Far-from-equilibrium superconductor in fluctuational regime

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We derive Ginzburg-Landau-type action for a two-dimensional disordered superconductor under far-from-equilibrium conditions in a fluctuational regime. Then, utilizing it, we calculate fluctuation-induced density of states and Maki-Thomson- and Aslamazov-Larkin-type contributions to the in-plane electrical conductivity. We apply our approach to a thin superconducting film sandwiched between a gate and a substrate, which have different temperatures and different electrochemical potentials.

I. INTRODUCTION

Most of the processes in physics and in the technological realm occur under far-from-equilibrium (FFE) conditions. At the same time, the theories of nonequilibrium behavior were mostly restricted to a small deviation from the equilibrium. A marked progress in the approaches to quantitative description of FFE physics is related to Keldysh technique-based methods.[1] However, mostly FFE systems have been studied within the nonequilibrium form of the mean-field theory (see, for example, Refs. 2 and 3). In this paper, we address the important question of what happens to the conventional second-order phase transition if the system under consideration is driven out of equilibrium. In equilibrium, when close enough to the second-order phase transition, the mean-field theory does not hold and the physics starts to be governed by fluctuations.[4,5] In our work, we construct a theory of fluctuations under FFE conditions. As the exemplary system, we consider a superconductor FFE in the fluctuational regime.

Our first notion is that, while in clean three-dimensional conventional superconductors the fluctuations are important only in a very narrow region around the superconducting transition line (usually within the $\sim 10^{-12}$ K temperature range), in high-temperature, low-dimensional, and organic superconductors, the fluctuation region is much wider. In particular, as early as in 1968, Aslamazov and Larkin, and, independently, Maki showed that in disordered thin superconducting films, the width of the fluctuation region, which is determined by the sheet resistance, grows noticeably as compared to that of bulk superconductors.[5–8] Moreover, it was demonstrated that not only thermodynamic but also dynamic characteristics of the low-dimensional systems are strongly influenced by the fluctuations close to an equilibrium (see, e.g., Refs. 5 and 9 for a review).

A quantitative approach to nonequilibrium fluctuation superconductivity was recently formulated in Refs. 10 and 11. Building on this approach, we develop further our original Keldysh technique enabling us to find nonequilibrium fluctuation contributions to the electrical conductivity of a superconductor above the (nonequilibrium) superconducting transition. We show that, by measuring the fluctuation corrections, one can infer the parameters of the nonequilibrium state of the superconductor from the experimental data. Further, while in

the equilibrium, the lifetime of the fluctuation-induced Cooper pairs is determined by the difference $T − T_c$, where T_c is the critical temperature,[5,9] we find that in FFE conditions, it is controlled by the parameters of the nonequilibrium density matrix of the system. For example, for a thin superconducting film sandwiched between the gate and the substrate (see Fig. 1), these parameters are the temperatures of the gate and of the substrate, and the gate voltage V_G.

The paper is organized as follows. In Sec. II, we introduce the model. In Sec. III, we derive the nonequilibrium Ginzburg-Landau- (GL-) type action, while in Sec. IV, we calculate fluctuation-induced corrections to the electrical conductivity of the normal metal. In Sec. V, we focus on the specific experimental realization shown in Fig. 1 and analyze the previously derived results for this concrete setup. In Sec. VI, we summarize our results and discuss their further applications. Technical details are relegated to the Appendix.

II. KELDYSH SIGMA-MODEL ACTION

The Keldysh partition function of a superconductor with the Hamiltonian H in the coherent state basis is defined as

$$\mathcal{Z} = \mathcal{N} \int D\bar{\psi} \, D\psi \, \exp\{i S[\bar{\psi},\psi]\}, \tag{1}$$

where

$$S[\bar{\psi},\psi] = \int_C dt \left\{ \int_{\mathbf{r}} (\bar{\psi}_\alpha i \partial_t \psi_\alpha) - H[\bar{\psi},\psi] \right\} \tag{2}$$
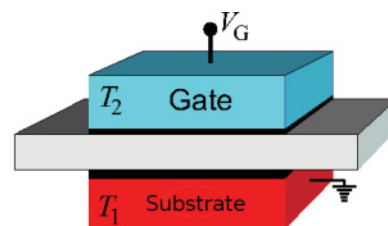

FIG. 1. (Color online) Thin superconducting film sandwiched between the substrate and the gate. The substrate temperature is T_1, the gate temperature is T_2, and the gate voltage is V_G. Changing T_1, T_2, and V_G, one can tune the nonequilibrium quasiparticle distribution in the film.

and $\mathcal{N}$ is a normalization constant. Hereafter, $\hbar = c = k_B = 1$. Here, $\mathcal{C}$ is the Keldysh contour and $\alpha \equiv \uparrow, \downarrow$ is the spin variable.

The Hamiltonian has the form $H = H_0 + H_{\text{int}}$, where the single-particle Hamiltonian

$$H_0 = \int_{\mathbf{r}} \bar{\psi}_\alpha \left[-\frac{(\nabla - ie\mathbf{A})^2}{2m} + U_{\text{dis}} + e\phi \right] \psi_\alpha. \qquad (3)$$

Here, $\mathbf{A}$, ϕ, and U_{dis} are vector, scalar, and disorder potentials and e is the electron charge; the tensor summation over the spin indices is implied. The interaction Hamiltonian describes the electron-electron interaction in the Cooper channel

$$H_{\text{int}} = -\frac{\lambda}{\nu} \int_{\mathbf{r}} \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow, \qquad (4)$$

where $\rho(\mathbf{r}) = \sum_\alpha \bar{\psi}_\alpha(\mathbf{r}) \psi_\alpha(\mathbf{r})$ is the local electron density and the superconductive coupling constant $\lambda > 0$. The disorder potential is assumed to be Gaussian distributed with the correlator

$$\langle U_{\text{dis}}(\mathbf{r}) U_{\text{dis}}(\mathbf{r}') \rangle = \frac{1}{2\pi\nu\tau} \delta(\mathbf{r} - \mathbf{r}'). \qquad (5)$$

Averaging over disorder and carrying out the standard decoupling in the four-fermion terms in the action via the Stratonovich-Hubbard fields $\{Q, \Delta\}$,[9] and integrating out the degrees of freedom with the energies higher than $1/\tau$, where τ^{-1} is the elastic scattering rate, we arrive at the Keldysh nonlinear σ-model partition function

$$Z = \int D[Q, \Delta] \exp\{i S[\check{Q}_\mathcal{K}, \check{\Delta}_\mathcal{K}]\}, \qquad (6)$$

$$S[\check{Q}_\mathcal{K}, \check{\Delta}_\mathcal{K}] = S_\Delta + S_\phi + S_Q. \qquad (7)$$

The nonlinear σ-model action S consists of three parts:

$$S_\Delta = -\frac{\nu}{2\lambda} \text{Tr}[\check{\Delta}_\mathcal{K}^\dagger \check{Y} \check{\Delta}_\mathcal{K}], \qquad (8)$$

$$S_\phi = \frac{e^2 \nu}{2} \text{Tr}[\check{\phi}_\mathcal{K} \check{Y} \check{\phi}_\mathcal{K}], \qquad (9)$$

$$S_Q = \frac{i\pi\nu}{4} \text{Tr}[D(\partial_{\mathbf{r}} \check{Q}_\mathcal{K})^2 - 4\check{\Xi}\partial_t \check{Q}_\mathcal{K} \\ - 4ie\check{\phi}_\mathcal{K} \check{Q}_\mathcal{K} + 4i\check{\Delta}_\mathcal{K} \check{Q}_\mathcal{K}], \qquad (10)$$

where $\check{Q}^2 = 1$. Here, D is the diffusion coefficient and ν is the bare single-particle density of states at the Fermi level per one spin projection. The action (7) holds while the effective temperatures (see Sec. IV) that follow from it are much smaller than $1/\tau$. The check mark above the field variables indicates that they are defined in the space that is the tensor product of the Keldysh and Nambu spaces. The former and latter are spanned by the Pauli matrices $\hat{\sigma}_i$ and $\hat{\tau}_i$, $i \in \{0, x, y, z\}$, respectively. So, $\check{Y} = \hat{\sigma}_x \otimes \hat{\tau}_0$, $\check{\Xi} = \hat{\sigma}_0 \otimes \hat{\tau}_z$. Multiplication in time space is implicitly assumed, and "Tr" includes an integration over real space. The subscript $\mathcal{K}$ denotes the gauge transformed fields

$$\check{\phi}_\mathcal{K} = \check{\phi} - \partial_t \check{\mathcal{K}}, \qquad (11)$$

$$\check{\mathbf{A}}_\mathcal{K} = \check{\mathbf{A}} + \nabla\check{\mathcal{K}}, \qquad (12)$$

$$\check{\mathcal{K}} = [k^{cl}\hat{\sigma}_0 + k^q\hat{\sigma}_x] \otimes \hat{\tau}_0. \qquad (13)$$

$\check{\mathbf{A}}$ and $\check{\phi}$ are defined in the same way as $\check{\mathcal{K}}$. Also,

$$\check{\Delta} = [\Delta^{cl}\hat{\sigma}_0 + \Delta^q\hat{\sigma}_x] \otimes \hat{\tau}_+ - \text{H.c.}, \qquad (14)$$

$$\check{\Delta}_\mathcal{K}(\mathbf{r}, t) = e^{ie\check{\Xi}\check{\mathcal{K}}(\mathbf{r}, t)} \check{\Delta} e^{-ie\check{\Xi}\check{\mathcal{K}}(\mathbf{r}, t)}. \qquad (15)$$

