Universal Fermi distribution of semiclassical nonequilibrium Fermi states

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When a classical device suddenly perturbs a degenerate Fermi gas, a semiclassical nonequilibrium Fermi state arises. Similar states appear when a charge is transferred into a Fermi gas through a localized contact. Semiclassical Fermi states are characterized by a Fermi energy or Fermi momentum that slowly depends on space and/or time. We show that the Fermi distribution of a semiclassical Fermi state has a universal nature. It is described by Airy functions regardless of the details of the perturbation.

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I. INTRODUCTION

Among various excitations of a degenerate Fermi gas, *coherent Fermi states* play a special role. A typical (not coherent) excitation of a Fermi gas consists of a finite number of holes below the Fermi level and a finite number of particles above it. A coherent Fermi state involves, instead, an infinite superposition of particles and holes arranged in such a manner that one can still think in terms of a Fermi sea with no holes in it. The Fermi edge of such state depends on time and space (Fig. 1). Later in the paper we will give a formal definition of Fermi coherent states. It is based on current (or Tomonaga) algebra.

Coherent Fermi states appear in numerous recent proposals about generating coherent quantum states in nanoelectronic devices and fermionic cold atomic systems. These states can be used to transmit quantum information, test properties of electronic systems and generate many-particles entangled states.

Coherent Fermi states can be obtained by various means. Below, we list two:

(i) A sudden perturbation of a Fermi gas. For example, a smooth potential well, the spatial extent of which much larger than the Fermi length, is applied to a Fermi gas. Fermions are trapped in the well. Then the well is suddenly removed. An excited state of the Fermi gas obtained in this way is a *coherent Fermi state*. This kind of perturbation is typical for various manipulations with cold fermionic atomic gases.

(ii) Nonstationary driven electronic systems. For example, a time-dependent voltage is applied through a point contact which results in a charge transferred into the Fermi gas. These systems have been intensively studied in recent papers.[1] This way to create and manipulate by coherent states in electronic systems has been proven to be realistic.[2]

Both examples show that Fermi coherent states are essentially nonequilibrium.

Although the realization of coherent states in electronic systems experimentally is more challenging than in atomic systems, numerous recent manipulations with nanoelectronic devices attempt to produce just such states. We will routinely talk about electrons rather than Fermi atoms.

From a theoretical standpoint, coherent Fermi states are an important concept, revealing fundamental properties of Fermi statistics.[3] Coherent Fermi states appeared in other disciplines not directly related to electronic physics. Examples include the theory of solitons,[4,5] crystal growth,[6,7] various determinantal stochastic processes,[8] and asymmetric diffusion processes.[9]

Unless special effort is made (see, e.g., Ref. 2), coherent Fermi states involve many electrons and are such that space-time gradients of the electronic density are much smaller than the Fermi scale. These state arise as a result of perturbing the Fermi gas by a classical device. We call them *semiclassical Fermi states*. They are the main object of this paper. We will show that semiclassical Fermi states show a great degree of universality. Semiclassical Fermi states must be distinguished from single electron states studied in Refs. 1 and 2. Those are also coherent states but of a different nature. We briefly discuss them below.

A general coherent Fermi state is a unitary transformation of the ground state $|0\rangle$ of a Fermi gas: $|U\rangle = U|0\rangle$, $U = e^{i\int \Xi(x)\rho(x)dx - i\int \Pi(x)v(x)dx}$, where ρ and v are operators of electronic density and velocity $[\rho(x), v(y)] = -\frac{\hbar}{m}\nabla\delta(x-y)$ and $\Xi(x)$ and $\Pi(x)$ are two real functions characterizing coherent state.

For the purpose of this paper it is sufficient to assume that the motion of electrons is one dimensional and chiral (electrons move to the right), although most of the results we discuss are not limited to one-dimensional electronic gases. In dimensions higher than one, Fermi coherent states represent a propagating front. In this case only the normal direction to the front matters. To this end, edge states in the integer Quantum Hall effect may serve as a prototype.

