Exact dispersion relation for nonlinear plasmonic waveguides

Ivan D. Rukhlenko,* Asanka Pannipitiya, and Malin Premaratne
Advanced Computing and Simulation Laboratory (AχL), Department of Electrical and Computer Systems Engineering, Monash University, Clayton, Victoria 3800, Australia.

Govind P. Agrawal
Institute of Optics, University of Rochester, Rochester, New York 14627, USA
(Received 2 September 2011; published 20 September 2011)

We derive an exact dispersion relation for the surface plasmon polaritons of a nonlinear plasmonic waveguide using exact field decomposition of TM waves. Our approach generalizes the known linear dispersion relations to the case of a medium nonlinearity of the form $\varepsilon_{\rm NL} = \varepsilon_L + \alpha|\mathbf{E}|^{2n}$. We apply the unique dispersion relation to a plasmonic waveguide with a Kerr-type nonlinearity ($n = 1$) and show that it enables backward-propagating modes. It also introduces critical points in the energy spectrum of surface plasmon polaritons that result in enhanced interaction of nonlinear modes with each other and external electromagnetic fields.

DOI: 10.1103/PhysRevB.84.113409 PACS number(s): 73.20.Mf, 42.65.Tg, 42.65.Wi

A problem of continuing interest to scientists working in the areas of nanoscience, plasmonics, and metamaterials is the achievement of full control over the optical energy flow on a nanoscale.[1–4] The need for such a control is dictated by the Internet-traffic demands and information processing challenges that need to be handled in the near future.[5] It is expected that some of these challenges will be met by nonlinear plasmonic waveguides in which digital signals in the form of surface plasmon polaritons (SPPs) are confined to nanoscale areas ($\sim$100 nm^2) and transmitted and altered at Tbit/s rates.[6–9] Managing such digital bit streams will be impossible without exploiting the strong and ultrafast optical nonlinearities of dielectrics and metals.[10–14] It turns out that nonlinearities cause drastic modification of the SPP energy spectrum; these features have to be thoroughly understood using a suitable nonlinear theory prior to being beneficially employed in practice.

During the past three decades, ample attention has been devoted to deriving nonlinear dispersion equations for the SPP modes of metal–dielectric interfaces and linear metallic slabs surrounded by different types of nonlinear media.[15–22] It is rather surprising that, to the best of our knowledge, no analytical expression for the SPP dispersion relation has been reported so far for a thin nonlinear waveguide surrounded by metals. The purpose of this Brief Report is to derive such a relation for a waveguide with the power-law nonlinearity and analyze it with emphasis on its impact on the performance of real, integrated plasmonic devices.

We consider a plasmonic waveguide with the geometry shown schematically in Fig. 1. It is composed of a nonlinear dielectric layer (medium 1) of thickness $2h$ that is embedded between two metallic slabs (medium 2) that are thick enough to be treated as semi-infinite. Since absorption losses associated with propagation through dielectrics and metals can in principle be compensated by gain (at any wavelength of interest), we neglect them in this study by taking a positive (real) permittivity ($\varepsilon_1 > 0$) for the dielectric and a negative permittivity ($\varepsilon_2 < 0$) for the metal. For definiteness, we assume that ε_1 depends on the electric field E_1 inside the dielectric layer as $\varepsilon_1 = \varepsilon_L + \alpha|\mathbf{E}_1|^{2n}$, where ε_L is the linear part, n is an integer, and α is a constant parameter. The choice $n = 1$ corresponds to the Kerr nonlinearity.

If we write the complex electric field of the transverse-magnetic (TM) SPPs inside a jth medium as $\mathbf{E}_j = (\hat{x}E_{xj} + i\hat{z}E_{zj})\exp(i\beta z)$, where β is the propagation constant and $\hat{s}$ is the unit vector along the direction s, its components can be calculated using the following two real equations (obtained from Maxwell's equations):

$$\frac{dE_{xj}}{dx} = \beta E_{zj} - \frac{d(\ln\varepsilon_j)}{dx}E_{xj}, \tag{1a}$$

$$\frac{dE_{zj}}{dx} = \left(k_j^2/\beta\right)E_{xj}, \tag{1b}$$

where $k_j = \beta(1-\varepsilon_j\vartheta^2)^{1/2}$, $\vartheta = k/\beta$, and $k = \omega/c$ is the free-space wave number at frequency ω (c is the speed of light in vacuum).

