Topological classification of interaction-driven spin pumps

Dganit Meidan, Tobias Micklitz, and Piet W. Brouwer
Dahlem Center for Complex Quantum Systems and Institut für Theoretische Physik, Freie Universität Berlin, DE-14195 Berlin, Germany
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When adiabatically varied in time, certain one-dimensional band insulators allow for the quantized noiseless pumping of spin even in the presence of strong spin-orbit scattering. These spin pumps are closely related to the quantum spin Hall system, and their properties are protected by a time-reversal restriction on the pumping cycle. In this paper we study pumps formed of one-dimensional insulators with a time-reversal restriction on the pumping cycle and a bulk energy gap which arises due to interactions. We find that the correlated gapped phase can lead to novel pumping properties. In particular, systems with d different ground states can give rise to $d+1$ different classes of spin pumps, including a trivial class which does not pump quantized spin and d nontrivial classes allowing for the pumping of quantized spin $\hbar/n$ on average per cycle, where $1 \leqslant n \leqslant d$. We discuss an example of a spin pump that transfers an average of spin $\hbar/2$ without transferring charge.

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I. INTRODUCTION

Finding ways to manipulate individual charges or spins at zero external bias, the idea that lies at the essence of pumping, holds promise for numerous applications. In a seminal work, Thouless observed that certain band insulators allow for the adiabatic pumping of quantized charge.[1] Transcending its relevance for practical applications, this observation sheds light on transport properties found in other systems. Notably, Laughlin's argument for the quantization of Hall conductance[2] can be formulated in terms of a quantized charge pump.[3] In accordance with the quantum Hall system, the adiabatic charge pump formed of a band insulator can be characterized by a $\mathbb{Z}$ topological invariant, which determines the quantized charge pumped in one cycle.

These ideas have found a recent extension to spin pumps with the discovery of the quantum spin Hall effect in time-reversal invariant systems.[4] In analogy to the quantum Hall state, it is possible to gain insight into the spin Hall state by studying a pump formed by placing the two-dimensional system on a cylinder with a circumference of a single unit cell and threading it with a magnetic flux which is varied in time.[1,3] Time-reversal symmetry of the two-dimensional Hamiltonian imposes a time-reversal restriction on the pumping cycle.[4] Similarly to its two-dimensional analog, spin pumps with a time-reversal restriction on their pumping cycle are characterized by a $\mathbb{Z}_2$ topological invariant. The nontrivial class of pumps allows for the symmetry-protected pumping of quantized spin, even in the presence of strong spin-orbit scattering.[5] Following the paradigm of the fractional quantum Hall effect,[6–9] a natural yet challenging question arises: can interactions give rise to spin pumping properties that cannot be found in noninteracting pumps?

In this paper we study the topological classification of spin pumps consisting of a family of one-dimensional insulators in which the bulk gap arises due to electron-electron interactions. Our classification is made with respect to the observable pumping properties of pumps that are weakly coupled to leads, not on the structure of the bulk insulating state. To this extent, the study of pumps based on their scattering matrix provides a powerful tool that allows one to obtain information on the strongly correlated system based on the phase shift acquired by the noninteracting scattering states. We find that the number of classes in the correlated system is larger than in the noninteracting case if the system has d different many-body ground states. In particular, a spin pump with d ground states gives rise to $d+1$ distinct classes which exhibit different pumping properties: For a weakly coupled pump, these are a trivial class, which does not pump quantized spin, and d nontrivial classes. The nontrivial classes include an integer spin pump that allows for the pumping of quantized spin $\hbar$ during a cycle, as well as $d-1$ "fractional" spin pumps that allow for the average pumping of quantized spin $\hbar/n$ with $1 < n \leqslant d$ per cycle. We discuss an example of a pump that transfers an average of spin $\hbar/2$ during a pumping cycle, without transferring charge.