In a chiral sector the operators of density and velocity are identical $v = \frac{\hbar}{m}\rho$. The contribution of the right sector is easy to take into account. In the chiral case the Fermi coherent state

$$|U\rangle = e^{i\int \Phi(x)\rho(x)dx}|0\rangle \tag{1}$$

is characterized by a single function Φ.[10]

The interpretation of the function Φ depends on the way the state is created. If it is created by a potential $V(x)$ which has been suddenly removed, then $eV(x) = \hbar v_F \nabla\Phi(x)$. If a coherent state is created by manipulating with a pointlike contact as in Refs. 1 and 2, then Φ is the action of a time-dependent gate voltage $eV(t) = -\hbar\frac{d}{dt}\Phi(x_0 - v_F t)$ applied through a point contact (located at x_0).

Coherent Fermi states are nonequilibrium states. They evolve in time, but as long as the propagation is ballistic and the velocity of excited electrons can be approximated by the Fermi velocity, the evolution is merely a Galilean shift of

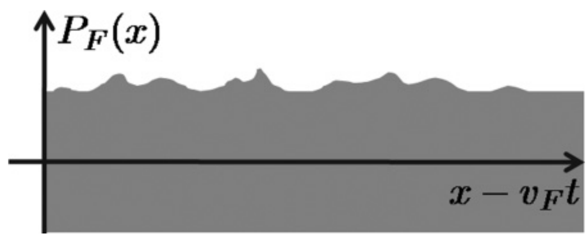

FIG. 1. Modulated Fermi edge. The Wigner functions are almost 1 below the edge (shaded area) and vanish above the edge, featuring a universal character in the "surf."

a coordinate $x \to x - v_F t$. We assume these conditions and treat x as a coordinate in case of a sudden perturbation, or as $v_F t$ in the case of manipulation with a voltage applied to a point contact. (The momentum dependence of the velocities of electrons can be only ignored before the occurrence of a gradient catastrophe.[11] The evolution of coherent states in a dispersive Fermi gas was studied in Ref. 12.)

Fermi coherent states feature inhomogeneous electric density $\rho(x)$ and current $I(x)$. Their expectation values give a meaning to the functions Ξ and Π. In the chiral state, where the electric current and the electronic density are proportional, their expectation values are gradients of Φ,

$$\langle U|\rho(x)|U\rangle = \frac{1}{ev_F}\langle U|I(x)|U\rangle = \rho_0 + \nabla\Phi. \quad (2)$$

Here ρ_0 is the density of electrons in the ground state. Consequently $\hbar\nabla\Phi(x)$ plays the role of the space-time modulation of the Fermi edge. Measuring the momentum from the Fermi momentum of the ground state, we will see that all states with momenta (or energy) less than $P_F(x) = \hbar\nabla\Phi(x)$ (or $E_F(x) = v_F\hbar\nabla\Phi$) are occupied, as shown in Fig. 1. There are no holes in the Fermi sea. We refer to the function $P_F(x)$ as a modulated Fermi edge and to the region in phase space around $P_F(x)$ as the "Fermi surf."

The question we address in this paper is as follows: What is the Fermi distribution in the Fermi surf of a modulated Fermi edge?

We show that, quite interestingly, the semiclassical Fermi surf features a universal Fermi distribution.

Close to the surf, the Wigner function (11) is described by the function $\mathrm{Ai}_1(s) = \int_s^\infty \mathrm{Ai}(s')ds'$:

$$n_F(x, p + P_F(x)) \approx \mathrm{Ai}_1\left(2^{2/3}\kappa p\right), \quad (3)$$

where the scale $\kappa = |\hbar^2 P_F''(x)/2|^{-1/3}$, and the offset $P_F(x)$ are the only information about the state that enters the formula. This formula holds close to any point where the Fermi surf is concave $P_F''(x) < 0$. If the Fermi surf is convex rather than concave, then particle-hole symmetry $n_F(p,x) \to 1 - n_F(-p,x)$ provides the result for the Wigner function. The Wigner function is plotted in Fig. 2.