Similar to the linear case, the dispersion relation for a nonlinear waveguide is obtained by matching the general solutions of Eq. (1) at the two metal–dielectric interfaces located at $x = \pm h$. In the case of the symmetric waveguide in Fig. 1, only the interface at $x = h$ is needed. Inside the metallic layer ($x \geqslant h$), the solution is $E_{x2} = \widetilde{E}_{x2}\exp[-k_2(x-h)]$ and $E_{z2} = -(k_2/\beta)E_{x2}$. The requirement of continuity of the tangential component of $\mathbf{E}$ and the normal component of the electric displacement vector across the interface $x = h$ yields $\widetilde{E}_{z1} = -(k_2/\beta)\widetilde{E}_{x2}$ and $\widetilde{\varepsilon}_1\widetilde{E}_{x1} = \varepsilon_2\widetilde{E}_{x2}$, where $\widetilde{\varepsilon}_1 = \varepsilon_L + \alpha\widetilde{E}_1^{2n}$ and a tilde over a variable denotes its value at $x = h$. Combining these two equations and assuming $\widetilde{E}_{x1} > 0$ for $x > 0$, we get the first key relation for the derivation of the dispersion equation: $\widetilde{E}_{x1} = \gamma\widetilde{E}_1$, where $\gamma = (1+\eta^2)^{-1/2}$ and $\eta = (\widetilde{\varepsilon}_1/|\varepsilon_2|)(k_2/\beta)$.

Inside the nonlinear medium, Eq. (1) can be solved in quadratures owing to the fact that the electric field components obey a conservation law.[23] It is not difficult to find this law by integrating the equation

$$\frac{d}{dx}\left(\frac{k_1^2}{\beta^2}E_{x1}\right)^2 = \frac{dE_{x1}^2}{dx} - \varepsilon_1\vartheta^2\frac{dE_1^2}{dx},$$

which mathematically follows from Eq. (1). A little algebra shows that

$$(\varepsilon_1 + n\varepsilon_L)E_1^2 - \varepsilon_1(2-\varepsilon_1\vartheta^2)(n+1)E_{x1}^2 = C_{\pm}, \tag{2}$$

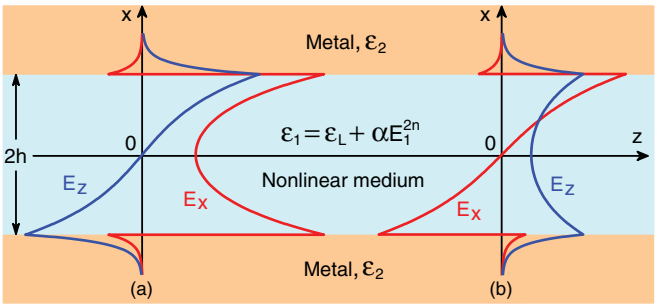

FIG. 1. (Color online) Schematic dependence of the electric field components E_x and E_z on the transverse coordinate x for (a) symmetric and (b) antisymmetric SPP modes of a nonlinear plasmonic waveguide, formed by a nonlinear dielectric layer of thickness $2h$ and permittivity ε_1 sandwiched between two metallic layers of permittivity ε_2.

where we marked the integration constant C with the subscript $\pm$ to stress that its value depends on the symmetry of the SPP mode, determined by the symmetry of the transverse electric field with respect to the reflection in the plane $x = 0$.[24] If we denote the amplitude of the electric field in this plane by E_0, then the constant may be written as

$$C_\pm = [\varepsilon_0 + n\varepsilon_L - \varepsilon_0(2 - \varepsilon_0\vartheta^2)(n+1)\sigma_\pm]E_0^2, \qquad (3)$$

where $\varepsilon_0 = \varepsilon_L + \alpha E_0^{2n}$, $\sigma_\pm = (1 \pm 1)/2$, and $\pm$ correspond to the modes with $E_{x1}(x) = \pm E_{x1}(-x)$.

