear Klein-Gordon equation has additional pulse soliton solutions. These modes are briefly discussed.

In the case of discrete breathers, we will develop a more general differential equation than either the nonlinear Klein-Gordon or Schrödinger equations to treat the system. The resulting equation handles the case of multiple large couplings between the SRR and is correct for arbitrary mode amplitude. A discussion of the solutions is given with a comparison of the solutions to the discrete system.

Our focus is on the continuum limit model of the one-dimensional array of loosely coupled SRRs. The formulation is obtained from the discrete model by applying a Taylor expansion in the lattice site indices, taking into account the magnetization of the metamaterial[20] to obtain the field behavior of the SRR array. The soliton solutions of the Klein-Gordon equation (envelope, breather, and pulse modes) for both bright and dark types are derived and studied for different cases of the system, and a comparison is then made of the dispersive properties of the discrete and continuum limit modes of the system. Despite voluminous work done on SRR-based metamaterials, to the best of our knowledge, no derivation of classical dynamics from the Hamiltonian of the system has been attempted. Hence, an attempt is made in this direction in the this paper.

The order of the paper is as follows. In Sec. II, the model of the one-dimensional chain of inductively coupled SRRs is described. A brief overview of the theory of the discrete system is given, the continuum limit is taken to obtain the Klein-Gordon equation for the system, and bright and dark soliton solutions are obtained with comparisons to appropriate modes in the nonlinear Schrödinger equation approximation. Discussions are given of dissipative effects in the system. In Sec. III, the intrinsic localized mode solutions of the discrete limit of the system are given, and their limiting forms in the continuum limit are discussed. A differential equation for the breather modes is developed for general couplings between the SRRs. These are compared to the bright and dark soliton solutions obtained in Sec. II. The effects of further neighbor and strong couplings between the SRRs are discussed. In Sec. IV, numerical results are presented to illustrate the equations generated in Secs. II and III. Section V contains conclusions.

II. THEORETICAL DEVELOPMENT

We consider a model of weakly coupled SRRs similar to that treated in Ref. 5 for the discrete limit. A brief review of the discrete model is given followed by discussions of its continuum limit and continuum limit solutions.

A typical SRR unit is a wire ring with a single cut made perpendicular to the plane of the ring so as to form a slit in the ring. The resulting unit then looks roughly like a letter C. The ring operates as an inductor-resistor and the slit as a capacitor so that the SRR is essentially an LRC resonant circuit. Adding Kerr nonlinear dielectric filling to the SRRs' capacitor slits makes the SRRs give a nonlinear response to applied electormagnetic fields. This comes from the field dependence of the Kerr permittivity ε which is given by:

$$\varepsilon(|E|^2) = \varepsilon_0 \left(\varepsilon_l + \alpha \frac{|E|^2}{E_c^2} \right). \tag{1}$$

Here, E is the electric field in the Kerr medium, E_c is a characteristic (large) electric field, ε_l is the linear part of the permittivity, ε_0 is the permittivity of vacuum, and $\alpha = +1\,(-1)$ corresponds to a self-focusing (self-defocusing) Kerr medium. The Kerr filled SRRs acquire a field-dependent capacitance described by:

$$C(E_g) = \varepsilon(|E_g|^2)A/d_g, \tag{2}$$

where A is the cross-sectional area of the SRR wire and slit, E_g is the electric field within and along the axis of the SRR slit, and d_g is the distance between the upper and lower edges of the SRR slit. From the general relationship for a voltage-dependent capacitance, $C(U) = Q/U$, and Eq. (2), the charge, Q, stored within the capacitor slit of an SRR is given by

$$Q = C_l \left(1 + \alpha \frac{U^2}{\varepsilon_l U_c^2}\right) U \quad (3)$$

where $U = d_g \mathrm{E}_g$, $C_l = \varepsilon_0 \varepsilon_l (A/d_g)$ is the linear capacitance, and $U_c = d_g \mathrm{E}_c$.

The SRR interacts as an LRC circuit with an externally applied EM field, giving a characteristic frequency response as determined by its inductive, capacitive, and resistive components. In the limit, considered here, of weak nonlinearity and very small resistance, the resonant frequency of the freestanding SRR is approximately that of its linear limit, i.e. $\omega_l = \frac{1}{\sqrt{LC_l}}$. In the presence of a time-dependent magnetic field of the form

$$H = H_0 \cos(\omega t) \quad (4)$$

applied perpendicular to the plane of the SRR loop, an induced electromotive force (emf) proportional to the area, S, of the SRR loop is generated within the loop. This emf is given by

$$\mathrm{emf} = \mu_0 \omega S H_0 \sin(\omega t). \quad (5)$$

In addition to interactions with external fields, the SRR can have self interactions and interactions with other SRRs. As discussed later, a time-varying current within a single SRR leads to a self-inductive interaction, and two or more neighboring SRR interact with one another through mutual inductive couplings. For the periodic arrays of SRRs we consider here, it was shown in Ref. 5 that only nearest neighbor SRR couplings are important.

The MMs under study are one-dimensional, discrete, periodic chains formed of identical nonlinear SRR units. The SRR units forming the one-dimensional chain are labeled consecutively by integers, n. In a first case, the loops of all of the SRRs lie in a common plane. The electromagnetic modes of interest have magnetic components perpendicular to the common plane of the SRRs, with electric components transverse to the SRR slits. Only the magnetic component excites an emf in the SRRs, resulting in an oscillating current in each SRR loop and the development of an oscillating voltage difference U (electric field E_g) across (within) the slits. The additional possibility of a small externally applied magnetic field of the form given in Eq. (4) is also taken into account in our calculations. Neighboring SRRs of the linear chain have their centers separated by a distance d such that $\{x_n = nd\}$ for n an integer gives the coordinates of the SRR along the x axis. Each SRR has a self-inductance, L, and a weak mutual inductance, M, between its nearest neighbor loops. However, more general, device-oriented, optimized models, with a greater range of L and M values must be taken into account in practical applications.[5]

In a second case, the loops of the SRR units in our first system are rotated by 90° about the axis perpendicular to the chain and in the common plane of the loops of the first system. This forms a one-dimensional chain of SRRs lying in parallel planes perpendicular to the axis of the chain and cutting the x axis at $\{x_n = nd\}$ for n an integer. Again the coupling between SRRs is inductive, and externally applied fields have magnetic components perpendicular to the planes of the SRRs. As shown in Ref. 5, the difference in the equations describing these two systems with differing SRR orientations is the sign of the mutual inductance, M. For the case in which the SRR are in a common plane $M < 0$, for the case in which the SRR lie in planes perpendicular to the axis of the one-dimensional array $M > 0$.