The Fermi occupation number (12) also displays universal behavior near a maximum of the surf, where the Fermi number reads

$$n_F(P_F^* + p) \approx \kappa\Delta\{[\mathrm{Ai}'(\kappa p)]^2 - (\kappa p)\mathrm{Ai}^2(\kappa p)\}. \quad (4)$$

In this formula the onset P_F^* is a maximum of the surf, and similarly κ is computed at the maximum; Δ is the momentum spacing ($2\pi\hbar/\Delta$ is the system volume). It is assumed that $\Delta \ll \hbar\nabla\Phi$. The behavior near a minimum of the surf follows due to the particle-hole symmetry. Figure 3 illustrates the universal regime.

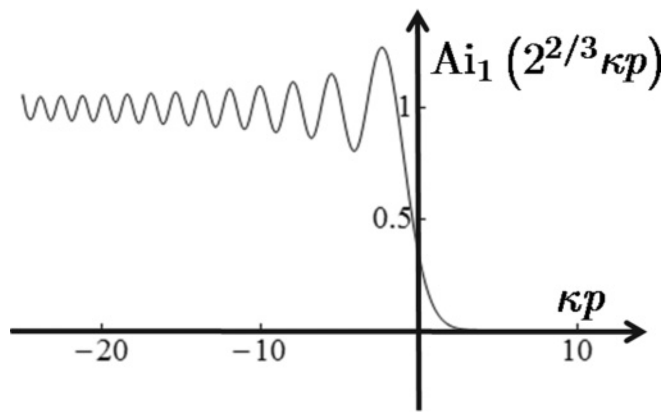

FIG. 2. Universal behavior of the Wigner function, Eq. (3).

The goal of the paper is to emphasize these simple, albeit universal, distributions. [It must not be confused with the Fermi distribution of a Fermi gas confined by a static potential, which has been studied in the literature on nuclear matter. Coincidentally, the Wigner function for a Fermi gas confined by a linear potential is identical to (3).[13]]

II. COHERENT FERMI STATES

The formal definition of Fermi coherent states starts with the current algebra (see, e.g., Ref. 14). To simplify the discussion and formulas we consider only one chiral (right) part of the current algebra.

Current modes are Fourier harmonics of the electronic density $\rho(x) = \sum_{k>0} e^{\frac{i}{\hbar}kx} J_k$. An electronic current mode $J_k = \sum_p c_p^\dagger c_{p+k}$ (we measure electronic momentum from the Fermi momentum) creates a superposition of particle-hole excitations with momentum k. Positive modes annihilate the ground state $|0\rangle$, a state where all momenta below the Fermi edge are filled: $J_k|0\rangle = 0,\ k > 0$. Negative modes of the chiral states are Hermitian conjugated to the positive modes $J_{-k} = (J_{+k})^\dagger$.

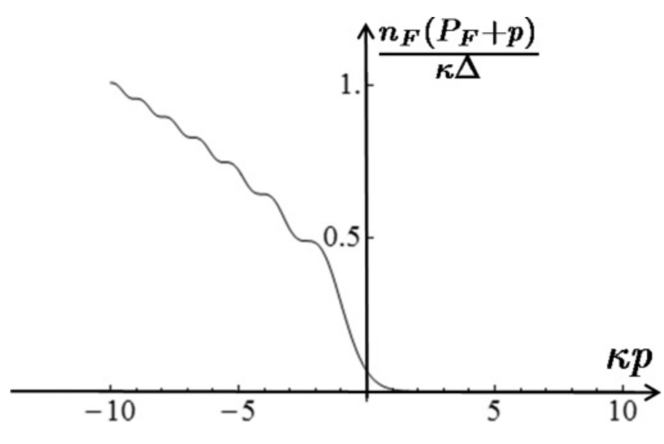

FIG. 3. Universal behavior of the Fermi number, Eq. (4).

Chiral currents obey a current (or Tomonaga) algebra:

$$[J_k, J_l] = \frac{k}{\Delta}\delta_{k+l,0}. \tag{5}$$

A Fermi coherent state $|U\rangle$ is defined as an eigenstate of positive current modes:

$$J_k|U\rangle = p_k|U\rangle, \quad k > 0. \tag{6}$$

As follows from (1), $p_k = \Delta\int e^{-\frac{i}{\hbar}kx}d\Phi(x)/(2\pi)$ are positive Fourier modes of the function $\Phi(x)$. Assuming that the total number of particles (or the component of the dc current) in the coherent state is the same as in the ground state, i.e., $\int d\Phi = 0$, we obtain

$$|U\rangle = Z^{-1/2}e^{\sum_{k>0}\frac{p_k}{k}J_{-k}}|0\rangle, \quad Z = e^{\sum_{k>0}\frac{1}{k}|p_k|^2}. \tag{7}$$