Equation (2) is an algebraic equation of degree $4n$ with respect to the unknown E_1, but it reduces to the $2n$-order equation with respect to E_1^2. Hence, it admits a general solution and allows E_{z1} to be analytically expressed as a function of E_{x1} only when $n = 1$ or $n = 2$. Since the left-hand side of Eq. (2) cannot be written in terms of a single variable ε_1, the replacement $\varepsilon_L \to \varepsilon_1$ in the linear conservation law would fail to restore its nonlinear analog. For the same reason, the nonlinear dispersion relation cannot be deduced through a similar replacement performed in the dispersion equation for a linear waveguide.

The nonlinear dispersion relation for SPP modes can now be obtained by integrating Eq. (1) while E_{z1} is treated as a function of E_{x1}. The result is given by the integral

$$\int_{E_0\sigma_\pm}^{\gamma\widetilde{E}_1} \frac{\varepsilon_1 + 2n\alpha E_{x1}^2 E_1^{2n-2}}{\varepsilon_1 - 2n(k_1/\beta)^2\alpha E_{x1}^2 E_1^{2n-2}} \frac{dE_{x1}}{E_{z1}} = \beta h. \qquad (4)$$

In writing this equation we have taken into account that $dE_{x1}/dx > 0$ for $x > 0$. The quantity $\widetilde{E}_1$, which enters the dispersion relation explicitly and through the parameter γ, can be calculated from the following $(6n+2)$-order algebraic equation:

$$[\widetilde{\varepsilon}_1 + n\varepsilon_L - \widetilde{\varepsilon}_1(2 - \widetilde{\varepsilon}_1\vartheta^2)(n+1)\gamma^2]\widetilde{E}_1^2 = C_\pm. \qquad (5)$$

For the Kerr-type nonlinearity ($n = 1$), this equation is a fourth-order polynomial with respect to $\widetilde{E}_1^2$ and can be solved exactly; for $n > 1$ its roots should be found numerically.

Equations (2)–(5) allow us to compute $\omega_{SPP} = ck$ to any desired precision, if values of β and E_0 are known. It is easy to verify that, when $\alpha = 0$, they reduce to the known dispersion relation of linear plasmonic waveguides. Indeed, Eqs. (2) and (3) in this case provide

$$E_{z1} = \sqrt{(1 - \varepsilon_L\vartheta^2)E_{x1}^2 + [1 - (2 - \varepsilon_L\vartheta^2)\sigma_\pm]E_0^2},$$

while Eq. (5) yields the relation

$$\left(\frac{\widetilde{E}_1}{E_0}\right)^2 = \frac{1 - \sigma_\pm(2 - \varepsilon_L\vartheta^2)}{1 - (2 - \varepsilon_L\vartheta^2)/(1 + \eta_L^2)},$$

with $\eta_L = (\varepsilon_L/|\varepsilon_2|)(k_2/\beta)$. Using these expressions in Eq. (4) and introducing a new parameter $q = \sqrt{1 - \varepsilon_L\vartheta^2}$, we come up with the expected result[3,6,24–26]

$$\tanh(q\beta h) = (\eta_L/q)^{\pm 1}.$$

For the purpose of illustrating the SPP dispersion peculiarities in nonlinear plasmonic waveguides, we focus on the Kerr nonlinearity for which Eqs. (2) and (5) can be solved analytically. From Eq. (2), we find that

$$E_{z1} = \left(\frac{b + \sqrt{b^2 + ac_\pm}}{\alpha a} - E_{x1}^2\right)^{1/2},$$

where $a = 2 + 4\vartheta^2\alpha E_{x1}^2$, $b = 4(1 - \varepsilon_L\vartheta^2)\alpha E_{x1}^2 - 2\varepsilon_L$, and $c_\pm = \alpha C_\pm + 4\varepsilon_L(2 - \varepsilon_L\vartheta^2)\alpha E_{x1}^2$. The roots of Eq. (5) are given by Ferrari's formula.[27] We assume that the plasmonic waveguide is characterized by the parameters $\varepsilon_L = 2.25$, $\varepsilon_2 = 1 - \omega_p^2/\omega^2$, and $\omega_p = 1.36 \times 10^{16}$ Hz, and consider two values of the nonlinear coefficient $\alpha = \pm 2 \times 10^{-16}$ m^2/V^2, with the plus and minus signs corresponding to self-focusing and self-defocusing nonlinear media, respectively.