Following Lazarides *et al.*,[5] the Hamiltonian of these two systems (in a modified notation, which brings it into closer analogy with systems of classical mechanics) is given by:

$$H = \sum_n \left[\frac{1}{2}\dot{q}_n^2 + \lambda \dot{q}_n \dot{q}_{n+1} + V(q_n)\right] \quad (6)$$

where $\frac{dq_n}{d\tau} = \dot{q}_n = i_n$ is the current in the SRR centered about $x = nd$, $\lambda = M/L$ is a coupling parameter, the connonical momenta are given by $p_n = \dot{q}_n + \lambda(\dot{q}_{n+1} + \dot{q}_{n-1})$, $V(q_n) = -\int_0^{q_n} f(q_n')dq_n'$ is the nonlinear on-site potential, and we make the approximation that $f(q_n) \approx -[q_n - (\frac{\alpha}{\varepsilon_l})q_n^3 + 3(\frac{\alpha}{\varepsilon_l})^2 q_n^5]$. Here, from Eq. (3), we have defined a dimensionless time $\tau = t\omega_l$ and dimensionless charge $q_n = Q_n/Q_c$ where $Q_c = C_l \mathrm{U}_c$. As a further generalization, in the presence of an external field of the type in Eq. (4), the Hamiltonian in Eq. (6) applies where now the connonical momenta are given by $p_n = \dot{q}_n + \lambda(\dot{q}_{n+1} + \dot{q}_{n-1}) + \Lambda\cos(\Omega\tau)$ with $\Lambda = \frac{\mu_0 S H_0 \omega_l}{U_c}$ and $\Omega = \omega/\omega_l$. The definitions of the other terms in the Hamiltonian are unchanged in the presence of the external field.

The Langrangian for the system with general external field interactions is

$$L = \sum_n \left[\frac{1}{2}\dot{q}_n^2 + \lambda \dot{q}_{n+1}\dot{q}_n - V(q_n) + \Lambda \dot{q}_n \cos(\Omega\tau)\right]. \quad (7)$$

From Eq. (7) the connonical momenta, $p_n = \frac{\partial L}{\partial \dot{q}_n} = \dot{q}_n + \lambda(\dot{q}_{n+1} + \dot{q}_{n-1}) + \Lambda\cos(\Omega\tau)$, and the equations of motion,

$$\ddot{q}_n + \lambda(\ddot{q}_{n+1} + \ddot{q}_{n-1}) - f(q_n) - \Lambda\Omega\sin(\Omega\tau) = 0, \quad (8)$$

are obtained, consistent with the Hamiltonian formulation given above.

In the continuum limit, we make the expansion $\ddot{q}_{n\pm1} = [\ddot{q} \pm d\frac{\partial \ddot{q}}{\partial x} + \frac{1}{2}d^2\frac{\partial^2 \ddot{q}}{\partial x^2} + \cdots]_{x=nd}$ where $q = q(x)$ is the generalized continuum variable of q_l. Retaining terms of order q^3 in $f(q_n)$, Eq. (8) in this approximation is given by

$$\ddot{q} + ad^2\frac{\partial^2 \ddot{q}}{\partial x^2} + b\left[q - \frac{\alpha}{\varepsilon_l}q^3\right] - b\Lambda\Omega\sin(\Omega\tau) = 0 \quad (9)$$

where $a = \lambda/(1 + 2\lambda)$ and $b = 1/(1 + 2\lambda)$ and is then rewritten as

$$[1 + ad^2 D]\ddot{q} = -b\left[q - \frac{\alpha}{\varepsilon_l}q^3\right] + b\Lambda\Omega\sin(\Omega\tau) \quad (10)$$

TABLE I. (a) Parameters for the solutions of the Klein-Gordon and nonlinear Schrödinger equations that are of the general form $q(x,\tau) = \sqrt{\frac{\varepsilon_l}{\alpha}} a_0 \text{sech}[\beta x - \delta\tau] \exp[i(px - \omega\tau)] \exp[-\frac{\gamma}{2}\tau] + cc$. (b) Parameters of the solutions of the Klein-Gordon and nonlinear Schrödinger equations that have solutions of the general form $q(x,\tau) = \sqrt{\frac{\varepsilon_l}{\alpha}} a_0 \tanh(\beta x - \delta\tau) \exp[i(px - \omega\tau)] \exp[-\frac{\gamma}{2}\tau] + cc$.

	p	ω	β^2	δ
K G $\gamma = 0$	k	$\omega^2 = b\left[1 - \frac{3}{2}a_0^2 + ak^2\right]$	$\frac{3}{2}\frac{a_0^2}{a(1-abk^2/\omega^2)}$	$ab\left(\frac{k\beta}{\omega}\right)$
K G $\gamma \neq 0$	k	$\omega^2 = b\left[1 - \frac{3}{2}a_0^2 + ak^2\right] - \frac{\gamma^2}{4}$	$\frac{3}{2}\frac{a_0^2}{a(1-abk^2/\omega^2)}$	$ab\left(\frac{k\beta}{\omega}\right)$
NLS $\gamma = 0$	$k - d$	$\omega = \phi - d\frac{abk}{\phi} - \frac{3}{4}\frac{b}{\phi}a_0^2 + \frac{ab(1-abk^2/\phi^2)}{2\phi}d^2$ where $\phi^2 = b + abk^2$	$\frac{3}{2}\frac{a_0^2}{a(1-abk^2/\phi^2)}$	$ab\left(\frac{\beta}{\phi}\right)\left[k - d + \frac{abk^2}{\phi^2}d\right]$
K G $\gamma = 0$	k	$\omega^2 = b\left[1 - 3a_0^2 + ak^2\right]$	$\frac{3}{2}\frac{a_0^2}{a\left(\frac{abk^2}{\omega^2}-1\right)}$	$ab\left(\frac{k\beta}{\omega}\right)$
K G $\gamma \neq 0$	k	$\omega^2 = b\left[1 - 3a_0^2 + ak^2\right] - \frac{\gamma^2}{4}$	$\frac{3}{2}\frac{a_0^2}{a\left(\frac{abk^2}{\omega^2}-1\right)}$	$ab\left(\frac{k\beta}{\omega}\right)$
NLS $\gamma = 0$	$k - d$	$\omega = \phi - \frac{abkd}{\phi} - \frac{3}{2}\frac{ba_0^2}{\phi} - \frac{ab(abk^2/\phi^2-1)}{2\phi}d^2$ where $\phi^2 = b + abk^2$	$\frac{3}{2}\frac{a_0^2}{a\left(\frac{abk^2}{\phi^2}-1\right)}$	$ab\left(\frac{\beta}{\phi}\right)\left[k - d + \frac{abk^2}{\phi^2}d\right]$

where $D = \frac{\partial^2}{\partial x^2}$. Using the formal relationship $\frac{1}{[1+ad^2D]} = 1 - ad^2D + \cdots$ (i.e. for modes slowly varying in space and/or for $\lambda \ll 1$) the nonlinear Klein-Gordon equation is obtained as

$$\frac{\partial^2 q}{\partial \tau^2} - ab\frac{\partial^2 q}{\partial x^2} + b\left[q - \frac{\alpha}{\varepsilon_l}q^3\right] - b\Lambda\Omega\sin(\Omega\tau) + \gamma\frac{\partial q}{\partial \tau} = 0 \tag{11}$$

where x is rescaled to be expressed in units of d (i.e. $x/d \rightarrow x$) and a term in γ is added to include the possibility of dissipation in the system. In the following, we shall discuss the rich variety of solutions of Eq. (11). These include: bright and dark breather and envelope solitons and pulse soliton modes.