$\check{Q}_\mathcal{K}$ is defined in the same way. The quantum (q) and classical (cl) components are defined as half-sum and half-difference of the field values at the lower and upper branches of the Keldysh time contour. The field Δ^{cl} becomes the superconducting order parameter on the mean-field (saddle-point) level, while the saddle-point equation for $\check{Q}$ produces the Usadel quasiclassical equations, where $\check{Q}$ plays the role of the quasiclassical Green's function. The covariant spatial derivative is given by

$$\partial_{\mathbf{r}} \check{Q}_\mathcal{K} = \nabla_{\mathbf{r}} \check{Q}_\mathcal{K} - ie[\check{\Xi}\check{\mathbf{A}}_\mathcal{K}, \check{Q}_\mathcal{K}]. \qquad (16)$$

III. GINZBURG-LANDAU ACTION

First, let us consider the simplified situation ignoring the interactions. Then, the metallic saddle point of Eq. (7) is[12–14]

$$\check{\Lambda} = \check{\mathcal{U}}\check{\Lambda}_0\check{\mathcal{U}}^{-1}, \qquad \check{\Lambda}_0 = \hat{\sigma}_z \otimes \hat{\tau}_z, \qquad (17)$$

$$\check{\mathcal{U}}_{t,t'}(\mathbf{r}) = \check{\mathcal{U}}_{t,t'}^{-1}(\mathbf{r}) = \begin{pmatrix} \delta_{t-t'-0}\hat{\tau}_0 & \hat{F}_{t,t'}(\mathbf{r}) \\ 0 & -\delta_{t-t'+0}\hat{\tau}_0 \end{pmatrix}, \qquad (18)$$

$$\hat{F}_{t,t'}(\mathbf{r}) = \begin{pmatrix} F_{t,t'}^e(\mathbf{r}) & 0 \\ 0 & F_{t,t'}^h(\mathbf{r}) \end{pmatrix}. \qquad (19)$$

By setting quantum components of electromagnetic potentials to zero, the equation for $F_{t,t'}^{e/h}(\mathbf{r})$ reads as

$$D\Big(\nabla^2 F^{e/h} \mp 2ie[\mathbf{A}_\mathcal{K}^{cl}, \nabla F^{e/h}] \mp ie[\nabla\mathbf{A}_\mathcal{K}^{cl}, F^{e/h}] \\ - \frac{e^2}{2}[\mathbf{A}_\mathcal{K}^{cl}, [\mathbf{A}_\mathcal{K}^{cl}, F^{e/h}]]\Big) - \overrightarrow{\partial_t} F^{e/h} + F^{e/h} \overleftarrow{\partial_t} \\ \mp ie[\phi_\mathcal{K}^{cl}, F^{e/h}] = 0. \qquad (20)$$

After the Wigner transformation, we can map $F^{e/h}$ to quasiparticle electron (hole) distribution functions: $F_\epsilon^{e/h}(\mathbf{r}, t) \equiv 1 - 2f_\epsilon^{e/h}(\mathbf{r}, t)$. In the leading order, with respect to time and energy partial derivatives of external potentials, the kinetic equations become

$$D\big[\nabla^2 F_\epsilon^{e/h} \mp 2e(\partial_t \mathbf{A}_\mathcal{K}^{cl})(\partial_\epsilon \nabla F_\epsilon^{e/h}) \\ + \tfrac{1}{2}e^2(\partial_t \mathbf{A}_\mathcal{K}^{cl})^2(\partial_\epsilon^2 F_\epsilon^{e/h})\big] - \partial_t F_\epsilon^{e/h} \\ \mp e\partial_t(D\nabla\mathbf{A}_\mathcal{K}^{cl} + \phi_\mathcal{K}^{cl})\partial_\epsilon F_\epsilon^{e/h} = 0. \qquad (21)$$

Note that one has to take into account adequate boundary conditions.

Having specified the metallic saddle-point solution, we consider the massless fluctuations around it. They can be parametrized as[15]

$$\check{Q}_\mathcal{K}(\mathbf{r}) = e^{-\check{W}(\mathbf{r})/2} \check{\Lambda}(\mathbf{r}) e^{\check{W}(\mathbf{r})/2}, \qquad \check{W} = \check{\mathcal{U}}\check{\mathcal{W}}\check{\mathcal{U}}^{-1}, \qquad (22)$$

$$\check{\mathcal{W}} = \begin{pmatrix} w\tau_+ - w^*\tau_- & w_0\tau_0 + w_z\tau_z \\ \bar{w}_0\tau_0 + \bar{w}_z\tau_z & \bar{w}\tau_+ - \bar{w}^*\tau_- \end{pmatrix}, \qquad (23)$$

such that $\check{W}\check{\Lambda} + \check{\Lambda}\check{W} = 0$. Here, we introduced four real fields $w^{\alpha}_{tt'}(\mathbf{r}), \bar{w}^{\alpha}_{tt'}(\mathbf{r})$ with $\alpha = 0, z$ representing diffusion degrees of freedom and two complex fields $w_{tt'}(\mathbf{r}), \bar{w}_{tt'}(\mathbf{r})$ for cooperon degrees of freedom.

Since the main goal of this section is the derivation of the nonequilibrium extension of GL-type action in the normal state but very close to the transition, in what follows we will concentrate on cooperon degrees of freedom only. By plugging $\check{Q}_{\mathcal{K}}$, given by Eq. (22), in the action Eq. (7) and expanding it up to the second order in cooperons, we get

$$S_Q = S_{w^2} + S_{w\Delta}, \tag{24}$$

$$S_{w^2} = i\frac{\pi\nu}{2}\text{Tr}\big[\mathbf{w}^{\dagger}_{t,t'}(\mathbf{r})\mathcal{C}^{-1}_{t,t'}(\mathbf{r})\mathbf{w}_{t',t}(\mathbf{r})\big], \tag{25}$$

$$S_{w\Delta} = -\pi\nu\,\text{Tr}[(-\Delta^{cl*} - F_h\Delta^{q*})w + (\Delta^{cl} + F_e\Delta^q)w^*$$
$$+ (\Delta^{cl*} - F_e\Delta^{q*})\bar{w} + (-\Delta^{cl} + F_h\Delta^q)\bar{w}^*]. \tag{26}$$

Here, $\mathbf{w}_{t,t'}(\mathbf{r}) = (w_{t,t'}(\mathbf{r}), \bar{w}_{t,t'}(\mathbf{r}))^T$ and[9]

$$\mathcal{C}^{-1}_{t,t'}(\mathbf{r}) = \begin{pmatrix} C^{-1}_{t,t'}(\mathbf{r}) & 0 \\ 0 & \bar{C}^{-1}_{t,t'}(\mathbf{r}) \end{pmatrix}, \tag{27}$$

$$C^{-1}_{t,t'}(\mathbf{r}) = -\partial_t + \partial_{t'} - ie\big[\phi^{cl}_{\mathcal{K}}(\mathbf{r},t) - \phi^{cl}_{\mathcal{K}}(\mathbf{r},t')\big]$$
$$- D\big[\nabla - ie\mathbf{A}^{cl}_{\mathcal{K}}(\mathbf{r},t) - ie\mathbf{A}^{cl}_{\mathcal{K}}(\mathbf{r},t')\big]^2, \tag{28}$$

$$\bar{C}^{-1}_{t,t'}(\mathbf{r}) = \partial_t - \partial_{t'} + ie\big[\phi^{cl}_{\mathcal{K}}(\mathbf{r},t) - \phi^{cl}_{\mathcal{K}}(\mathbf{r},t')\big]$$
$$- D\big[\nabla - ie\mathbf{A}^{cl}_{\mathcal{K}}(\mathbf{r},t) - ie\mathbf{A}^{cl}_{\mathcal{K}}(\mathbf{r},t')\big]^2. \tag{29}$$

Note that there are additional terms in the action (24), which are not presented here for the sake of brevity. They contain quantum components of electromagnetic potentials. Since quantum components of fields are the auxiliary source fields usually used to calculate observables by the appropriate differentiation of the Keldysh action, we omit the mentioned terms here. They are not important for the derivation of the GL action. We will discuss these terms in the next section when calculating different corrections to the Drude conductivity that arise due to superconducting fluctuations.

Now, the cooperon degrees of freedom can be integrated out from the Keldysh partition function (6). In this way, the GL action is generated:

$$iS_{\text{GL}}[\Delta^{cl}, \Delta^q] = -\tfrac{1}{2}\big\langle S^2_{w\Delta}\big\rangle_{iS_{w^2}} + iS_\Delta. \tag{30}$$

In what follows, we consider stationary distribution functions (i.e., $F_{t,t'} = F_{t-t'}$) and stationary electromagnetic fields. Also, we assume that they slowly vary in space as compared to the fluctuating order parameter $\Delta(\mathbf{r},t)$. Then, we obtain (see the Appendix)

$$S_{\text{GL}}[\Delta^{cl}, \Delta^q] = 2\nu\,\text{Tr}[\mathbf{\Delta}^{\dagger}_{\mathcal{K}}(\mathbf{r},t)\hat{L}^{-1}\mathbf{\Delta}_{\mathcal{K}}(\mathbf{r},t)], \tag{31}$$

where $\mathbf{\Delta} = (\Delta^{cl}, \Delta^q)^T$. The structure of the fluctuation propagator $\hat{L}$ is characteristic for bosons:

$$\hat{L}^{-1} = \begin{pmatrix} 0 & L^{-1}_A \\ L^{-1}_R & L^{-1}_K \end{pmatrix}. \tag{32}$$