Using normal ordering with respect to the ground state (where all positive modes of the current are placed to the right-hand side of negative modes) the unitary operator reads

$$\begin{aligned} e^{i\int\Phi(x)\rho(x)dx} &= Z^{-\frac{1}{2}} : e^{i\int\Phi(x)\rho(x)dx} : \\ &= Z^{-\frac{1}{2}}e^{\sum_{k>0}\frac{1}{k}p_kJ_{-k}}e^{-\sum_{k>0}\frac{1}{k}p_{-k}J_k}. \end{aligned} \tag{8}$$

It is a simple exercise involving the algebra of the current operators to show that the function $\nabla\Phi(x)$ is a nonuniform part of the density as is in (2). Alternatively, one can use Z as a generation function $\langle\rho(x)\rangle \equiv \langle U|\rho(x)|U\rangle = \rho_0 + 2\,\mathrm{Re}\sum_{k>0}ke^{ikx}\partial_{\bar{p}_k}\log Z$.

Coherent states obey the Wick theorem. The Wick theorem allows to compute a correlation function of any finite number of electronic operators, as a determinant over the one-fermionic function

$$K(x_1,x_2) \equiv \langle U|\psi^\dagger(x_1)\psi(x_2)|U\rangle,$$

where $\psi(x) = (\hbar/\Delta)^{1/2}\sum_p e^{\frac{i}{\hbar}px}c_p$ is an electronic operator. The one-fermionic function can be computed with the help of the formula

$$U\psi(x)U^{-1} = e^{-i\Phi(x)}\psi(x), \tag{9}$$

which leads to the expression

$$K(x_1,x_2) = \frac{e^{i\Phi(x_1)-i\Phi(x_2)-\frac{i}{\hbar}p_F(x_1-x_2)} - 1}{i(x_1-x_2)}, \tag{10}$$

valid for $\Delta|x_1 - x_2| \ll \hbar$. An equivalent object appears in random matrix theory, where it is often called Dyson's kernel. We adopt this name. As points merge, one recovers the density (2) $K(x,x) = \langle\rho(x)\rangle = \nabla\Phi$.

III. WIGNER FUNCTION AND FERMI OCCUPATION NUMBER

The Wigner function is defined as the Wigner transform of the Dyson kernel

$$n_F(x,p) = \frac{1}{2\pi}\int K\left(x+\frac{y}{2},x-\frac{y}{2}\right)e^{-\frac{i}{\hbar}py}\,dy. \tag{11}$$

The meaning of the Wigner function is simply away from the surf. There it has the meaning of the occupation of electrons in phase space (x,p): 1 below the edge, and 0 above (Fig. 1). At the surf the Wigner function is not necessarily positive.

The Fermi number

$$n_F(p) = \langle U|c_p^\dagger c_p|U\rangle = \frac{\Delta}{2\pi\hbar}\int n_F(x,p)\,dx \tag{12}$$

is the Wigner function averaged over space.

Combining (10) and (11) we write

$$n_F(x,p) = \frac{1}{2\pi i}\int e^{\frac{i}{\hbar}\int_{x-\frac{y}{2}}^{x+\frac{y}{2}}(P_F(x')-p)dx'}\frac{dy}{y-i0}, \tag{13}$$

where we denoted $P_F(x) = \hbar\nabla\Phi = \langle\rho(x)\rangle$ as in (2).

Below we evaluate the integral (13) semiclassically, bearing in mind that Φ is of a finite order as $\hbar \to 0$.

IV. UNIVERSAL FERMI SURF

A universal regime arises at the Fermi surf, $\kappa|p-P_F(x)|\simeq 1$. In this case it is sufficient to expand $P_F(x)$ in a Taylor series around extrema of $P_F(x)$ to second order $P_F(x) = P_F(x_*) + \frac{1}{2}P_F''(x_*)(x-x_*)^2 + \cdots$. Then the integral (13) becomes the Airy integral given in Eq. (3). Further integration over space yields (4).