Following the discussions of Bandyopadhyay *et al.*[17,18] the solutions of Eq. (11) for $\Lambda = 0$ and $\gamma = 0$ for breather and envelope soliton modes are obtained within the rotating wave approximation (RWA). We assume a solution of the form $q(x,\tau) = \sqrt{\frac{\varepsilon_l}{\alpha}}A(\beta x - \delta\tau)e^{i(kx-\omega\tau)} + cc$ for $\frac{\varepsilon_l}{\alpha} > 0$, substitute into Eq. (11), and use the RWA to retain only terms in $e^{-i\omega\tau}$. The resulting equation is

$$\frac{d^2A}{dy^2} + 2i\frac{\delta\omega - ab\beta k}{\delta^2 - ab\beta^2}\frac{dA}{dy} + \frac{b - \omega^2 + abk^2}{\delta^2 - ab\beta^2}A + \frac{3b}{ab\beta^2 - \delta^2}A|A|^2 = 0, \tag{12}$$

where $y = \beta x - \delta\tau$. Two solutions of interest of this equation for amplitude $\sqrt{\frac{\varepsilon_l}{\alpha}}a_0$ give:

$$q(x,\tau) = \sqrt{\frac{\varepsilon_l}{\alpha}}a_0\,\text{sech}[\beta x - \delta\tau]\exp[i(kx - \omega\tau)] + cc \tag{13a}$$

where $\omega^2 = b[1 - \frac{3}{2}a_0^2 + ak^2]$, $\beta^2 = \frac{3}{2}\frac{a_0^2}{a(1-abk^2/\omega^2)}$, $\delta = ab(\frac{k\beta}{\omega})$, and

$$q(x,\tau) = \sqrt{\frac{\varepsilon_l}{\alpha}}a_0\tanh[\beta x - \delta\tau]\exp[i(kx - \omega\tau)] + cc \tag{13b}$$

where $\omega^2 = b[1 - 3a_0^2 + ak^2]$, $\beta^2 = \frac{3}{2}\frac{a_0^2}{a(abk^2/\omega^2-1)}$, $\delta = ab(\frac{k\beta}{\omega})$. These results are summarized in Table I. The breather modes are obtained from the $\delta = 0$ solutions and the envelope modes have $\delta \neq 0$, giving a mode envelope which moves with a constant velocity along the x axis. Solutions for $\frac{\varepsilon_l}{\alpha} < 0$ are obtained by making a few simple changes in the above Eqs. (12) and (13).[21]

The case including $\gamma \neq 0$ is much more difficult, but a limiting form for $\tau \ll \frac{1}{\gamma}$ can be obtained by making a few changes in the parameters entering into Eqs. (13). For the sech form of the solution, Eq. (13a) is multiplied by $\exp(-\gamma\tau/2)$, and we redefine $\omega^2 = b[1 - \frac{3}{2}a_0^2 + ak^2] - \frac{\gamma^2}{4}$, $\beta^2 = \frac{3}{2}\frac{a_0^2}{a(1-abk^2/\omega^2)}$, and $\delta = ab(\frac{k\beta}{\omega})$. For the tanh form of the solution, Eq. (13b) is multiplied by $\exp(-\gamma\tau/2)$, and we redefine $\omega^2 = b[1 - 3a_0^2 + ak^2] - \frac{\gamma^2}{4}$, $\beta^2 = \frac{3}{2}\frac{a_0^2}{a(\frac{abk^2}{\omega^2}-1)}$, and $\delta = ab(\frac{k\beta}{\omega})$. These results are summarized in Table I. The approximation made to obtain these forms can be understood by substituting the forms into Eq. (11) and apply the RAW to the frequency terms. Upon substitution into Eq. (11) and multiplying by $\exp(\gamma\tau/2)$ an explicit time dependent factor of $\exp(-\gamma\tau)$ is retained only in the small nonlinear term of the resulting equation. For the conditions on time mentioned above, we may take $\exp(-\gamma\tau) \approx 1$ with the resulting equation being solved by the proposed sech and tanh forms. The results give an indication of the timescale over which a mode may be expected to exist and retain its shape as a bright or dark pulse before it decays into a combination of other modes in the system. It is important to note that in the expressions for β^2 and δ, ω now depends on γ, and the resulting forms give the leading dependence in γ.

For comparison, we look at results for the envelope and breather modes from the nonlinear Schrödinger equation, which is an approximation to the nonlinear Klein-Gordon equation and gives approximate solutions to the nonlinear Klein-Gordon equation.[22,23] The nonlinear Schrödinger equation is obtained from the nonlinear Klein-Gordon equation by treating it in a multiple scale perturbation theory approach within the context of the rotating wave approximation.[22,23] This approximation is briefly reviewed in the Appendix, where it is shown to be an expansion in the mode amplitude taken as a small parameter. The theory is correct to the third order in the small parameter of the mode amplitude, whereas our discussions above are valid for a general amplitude, as they are not based on perturbation theory in the mode amplitude. In the nonlinear Schrödinger equation approach, approximate sech and tanh forms similar to those discussed above are obtained and given in detail in the Appendix. The approximations closest to the nonlinear Klein-Gordon solutions in Eqs. (13a) and (13b) are obtained for $d = 0$ where d is defined in the Appendix. Taking $\omega = \phi - \frac{3}{4}\frac{b}{\phi}a_0^2$, $\beta^2 = \frac{3}{2}\frac{a_0^2}{a(1-\frac{abk^2}{\phi^2})}$, $\phi^2 = b(1 + ak^2)$, and $\delta = (\frac{ab\beta k}{\phi})$ in Eq. (13a) gives the nonlinear Schrödinger equation sech form mode. Taking $\omega = \phi - \frac{3}{2}\frac{b}{\phi}a_0^2$, $\beta^2 = \frac{3}{2}\frac{a_0^2}{a(\frac{abk^2}{\phi^2}-1)}$, and $\delta = (\frac{ab\beta k}{\phi})$ in Eq. (13b) gives the nonlinear Schrödinger equation tanh form mode. The most general forms for the sech and tanh modes of the nonlinear Schrödinger equation as discussed in the Appendix are listed in Table I. Notice that the expressions for the pulse width and velocity are similar to the nonlinear Klein-Gordon equation, but the nonlinear Klein-Gordon solutions are not restricted to small values of a_0.