Figure 2 shows dispersion relations $\omega_{SPP}(\beta)$ in the self-focusing case ($\alpha > 0$) in the case of two nonlinear

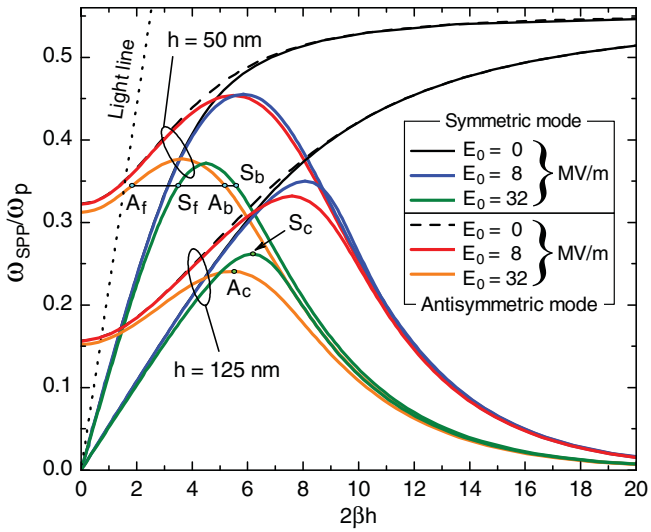

FIG. 2. (Color online) Dispersion curves of SPPs for three values of E_0 (electric field at mode center at $x = 0$) in the case of 100- and 250-nm-thick nonlinear plasmonic waveguides with a self-focusing Kerr nonlinearity ($\alpha > 0$). Thin circles marked by the letters S (symmetric mode) and A (antisymmetric mode) indicate specific points on dispersion curves corresponding to waves moving forward (f) and backward (b); two critical points are marked as A_c and S_c. The light line in the nonlinear medium with $\alpha = 0$ is shown as dotted. For material parameters, refer to the text.

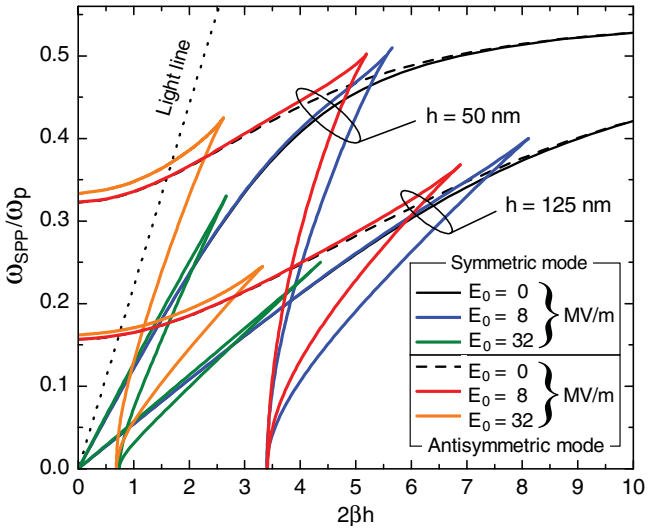

FIG. 3. (Color online) Same as Fig. 2 except that SPP dispersion curves are for 100- and 250-nm-thick nonlinear plasmonic waveguides with a self-defocusing Kerr nonlinearity ($\alpha < 0$). All parameter values are identical to those used for Fig. 2.

waveguides of thicknesses 100 and 250 nm. Black curves correspond to a linear waveguide (or weak-intensity SPP modes propagating in a nonlinear waveguide). It is evident that the presence of self-focusing nonlinearity profoundly alters the dispersion of SPPs, giving birth to backward-propagating modes and critical points in the SPP density of states. The backward-propagating modes exhibit a negative group velocity and are described by portions of the dispersion curves with negative slopes. As shown by a thin horizontal line, a nonlinear plasmonic waveguide can support up to four SPP modes at a single frequency ω: two forward-propagating modes (S_f and A_f) and two backward-propagating modes (S_b and A_b); the wave numbers β_q corresponding to these modes are the roots of the dispersion relation $\omega_{SPP}(\beta) = \omega$. We note that a steep growth in the intensity of the electromagnetic field with increasing β may prevent the backward-propagating modes from being accessible experimentally for large values of β.