The indexes R, A, and K denote the retarded, advanced, and Keldysh propagators, respectively. The cl-cl component of $\hat{L}$ is

zero. This is expected since, for $\Delta^q = 0$, the field on the upper and lower branches of the Keldysh contour is the same, and therefore the corresponding actions cancel each other, resulting in $S_{\text{GL}}(\Delta^{cl}, 0) = 0$. The Keldysh propagator is responsible for the Gaussian noise term in the time-dependent GL (TDGL) equation, and it also carries information about distribution of electrons and holes. The details of the derivation, as well as general formulas for the fluctuating propagators (that are valid if the system is far from the transition) are presented in the Appendix. Close to the transition into the superconducting state, the propagators become

$$L^{-1}_K = i\frac{\pi}{2}\big(1 - \tilde{F}^h_0\tilde{F}^e_0\big), \tag{33}$$

$$L^{-1}_{R/A} = \frac{\pi}{8T_e}\bigg\{ -(\tau_{\text{GL}}z_{\text{cp}})^{-1} + \big[\mp 4iT_e\tilde{F}^R_0$$
$$+ D\big(\nabla - 2ie\mathbf{A}^{cl}_{\mathcal{K}}\big)^2 \mp \partial_t \mp 2ie\phi^{cl}_{\mathcal{K}}\big]$$
$$\times \bigg(1 \pm i\frac{T_e}{\Omega}\bigg)\bigg\}. \tag{34}$$

Here, $\tilde{F}^{e/h}_{\epsilon} = F^{e/h}_{\epsilon \pm e\phi^{cl}_{\mathcal{K}}}$ denotes the gauge-invariant distribution function, while other parameters appearing in the retarded and advanced propagators are the functionals of $F^R_{\epsilon} = (F^h_{\epsilon} - F^e_{-\epsilon})/2$:

$$T^{-1}_e = 2\frac{d\tilde{F}^R_{\epsilon}}{d\epsilon}\bigg|_{\epsilon=0}, \tag{35}$$

$$\Omega^{-1} = \frac{2}{\pi}\fint d\epsilon\frac{\tilde{F}^R_{\epsilon} - \tilde{F}^R_0}{\epsilon^2}, \tag{36}$$

$$z^{-1}_{\text{cp}} = 1 + \bigg(\frac{T_e}{\Omega}\bigg)^2. \tag{37}$$

They are strongly drive dependent, as will be demonstrated in Sec. V. The symbol $\fint$ denotes the principal value of the integral. Note that the existence of effective temperature T_e does not imply the local equilibrium form of excitation distribution functions.

The nonequilibrium GL relaxation rate is defined as

$$\tau^{-1}_{\text{GL}} = -\frac{4}{\pi}z_{\text{cp}}T_e\int^{+\omega_D}_{-\omega_D}d\epsilon\frac{\tilde{F}^R_{\epsilon} - \tanh\big(\frac{\epsilon}{2T_c}\big)}{\epsilon} + 4z_{\text{cp}}\frac{T^2_e}{\Omega}\tilde{F}^R_0, \tag{38}$$

where ω_D is the Debye energy. The GL relaxation rate represents the inverse lifetime of Cooper pairs, and, therefore, it vanishes at the transition to the superconducting state. In an equilibrium, the density matrix is parametrized by the temperature, and the condition $\tau^{-1}_{\text{GL}} = 0$ tells us that the transition occurs at $T = T_c$. In a general nonequilibrium case, additional parameters may appear, e.g., voltage drop and temperatures of the thermal baths that are in contact with the system. Then, the condition $\tau^{-1}_{\text{GL}} = 0$ defines the phase-transition surface in the parameter space. We find that any distribution function F_{ϵ} at the phase-transition surface satisfies

$$\int^{+\omega_D}_{-\omega_D}d\epsilon\frac{\tilde{F}^R_{\epsilon} - \tanh\big(\frac{\epsilon}{2T_c}\big)}{\epsilon} - \pi\frac{T_e}{\Omega}\tilde{F}^R_0 = 0. \tag{39}$$

By tuning the external electromagnetic fields and/or temperature of thermal baths, one can control the distance from the phase-transition surface. We point out that the theory presented above is valid only for the systems (in the normal state) close to the transition, i.e., when $(\tau_{\mathrm{GL}} T_e)^{-1} \ll 1$. The difference between Eqs. (34) and (38) and the work in Ref. 10 is the appearance of Ω^{-1} and presence of terms $\sim \tilde{F}_0^R$ (results of Ref. 10 are restricted to the nonequilibrium states generated by the voltage between the leads or gates in a certain way). In general, Ω^{-1} is nonzero, as discussed in Sec. V.

In an equilibrium, the following relations hold: $F^{e/h} = \tanh\left(\frac{\epsilon}{2T}\right)$, $T_e = T$, $\Omega = 0$, $z_{\mathrm{cp}} = 1$, and we reproduce the standard GL action. It is given by Eq. (31), but with the propagators

$$\left(L_{eq}^{-1}\right)_K = i\frac{\pi}{2}, \tag{40}$$

$$\left(L_{eq}^{-1}\right)_{R/A} = \frac{\pi}{8T}\left[-\left(\tau_{\mathrm{GL}}^{eq}\right)^{-1} + D\nabla^2 \mp \partial_t\right], \tag{41}$$

$$\left(\tau_{\mathrm{GL}}^{eq}\right)^{-1} = \frac{8}{\pi}(T - T_c). \tag{42}$$

In an equilibrium, the Keldysh fluctuation propagator satisfies the relation following from the fluctuation-dissipation theorem

$$\left(L_{eq}^{-1}\right)_K = \coth\left(\frac{\omega}{2T}\right)\left[\left(L_{eq}^{-1}\right)_R(\mathbf{q},\omega) - \left(L_{eq}^{-1}\right)_A(\mathbf{q},\omega)\right] \tag{43}$$

for $\omega \ll T$. We find that, in a general case, this relation is violated for a system out of the equilibrium.

Next, we derive the TDGL equation. After the term $\sim |\Delta^q|^2$ in Eq. (31) is decoupled by introduction of the Hubbard-Stratonovich field $\zeta(\mathbf{r},t)$, one differentiates (31) with respect to Δ^{q*} and obtains

$$\frac{8T_e}{\pi}L_R^{-1}\Delta_{\mathcal{K}}^{cl} + \zeta = 0, \tag{44}$$

$$\langle \zeta(\mathbf{r},t)\zeta^*(\mathbf{r}',t')\rangle = \frac{16}{\pi\nu}T_e^2\left(1 - \tilde{F}_0^h\tilde{F}_0^e\right)\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{45}$$

The field ζ has a meaning of the Gaussian noise. The superconducting order parameter changes under a gauge transformation as $\Delta_{\mathcal{K}}^{cl} = \Delta^{cl}e^{2iek^{cl}}$. However, the TDGL equation for the order parameter is gauge invariant since the presence of the gauge field k^{cl} in L_R^{-1} (through $\mathbf{A}_{\mathcal{K}}^{cl}$ and $\phi_{\mathcal{K}}^{cl}$) compensates the change of the order-parameter phase.

Using the Keldysh formalism, derivation of the GL close-to-equilibrium action was performed in Ref. 9. A special gauge was utilized where $D\nabla\mathbf{A}_{\mathcal{K}} + \phi_{\mathcal{K}} = 0$ and found that the scalar potential does not appear in the TDGL equation for the order parameter. In Ref. 9, it was used as $F_\epsilon = \tanh(\epsilon/(2T))$. By looking at kinetic equation (21), it becomes obvious that the mentioned gauge may indeed simplify the calculation. However, in the presence of external potentials, $\tanh[\epsilon/(2T)]$ is not the solution of the kinetic equation even in this gauge, due to boundary conditions. As a result, the scalar electromagnetic potential does appear in the retarded

propagator (34) and therefore also appears in the TDGL equation for the order parameter (44), contrary to the statements in Ref. 9.

Although the higher-order terms in the superconducting order parameter in the GL action are unimportant for our analysis of superconducting fluctuations, we will state them for completeness. We focus only on local fourth-order terms of the form $\Delta^{q*}\Delta^{cl}|\Delta^{cl}|^2$ and find

$$S_{\Delta^4} = -\pi\nu\int_0^\infty d\tau\,\tau\,\tilde{Y}(\tau)$$
$$\times \mathrm{Tr}\left[\Delta_{\mathcal{K}}^{q*}(\mathbf{r},t)\Delta_{\mathcal{K}}^{cl}(\mathbf{r},t)\left|\Delta_{\mathcal{K}}^{cl}(\mathbf{r},t)\right|^2\right] + \mathrm{c.c.}, \tag{46}$$

$$\tilde{Y}(\tau) = \int_{-\infty}^\infty \frac{d\epsilon}{2\pi}\frac{\tilde{F}_\epsilon^R}{\epsilon + i0}e^{i\epsilon\tau}. \tag{47}$$

In the equilibrium, the action reduces to[9] $S_{\Delta^4} = -7\nu\zeta(3)\mathrm{Tr}[\Delta_{\mathcal{K}}^{q*}(\mathbf{r},t)\Delta_{\mathcal{K}}^{cl}(\mathbf{r},t)|\Delta_{\mathcal{K}}^{cl}(\mathbf{r},t)|^2 + \mathrm{c.c.}]/(4\pi^2 T^2)$.

IV. CORRECTIONS TO ELECTRICAL CONDUCTIVITY

In this section, we derive corrections to the Drude conductivity arising in thin films due to superconducting fluctuations. The system is close to the transition, but under FFE conditions.