In addition to the bright and dark breather and envelope pulse solutions discussed above, another set of solutions, known as pulse solitons, exist for the nonlinear Klein-Gordon equation in Eq. (11). These modes are not found in the nonlinear Schrödinger equation. In addition, they do not require the RWA approximation and are exact solutions of the nonlinear Klein-Gordon equation. They are pure pulses that are not modulated by an $\exp[-i\omega\tau]$ time dependence or an $\exp[i(kx - \omega\tau)]$ space time dependence as are the breather and envelope solitions, respectively. A first solution is a dark soliton-type given by

$$q(x,\tau) = A\tanh(Bx - C\tau) \tag{14}$$

where substitution into Eq. (11) gives $A = \pm\sqrt{\frac{\varepsilon_l}{\alpha}}$, and the coefficients B and C are related by $2[C^2 - abB^2] = b$ so that $C = \pm\sqrt{b(\frac{1}{2} + aB^2)}$. The velocity, $\frac{C}{B}$, of the excitation is, consequently, related to its width, $\frac{2}{B}$. A second solution is a bright soliton solution given by

$$q(x,\tau) = A\text{sech}(Bx - C\tau) \tag{15}$$

where substitution into Eq. (11) gives $A = \mp\sqrt{\frac{2\varepsilon_l}{\alpha}}$. Here, B and C are now related by $[C^2 - abB^2] = -b$ so that $C = \pm\sqrt{b(aB^2 - 1)}$. Again, the velocity, $\frac{C}{B}$, of the excitation is related to its width, $\frac{2}{B}$.

The behavior of the pulse modes in the presence of dissipation can be handled in a similar manner to the envelope and breather modes. For the case that $\gamma \neq 0$, for $\tau \ll \frac{1}{\gamma}$, the dark soliton solution in Eq. (14) changes so that the tanh function becomes multiplied by $\exp[-\frac{\gamma}{2}\tau]$. In addition, A changes to $A = \pm\sqrt{\frac{\varepsilon_l}{\alpha}(1 - \frac{1}{b}\frac{\gamma^2}{4})}$, and B and C are related by $C^2 = b[\frac{1}{2} + aB^2] - \frac{\gamma^2}{8}$. Likewise, for $\gamma \neq 0$ in the limit $\tau \ll \frac{1}{\gamma}$ the bright soliton solution in Eq. (15) changes so that the sech function becomes multiplied by $\exp[-\frac{\gamma}{2}\tau]$ and A, B, and C are given by $A = \pm\sqrt{2\frac{\varepsilon_l}{\alpha}(1 - \frac{1}{b}\frac{\gamma^2}{4})}$ and $C^2 = b[aB^2 - 1] + \frac{\gamma^2}{4}$. These later forms indicate the time over which the pulses retain their basic initial forms. It should be noted that, while the pulse modes are formal solutions of the nonlinear Klein-Gordon equation in Eq. (11), the expansion of $f(q_n)$ given below Eq. (6) give significant corrections from the q_n^5 terms for the case $\gamma = 0$ due to the magnitude of the pulse amplitude. As $\gamma \neq 0$ tends to decrease the mode amplitude, it may be possible for this case to observe pulses which decay in the system.

Another exact solution case of Eq. (11) is that in which $\gamma \neq 0$.[17,18] This mode is again not present in the nonlinear Schrödinger equation approximation. A solution in this limit exists and is of the form:

$$q(x,\tau) = \frac{A}{\sqrt{1 + B\exp[Cx - D\tau]}}. \tag{16}$$

Upon substitution into Eq. (11), it is found that $A = \pm\sqrt{\frac{\varepsilon_l}{\alpha}}$, $C = \pm\sqrt{\frac{4b}{a\gamma^2}}$, and $D = -\frac{2b}{\gamma}$, and B fixes the initial conditions at $x = 0$ and $\tau = 0$. At $\tau = 0$ and $a > 0$, the characteristic width of the excitation is $\frac{1}{2}\sqrt{\frac{a\gamma^2}{b}}$ with a rate of decay given by $\sqrt{ab}$. For $a < 0$, however, there are no real solutions for $q(x, \tau)$.

III. RELATIONSHIP TO DISCRETE MODES

In this section, the intrinsic localized (breather) modes of the discrete limit of the system are briefly discussed. These are stationary modes which are periodically modulated in time. This is followed by a development of the continuum limit for the intrinsic localized modes in terms of a differential equation and a comparison with the bright and dark soliton modes obtained in the previous section. The continuum limit equation developed covers the behavior of the system over a parameter range that is different than that treated by the nonlinear Klein-Gordon equation or its nonlinear Schrödinger equation approximation.

The discrete breathers are obtained by looking for time-dependent solutions of Eq. (8) of the form $q_n = q_{n,0}e^{-i\omega\tau} + cc$

and retaining the terms in $e^{\pm i\omega\tau}$, i.e. applying the RWA. This reduces Eq. (8) to

$$-\omega^2[q_{n,0}+\lambda(q_{n+1,0}+q_{n-1,0})]+q_{n,0}-\frac{3\alpha}{\varepsilon_l}|q_{n,0}|^2 q_{n,0}=0. \tag{17}$$

A good analytical approximation to the breather modes of Eq. (17) and their dispersion relation is obtained using methods developed by Sievers *et al.*[24–26] In this method a pulse form with adjustable parameters is assumed, and the parameters are determined which give the best approximate solution of Eq. (17).

Two types of discrete breathers are encountered. The first type is a pulse centered at $n=0$ and taken to be of the form $q_{0,0}=\alpha_0$ and $q_{n,0}=\alpha_0 A e^{-(|n|-1)r}$ for $n\neq 0$. The second type is a pulse centered at $n=0$ and taken to be of the form $q_{0,0}=\alpha_0$ and $q_{n,0}=\alpha_0(-1)^n e^{-(|n|-1)r}$ for $n\neq 0$. In the method of Sievers *et al.*, the parameters A, r, and ω are determined as functions of the pulse height, α_0, by choosing them to satisfy Eq. (17) for $n=0$, 1, and $n\to\infty$. This method is effective for highly localized pulses and for our system yields three nonlinear equations for A, r, and ω:

$$\frac{3\alpha}{\varepsilon_l}\alpha_0^2 \pm 2\lambda A \mp 2\lambda\left[1-\frac{3\alpha}{\varepsilon_l}\alpha_0^2\right]\cosh(r)=0, \tag{18a}$$

$$\lambda(1+Ae^{-r}) \pm \frac{3\alpha}{\varepsilon_l}\alpha_0^2 A^3 - 2\lambda\left[A-\frac{3\alpha}{\varepsilon_l}\alpha_0^2 A^3\right]\cosh(r)=0, \tag{18b}$$

and

$$\omega^2=[1\pm 2\lambda\cosh(r)]^{-1}. \tag{18c}$$

Here, the upper (lower) signs are for the first (second) type of excitations discussed above Eq. (18).