The critical points[28] of the SPP energy spectrum correspond to the peaks of nonlinear dispersion curves in Fig. 2 and are obtained by setting $\partial\omega_{SPP}/\partial\beta = 0$. They arise in the event that the total energy flow of the plasmon mode becomes zero. The quantity of primary interest is the density of states, defined as

$$\mathcal{D}(\omega) = \int_0^\infty \frac{d\beta}{2\pi}\,\delta[\omega - \omega_{SPP}(\beta)] = \sum_q \frac{\beta_q}{2\pi}\left|\frac{\partial\omega_{SPP}}{\partial\beta'}\right|^{-1}_{\beta'=\beta_q},$$

because it becomes singular at these points. Simple algebra shows that $\mathcal{D}(\omega)$ diverges near the critical frequency ω_c as $(\omega_c - \omega)^{-1/2}$. In practice, divergency is avoided by metal losses, but the density of states near ω_c can become very large. As a result, coherent coupling between SPPs and optical emitters can be enhanced by a large factor at $\omega = \omega_c$, limited solely by metal losses.[29,30] The same physics also intensifies interaction between SPPs of different modes and leads to an enhanced interaction of SPPs with the external fields. The comparison of different curves in Fig. 2 suggests that the positions of critical points may be changed by varying either the thickness of the nonlinear layer or the intensity of the input beam used to excite the SPPs.

The situation is significantly different in the self-defocusing case ($\alpha < 0$), as is apparent from the dispersion relations plotted in Fig. 3. The self-defocusing nonlinearity still doubles the number of SPP modes at certain frequencies, but it gives rise neither to backward-propagating modes nor to the critical points (the density of states corresponding to the needle-shaped sections of the dispersion curves is finite). Another fundamental difference of the curves in Fig. 3 from those in Fig. 2 is that they exhibit a cutoff in the reciprocal space for large β. The cutoff frequency and wave number strongly depend on E_0 and h, as do the positions of critical points. Similar to the case of $\alpha > 0$, the modes of a specific symmetry that coexist at the same frequency differ by the electromagnetic field intensities associated with them. It may also interest the reader to note the difference between SPP dispersion in a metallic slab surrounded by a dielectric, and that in a gap plasmonic waveguide.[15,16] In the first instance, the nonlinear dispersion relations only marginally differ from their linear analogs, whereas in our case a drastic reconstruction of the SPP energy spectrum occurs even for small nonlinear coefficients, because of the different heterostructure topology.

In summary, SPP dispersion relations have been derived for plasmonic waveguides with an arbitrary power-law nonlinearity within the dielectric core. When applied to a Kerr medium as an example, the dispersion relations reveal in the case of a self-focusing nonlinearity the existence of backward-propagating modes as well as the critical points in their energy spectra where the density of states is enhanced. In the case of a self-defocusing nonlinearity, backward-propagating modes do not exist. Moreover, the density of states is not enhanced at the critical points, which now indicate a power-dependent cutoff of the forward-propagating modes. We note in passing that Eq. (4) can also be used to study symmetry-breaking bifurcations of SPP modes,[31,32] if the assumption of a particular mode symmetry in Eq. (2) is discarded. This topic requires further investigation.

This work was supported by the Australian Research Council, through its Discovery Grant scheme under Grant No. DP110100713. The work of G.P.A. was also supported by the US National Science Foundation (ECCS-0801772).

*ivan.rukhlenko@monash.edu

[1] J. A. Schuller, E. S. Barnard, W. Cai, Y. C. Jun, J. S. White, and M. L. Brongersma, Nat. Mater. **9**, 193 (2010).

[2] C.-P. Huang and Y.-Y. Zhu, Act. Passive Electron. Compon. **2007**, 30946 (2007).

[3] S. A. Maier, *Plasmonics: Fundamentals and Applications* (Springer, Berlin, 2007).

[4] J. B. Pendry, D. Schurig, and D. R. Smith, Science **312**, 1780 (2006).