First, we briefly explain the origin of different fluctuation-induced contributions. Fluctuation Cooper pairs carry charge and directly contribute to the electrical conductivity, determining the Aslamazov-Larkin (AL) contribution.[6,7] Since quasiparticles are involved in Cooper pairing, effectively the number of carriers participating in the single-electron charge transfer is decreased. The fluctuation pseudogap opens at the Fermi level in the single-particle spectrum, and results in diminishing of the Drude conductivity for the so-called density of states (DOS) correction.[5] The third contribution to conductivity, the (anomalous) Maki-Thompson[8,16] (MT) correction, is of purely quantum origin. It arises due to coherent scattering of the electrons, forming a fluctuation Cooper pair, on the elastic impurities. As a result, the diffusion coefficient changes and, therefore, the Drude conductivity is influenced.

We are interested in linear response to the in-plane electric field, while the system is strongly driven out of equilibrium due to contacts with thermal baths at different temperatures between which it is sandwiched, or due to an electric field perpendicular to the plane. We focus on the regime where fluctuations can be treated perturbatively, meaning that the below derived corrections are small compared to the Drude conductivity σ_D. We take into account terms linear in the Ginzburg number $G_i = (\nu Dd_f)^{-1}$, i.e., $\delta\sigma/\sigma_D \sim G_i$, where d_f is film thickness. The in-plane dc conductivity is given by

$$\sigma_{xx} = -\frac{1}{2}\lim_{\Omega \to 0}\left(\frac{1}{\Omega}\frac{\delta^2 Z}{\delta\mathbf{A}_{\mathcal{K},x}^{cl}(\Omega)\delta\mathbf{A}_{\mathcal{K},x}^q(-\Omega)}\right)\Bigg|_{\mathbf{A}_{\mathcal{K}}^q = 0}, \tag{48}$$

where Z is the Keldysh partition function [Eq. (6)]. The terms in the action, which are linear or quadratic in the vector potential, arise from the first term of Eq. (10) and read as

$$iS_A = -\frac{i\pi \nu e D}{2}\text{Tr}\{[\check{W},(\boldsymbol{\nabla}\check{W})]\breve{\Xi}\check{\mathbf{A}}_{\mathcal{K}}\}, \qquad (49)$$

$$iS_{A^2} = \frac{\pi \nu D e^2}{4}\text{Tr}\{[\breve{\Xi}\check{\mathbf{A}}_{\mathcal{K}},\check{\Lambda}][\breve{\Xi}\check{\mathbf{A}}_{\mathcal{K}},\check{\Lambda}] + 2\breve{\Xi}\check{\mathbf{A}}_{\mathcal{K}}\check{\Lambda}\breve{\Xi}\check{\mathbf{A}}_{\mathcal{K}}\check{\Lambda}\check{W}^2$$
$$+ 2\breve{\Xi}\check{\mathbf{A}}_{\mathcal{K}}\check{\Lambda}\check{W}\breve{\Xi}\check{\mathbf{A}}_{\mathcal{K}}\check{\Lambda}\check{W}\}, \qquad (50)$$

respectively. The first term of Eq. (50) does not contain superconducting fluctuations and, therefore, after differentiation over classical and quantum components of the vector potential and averaging over $\hat{Q}$ and Δ fluctuations [see Eq. (48)], it gives the Drude conductivity. The other two terms in (50) produce the MT and DOS correction to the conductivity. The

part of the action linear in the vector potential (S_A) gives the AL correction. The Drude conductivity is

$$\sigma_D = -\frac{\pi \nu D e^2}{4}\lim_{\Omega \to 0}\left\{\frac{1}{\Omega}\text{Tr}(\check{\Lambda}_\epsilon \hat{\sigma}_x \check{\Lambda}_{\Omega+\epsilon} + \hat{\sigma}_x \check{\Lambda}_\epsilon \check{\Lambda}_{-\Omega+\epsilon})\right\}$$
$$= -\frac{\pi \nu D e^2}{2}\lim_{\Omega \to 0}\frac{\text{Tr}(F^e_{\epsilon-\Omega} - F^e_{\epsilon+\Omega} + F^h_{\epsilon-\Omega} - F^h_{\epsilon+\Omega})}{\Omega}$$
$$= 2\nu D e^2. \qquad (51)$$

In Eq. (48), all the quantum components of vector potential are set to zero after differentiation. Therefore, after differentiation with respect to $\mathbf{A}^{cl}$ and $\mathbf{A}^q$, in calculation of σ we can use the action (24) derived in the previous section. We assume the stationary situation ($F^{e/h}_{t,t'} = F^{e/h}_{t-t'}$). The in-plane classical components of the vector potential can be set to zero since we are interested in the linear response to the in-plane electric field. Then, we find the saddle-point configuration of $\mathbf{w}$ for the action (24) and, using the GL-type action (31), we obtain

$$\langle\langle w_{1,2}(\mathbf{q})w^*_{3,4}(-\mathbf{q})\rangle\rangle_{Q,\Delta} = \frac{2i}{\nu}\delta_{\epsilon_1-\epsilon_2,\epsilon_4-\epsilon_3}\frac{-L^{-1}_K L_{A,1-2}L_{R,1-2} + F^h_3 L_{R,1-2} + F^e_1 L_{A,1-2}}{\{D\mathbf{q}^2 - i(\epsilon_1+\epsilon_2)\}\{D\mathbf{q}^2 - i(\epsilon_3+\epsilon_4)\}}, \qquad (52)$$

$$\langle\langle \bar{w}_{1,2}(\mathbf{q})\bar{w}^*_{3,4}(-\mathbf{q})\rangle\rangle_{Q,\Delta} = \frac{2i}{\nu}\delta_{\epsilon_1-\epsilon_2,\epsilon_4-\epsilon_3}\frac{-L^{-1}_K L_{A,1-2}L_{R,1-2} - F^h_2 L_{A,1-2} - F^e_4 L_{R,1-2}}{\{D\mathbf{q}^2 + i(\epsilon_1+\epsilon_2)\}\{D\mathbf{q}^2 + i(\epsilon_3+\epsilon_4)\}}, \qquad (53)$$

$$\langle\langle \bar{w}_{1,2}(\mathbf{q})w^*_{3,4}(-\mathbf{q})\rangle\rangle_{Q,\Delta} = \frac{2i}{\nu}\delta_{\epsilon_1-\epsilon_2,\epsilon_4-\epsilon_3}\frac{L^{-1}_K L_{A,1-2}L_{R,1-2} + F^h_2 L_{A,1-2} - F^h_3 L_{R,1-2}}{\{D\mathbf{q}^2 + i(\epsilon_1+\epsilon_2)\}\{D\mathbf{q}^2 - i(\epsilon_3+\epsilon_4)\}}, \qquad (54)$$

$$\langle\langle w_{1,2}(\mathbf{q})\bar{w}^*_{3,4}(-\mathbf{q})\rangle\rangle_{Q,\Delta} = \frac{2i}{\nu}\delta_{\epsilon_1-\epsilon_2,\epsilon_4-\epsilon_3}\frac{L^{-1}_K L_{A,1-2}L_{R,1-2} + F^e_4 L_{R,1-2} - F^e_1 L_{A,1-2}}{\{D\mathbf{q}^2 - i(\epsilon_1+\epsilon_2)\}\{D\mathbf{q}^2 + i(\epsilon_3+\epsilon_4)\}}. \qquad (55)$$

Here, the angular brackets denote averaging over fluctuations of $\check{Q},\Delta^{cl/q}$. $F^{e/h}_i = F^{e/h}_{\epsilon_i}$ and $w_{i,j} = w_{\epsilon_i,\epsilon_j}$ where $i,j = 1,\ldots,4$. Also, $L_{A/R,i-j} = [L^{-1}_{R/A}(\mathbf{q},\epsilon_i - \epsilon_j)]^{-1}$.

Now, we can proceed to the calculation of different corrections to the Drude conductivity. We start with the DOS correction

$$\delta\sigma^{\text{DOS}} = -\frac{\nu D e^2}{8\pi}\int_{\mathbf{q},\epsilon_3,\epsilon_4}\Big[\big(\partial_{\epsilon_3}F^e_3\big)\langle\langle w_{3,4}(\mathbf{q})w^*_{4,3}(-\mathbf{q}) + \bar{w}_{3,4}(\mathbf{q})\bar{w}^*_{4,3}(-\mathbf{q})\rangle\rangle_{Q,\Delta}$$
$$+ \big(\partial_{\epsilon_3}F^h_3\big)\langle\langle w^*_{3,4}(\mathbf{q})w_{4,3}(-\mathbf{q}) + \bar{w}^*_{3,4}(\mathbf{q})\bar{w}_{4,3}(-\mathbf{q})\rangle\rangle_{Q,\Delta}$$
$$- F^h_4 \partial_\Omega\langle\langle w_{3,4}(\mathbf{q})w^*_{4+\Omega,3+\Omega}(-\mathbf{q})\rangle\rangle\big|_{\Omega=0} - F^e_4 \partial_\Omega\langle\langle w^*_{3,4}(\mathbf{q})w_{4+\Omega,3+\Omega}(-\mathbf{q})\rangle\rangle\big|_{\Omega=0}$$
$$+ F^e_3 \partial_\Omega\langle\langle \bar{w}_{3,4}(\mathbf{q})\bar{w}^*_{4-\Omega,3-\Omega}(-\mathbf{q})\rangle\rangle\big|_{\Omega=0} + F^h_3 \partial_\Omega\langle\langle \bar{w}^*_{3,4}(\mathbf{q})\bar{w}_{4-\Omega,3-\Omega}(-\mathbf{q})\rangle\rangle\big|_{\Omega=0}\Big], \qquad (56)$$