We next use the expansion of q_n and $q_{n\pm1}$ in terms of the continuous variable x, as made in going from Eqs. (8) to (9) in Sec. II, to investigate the continuum limit form of Eq. (17) and its solutions. This is valid for broad pulse forms. In this limit, Eq. (17) becomes

$$\omega^2\lambda\frac{\partial^2 q}{\partial x^2}-[1-(1+2\lambda)\omega^2]q+\frac{3\alpha}{\varepsilon_l}|q|^2 q=0 \tag{19}$$

where x is again rescaled to be measure in units of d. Notice that unlike in the derivation of Eq. (11), a weak coupling approximation for λ is not made in obtaining Eq. (19) from Eqs. (17) and (19) holds for arbitrary coupling strength λ. Consequently, for the case of the breather modes, Eq. (19) is mathematically more general than Eq. (11), as its solution does not formally restrict the range of λ.

Equation (19) yields a first type of solution of the form

$$q(x)=\pm\sqrt{-\frac{\varepsilon_l[(1+2\lambda)\omega^2-1]}{3\alpha}}\tanh\left\{\left[\frac{(1+2\lambda)\omega^2-1}{2\lambda\omega^2}\right]^{1/2}x\right\}\exp[-i\omega\tau] \tag{20a}$$

and a second type of solution of the form

$$q(x)=\pm\sqrt{\frac{2\varepsilon_l[1-(1+2\lambda)\omega^2]}{3\alpha}}\operatorname{sech}\left\{\left[\frac{1-(1+2\lambda)\omega^2}{\lambda\omega^2}\right]^{1/2}x\right\}\exp[-i\omega\tau]. \tag{20b}$$

For $-\frac{\varepsilon_l}{\alpha}$, $\lambda>0$, Eq. (20a) gives real values of $q(x)$ when $[1+2\lambda]^{-1}<\omega^2$, and for $-\frac{\varepsilon_l}{\alpha}$, $\lambda<0$ when $[1+2\lambda]^{-1}>\omega^2$. For $\frac{\varepsilon_l}{\alpha}$, $\lambda>0$, Eq. (20b) gives real values of $q(x)$ when $[1+2\lambda]^{-1}>\omega^2$, and for $\frac{\varepsilon_l}{\alpha}$, $\lambda<0$ when $[1+2\lambda]^{-1}<\omega^2$. Both of these modes, which are stationary and have frequency modulated envelopes, are distinct from the envelope modes discussed in Sec. II. In addition, to exist as independent, nonresonant modes, they must have frequencies outside the band of MI waves that occur in the interval $[1+2|\lambda|]^{-1}\leqslant\omega^2\leqslant[1-2|\lambda|]^{-1}$. The solution in Eq. (20b) is the limiting form of the first type of intrinsic localized mode solution considered below Eq. (17). There is no continuum limit of the second type of intrinsic localized mode solution considered below Eq. (17).

IV. RESULTS AND DISCUSSION

We shall study the behavior of the various modal solutions of the nonlinear SRRs arrays obtained above, using parameters based on those found in Ref. 5. Comparisons are made of the different types of continuum limit excitations, in which the excitation wave functions change slowly over the position on the chain, and the discrete limit excitations[5] in which the wave functions of the excitations are rapidly varying functions of the position on the chain. The parameters used for the generation of the numerical results are taken from Ref. 5 with an allowance made for a slight change in notation between the two papers. A primary focus will be on the breather modes, which are stationary on the chain.

A. Continuum limit

The envelope and breather solutions of the nonlinear Klein-Gordon equation have dispersion relations $\omega^2=b[1-\frac{3}{2}a_0^2+ak^2]$ and $\omega^2=b[1-3a_0^2+ak^2]$ for the sech and tanh modes, respectively. These expressions are valid for arbitrary a_0, and should be compared with the dispersion relation of the long wavelength limit of the plain wave modes given by $\omega^2=b[1-a_0^2+ak^2]$. Consequently, the frequencies of these modes fall below those of the plain wave modes. In all cases, the dispersion relations depend only on

$b = 1/(1-\lambda)$ where λ is given by the mutual inductance and is assumed to be small. From Table I, it is noted that the nonlinear Schrödinger equation gives the leading term in the expansion of ω in a small parameter a_0. In the presence of dissipation, a factor of $-\frac{\gamma^2}{4}$ is added to the expression for ω^2 from the nonlinear Klein-Gordon equation for both the tanh and sech solitons. For the systems discussed below, it was shown in Ref. 16 that $\gamma \approx 0.0016$ gives a reasonable approximation to the dissipative effects of the system so that, for that particular realization of the SRR system, the correction from γ is small and can be accounted for by the leading order terms discussed earlier.

In Sec. III, a more general continuum equation for the breather modes was obtained in Eq. (19). Equation (19) applies to the case of general λ, whereas the results from Eq. (11) require λ to be small. In addition, Eq. (19) can be generalized to treat further neighbor couplings. In this case, Eq. (19) becomes

$$\omega^2 b_1 \frac{\partial^2 q}{\partial x^2} - [1-(1+2b_2)\omega^2]q + \frac{3\alpha}{\varepsilon_l}|q|^2 q = 0 \qquad (21)$$

where $b_1 = \sum_{j=1}^{\infty} j^2\lambda_j$ and $b_2 = \sum_{j=1}^{\infty} \lambda_j$ and λ_j for $j = 1, 2,$ 3, etc. at the first, second, etc. neighbor couplings. Solutions in tanh and sech forms similar to those in Eq. (20) are readily obtained. Consequently, Eqs. (19) and (21) are generalizations of the nonlinear Klein-Gordon equation to handle general couplings, and their solutions reduce to those in Sec. II in the small coupling limit. We shall focus on a comparison of the results of Eq. (19) with the discrete theory of the breather mode in discussions in the next subsection.

The other types of nondissipative modes arising as exact solutions of the nonlinear Klein-Gordon equation are the pulsed modes in Eqs. (14) and (15). These are not modulated by plain wave forms but are pure pulses that travel along the x axis. The condition for the observation of both the tanh and sech modes is that the pulse amplitudes must be small enough so that $A \ll 1$.

For the case in which $\Lambda = 0$ and $\gamma \neq 0$, a dissipative solution of the nonlinear Klein-Gordon equation has been given in Eq. (16). The solution involves an exponential factor, $\exp[Cx - D\tau]$, that sets the space-time dependence of the excitation. A natural characterization of the wave function from this form is given by the width, $\frac{1}{C}$, of the region over which the variation of the exponential factor is significant at $\tau = 0$; the rate of change of the exponential factor in time, $\frac{D}{C}$; and the amplitude, $A = \pm\sqrt{\frac{\varepsilon_l}{\alpha}}$, which only depends on the linear part of the Kerr dielectric constant.