II. $\mathbb{Z}_2$ CLASSIFICATION OF NONINTERACTING SPIN PUMPS

To set the stage, we briefly review the $\mathbb{Z}_2$ classification of noninteracting spin pumps. We then show that this classification is naturally extended to interacting systems. We consider a family of one-dimensional Hamiltonians with a bulk energy gap that depends continuously on a cyclic pumping parameter t and satisfy

$$H(t+T) = H(t), \quad H(-t) = \Theta H(t) \Theta^{-1}, \tag{1}$$

where Θ is the time-reversal operator. We assume that $H(t)$ does not possess any additional discrete symmetries. Due to the time-reversal restriction (1), such a system cannot pump charge but it may pump spin. These pumps are related to quantum spin Hall systems (class AII in the classification of Ref. 10) upon placing the two-dimensional system on a cylinder threaded by magnetic flux. [This connection becomes evident upon the identification $(k_x,k_y) \to (k_x,t)$.[4]]

When coupling the system to one-dimensional noninteracting leads, the transport properties of the open system at time t are determined from the scattering matrix. Provided that the system's size exceeds the attenuation length associated with the bulk energy gap, the scattering matrix decouples into two unitary 2×2 reflection matrices $\mathcal{R}_\alpha$, for left and right leads

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$\alpha = L/R$, respectively. The average spin injected into lead α during the pumping cycle,[11]

$$\vec{S}_\alpha = \frac{\hbar}{2\pi}\int_0^T dt\, \mathrm{Im}\,\mathrm{tr}([d\mathcal{R}_\alpha/dt]\mathcal{R}_\alpha^\dagger\vec{\sigma}), \tag{2}$$

is invariant under a U(1) gauge transformation and depends only on the particle-hole symmetric $\mathrm{SU}(2) = \mathrm{U}(2)/\mathrm{U}(1) \simeq S^3$ part of the reflection matrix, denoted by $\tilde{\mathcal{R}}$, where we have dropped the lead index α for brevity. Hence, the pumping cycle can be visualized as a loop which $\tilde{\mathcal{R}}(t)$ forms on the three sphere S^3.

The symmetry constraints (1) lead to similar constraints on the reflection matrix

$$\tilde{\mathcal{R}}(t) = \sigma_2\tilde{\mathcal{R}}^T(-t)\sigma_2, \quad \tilde{\mathcal{R}}(t+T) = \tilde{\mathcal{R}}(t). \tag{3}$$

These constraints ensure the existence of two time-reversal invariant moments (TRIM) $t_1 = 0$ and $t_2 = T/2$, at which $\tilde{\mathcal{R}}(t_i) = \pm\mathbb{1}$, which corresponds to the occurrence or absence of a pair of resonances that occur precisely at the TRIM in the presence of particle-hole symmetry, which fixes the U(1) part of $\mathcal{R}$ to be unity.[12] [For a generic U(1) phase, the pair is symmetrically split around the TRIM and related by time reversal.] Following Ref. 5, the parity of resonance pairs around the TRIM defines a $\mathbb{Z}_2$ index:

$$\tilde{\mathcal{R}}(t_1)\tilde{\mathcal{R}}(t_2) = \pm\mathbb{1}^{z_2}. \tag{4}$$

Any loop $\tilde{\mathcal{R}}(t)$ on the three sphere characterized by $z_2 = 0$ can be contracted onto a single point and, hence, corresponds to a trivial pump. Alternatively, paths with $z_2 = 1$ cannot be contracted.[5]