where $\int_{\mathbf{q},\epsilon_3,\epsilon_4} = \int d\mathbf{q}/(2\pi)^2\int_{-\infty}^\infty d\epsilon_3 d\epsilon_4$. Note that the terms which in the limit $\Omega \to 0$ behave as $1/\Omega$ will be canceled out with similar terms from the other corrections (MT and AL), and that is why they are omitted here. The main contribution close to the transition reads as

$$\delta\sigma^{\text{DOS}} \approx -\frac{D e^2}{4\pi}\text{Im}\left[\int_{\epsilon,\omega,\mathbf{q}}\frac{L^{-1}_K\big(\partial_\epsilon F^e_\epsilon |L^A_+|^2 + \partial_\epsilon F^h_\epsilon |L^A_-|^2\big)}{\{D\mathbf{q}^2 - i(2\epsilon - \omega)\}^2}\right], \qquad (57)$$

where $L^A_\pm = L_A(\mathbf{q},\pm\omega)$. We obtain the DOS correction in a thin film:

$$\delta\sigma^{\text{DOS}} \approx -\frac{7e^2\zeta(3)}{\pi^4 d_f}\frac{T_e T_{\text{cp}}}{T^2_{\text{DOS}}}\log\left(\frac{T_e}{\tau^{-1}_{\text{GL}}}\right), \qquad (58)$$

up to logarithmic accuracy. Here, d_f is the film thickness and the new characteristic temperatures are

$$T_{\text{cp}} = T_e z_{\text{cp}}\big(1 - \tilde{F}^h_0 \tilde{F}^e_0\big), \qquad (59)$$

$$\frac{1}{T_{\text{DOS}}^2} = -\frac{\pi^2}{7\zeta(3)}\text{Re}\left[\int d\epsilon \frac{\partial_\epsilon \tilde{F}_\epsilon^L}{(\epsilon + i0)^2}\right], \tag{60}$$

$\tilde{F}_L = (\tilde{F}_\epsilon^e + \tilde{F}_\epsilon^h)/2$. The important contribution in Eq. (57) comes from small momenta and, therefore, we safely cut the momentum integration at the upper limit $Dq_{\max}^2 \sim T_e$. The main contribution in the DOS, MT, and AL corrections comes from frequencies $\omega - 2e\phi + \epsilon_0 \lesssim \tau_{\text{GL}}^{-1}$. Here and in the following, we assume that the system is close to the transition, such that the characteristic energy scales of $\tilde{F}_\epsilon^{e/h}$ are much greater than τ_{GL}^{-1} and $|\epsilon_0|/2$; $\epsilon_0 = -4T_e\tilde{F}_0^R + \tau_{\text{GL}}^{-1}T_e/\Omega$. If the system is far from the transition and these conditions are not satisfied, then one can start calculation from Eq. (56) and use general fluctuation propagators (see the Appendix) that are not restricted to low frequencies.

Next, we focus on the MT correction to the conductivity. It is given by

$$\delta\sigma^{\text{MT}} = -\frac{\nu De^2}{8\pi}\int_{\mathbf{q},\epsilon_2,\epsilon_3}\left[\partial_{\epsilon_2}F_2^e\langle\langle\bar{w}_{2,3}(\mathbf{q})w_{3,2}^*(-\mathbf{q})\rangle\rangle\right.$$
$$\left. + \partial_{\epsilon_2}F_2^h\langle\langle\bar{w}_{2,3}^*(\mathbf{q})w_{3,2}(-\mathbf{q})\rangle\rangle\right]. \tag{61}$$

The main contribution close to the transition is

$$\delta\sigma^{\text{MT}} \approx \frac{-iDe^2}{4\pi}\int_{\mathbf{q},\epsilon,\omega}L_K^{-1}\frac{\partial_\epsilon F_\epsilon^e|L_-^A|^2 + \partial_\epsilon F_\epsilon^h|L_+^A|^2}{D^2\mathbf{q}^4 + (2\epsilon + \omega)^2} \tag{62}$$

$$\approx -\frac{e^2}{\pi d_f}T_{\text{cp}}\tau_{\text{GL}}\ln\left(\frac{\tau_{\text{GL}}}{\tau_\phi}\right) + \delta\sigma^{\text{DOS}}, \tag{63}$$

where we cut off the infrared divergency in the momentum integration by introduction of the finite dephasing time $Dq_{\min} \sim \tau_\phi$, $\tau_{\text{GL}} \ll \tau_\phi$.[5] There are many phase-breaking sources, such as the electron scattering on phonons or paramagnetic impurities, or superconducting fluctuations.[5] The nonequilibrium conditions may affect also the equilibrium phase-breaking time. However, we leave this problem for future studies. By treating energy ϵ as a complex number in (62), we obtain the first term in Eq. (63) from the poles of the integrand determined by zeros of the denominator. The second term in Eq. (63) comes from the poles of distribution functions $\tilde{F}^{e/h}$. Note that the first term is positive, while the second one is negative. Then, the DOS correction is effectively doubled, although the first term in $\delta\sigma^{\text{MT}}$ is the dominant one close to the transition.

Next, we calculate the AL correction to the conductivity

$$\delta\sigma^{\text{AL}} = -\frac{1}{2}\lim_{\Omega\to 0}\left[\frac{1}{\Omega}\left\langle\!\left\langle\frac{\delta(iS_A)}{\delta A_x^{cl}(\Omega)}\frac{\delta(iS_A)}{\delta A_x^q(-\Omega)}\right\rangle\!\right\rangle_{Q,\Delta}\right] \tag{64}$$

$$= -\frac{(\pi\nu De)^2}{2(2\pi)^4}\lim_{\Omega\to 0}\left\{\frac{1}{\Omega}\int_{\mathbf{r}_1,\mathbf{r}_2,\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}\langle\!\langle[w_{1,2}(\mathbf{r}_1)\nabla_x w_{2,1+\Omega}^*(\mathbf{r}_1) - w_{1,2}^*(\mathbf{r}_1)\nabla_x w_{2,1+\Omega}(\mathbf{r}_1) + w \to \bar{w}]\right.$$
$$\times\left[-F_{3-\Omega}^e w_{3,4}(\mathbf{r}_2)\nabla_x w_{4,3-\Omega}^*(\mathbf{r}_2) + F_{3-\Omega}^h w_{3,4}^*(\mathbf{r}_2)\nabla_x w_{4,3-\Omega}(\mathbf{r}_2)\right.$$
$$\left.\left. + F_3^e\bar{w}_{3,4}(\mathbf{r}_2)\nabla_x\bar{w}_{4,3-\Omega}^*(\mathbf{r}_2) - F_3^h\bar{w}_{3,4}^*(\mathbf{r}_2)\nabla_x\bar{w}_{4,3-\Omega}(\mathbf{r}_2)]\rangle\!\rangle_{Q,\Delta}\right\}. \tag{65}$$

We find that, close to the transition, the main contribution assumes the form

$$\delta\sigma^{\text{AL}} \approx \frac{(eD)^2\pi}{16d_f}\left(1 + i\frac{T_e}{\Omega}\right)\frac{1}{T_{\text{AL}}^2}$$
$$\times\int_{\mathbf{q},\omega}q^2 L_K^{-1}|L_A(\mathbf{q},\omega-\epsilon_0)|^2\frac{\partial}{\partial_\omega}[L_R(\mathbf{q},\omega-\epsilon_0)], \tag{66}$$

where the new characteristic temperature is given by

$$\frac{1}{T_{\text{AL}}^2} = \frac{4z_{\text{cp}}}{\pi^2}\left(\frac{1}{T_a^2} - \frac{T_e}{\Omega}\frac{1}{T_aT_b}\right), \tag{67}$$

$$\frac{1}{T_a} = \text{Im}\left[\int d\epsilon\frac{\tilde{F}_\epsilon^R}{(\epsilon - i0)^2}\right], \tag{68}$$

$$\frac{1}{T_b} = \text{Re}\left[\int d\epsilon\frac{\tilde{F}_\epsilon^R}{(\epsilon - i0)^2}\right]. \tag{69}$$

Performing the remaining integration over $\mathbf{q},\omega$, we obtain, in the quasi-two-dimensional case,

$$\delta\sigma^{\text{AL}} \approx \frac{e^2}{2\pi d_f}T_{\text{cp}}\tau_{\text{GL}}\frac{T_e^2}{T_{\text{AL}}^2}. \tag{70}$$

We conclude that all the fluctuation-induced corrections to the conductivity behave differently as a function of τ_{GL}. For a thin film close to the superconducting transition, the MT is the most important one. Moreover, each correction is parametrized by a different combination of the effective temperatures: T_e, T_{cp}, T_{DOS}, and T_{AL}. These temperatures are strongly drive dependent, as will be shown in the next section when considering a concrete example. In the equilibrium, $F_\epsilon^{e/h} = \tanh\epsilon/2T$ and $T_e = T_{\text{cp}} = T_{\text{DOS}} = T_{\text{AL}} = T$. Then, we reproduce the well-known results for the DOS, MT, and AL corrections to the conductivity.

Note that the calculation of the DOS, MT, and AL corrections in the equilibrium within the Keldysh formalism was done in Ref. 9. In their derivation of DOS correction, they have missed the last four terms from Eq. (56), which give important contributions in the final result. However, after some canceling mistakes, they surprisingly obtained the correct final result.

In this section, we have focused on the derivation of the fluctuation conductivity corrections close to the transition. They are the most pronounced in that region, but nevertheless they can be still significant also far from the transition.[5] Then, the derived GL-type theory is not applicable. One has to take

into account high frequencies and short-wave contributions in fluctuating propagators. However, this can be done within the above-developed approach. Namely, the expressions for the corrections given by Eqs. (56), (61), and (65) are valid also far from the transition. Then, in Eqs. (52)–(55), one has to use the general expressions for the fluctuating propagators, which are given in the Appendix .