In this regime the Dyson kernel in the momentum space $K_{p_1,p_2} \equiv \langle U|c_{p_1}^\dagger c_{p_2}|U\rangle$ reads

$$K_{p_1,p_2} \approx \Delta\frac{\mathrm{Ai}(\kappa p_1)\mathrm{Ai}'(\kappa p_2) - \mathrm{Ai}(\kappa p_2)\mathrm{Ai}'(\kappa p_1)}{(p_1-p_2)}. \tag{14}$$

This is the celebrated Airy kernel appearing in numerous problems as the limiting shape of crystals,[6] asymmetric diffusion,[9] edge distribution of eigenvalues of random matrices,[15] etc.

The Fermi number (4) can be directly obtained from the kernel by taking a limit $p_1 \to p_2$ in (14). At large positive momenta ($\kappa p \to +\infty$) the Fermi number behaves as $n_F(p) \sim \frac{\Delta}{8\pi p}e^{-\frac{4}{3}(\kappa p)^{3/2}}$, and as $\sim\frac{\Delta}{\pi}(\kappa\sqrt{-p\kappa} - \frac{1}{4p}\cos[\frac{4}{3}(-\kappa p)^{3/2}])$ for large negative momenta within the surf.

V. TAILS OF FERMI DISTRIBUTION

Away from the universal regime of the surf the Fermi distribution can be computed within a saddle-point approximation. The saddle point of the integral (13) is

$$P_F\left(x+\frac{y}{2}\right) + P_F\left(x-\frac{y}{2}\right) = 2p. \tag{15}$$

It has pairs of solutions $\pm y_*(x,p)$. Let $P_{\max} = \max[P_F(x)]$ and $P_{\min} = \min[P_F(x)]$ be adjacent extrema of the surf. Without loss of generality we may assume that (x,p) is outside the Fermi sea $p > P_F(x)$. The particle-hole symmetry $n_F \to 1 - n_F$ helps to recover the case when the momentum is inside the sea. If p is in the surf, $p \in (P_{\min},P_{\min})$, then some saddle-point pairs of (15) may be real. Their contribution produces oscillatory features with a suppressed amplitude. If p hovers above the surf, $p > P_{\max}$, then the saddle points are imaginary. Their contributions are exponentially small.

Between two adjacent extrema the Wigner function reads

$$n_F(x,p) \approx \sqrt{\frac{\hbar\left|\frac{dy_*}{dp}\right|}{8\pi|y_*|^2}}\begin{cases} 2\sin\left(\Omega - \frac{\pi}{4}\right), & p \in (P_{\min},P_{\max}), \\ e^{-|\Omega|}, & p > P_{\max}, \end{cases}$$

where $\hbar\Omega = -\int_{P_F(x)}^{p}y_*(x,p')dp'$. In the surf it is half the action of a semiclassical periodic orbit—the area of the graph $y_*(x,p)$ vs p. In the universal regime, when one approximates $y_*(x,p) \approx \hbar\kappa^{3/2}(p - P_F(x))^{1/2}$, this equation reproduces the asymptotes of Eq. (3): $n_F(x, p + P_F(x)) \sim (8\pi)^{-1/2}(\kappa p)^{-3/4} e^{-3(\kappa p)^{3/2}}$.

VI. PERIODIC CURRENT

A Fermi coherent state with a periodic current is an instructive example. It corresponds to "quantum pumping"—periodic transfer of a charge through the system by applying a periodic voltage through a point contact. Setting $\langle \rho(x)\rangle = \rho_0 + (n/\ell)\cos(x/\ell)$, the Dyson kernel in the momentum representation becomes the integer Bessel kernel

$$K_{p_1,p_2} = \frac{n}{2}\frac{m_1 J_{m_1}(n) J'_{m_2}(n) - m_2 J_{m_2}(n) J'_{m_1}(n)}{m_1 - m_2}, \qquad (16)$$

where $m_{1,2} = \ell p_{1,2}/\hbar$ are integers. Here and in Eq. (17) below momenta are assumed to be integers in units of $\hbar/\ell$. The Fermi number is $(m = p\ell/\hbar)$

$$n_F(p) = \frac{1}{2} - \mathrm{sign}(p)\left[\frac{J_0^2(n)}{2} + \sum_{j=1}^{m} J_j^2(n)\right]. \qquad (17)$$