III. GENERALIZATION TO INTERACTING SYSTEMS

Extending the considerations presented above to systems in which the gap arises due to many-body interactions requires that the description of transport in terms of a unitary scattering matrix remains meaningful. As interactions may lead to inelastic scattering, the unitarity of the scattering matrix is not ascertained in general. In the presence of a finite energy gap Δ for bulk excitations, bulk charge and spin excitations are absent at sufficiently low temperatures $\beta^{-1} \ll \Delta$. Nonetheless, inelastic scattering can still arise due to ground-state degeneracies or mid-gap states at the edge of the wire. Inelastic scattering involving the excitation of boundary states at energy $\epsilon < \Delta$ are suppressed in the weak-coupling limit $\Gamma \ll \epsilon$. A more subtle effect may arise in the presence of (Kramers) degenerate edge states, where a Kondo effect may develop. For the purpose of classification, we may restrict the discussion to degeneracies protected by time-reversal symmetry. Such degenerate states generically occur at a finite distance μ from the Fermi energy in the leads if there are no additional discrete symmetries (such as particle-hole symmetry). This leads to an exponentially small Kondo temperature $\beta_{\mathrm{K}}^{-1} \sim e^{-\mu/\Gamma}$.[13] In the perturbative limit, $\beta_{\mathrm{K}} \gg \beta$, transitions between the Kramers degenerate pair occur at a rate $\Gamma^2/(\mu^2\nu_0) \ll \Gamma$, where ν_0 is the density of states in the lead. Therefore, in order to avoid inelastic scattering from transitions between boundary states, we restrict our analysis throughout this work to weak coupling, and operate the pump in the limit $\beta_{\mathrm{K}}^{-1}, \Gamma^2/(\mu^2\nu_0) \ll \hbar/\beta$, $\hbar/T \ll \Gamma \ll \Delta$, where a scattering-matrix description is appropriate. [We remark that inelastic scattering may also occur due to transitions between (nearly degenerate) bulk ground states. However, the typical transition rates for such processes are exponentially smaller than the edge excitations due to orthogonality of the many-body ground states. This is a manifestation of the fact that in $1+1$ dimensions it is possible to spontaneously break a discrete symmetry.[14] Hence, inelastic scattering arising from bulk degeneracies is absent in the above limit.]

We note that while a finite chemical potential is imperative to ensure the scattering matrix remains unitary, for the purpose of classification, we need only consider the particle-hole symmetric part of the reflection matrix, $\tilde{\mathcal{R}}$.[5] Hence, similar to their noninteracting counterparts, interacting spin pumps may be classified by the topology of the loop which $\tilde{\mathcal{R}}(t)$ forms on S^3.

While the existence of a bulk energy gap ensures the description in terms of a reflection matrix remains valid, its many-body nature gives rise to a richer variety of classes. Notably, interactions can change the Fermi sea of noninteracting electrons into multiple many-particle ground states. The (near) degeneracy of these states is not protected by symmetry, and is therefore split by small perturbations, such as the coupling to the leads, unlike in non-Abelian systems, which have a topologically nontrivial $H(t)$ at all times. In the following we show that multiple ground states may alter both the periodicity and the time-reversal restriction given in Eq. (3), thus modifying the classification of pumps.

To understand the implication of d nearly degenerate ground states, we note that the ground state of the macroscopic system spontaneously breaks the symmetry of the Hamiltonian. The reflection matrix will therefore depend on the specific ground state ϕ_a that the system is prepared in at the beginning of the cycle, $\tilde{\mathcal{R}}_a(t) = \tilde{\mathcal{R}}[\phi_a(t)]$, typically the true ground state. The periodicity of the Hamiltonian ensures that the ensemble of ground states $\{\phi_a\}_{a=1,\ldots,d}$ is restored after a period T, but the system need not return to the ground state ϕ_a it was in at $t=0$ (as discussed above, relaxation to the true ground state occurs at times exponentially larger than T). There are d possible scenarios: After a full cycle of the pump, the system may either return to the original ground state ϕ_a, or it may evolve to one of the $d-1$ other ground states. The latter scenario will result in an extended periodicity of the pump. In particular, the reflection matrix $\tilde{\mathcal{R}}_a(t)$ can have d different periods $\tilde{\mathcal{R}}_a(t+nT) = \tilde{\mathcal{R}}_a(t)$, where $1 \leqslant n \leqslant d$. (In the presence of a symmetry that relates the ground states to each other, n would typically divide d, however, in the absence of such a symmetry, there is no fundamental reason why this should be the case.)