V. EXPERIMENTAL REALIZATION

In this section, we propose an experimental setup where our predictions could be tested. The setup is shown in Fig. 1, where the superconducting film is sandwiched between the substrate and the gate and is separated from them by the interface barriers with the resistances R_1 and R_2, respectively. We consider the stationary situation and assume that the Thouless energy corresponding to diffusion across the film $E_T^\perp = D/d_f^2$ well exceeds all the effective temperatures. Then, the current across the interface separating the substrate and film is

$$I = \frac{1}{4eR_1} \int d\epsilon \{F^e(\epsilon) - F_S^e(\epsilon) - F^h(\epsilon) + F_S^h(\epsilon)\}, \quad (71)$$

and a similar equation holds for the interface between the film and the gate (here the subscript S denotes the substrate). From the continuity equation for the current follows $F^{e/h}(\epsilon) = x F_S^{e/h}(\epsilon) + (1-x)F_G^{e/h}(\epsilon)$, where $x = R_2/(R_1 + R_2)$. Here, $F_S^{e/h}(\epsilon) = \tanh[\epsilon/(2T_1)]$ and $F_G^{e/h} = \tanh[(\epsilon \mp eV_G)/(2T_2)]$ denote distributions in the substrate and in the gate. Then, the gauge-invariant distribution in the film assumes the form

$$\tilde{F}^{e/h}(\epsilon) = x \tanh\left[\frac{\epsilon \pm (1-x)eV_G}{2T_1}\right] \quad (72)$$

$$+ (1-x)\tanh\left[\frac{\epsilon \mp xeV_G}{2T_2}\right], \quad (73)$$

in the case of very resistive interfaces, i.e., when the resistance of the film can be neglected: $R_{tot} \approx R_1 + R_2$. Next, we calculate parameters appearing in the GL-type action [Eq. (31)] and demonstrate that they are strongly drive dependent.

A. Ginzburg-Landau relaxation time

In this section, we analyze the GL relaxation rate under FFE conditions [given by Eq. (38)] for the setup shown in Fig. 1. The GL relaxation rate is a very important parameter since many quantities influenced by superconducting fluctuations are singular functions of it. One example is fluctuation-induced corrections to the conductivity, which are analyzed in the previous section. While in the equilibrium ($T_1 = T_2$ and $V_G = 0$) and close to the transition, it behaves as (42); far from equilibrium, we find

$$T_e = \left[\frac{x}{T_1 \mathrm{ch}^2 \frac{(1-x)eV_G}{2T_1}} + \frac{(1-x)}{T_2 \mathrm{ch}^2 \frac{xeV_G}{2T_2}}\right]^{-1}, \quad (74)$$

$$\Omega^{-1} = \frac{2x}{T_1 \pi^2} \mathrm{Im}\left[\Psi'\left(\frac{1}{2} - i\frac{eV_G(1-x)}{2\pi T_1}\right)\right]$$
$$+ \frac{2(1-x)}{T_2 \pi^2} \mathrm{Im}\left[\Psi'\left(\frac{1}{2} + i\frac{eV_G x}{2\pi T_2}\right)\right], \quad (75)$$

$$\tau_{GL}^{-1} = \frac{8}{\pi} z_{cp} T_e \left\{x \, \mathrm{Re}\left[\Psi\left(\frac{1}{2} + i\frac{(1-x)eV_G}{2\pi T_1}\right)\right]\right.$$
$$+ (1-x)\mathrm{Re}\left[\Psi\left(\frac{1}{2} + i\frac{xeV_G}{2\pi T_2}\right)\right] + 2\log 2$$
$$\left. + x\log\frac{T_1}{T_2} - \log\frac{T_c}{T_2} + \gamma\right\} + 4z_{cp}\frac{T_e^2}{\Omega}\tilde{F}_0^R, \quad (76)$$

where $\Psi(z)$ is the digamma function defined as $\Psi(z) = \Gamma'(z)/\Gamma(z)$, where $\Gamma(z)$ is the gamma function. γ is the Euler constant and definitions for z_{cp} and $\tilde{F}^R$ are given in Sec. III. Note that the theory presented in the previous chapters is valid only above ($\tau_{GL} > 0$), and very close to the transition. However, the obtained expression for GL relaxation rate might be valid also below the transition. Also, all the expressions are valid for sufficiently small voltage drop. The system can be driven from equilibrium due to finite difference of the gate and the substrate temperatures and/or due to gate voltage. First, we start with the zero voltage case. Then, the general expressions (74)–(76) simplify significantly, and we obtain $\Omega^{-1} = 0$, $z = 1$, and

$$\tau_{GL}^{-1} = \frac{8}{\pi} T_e \left(x\log\frac{T_1}{T_2} - \log\frac{T_c}{T_2}\right). \quad (77)$$

Simple analysis shows that τ_{GL}^{-1} is negative (positive) when both temperatures T_1 and T_2 are smaller (greater) than the critical temperature T_c and can be either positive or negative when one of the temperatures is greater and another is smaller than T_c (see Fig. 2). Looking at lower part of Fig. 2, one notices that the GL relaxation rate can take rather different values than in the equilibrium. The dotted line denotes the equilibrium situation. In the equilibrium, we reproduce (42) when the system is close to the transition ($T - T_c \ll T_c$).
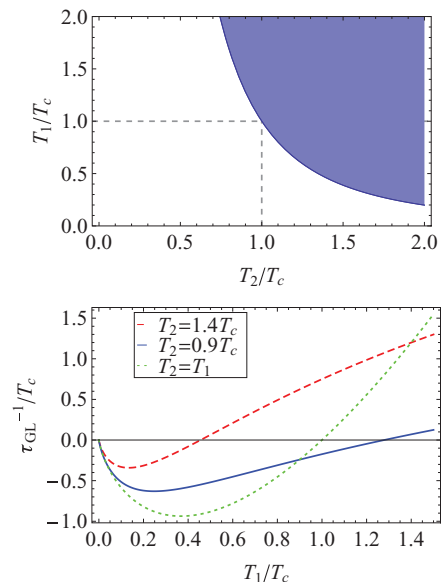

FIG. 2. (Color online) The upper figure shows two regions in which τ_{GL} has different signs for fixed $x = 0.3$. In the blue (shaded) region, it is greater than zero, and in the white region, it is smaller than zero. The lower figure shows τ_{GL}^{-1} dependence on T_1 for fixed T_2 and $x = 0.3$.
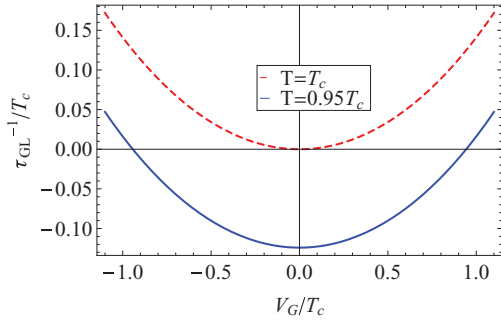
FIG. 3. (Color online) GL relaxation rate dependence on the gate voltage for $T_1 = T_2 = T$ and $x = 0.5$. One distinguishes quadratic and linear dependence on the gate voltage around the transition for $T = T_c$ and $T = 0.95T_c$, respectively.

Next, we switch on the gate voltage. Since the expression for the GL relaxation rate is rather complicated, we first analyze the case $T_1 = T_2 = T$. Then, for $V_G \leqslant T$, we obtain

$$\tau_{\text{GL}}^{-1} \approx -\frac{8T}{\pi} \log \frac{T_c}{T} - \frac{2V_G^2}{T\pi} x(1-x) \left[\log \frac{T_c}{T} - \frac{7}{\pi^2} \zeta(3) \right], \tag{78}$$

where $\zeta(x)$ is the Riemann zeta function. Then, one sees that, for $T = T_c$ and $V_G = 0$, the system is at the transition. It moves away from the transition by increasing the gate voltage, as it is shown in Fig. 3. The GL relaxation rate increases quadratically with V_G and assumes the form $\tau_{\text{GL}}^{-1} = 14V_G^2 x(1-x)\zeta(3)/(T_c\pi^3)$. On the other hand, if for zero gate voltage, the system is below but close to the transition [$|\tau_{\text{GL}}^{-1}(V_G = 0, T)| \ll T < T_c$], then at some critical finite voltage V_c it will be at the transition

$$V_c = T \sqrt{\frac{4}{x(1-x)} \frac{\log \frac{T_c}{T}}{\frac{7}{\pi^2}\zeta(3) - \log \frac{T_c}{T}}}. \tag{79}$$

In this case, one obtains linear behavior in $V - V_c$ around the transition for the fixed temperature $\tau_{\text{GL}}^{-1} \approx \frac{8}{\pi} \sqrt{\log(\frac{T_c}{T}) x(1-x) [\frac{7}{\pi^2}\zeta(3) - \log \frac{T_c}{T}]}(V - V_c)$. This situation is illustrated in Fig. 3. However, for some choice of parameters, the situation can be more complicated, as it is shown in Fig. 4.
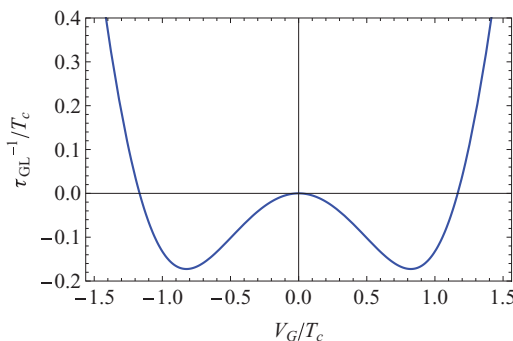

FIG. 4. (Color online) GL relaxation rate dependence on the gate voltage for $x = 0.3$, $T_2 = 2T_c$, and $T_1 = T_2^{((x-1)/x)}T_c^{(1/x)} \approx 0.2T_c$.