[5] D. A. B. Miller, IEEE Proc. **97**, 1166 (2009).

[6] A. Pannipitiya, I. D. Rukhlenko, M. Premaratne, H. T. Hattori, and G. P. Agrawal, Opt. Express **18**, 6191 (2010).
[7] K. F. MacDonald, Z. L. Samson, M. I. Stockman, and N. I. Zheludev, Nat. Photonics **3**, 55 (2009).
[8] E. Verhagen, L. Kuipers, and A. Polman, Nano Lett. **7**, 334 (2007).
[9] N.-C. Panoiu and J. R. M. Osgood, Nano Lett. **4**, 2427 (2004).
[10] A. R. Davoyan, I. V. Shadrivov, and Y. S. Kivshar, in *Quantum Electronics and Laser Science Conference*, OSA Technical Digest (CD) (Optical Society of America, 2010), p. QThH5.
[11] J. M. Dudley and R. W. Boyd, Opt. Express **15**, 5237 (2007).
[12] Y. Nakayama, P. J. Pauzauskie, A. Radenovic, R. M. Onorato, R. J. Saykally, J. Liphardt, and P. Yang, Nature (London) **447**, 1098 (2007).
[13] G. A. Wurtz, R. Pollard, and A. V. Zayats, Phys. Rev. Lett. **97**, 057402 (2006).
[14] V. R. Almeida, C. A. Barrios, R. R. Panepucci, and M. Lipson, Nature (London) **431**, 1081 (2004).
[15] L. Bing-Can, Y. Li, L. Zhi-Xin, and Z. Kai, Chin. Phys. B **19**, 097303 (2010).
[16] H. Yin, C. Xu, and P. M. Hui, Appl. Phys. Lett. **94**, 221102 (2009).
[17] A. D. Boardman, A. A. Maradudin, G. I. Stegeman, T. Twardowski, and E. M. Wright, Phys. Rev. A **35**, 1159 (1987).
[18] D. Mihalache, G. I. Stegeman, C. T. Seaton, E. M. Wright, R. Zanoni, A. D. Boardman, and T. Twardowski, Opt. Lett. **12**, 187 (1987).
[19] J. Ariyasu, C. T. Seaton, G. I. Stegeman, A. A. Maradudin, and R. F. Wallis, J. Appl. Phys. **58**, 2460 (1985).
[20] G. I. Stegeman, C. T. Seaton, J. Ariyasu, R. F. Wallis, and A. A. Maradudin, J. Appl. Phys. **58**, 2453 (1985).
[21] M. Y. Yu, Phys. Rev. A **28**, 1855 (1983).
[22] V. M. Agranovich, V. S. Babichenko, and V. Ya. Chernyak, Sov. Phys. JETP Lett. **32**, 512 (1981).
[23] R. I. Joseph and D. N. Christodoulides, Opt. Lett. **12**, 826 (1987).
[24] J. A. Dionne, L. A. Sweatlock, H. A. Atwater, and A. Polman, Phys. Rev. B **73**, 035407 (2006).
[25] Y. Kurokawa and H. T. Miyazaki, Phys. Rev. B **75**, 035411 (2007).
[26] J. A. Dionne, L. A. Sweatlock, H. A. Atwater, and A. Polman, Phys. Rev. B **72**, 075405 (2005).
[27] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems and Formulas for Reference and Review*, 2nd ed. (McGraw-Hill, New York, 1968).
[28] F. Bassani and G. P. Parravicini, *Electronic States and Optical Transitions in Solids*, 1st ed. (Pergamon, Oxford, New York, 1975).
[29] A. V. Fedorov, A. V. Baranov, I. D. Rukhlenko, T. S. Perova, and K. Berwick, Phys. Rev. B **76**, 045332 (2007).
[30] A. V. Fedorov, A. V. Baranov, I. D. Rukhlenko, and S. V. Gaponenko, Phys. Rev. B **71**, 195310 (2005).
[31] A. R. Davoyan, Phys. Lett. A **375**, 1615 (2011).
[32] A. R. Davoyan, I. V. Shadrivov, and Y. S. Kivshar, Opt. Express **16**, 21209 (2008).