The rate of decay of the solution in Eq. (16) for $a > 0$ is independent of the width of the excitation. For $\lambda = 0.02$ and 0.05, the rate is $\sqrt{ab} = 0.1360$ and 0.2033, respectively, and shows an increase with an increase of the coupling between the SRR. Depending on the value of B from the boundary conditions, the solution in Eq. (16) reaches a variety of $x \to \infty$ and/or $\tau \to \infty$ limits.

B. Discrete results

The set of nonlinear equations, Eq. (18), is solved numerically for the wave function parameters A and e^{-r} and the modal frequencies ω as functions of the wave function peak height or amplitude, α_0. In Fig. 1, results are presented for ω versus $\sqrt{|\frac{3\alpha}{\varepsilon_l}|}\alpha_0$ for the case in which $\varepsilon_l = 6.0$, $\lambda = \pm 0.02$ and for $\alpha = \pm 1$. (Here, we choose parameters in our theory which, following a slight change of notation, give a comparison with results in Ref. 5.) For the cases $\alpha = 1$ and $\lambda = \pm 0.02$, ω is found to be a mildly monotonically decreasing function of $\sqrt{|\frac{3\alpha}{\varepsilon_l}|}\alpha_0$, and for the cases $\alpha = -1$ and $\lambda = \pm 0.02$, ω is a mildly increasing function of $\sqrt{|\frac{3\alpha}{\varepsilon_l}|}\alpha_0$. The gap between the sets of increasing and decreasing solutions at $\sqrt{|\frac{3\alpha}{\varepsilon_l}|}\alpha_0 = 0$ contains the region of MI plane wave solutions.

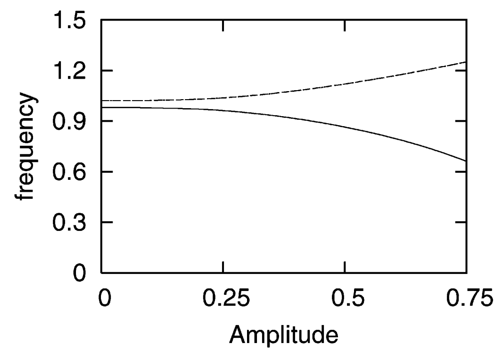

FIG. 1. Plot of the frequency dispersion of the intrinsic localized modes as a function of the peak height or amplitude, $\sqrt{|\frac{3\alpha}{\varepsilon_l}|}\alpha_0$. The upper (monotonically increasing) curves are for $\alpha = -1$ and $\lambda = \pm 0.02$, and the lower (monotonically decreasing) curves are for $\alpha = 1$ and $\lambda = \pm 0.02$.

We note that the method in Eqs. (17) and (18) is best for small nonlinearities (note: Eq. (3) is solved to the third order in the charge), whereas the essentially exact numerical solutions in Ref. 5 were evaluated for large charges and currents. Nevertheless, in the range of our plot in Fig. 1, results from Ref. 5 give points for $\sqrt{|\frac{3\alpha}{\varepsilon_l}|}\alpha_0 = 0.566$ at frequencies $\omega = 0.824$ and 1.151. These two points are in good agreement with the more general results of our theory presented in Fig. 1.

Results for the continuum limit of the intrinsic localized modes from Eqs. (19) and (20) are shown in Fig. 2. These represent broad pulses and transition widths that are not as highly localized along the chain as are those for the solution in Fig. 1. In Fig. 2(a), the frequency of the tanh and sech excitations are presented as functions of their widths for $\varepsilon_l = \pm 6$ and $\lambda = \pm 0.02$. (Note: the width of the sech and tanh modes are defined as the separations between the -1 and $+1$ arguments of these functions, respectively.) The two sets of upper curves and the two sets of lower curves are for sech and tanh modes with various combinations of λ and $\frac{\varepsilon_l}{\alpha}$. Outside the frequency band of MI modes, the tanh solutions in Fig. 2(a) are only found to exist for widths that are <1.41 and are, consequently, not practically realized in the system. The sech solutions exist outside the MI frequency band for the entire range of Fig. 2(a) widths shown. In Fig. 2(b), the frequency of the tanh and sech excitations are presented as a function of their heights for $\varepsilon_l = \pm 6$ and $\lambda = \pm 0.02$. Here, there is a general clustering of the sech and tanh solution by λ and $\frac{\varepsilon_l}{\alpha}$, but now the curves are resolved on the scale of the plot.

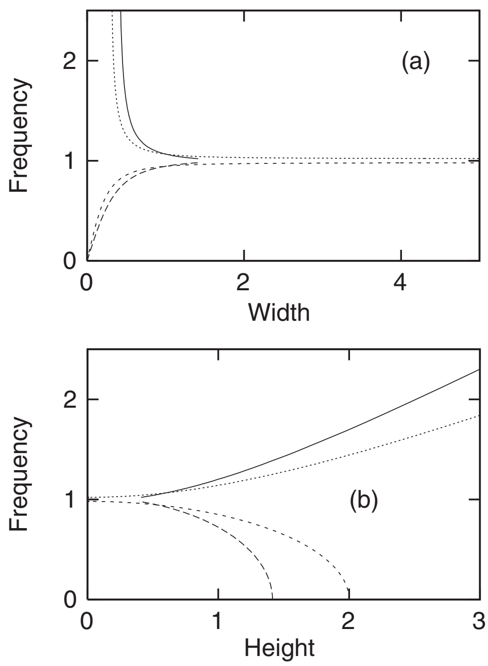

FIG. 2. Plot for continuum limit modes of Eqs. (20a) and (20b). Results are for (a) frequency versus width for $\lambda = -0.02$, $\frac{\varepsilon_l}{\alpha} = -6.0$ sech and $\lambda = 0.02$, $\frac{\varepsilon_l}{\alpha} = -6.0$ tanh modes (upper curves with the tanh results above the sech results) and $\lambda = 0.02$, $\frac{\varepsilon_l}{\alpha} = 6.0$ sech and $\lambda = -0.02$, $\frac{\varepsilon_l}{\alpha} = 6.0$ tanh modes (lower curves with the sech results above the tanh results) and (b) frequency versus height curves for (from top to bottom at height = 1.0) $\lambda = 0.02$, $\frac{\varepsilon_l}{\alpha} = -6.0$ tanh mode; $\lambda = -0.02$, $\frac{\varepsilon_l}{\alpha} = -6.0$ sech mode; $\lambda = 0.02$, $\frac{\varepsilon_l}{\alpha} = 6.0$ sech mode; and $\lambda = -0.02$, $\frac{\varepsilon_l}{\alpha} = 6.0$ tanh mode.

The tanh solutions only exist in the plot for heights >0.39. An interesting feature of the continuum limit solutions is that one can set the frequency, which must be taken outside the band of MI modes, and generally find solutions for both tanh and sech modes. In addition, it is found that the dispersion relations of the continuum limit pulse mode sech solutions join naturally with the intrinsic localized mode dispersion relations.