This formula allows to compare the asymptotes near the edges to the universal expression above. Using the homogeneous asymptote of the Bessel function $J_m(m - (m/2)^{1/3}\zeta) \sim (2/m)^{1/3}\mathrm{Ai}(\zeta)$ at large m, one recovers (14) and (4). Figure 4 shows the result of (17) and its fit with the asymptotes.

VII. HOLOMORPHIC FERMIONS AS COHERENT STATES

To contrast *semiclassical* coherent Fermi states, which were the subject of the development above, and *quantum* coherent Fermi states, we briefly discuss special, quantum, coherent states known as holomorphic fermions.[16]

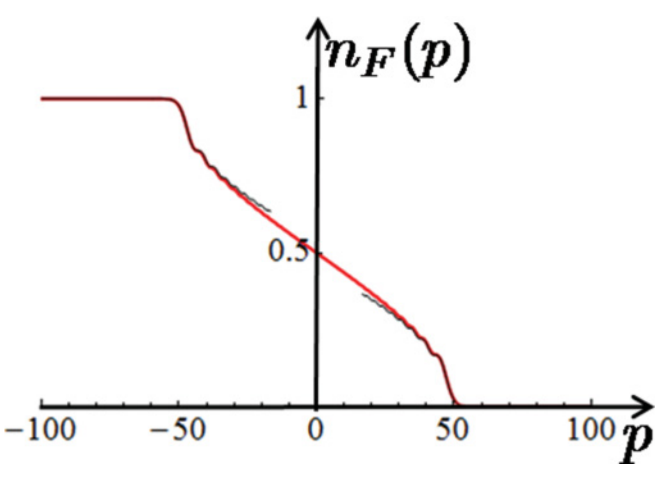

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FIG. 4. (Color online) Universal asymptotes of the Fermi number. The graph (red/gray curve) shows the Fermi number in units of $\hbar/\ell$ for the example $\langle\rho(x)\rangle = \rho_0 + (n/\ell)\cos(x/\ell)$ computed from Eq. (17). Black curves are the asymptotic forms obtained from the universal Fermi number formula (4) depicted and magnified on Fig. 3.

Holomorphic fermions are defined as a superposition of fermionic modes $\psi(z) = \sum_p e^{\frac{i}{\hbar}pz} c_p$, with a complex "coordinate" $\mathrm{Im}\, z < 0$.

Holomorphic fermions are coherent states since they can be represented as an exponent of a Bose field displacement of electrons $\varphi(z) = \Delta \sum_{k\neq 0} \frac{1}{ik} e^{\frac{i}{\hbar}kz} J_k$:[4,14]

$$\psi(z) = c_{p_F} :e^{i\varphi(z)}: . \qquad (18)$$

A function Φ for a string of fermions $\prod_{i=1}^n [\psi^\dagger(z_i)\psi(\zeta_i)]|0\rangle$ is $e^{i\Phi(x)} = \prod_{i=1}^n \frac{x - z_i}{x - \bar z_i}\frac{x - \bar\zeta_i}{x - \zeta_i}$. The density (or current) of these states consists of Lorentzian peaks, each carrying a unit electronic (positive or negative) charge $\langle\rho(x)\rangle - \rho_0 = \sum_i \mathrm{Im}\,(\frac{1}{x - z_i} - \frac{1}{x - \zeta_i})$, so that the state is a set of single electronic pulses. For possible applications of these states in nanodevices, see Refs. 1 and 2. As the complex coordinate approaches the real axis $\zeta = x - i0$, a holomorphic fermion operator becomes an electronic operator $\psi(z) \to \psi(x)$ as its density becomes a delta function.