B. Corrections to the electrical conductivity

In this section, we examine fluctuation-induced corrections to the in-plane conductivity for the setup in Fig. 1. We start with the DOS correction, Eq. (58). We find that the characteristic energy scale T_{DOS} [Eq. (60)], parametrizing the DOS correction reads as

$$\frac{1}{T_{\text{DOS}}^2} = -\frac{1}{14\zeta(3)} \frac{x}{T_1^2} \text{Re}\left[\Psi''\left(\frac{1}{2} - \frac{ieV_G(1-x)}{2T_1\pi}\right) \right]$$
$$- \frac{1}{14\zeta(3)} \frac{1-x}{T_2^2} \text{Re}\left[\Psi''\left(\frac{1}{2} - \frac{ieV_G x}{2T_2\pi}\right) \right]. \tag{80}$$

Now, we have the analytic expression for $\delta\sigma_{\text{DOS}}$ as a function of T_1, T_2, V_G, and x. In Fig. 5(a), we plot the dependence on the gate and substrate temperatures for zero voltage and $x = 0.5$. In Fig. 5(b), we plot the dependence on gate voltage for $T_1 = T_2$ and $x = 0.5$.

Next, we analyze the AL correction [Eq. (70)]. We find that the characteristic temperatures T_a and T_b [Eqs. (68) and (69)] for the given setup become

$$\frac{1}{T_a} = \text{Re}\left[\frac{x}{T_1\pi} \Psi'\left(\frac{1}{2} + \frac{ieV_G(1-x)}{2\pi T_1}\right) \right]$$
$$+ \text{Re}\left[\frac{1-x}{T_2\pi} \Psi'\left(\frac{1}{2} - \frac{ieV_G x}{2\pi T_2}\right) \right], \tag{81}$$

$$\frac{1}{T_b} = -\text{Im}\left[\frac{x}{T_1\pi} \Psi'\left(\frac{1}{2} + \frac{ieV_G(1-x)}{2\pi T_1}\right) \right]$$
$$- \text{Im}\left[\frac{1-x}{T_2\pi} \Psi'\left(\frac{1}{2} - \frac{ieV_G x}{2\pi T_2}\right) \right]. \tag{82}$$

Then, we have the analytic form of the AL correction. Its dependence on system parameters is illustrated in Figs. 5(c) and 5(d). All the temperatures that appear in $\delta\sigma_{\text{MT}}$ are already calculated. We plot just the first term in Eq. (63), the so-called anomalous part of the MT correction, in Fig. 5(e) and Fig. 5(f) since the second one is equal to $\delta\sigma_{\text{DOS}}$. The dephasing rate is taken to be $\tau_\phi^{-1} = 10^{-3} T_c$. Comparing all the corrections to the conductivity, one sees that the most important one close to the transition is the MT correction.

VI. CONCLUSIONS AND DISCUSSION

We have derived GL-type theory valid close to the transition into the superconducting state under FFE conditions. We considered the stationary situation and electromagnetic fields slowly varying (with respect to the superconducting order parameter) in space. We found that the parameters appearing in the GL-type action are functionals of electron and hole nonequilibrium distribution functions. Close to an equilibrium, we reproduced known results and showed that the time-dependent GL equation, which is frequently found in literature,[5,17] is correct, contrary to the findings in Ref. 9.

Utilizing the theory, we studied the influence of fluctuations on the electrical conductivity in the FFE situation. We demonstrated that different fluctuation-induced corrections are controlled by different effective temperatures. These temperatures are drive dependent and carry information about temperatures and electrochemical potentials of the reservoirs that are in
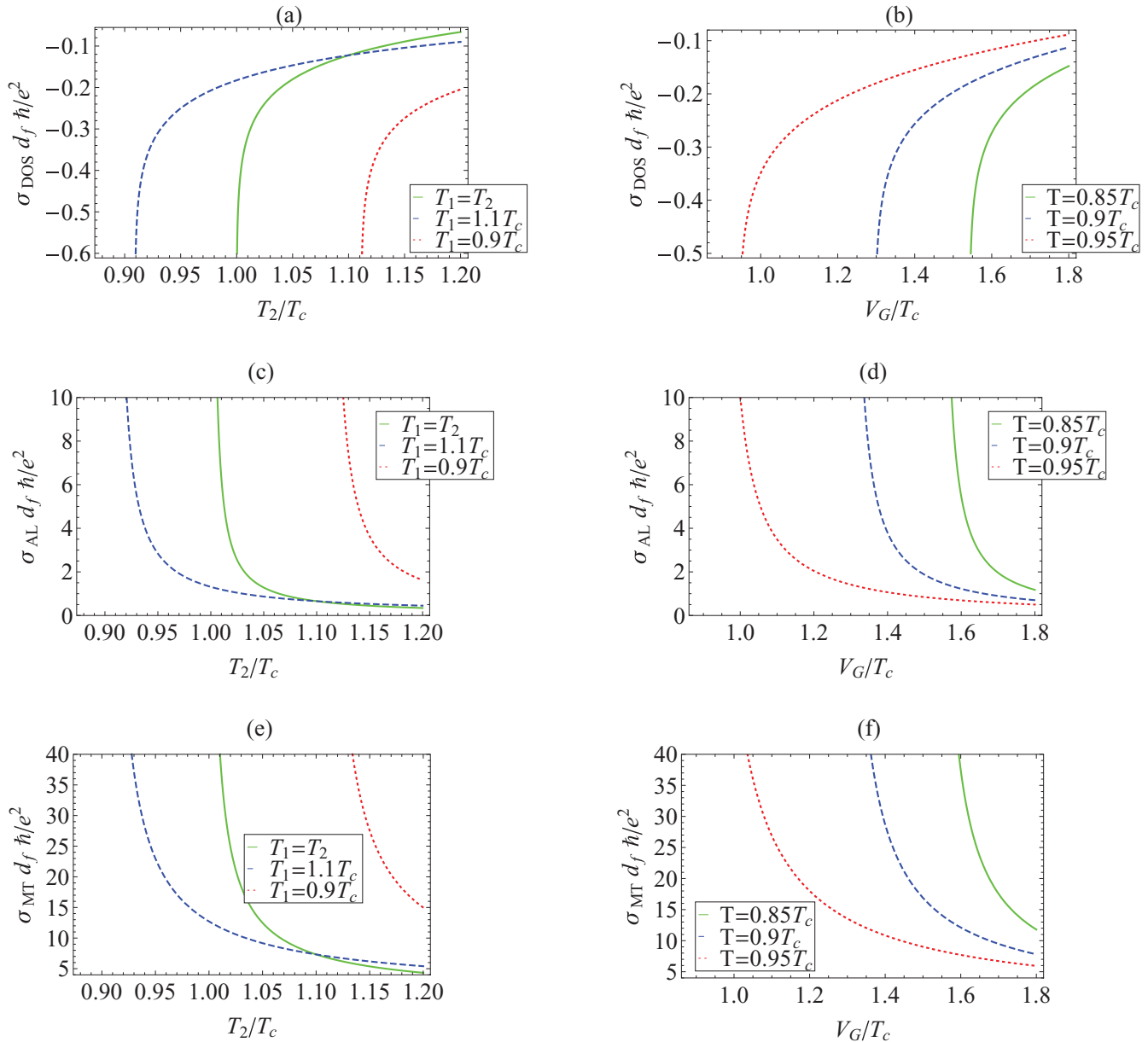
FIG. 5. (Color online) DOS, AL, and MT corrections to the in-plane electric conductivity as a function of the gate and the substrate temperatures for zero gate voltage $x = 0.5$ and $\tau_\phi^{-1} = 10^{-3}T_c$ are shown in (a), (c), and (e), respectively. DOS, AL, and MT corrections to electric conductivity as a function of gate voltage for $T_1 = T_2 = T$, $x = 0.5$, and $\tau_\phi^{-1} = 10^{-3}T_c$ are shown in (b), (d), and (f), respectively.

contact with the system. We proposed the experimental setup where our predictions could be tested (see Fig. 1).

The approach developed in this paper allows us to analytically treat many other important questions, for example, the influence of superconducting fluctuations on the thermal conductivity under FFE conditions. This question, close to an equilibrium, was a controversial and puzzling issue for a long time (see Ref. 18 and references therein). Hopefully, the final solution is that singular contributions of the the DOS and MT correction to the thermal conductivity cancel each other, while the AL contribution is finite.[18] Therefore, experimentally detected structure in the thermal conductivity,[19–22] which previously was believed to be explained by these corrections, needs a new explanation. However, since we have found that different corrections to the electrical conductivity are

characterized by different effective temperatures under FFE, it is likely that this is the case also with thermal conductivity. Then, the MT and DOS corrections do not cancel each other but instead produce a nontrivial result, which might explain the experiments.