V. CONCLUSION

A nonlinear Klein-Gordon equation, Eq. (11), was derived based on the Hamiltonian formulation of a one-dimensional periodic array of SRRs,[5] taking the continuum limit in which the modes of the system change slowly over adjacent lattice sites.[17–19] The conditions needed for the derivation of the nonlinear Klein-Gordon equation are that the pulse is broad so that the continuum limit is taken, and the coupling between the SRRs is weak. The solutions of the nonlinear Klein-Gordon equation include: bright and dark envelope solitons, bright and dark breather modes, bright and dark pulse solitons, and decaying modes of the form of Eq. (16). In addition to discussions of the solutions of the ideal system in the absence of dissipation, the leading order corrections in γ for the Klein-Gordon breather, envelope, and pulse solitons are obtained. These give the time interval over which such modes propagate without a significant change of shape.

For the case of envelope and breather modes, a comparison of the solutions of the nonlinear Klein-Gordon equation within the RWA with those of the nonlinear Schrödinger equation approximation to the Klein-Gordon modes was made.[22,23] The nonlinear Schrödinger equation for these modes is obtained by applying the RWA and a multiscale perturbation theory in a small parameter related to the mode amplitude of the Klein-Gordon equation. The solutions of the nonlinear Schrödinger equation are small amplitude approximations to the solutions of the modes of the nonlinear Klein-Gordon equation. Only envelope and breather modes are obtained from the nonlinear Schrödinger equations approximation.

We have also focused on the breather solutions, alternatively known as intrinsic localized modes. For the discrete system, Eq. (17) for the breather modes was studied. This equation is obtained directly from the general discrete system equation in Eq. (8) by applying breather boundary conditions. We calculated the dispersion relation and mode profile as functions of the amplitude of the localized peak amplitude using a theory of Sievers *et al.*[24,25] The dispersion relation, which is obtained as a solution of a simple set of three nonlinear equations, agrees well with two dispersion relation points, which are available from the essentially exact results presented in Ref. 5. A differential equation, Eq. (19), for the continuum limit of the discrete breather equations, Eq. (17), was also obtained. This equation is more general than the Klein-Gordon equation for the breather modes, as aside from the continuum limit and lack of restrictions on the mode amplitude, it does not place restrictions on the coupling between the SRRs in the chain. Consequently, discussions were also given as to how to generalize the equations to handle many neighbor couplings. Equation (19) is also found to be a consistent limiting form of the discrete breather obtained using the Sievers *et al.* formulation.[25] In the continuum limit of the equations for the intrinsic localized modes, Eq. (20), an additional tanh-type modal solution appears which is distinct from the pulse-type modes associated with intrinsic localized modes of the discrete limit in Eq. (17).

A future direction of this work will be on the existence of bisolitons and other types of multiple solitons in the SRR system studied in this paper.[27–39] Before these can be addressed, however, a thorough understanding of the various types of single excitations that are available due to the nonlinearity of the systems is needed.

ACKNOWLEDGMENTS

ARM would like to thank Michael Thorpe and the Department of Physics, Arizona State University, where much of this work was done, for hospitality during his sabbatical leave there.

APPENDIX

For comparison with the nonlinear Klein-Gordon solution, we look at the multiple-scale perturbation theory applied to Eq. (11).[22,23] This approach results in the nonlinear Schrödinger equation formulation of the problem.[22,23] In this development, the RWA is used along with a small parameter proportional to the mode amplitude and a multiple-scale treatment of the resulting equations. This results in a nonlinear Schrödinger equation describing the small amplitude modes of the system in a perturbation theory which does not exhibit spurious singularities. Specifically, in Eq. (11) we take $q(x,\tau) = \sqrt{\frac{\varepsilon_l}{\alpha}}\theta(x,\tau)$ for $\frac{\varepsilon_l}{\alpha} > 0$, write $\theta = \varepsilon\phi_0 + \varepsilon^2\phi_1 + \varepsilon^3\phi_2 + \cdots$ as an expansion in a small parameter ε, and express the space and time derivatives in the multiple scale forms $\frac{\partial}{\partial\tau} = \sum_{i=0}^{\infty}\varepsilon^i\frac{\partial}{\partial T_i}$ where $T_i = \varepsilon^i\tau$ and $\frac{\partial}{\partial x} = \sum_{i=0}^{\infty}\varepsilon^i\frac{\partial}{\partial X_i}$ where $X_i = \varepsilon^i x$. After applying the RWA and some algebra, the terms in this expansion of the first through third orders in ε give :

$$\theta = \varepsilon A(\xi_1,\tau_2)e^{i(kx-\omega\tau)} + cc \tag{A1}$$

where A satisfies the nonlinear Schrödinger equation

$$i\frac{\partial A}{\partial\tau_2} + P\frac{\partial^2 A}{\partial\xi_1^2} + Q|A|^2A = 0 \tag{A2}$$

with $\tau_2 = \varepsilon^2\tau, \xi_1 = \varepsilon(x - v_g\tau), v_g = \frac{abk}{\omega}, P = \frac{ab-v_g^2}{2\omega}, Q = \frac{3b}{2\omega}$, and $\omega^2 = b[1 + ak^2]$. Equation (A2) has a solution of the general form:

$$A = a_0 \text{ sec } h(B\xi_1 - C\tau_2)\exp[-i(d\xi_1 - e\tau_2)]. \tag{A3}$$

Upon substitution in Eq. (A2), we find for $a_\epsilon = \varepsilon a_0$ and $d_\epsilon = \varepsilon d$ that

$$q(x,\tau) = \sqrt{\frac{\varepsilon_l}{\alpha}}a_\varepsilon\text{sech}\left(\left[\frac{3}{2}\frac{1}{a\left(1-\frac{abk^2}{\omega^2}\right)}\right]^{1/2}a_\varepsilon\left\{x - \left[\frac{abk}{\omega} - \frac{ab(1-abk^2/\omega^2)}{\omega}d_\varepsilon\right]\tau\right\}\right)$$
$$\times\exp\left\{i\left[(k-d_\varepsilon)x - \left\{\omega - \frac{abkd_\varepsilon}{\omega} - \frac{ab(1-abk^2/\omega^2)}{2\omega}\left[\frac{3}{2}\frac{a_\varepsilon^2}{a(1-abk^2/\omega^2)} - d_\varepsilon^2\right]\right\}\tau\right]\right\} + cc \tag{A4}$$

satisfying the Klein-Gordon equation in the RAW to third order in ε^3. A second solution is of the general form:

$$A = a_0\tanh(B\xi_1 - C\tau_2)\exp[-i(d\xi_1 - e\tau_2)]. \tag{A5}$$

Upon substitution in Eq. (A2), and using the notation in Eq. (A4), it gives

$$q(x,\tau) = \sqrt{\frac{\varepsilon_l}{\alpha}}a_\varepsilon\tanh\left(\left[\frac{3}{2}\frac{1}{a\left(\frac{abk^2}{\omega^2}-1\right)}\right]^{1/2}a_\varepsilon\left\{x - \left[\frac{abk}{\omega} + \frac{ab(abk^2/\omega^2-1)}{\omega}d_\varepsilon\right]\right\}\tau\right)$$
$$\times\exp\left\{i\left[(k-d_\varepsilon)x - \left\{\omega - \frac{abkd_\varepsilon}{\omega} - \frac{ab(abk^2/\omega^2-1)}{2\omega}\left[\frac{3a_\varepsilon^2}{a(abk^2/\omega^2-1)} + d_\varepsilon^2\right]\right\}\tau\right]\right\} + cc \tag{A6}$$

satisfying the Klein-Gordon equation in the RAW to order ε^3.