Due to the multiplicity of the ground state, the time-reversal restriction (1) applies to the ensemble of ground states only; it does not directly imply the relation (3) on the reflection matrix $\tilde{\mathcal{R}}_a$ of a particular physical realization. To find the corresponding time-reversal restriction of a general nT periodic pump, we look at all the TRIM of the Hamiltonian during the extended cycle, $t_k = kT/2$ for $0 \leqslant k \leqslant 2n$. If the ground state is *not* time-reversal symmetric at any of these points $\phi_a(t_k) \neq \Theta\phi_a(t_k)\Theta^{-1}$, the reflection matrix does not have any restrictions on the pumping cycle arising from time-reversal symmetry. Such a loop $\tilde{\mathcal{R}}_a(t)$ can be contracted

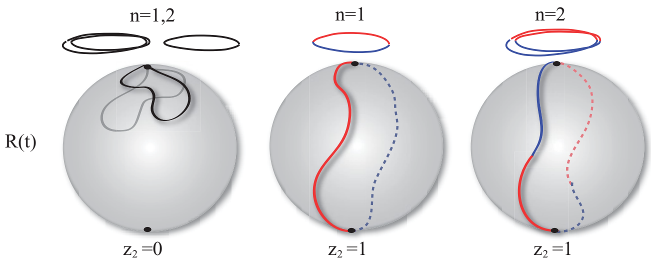

FIG. 1. (Color online) The three different classes of pumps for systems with two ground states corresponding to (left to right) the trivial class with $z_3 = n \times 0 = 0$, the integer pump with $z_3 = 1 \times 1$, and the fractional pump with $z_3 = 2 \times 1$. Here, the upper circles depict the extended cycle in parameter space, and for illustration purposes we restrict the reflection matrix to the two-sphere.

onto a single point, and is therefore in the trivial class of pumps. Conversely, if there exists a point $t_k \equiv 0$ at which $\phi_a(0) = \Theta\phi_a(0)\Theta^{-1}$, then $\phi_a(-t) = \Theta\phi_a(t)\Theta^{-1}$ and consequently, $\tilde{\mathcal{R}}_a(-t) = \Theta\tilde{\mathcal{R}}_a(t)\Theta^{-1}$ for all t.[15] Combined with the extended periodicity nT of the reflection matrix, this ensures that the loop $\tilde{\mathcal{R}}_a(t)$ is restricted by exactly two TRIM, $t_i = 0, nT/2$.[16] The existence of two TRIM allows one to distinguish two classes of loops for *each* nT-periodic pump, Eq. (4): A trivial pump which forms a loop that can be contracted to a single point, and a non-trivial pump that completes an uncontractable loop after n cycles of the pump. Hence, spin pumps with a d ground states can be classified by a $\mathbb{Z}_{d+1}$[17] index

$$z_{d+1} = n\, z_2 \in \{0,1,\ldots,d\}, \tag{5}$$

with $1 \leqslant n \leqslant d$ and $z_2 = 0,1$. This index discerns a trivial class of pumps that do not pump quantized spin, from d nontrivial classes (Fig. 1).

IV. PUMPING OF QUANTIZED SPIN

The $\mathbb{Z}_{d+1}$ classification of the reflection matrix (5) has a direct effect on the spin pumped during a cycle. In the weak-coupling limit, the d nontrivial pumps characterized by $z_2 = 1$ allow, in contrast to their trivial counterpart, for the noiseless pumping of quantized spin, even in the absence of a fixed spin quantization axis during the entire pumping cycle.[5] The extended periodicity nT, on the other hand, determines the averaged spin that is pumped during a cycle.

The class of nontrivial pumps that traverse a single resonance during the extended pumping cycle nT can be described by a rotation around a fixed axis $\vec{e}_\varphi(t_i)$:[5]

$$\tilde{\mathcal{R}}_a(t) = e^{i\varphi(t)\vec{e}_\varphi(t_i)\cdot\vec{\sigma}}. \tag{6}$$