APPENDIX: GINZBURG-LANDAU ACTION

In this appendix, we present a detailed derivation of the nonequilibrium GL-type action [Eq. (31)] valid in the normal state, but very close to the superconducting transition. We start from Eq. (30). In the following, we consider stationary distribution functions and stationary electromagnetic fields, slowly varying in space with respect to the fluctuating superconducting order parameter Δ^{cl}. Then,

$$\langle w_{\epsilon_1,\epsilon_2}(\mathbf{q_1}) w^*_{\epsilon_3,\epsilon_4}(\mathbf{q_2}) \rangle_{iS_{w^2}}$$
$$= \frac{2(2\pi)^{d+2}}{\pi \nu} \frac{\delta(\epsilon_1 - \epsilon_4)\delta(\epsilon_3 - \epsilon_2)\delta(\mathbf{q_1} + \mathbf{q_2})}{D(\mathbf{q_1} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2 - i(\epsilon_2 + \epsilon_4)}, \quad (A1)$$

$$\langle \bar{w}_{\epsilon_1,\epsilon_2}(\mathbf{q_1}) \bar{w}^*_{\epsilon_3,\epsilon_4}(\mathbf{q_2}) \rangle_{iS_{w^2}}$$
$$= \frac{2(2\pi)^{d+2}}{\pi \nu} \frac{\delta(\epsilon_1 - \epsilon_4)\delta(\epsilon_3 - \epsilon_2)\delta(\mathbf{q_1} + \mathbf{q_2})}{D(\mathbf{q_1} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2 + i(\epsilon_2 + \epsilon_4)}, \quad (A2)$$

where d denotes the dimension. The average values of all other two-field combinations give zero contribution. Then, after some algebra we obtain

$$S_{\text{GL}} = 2\nu \int \frac{d\omega}{2\pi} \frac{d\mathbf{q}}{(2\pi)^d} \big[\Delta^{q*}_{\mathcal{K},-} L_R^{-1} \Delta^{cl}_{\mathcal{K},+} + \Delta^{cl*}_{\mathcal{K},-} L_A^{-1} \Delta^{q}_{\mathcal{K},+}$$
$$+ \Delta^{q*}_{\mathcal{K},-} L_K^{-1} \Delta^{q}_{\mathcal{K},+} \big], \quad (A3)$$

where $\Delta_{\mathcal{K},\mp} = \Delta_{\mathcal{K}}(\mp\mathbf{q}, \mp\omega)$. The general formula for the retarded part of the superconductive fluctuation propagator is

$$L_R^{-1}(\mathbf{q},\omega) = -\frac{1}{\lambda} - i \int_{-\omega_D}^{\omega_D} d\epsilon \frac{F^R_{\epsilon_-}}{D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2 - 2i\epsilon}, \quad (A4)$$

where $\epsilon_- = \epsilon - \omega/2$, $F^R_\epsilon = (F^h_\epsilon - F^e_{-\epsilon})/2$, and $(L_A^{-1}(\mathbf{q},\omega))^* = L_R^{-1}(\mathbf{q},\omega)$. In the following, we show that L_R^{-1} can be written in the form of Eq. (34). Taking into account that the gauge-invariant distribution function is $\tilde{F}^{e/h}_\epsilon = F^{e/h}_{\epsilon \pm e\phi^{cl}_{\mathcal{K}}}$ and by replacing $\varepsilon = \epsilon_- + e\phi_{\mathcal{K}}$, one obtains

$$L_R^{-1} = -\frac{1}{\lambda} + \int_{-\omega_D}^{\omega_D} d\varepsilon \frac{\tilde{F}^R_\varepsilon - \tilde{F}^R_0}{2\varepsilon} - i \int_{-\omega_D}^{\omega_D} d\varepsilon \left[\frac{\tilde{F}^R_\varepsilon}{D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2 - 2i\varepsilon - i\omega + 2ie\phi^{cl}_{\mathcal{K}}} + \frac{\tilde{F}^R_\varepsilon - \tilde{F}^R_0}{2i\varepsilon} \right]. \quad (A5)$$

Here we have taken into account that Debye frequency $\omega_D \gg \omega/2 - e\phi^{cl}_{\mathcal{K}}$. Also, we have added and subtracted the term $\int_{-\omega_D}^{\omega_D} d\varepsilon (\tilde{F}^R_\varepsilon - \tilde{F}^R_0)/(2\varepsilon)$ from Eq. (A4). Let us concentrate on the last two terms in Eq. (A5) and denote their sum as l_R^{-1}:

$$l_R^{-1} = \frac{-i}{4} \int_{-\infty}^{\infty} d\varepsilon \left\{ \frac{(\tilde{F}^R_\varepsilon - \tilde{F}^R_0)[D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2 - i\omega - 2ie\phi^{cl}_{\mathcal{K}}]}{\varepsilon[\varepsilon + iD(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2/2 + \omega/2 - e\phi^{cl}_{\mathcal{K}}]} + \frac{2i\tilde{F}^R_0}{\varepsilon + iD(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2/2 + \omega/2 - e\phi^{cl}_{\mathcal{K}}} \right\}. \quad (A6)$$

The value of the first term in Eq. (A6) is determined by the poles of the function $\tilde{F}^R_\varepsilon$ with a positive imaginary part. Then, very close to the superconductor-metal transition, it becomes

$$l_R^{-1} = \frac{-i}{4} \big[D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2 - i\omega + 2ie\phi^{cl}_{\mathcal{K}} \big]$$
$$\times \int_{-\infty}^{\infty} d\varepsilon \frac{(\tilde{F}^R_\varepsilon - \tilde{F}^R_0)}{\varepsilon(\varepsilon + i0)} - \frac{i\pi \tilde{F}^R_0}{2}, \quad (A7)$$

since $\omega/2 + e\phi^{cl}_{\mathcal{K}}$, as well as $D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2$, is much smaller than any relevant scale of the distribution functions. Taking into account that $\int_{-\omega_D}^{\omega_D} d\varepsilon \tanh(\varepsilon/2T)/\varepsilon \approx 2\log(4\omega_D e^\gamma/2\pi T)$ for $\omega_D \gg T$, where γ is the Euler constant, $T_c = 2\omega_D e^{\gamma - 1/\lambda}/\pi$, and Sokhotsky's formula $(\varepsilon + i0)^{-1} = -i\pi\delta(\varepsilon) + \mathcal{P}(\varepsilon^{-1})$, one obtains

$$L_R^{-1} = \int_{-\omega_D}^{\omega_D} d\varepsilon \frac{\tilde{F}^R_\varepsilon - \tanh\left(\frac{\varepsilon}{2T_c}\right)}{2\varepsilon} + \big[-D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2 + i\omega$$
$$- 2ie\phi^{cl}_{\mathcal{K}} \big] \left(+\frac{\pi}{4} \frac{d\tilde{F}^R}{d\varepsilon}\bigg|_0 + i \int d\varepsilon \frac{\tilde{F}^R_\varepsilon - \tilde{F}^R_0}{4\varepsilon^2} \right) - \frac{i\pi \tilde{F}^R_0}{2}. \quad (A8)$$

After introducing T_e, Ω, and τ_{GL} as given by Eqs. (35)–(38), one arrives at Eq. (34).

Note that the term $\sim \Delta^{cl*}\Delta^{cl}$ vanishes. This is the expected property of the action (see the explanation in the main text). Taking into account that ∂_t in Eq. (7) is just a symbol standing instead of a matrix in the discrete time space, and that the sum of the retarded and the advanced Green's functions taken at the same time vanishes, we find that the term $\sim \Delta^{cl*}\Delta^{cl}$ is zero, while

$$L_K^{-1} = i \int d\epsilon \frac{1 - \frac{1}{2}(F^h_{\epsilon_-} F^e_{\epsilon_+} + F^h_{-\epsilon_+} F^e_{-\epsilon_-})}{D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2 - 2i\epsilon}. \quad (A9)$$

Very close to the transition, it reduces to

$$L_K^{-1} = -\frac{1}{2} \int d\epsilon \frac{1 - \frac{1}{2}(F^h_{\epsilon_-} F^e_{\epsilon_+} + F^h_{-\epsilon_+} F^e_{-\epsilon_-})}{\epsilon + i0} \quad (A10)$$

$$= \frac{i\pi}{2}\big(1 - \tilde{F}^h_{-\omega/2 + e\phi^{cl}_{\mathcal{K}}} \tilde{F}^e_{\omega/2 - e\phi^{cl}_{\mathcal{K}}}\big) \quad (A11)$$

$$\approx \frac{i\pi}{2}\big(1 - \tilde{F}^h_0 \tilde{F}^e_0\big). \quad (A12)$$

Note that, after applying the Sokhotsky's formula, the term $\oint d\epsilon (F^h_{\epsilon_-} F^e_{\epsilon_+} + F^h_{-\epsilon_+} F^e_{-\epsilon_-})/\epsilon$ gives zero contribution, since the function under the integral is the odd function of ϵ.

In the above calculation, we used that, close to the transition, $\omega/2 - e\phi^{\mathrm{cl}}_{\mathcal{K}}$ as well as $D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2$ is much smaller than any relevant scale of the distribution functions. Let us demonstrate the importance of the mentioned combination of the momenta and frequency and gauge fields by considering a simple example. Let us consider an equilibrium situation and denote the order parameter by $\Delta_0(\omega, \mathbf{q})$ in this case. It satisfied $D\mathbf{q}^2 \sim \omega \sim (\tau^{\mathrm{eq}}_{\mathrm{GL}})^{-1} \ll T$. Next, we turn on a constant scalar potential ϕ. Then, the order parameter becomes $\Delta = \Delta_0 \exp(-2ie\phi t)$, i.e., $\Delta(\omega, \mathbf{q}) = \Delta_0(\omega - 2e\phi, \mathbf{q})$. Then, $\omega - 2e\phi \sim (\tau^{\mathrm{eq}}_{\mathrm{GL}})^{-1} \ll T$. Similarly, it can be shown that the combination $D(\mathbf{q} - 2e\mathbf{A}^{cl}_{\mathcal{K}})^2$ has to be compared with characteristic energies of the distribution function. That is why it is necessary to introduce the gauge-invariant distribution functions during the calculation, as we did above.

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