*arthur.mcgurn@wmich.edu

[1] V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968).
[2] D. R. Smith, W. J. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, Phys. Rev. Lett. **84**, 4184 (2000).
[3] J. B. Pendry, A. J. Holden, D. J. Roberts, and W. J. Stewart, J. Phys. Condens. Matter **10**, 4785 (1998).
[4] J. B. Pendry, A. J. Holden, D. J. Roberts, and W. J. Stewart, IEEE Trans. Micr. Theory Tech. **47**, 2075 (1999).
[5] M. Eleftheriou, N. Lazarides, and G. P. Tsironis, Phys. Rev. E **77**, 036608 (2008).
[6] G. V. Eleftheriades, Materials Today **12**, 30 (2009).
[7] E. Shamonina, V. A. Kalinin, K. H. Ringhofer, and L. Solymar, J. Appl. Phys. **92**,6252 (2002).
[8] M. J. Freire, R. Marqués, F. Medina, M. A. G. Laso, and F. Martín, Appl. Phys. Lett. **85**, 4439 (2004).
[9] O. Sydoruk, O. Zhuromskyy, E. Shamonina, and L. Solymar, Appl. Phys. Lett. **87**, 072501 (2005).
[10] R. R. A. Syms, E. Shamonina, and L. Solymar, Eur. Phys. J. B **46**, 301 (2005).
[11] A. A. Zharov, I. V. Shadrivov, and Y. S. Kivshar, Phys. Rev. Lett. **91**, 037401 (2003).
[12] S. O'Brien, D. McPeake, S. A. Ramakrishna, and J. B. Pendry, Phys. Rev. B **69**, 241101 (2004).
[13] M. Lapine, M. Gorkunov, and K. H. Ringhofer, Phys. Rev. E **67**, 065601 (2003).
[14] I. V. Shadrivov, S. K. Morrison, and Y. S. Kivshar, Opt. Express **14**, 9344 (2006).
[15] I. V. Shadrivov, A. N. Reznik, and Y. S. Kivshar, Physica B **394**, 180 (2007).
[16] I. Kourakis, N. Lazarides, and G. P. Tsironis, Phys. Rev. E **75**, 067601 (2007); M. I. Molina, N. Lazarides, and G. P. Tsironis, *ibid.* **80**, 046605 (2009).
[17] A. K. Bandyopadhyay, P. C. Ray, and V. Gopalan, J. Phys. Condens. Matter **18**, 4093 (2006).
[18] A. K. Bandyopadhyay, P. C. Ray, and V. Gopalan, Eur. Phys. J. B **65**, 525 (2008).
[19] A. K. Bandyopadhyay, P. C. Ray, L. Vu-Quoc, and A. R. McGurn, Phys. Rev. B **81**, 064104 (2010).
[20] N. A. Zharova, I. V. Shadrivov, A. A. Zharov, and Y. S. Kivshar, Opt. Express **13**, 1291 (2005).
[21] The results for $\frac{\varepsilon_l}{\alpha} < 0$ are obtained by changing +3 in Eq. (12) to −3 and taking a solution of the form $q(x,\tau) = \sqrt{|\frac{\varepsilon_l}{\alpha}|}A(\beta x - \delta\tau)e^{i(kx-\omega\tau)} + cc$ for $A(\beta x - \delta\tau) = a_0$ sech$(\beta x - \delta\tau)$ or a_0 tanh$(\beta x - \delta\tau)$. For the sech solution: ω^2, β^2, and δ are obtained from those below Eq. (13a) by replacing +3 by −3. For

the tanh solution: ω^2, β^2, and δ are obtained from those below Eq. (13b) by replacing $+3$ by -3.
[22]T. Dauxois and M. Peyrard, *Physics of Solitons* (Cambridge University Press, Cambridge, 2006), Chap. 3, pp. 75–116.
[23]M. Remoissenet, Phys. Rev. B **33**, 2386 (1986).
[24]A. J. Sievers and J. B. Page, in *Dynamical Properties of Solids*, edited by G. K. Horton and A. A. Maradudin (North-Holland, Amsterdam, 1995), p. 137.
[25]A. J. Sievers and S. Takeno, Phys. Rev. Lett. **61**, 970 (1988).
[26]A. J. Sievers and S. Takeno, Phys. Rev. B **39**, 3374 (1989).
[27]A. Maruta, T. Inoue, Y. Nonaka, and Y. Yoshika, IEEE J. Sel. Top. Quantum Electron. **8**, 640 (2002).
[28]M. Stratmann, T. Pagel, and F. Mitschke, Phys. Rev. Lett. **95**, 143902 (2005).
[29]I. Gabitov and S. K. Turitsyn, Opt. Lett. **21**, 327 (1996).
[30]P. M. Lushnikov, Opt. Lett. **26**, 1535 (2001).
[31]L. F. Mollenauer and J. P. Gordon, *Solitons in Optical Fibers*, (Academic, London, 2006), pp. 103–126.
[32]V. Petviashvili and O. Pokhotelov, *Solitary waves in Plasmas and in the Atmosphere* (Gordon & Breach, Philadelphia, 1992), p. 248.
[33]A. Hasegawa and Y. Kodama, *Solitons in Optical Communications* (Clarendon Press, Oxford, 1995), pp. 151–172.
[34]A. Maruta, Y. Nonaka, and T. Inoue, Electron. Lett. **22**, 1357 (2001).
[35]M. J. Ablowitz and G. Biondini, Opt. Lett. **23**, 1668 (1998).
[36]I. Gabitov, R. Indik, L. F. Mollenauer, M. Shkarayev, M. Stepanov, and P. M. Lushnikov, Opt. Lett. **32**, 605 (2007).
[37]M. Shkarayev and M. G. Stepanov, Physica D: Nonlinear Phenomena **238**, 840 (2009).
[38]V. Vsevolod, P. Afanasjev, L. Chu, and Y. S. Kivshar, Opt. Lett. **22**, 1388 (1997).
[39]Y. S. Kivshar, M. Haelterman, and A. P. Sheppard, Phys. Rev. E **50**, 3161 (1994).