Here $t_i = 0$ or $nT/2$ is the TRIM at which the resonance occurs and $\vec{\sigma}$ are the spin Pauli matrices. Consequently, the average spin per cycle T injected into lead α by these pumps is a fraction $1/n$ of the total spin pumped in the noninteracting case

$$\langle\vec{S}\rangle_T = \frac{\hbar}{2\pi n}\vec{e}_\varphi(t_i)\int_0^{nT} dt\,\dot{\varphi}(t) = \frac{\hbar}{n}\vec{e}_\varphi(t_i). \tag{7}$$

Conversely, a trivial pump either remains insulating during the entire cycle, or traverses two resonances at the TRIM. In the weak-coupling limit, the former group can be approximated by a constant reflection matrix $\tilde{\mathcal{R}}_a(t) \approx \mathbb{1}$, and thus does not pump spin, while the latter cannot be described by a *time-independent* spin direction that would lead to a quantized spin pumped. The $\mathbb{Z}_{d+1}$ classification (5) of interacting pumps together with the average fractional spin pumped during a cycle (7) constitute the main result of this paper.

The extended periodicity of the pumping cycle in the fractional spin pumps is reminiscent of the Aharonov-Bohm periodicity in a ring made of a material in the fractional quantum Hall state at $\nu = 1/3$: The ground state of the $\nu = 1/3$ is (nearly) threefold degenerate. Threading the ring by a single flux quantum $\phi_0 = h/e$ interchanges these ground states and the system returns to its initial state after the flux changes by $3\phi_0$, giving rise to an Aharonov-Bohm periodicity of $3\phi_0$.[6–9] We note that pumps constructed out of different physical systems with different ground-state degeneracies but with the same periodicity nT pump the same average spin per cycle. This is related to the observation that topological orders in fractional quantum Hall states cannot be characterized by the Hall conductance alone.

V. EXAMPLE

As an example of a spin pump characterized by $z_3 = 2$, we study the one-dimensional system with a half-filled energy band where interactions give rise to a bulk gap,

$$\begin{aligned} H_{\text{int}} = \sum_i & U n_{i,\uparrow}n_{i,\downarrow} + [U/2 - \delta V(t)]n_i n_{i+1} \\ & + U_H(t)(n_{i,\downarrow}\psi^\dagger_{i\uparrow}\psi_{i+1,\uparrow} - n_{i,\uparrow}\psi^\dagger_{i\downarrow}\psi_{i+1,\downarrow} + \text{H.c.}). \end{aligned} \tag{8}$$

Here $\delta V(t) = \delta V\,\cos(2\pi t/T)$, $U_H(t) = U_H\,\sin(2\pi t/T)$ set the strength of the time-dependent interaction terms in the Hamiltonian. The ground state of this system adiabatically switches between spin- and charge-ordered insulating and interaction-driven dimerized phases (see Fig. 2). For $U_H = 0$

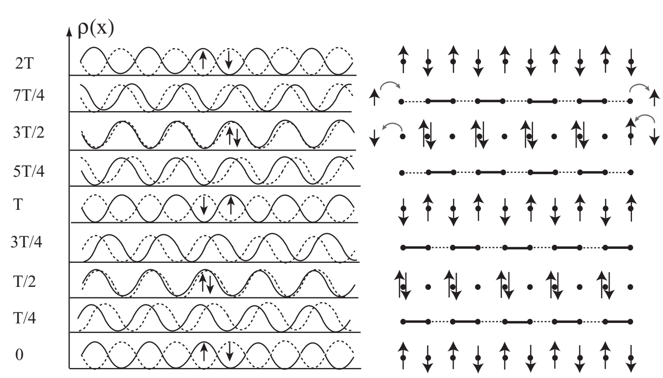

FIG. 2. An interaction-driven pump described by the Hamiltonian in Eq. (8) characterized by a $z_3 = 2$ index. The two ground states at $t = 0$ and $t = T$ can be obtained from one another by creating a single spin flip in the bulk which then propagates to the edges of the wire. The system returns to its original state after two pumping cycles. After a single pumping cycle the spin-density waves for the the two orientations have shifted by half a wavelength, corresponding to the transfer of $1/2[\hbar/2 - (-\hbar/2)]$ spin from the left to the right edge of the system, without transferring charge.

and $\delta V(t) > 0$ the electrons occupy different sites and the system is in the spin-density wave (SDW) ground state. At $\delta V(t) < 0$ the electrons pair up on the same site resulting in a charge-density wave (CDW) ground state. The third interaction term U_H breaks time-reversal symmetry and can, e.g., be generated in the presence of a staggered magnetic field and alternating bond strength.[18] Similar models have been studied in Refs. 19–21.