Coherent states formed by a single holomorphic fermion carry a unit charge in contrast to semiclassical Fermi states. The Wigner function of this state follows from (10),

$$n_F(x,p) = \frac{1}{2\pi i}\int \frac{x + \frac{y}{2} - z}{x + \frac{y}{2} - \bar z}\frac{x - \frac{y}{2} - \bar z}{x - \frac{y}{2} - z} e^{-\frac{i}{\hbar}py}\frac{dy}{y - i0},$$

where $z = X - ia$, X is a real coordinate of fermion, and $|a|$ is its width ($|a|\Delta \ll \hbar$). Evaluating this integral we obtain

$$n_F(x,p) = \Theta(-p) - \Theta(-pa)a\frac{\sin 2p(x - X)}{x - X}e^{\frac{2}{\hbar}pa},$$
$$n_F(p) = \Theta(-p) - \Theta(-pa)\frac{2a\Delta}{\hbar}e^{\frac{2}{\hbar}pa},$$

Noticeable features of this distribution are as follows: the Fermi function jumps on the Fermi edge; beyond the Fermi edge, the Fermi number and the Wigner functions decay exponentially, and if the complex coordinate of the fermion is in the lower half plane, $a > 0$, a holomorphic fermion $\psi(z)$ acts as an annihilation operator, creating a dent on the Fermi surf of the unit area $\hbar$. Vice versa, if $a < 0$, the holomorphic fermion creates a bump of a unit area; and the Wigner function features a Friedel's type oscillation with a distance $x - X$ to the center of the fermion.

Summing up, we showed that an important class of nonequilibrium semiclassical states of Fermi gas created by manipulation with Fermi gas by classical instruments (i) can be still characterized by a Fermi sea with a space-time-modulated Fermi edge, and (ii) a steplike Fermi distribution is smeared in a universal manner described by Airy functions.

ACKNOWLEDGMENTS

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[1]D. A. Ivanov, H. W. Lee, and L. S. Levitov, Phys. Rev. B **56**, 6839 (1997); J. Keeling, I. Klich, and L. Levitov, Phys. Rev. Lett., **97** 116403 (2006); J. Keeling, A. V. Shytov, and L. S. Levitov, *ibid.* **101**, 196404 (2008).
[2]G. Fève, A. Mahé, J.-M. Berroir , T. Kontos, B. Plaçais, D. C. Glattli, A. Cavanna, B. Etienne, and Y. Jin, Science **316**, 1169 (2007).
[3]For coherent states for finite dimensional fermionic system, see A. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986).
[4]Tetsuji Miwa, Michio Jimbo, and Etsuro Date, *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras* (Cambridge University Press, Cambridge, UK, 2000).
[5]E. Bettelheim, A. G. Abanov, and P. Wiegmann, J. Phys. A **40**, F193 (2007).
[6]P. L. Ferrari and H. Spohn, J. Stat. Phys. **113**, 1 (2003).
[7]A. Okounkov and N. Reshetikhin, J. Amer. Math. Soc. **16**, 581 (2003).
[8]K. Johansson, in *Mathematical Statistical Mechanics*, edited by A. Bovier, F. Dunlop, A. C. D. van Enter, and F. den Hollander, Proceedings of the 83rd Les Houches Summer School, 2005 (Elsevier, Amsterdam, 2006), p. 1.
[9]A. Borodin, P. L. Ferrari, and T. Sasamoto, Commun. Math. Phys. **283**, 417449 (2008).
[10]M. Stone, Phys. Rev. B **42**, 8399 (1990).
[11]E. Bettelheim, A. G. Abanov, and P. Wiegmann, Phys. Rev. Lett. **97**, 246402 (2006); E. Bettelheim, Y. Kaplan, and P. Wiegmann, *ibid*. **106**, 166804 (2011).
[12]E. Bettelheim, A. G. Abanov, and P. Wiegmann, J. Phys. A **41**, 392003 (2008).
[13]M. Durand, P. Schuck, and X. Vinas, Phys. Rev. A **36**, 1824 (1987).
[14]*Bosonization*, edited by M. Stone (World Scientific, Singapore, Singapore, 1994).
[15]C. A. Tracy and H. Widom, Commun. Math. Phys. **159**, 151 (1994).
[16]Holomorphic fermions have been introduced in Ref. 3. For physical applications including shot noise of these states, see recent papers (Ref.1). Experimental realization of these states have been reported in Ref. 3.