Interacting systems in one dimension are conveniently described in bosonization formalism. Here, the opening of the excitation gap arises due to the pinning of the bosonic degrees of freedom. In our example the pinning of spin degree of freedom ϕ_σ follows from the bosonized expression for H_{int},

$$H_{\text{int}}[\phi_\sigma] \sim \delta V(t) \cos 4\phi_\sigma + U_H(t) \sin 4\phi_\sigma. \tag{9}$$

The *four* multiples of the bosonic phase $4\phi_\sigma$ reflect the nature of the gap arising due to interaction terms containing four fermion operators. This results in a doubled periodicity compared to single-particle gapped phases and consequently, two ground states. For $\delta V = U_H$ these correspond to the pinning of the bulk phase at

$$2\phi^{(1,2)}_{\sigma,\,\text{min}}(t) = \pm\pi/2 + \pi t/T. \tag{10}$$

These two ground states are interchanged after one period of the pump, T, and the system corresponding to a particular ground state is recovered after $2T$. In addition to the bulk pinning, weak coupling to the leads pin the bosonic phase at the edge of the wire. This single-particle potential contains *two* multiples of the bosonic phase $2\phi_\sigma$. Due to the time dependence of the bulk pinning potential, Eq. (10), the phase $2\phi_\sigma$ changes by 2π during the extended cycle, $2T$, while the boundary pinning remains unchanged. As a result, during the course of the extended cycle, $2T$, the bosonic phase develops a single kink close to the edge, giving rise to a resonance in the tunneling density of states at the edge of the wire.[22]

Figure 2 illustrates the ground state of the system described by Eq. (8). Due to the ground-state multiplicity, the system returns to its original state after two pumping cycles. (A similar observation was made in Ref. 21.) After a single pumping cycle the spin-density waves for the two orientations have shifted by half a wavelength, resulting in the transfer of a spin of $\hbar/2$ from the left to the right edge of the system, without transferring charge.

VI. SUMMARY

We have studied the topological classification of one-dimensional insulators with a time-reversal restriction on the pumping cycle, in which the bulk excitation gap arises due to electron-electron interaction. Our classification holds in the weak-coupling limit, where common sources of inelastic scattering can by avoided and a description in terms of a unitary scattering matrix is possible. We found that a system with d many-body ground states can give rise to $d+1$ different classes of spin pumps. These include a trivial spin pump, which does not pump quantized spin, a quantized integer spin pump, and $d-1$ fractional spin pumps that allow for the average pumping of fractional spin $\hbar/n$ through the insulator. Recent works show that interactions may lead to new fractional topological insulators in the presence of a ground-state degeneracy.[23,24] The relation of our findings to the existence of a fractional quantum spin Hall state remains an interesting question.

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[16] To show that there are exactly 2 TRIM for an nT-periodic reflection matrix, we assume there is an additional $t_k \neq 0, nT/2$ such that $\tilde{\mathcal{R}}(t_k - t) = \Theta\tilde{\mathcal{R}}(t_k + t)\Theta^{-1}$; this implies $\tilde{\mathcal{R}}(2t_k - t') = \Theta\tilde{\mathcal{R}}(t')\Theta^{-1} = \tilde{\mathcal{R}}(-t')$, where the last equality follows from the time-reversal restriction around $t = 0$. This is in contradiction to the assumption that the system is $nT \neq 2t_k$ periodic.
[17] Z_{d+1} refers to the set of numbers; no multiplication rules implied.
[18] Such interaction terms are the relevant gap-opening perturbations in a Luttinger liquid with interaction parameters $K_\sigma < K_\rho/3 < 1$, as will be discussed in a future publication.
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