Decomposition of fractional quantum Hall model states: Product rule symmetries and approximations

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We provide a detailed description of a product rule structure of the monomial (Slater) expansion coefficients of bosonic (fermionic) fractional quantum Hall (FQH) states derived recently, which we now extend to spin-singlet states. We show that the Haldane-Rezayi spin-singlet state can be obtained without exact diagonalization through a differential equation method that we conjecture to be generic to other FQH model states. The product rule symmetries allow us to build approximations of FQH states that exhibit *increasing* overlap with the exact state (as a function of system size) even though our approximation omits more than half of the Hilbert space. We show that the product rule is valid for any FQH state that can be written as an expectation value of parafermionic operators.

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I. INTRODUCTION

Our understanding of the physics of the fractional quantum Hall effect (FQHE) has benefited greatly from the existence of model wave functions. Laughlin's trial wave function for the $\nu = 1/3$ filled FQH state provided the first paradigm to understand the emergent behavior of interacting electrons in a strong magnetic field.[1] The current understanding of trial wave functions predominantly uses the conformal field theory (CFT) connection first proposed in Ref. 2. For every existing CFT, one can build a FQH trial wave function by taking the expectation value of branch-cut free primary fields in the CFT. The Read-Rezayi (RR) states are a product of this line of reasoning.[3] Both spin-polarized as well as spin-singlet states can be obtained this way, the most prominent examples of which are the Haldane-Rezayi (HR), the non-Abelian spin-singlet (NASS), and the Halperin states.[4–6] FQH trial wave functions are essential to understanding the physically important concepts of fractional Abelian (in the Laughlin and composite fermion states[7–9]) and non-Abelian statistics [in the Moore-Read (MR)[2] and RR states[3]].

The central drawback of the CFT-motivated trial-wave-function approach is the lack of both an explicit decomposition of a trial state in a second quantized many-body basis and of a first quantized closed-form expression for the state. As a consequence, Monte Carlo methods, while useful for Laughlin states,[1] can not be applied for most non-Abelian states. Any quantitative analysis of these trial states has, hence, so far relied on exact diagonalization (ED) methods.[10,11] There, one starts with a trial Hamiltonian and generates the (lowest) Landau level (LLL) Hilbert space. The computational effort of diagonalization depends algebraically on the Hilbert-space dimension, which grows factorially with system size. This sets the size limit that is reachable from ED. It is hence essential to use all available symmetries contained in the trial state and in the associated trial Hamiltonian to find the smallest subblock structure of the Hamiltonian matrix in terms of the noninteracting basis. One symmetry is the reflection of angular momentum $L_z \to -L_z$, which, for a sphere geometry, is equivalent to the indistinguishability of the north and south poles. Other symmetries such as total L^2 multiplet structure exist in some cases. However, they are rather obvious in general and do not gain us deep insight into the structure of the FQH states. For the Laughlin $1/3$ state, previous attempts[12,13] to calculate the weights of the free many-body wave functions in the full interacting state failed. These works[12,13] obtain only $O(1/N!)$ of the $O(N!)$ coefficients, and hence represent a set of measure zero of the Laughlin-state expansion. In a recent paper, two of us found that a large series of FQH trial states obey a type of symmetry for their free basis expansion coefficients.[14] The symmetry relates a subset of the coefficients of the expansion in free many-body states of a given FQH state to products of state coefficients from a smaller system size. This was developed for bosonic and spin-polarized fermionic states. In particular, it was observed in Ref. 14 that the overlap of the exact FQH state with the state approximated by the "product rule" symmetry increases with system size asymptotically toward unity.

In this paper, we give a detailed account of a general differential equation method used in Ref. 14 to access the monomial (Slater) decomposition of bosonic (fermionic) FQH states. We provide a detailed description of the very condensed derivation in Ref. 14 of the expansion coefficients for bosonic and polarized fermionic states. From there, we explain how the trial state can be numerically generated at a level intended for the novice reader. Next, we present an extended proof of the product rule symmetry for FQH trial states (and for all Jack polynomials in general). We then generalize the product rule symmetry, which allows us to generate even more expansion coefficients than previously allowed. We also generalize the whole approach to spinful trial states, and illustrate it in detail for the HR state. We first derive an annihilation operator for the HR state from which we develop a recurrence relation for the expansion coefficients. We investigate the product rule symmetry analog for spinful states and extract the entanglement spectrum of the HR state. For a spinful trial state, we find that the particle number N, angular momentum L, and spin multiplet S are the quantum numbers of the reduced density matrix subblocks.

This paper is organized as follows. In Sec. II, we discuss the recurrence relation of Jack polynomials that leads to the monomial expansion coefficients for bosonic FQH trial states. We elaborate on numerical subtleties for certain negative Jack parameters α. In these cases, denominator divergences appear in the recurrence formula; they are accompanied by a (at least) similarly vanishing numerator. In Sec. III, the Slater expansion coefficients of spin-polarized fermionic FQH states are derived from a fermionic version of the Laplace-Beltrami operator. This is the expanded version of previous calculations presented in Ref. 14. The approach is used to develop a recurrence formula for fermionic FQH trial states. In Sec. IV, we provide a largely expanded proof of the product rule symmetry. For non-Abelian bosonic states, we extend the product rule to treat general cases of cutting through a multiply occupied root partition orbital. In Sec. V, we generalize the entire approach to the spinful HR state. We derive the recurrence formula, show the product rule property, and compute the entanglement spectrum of this spinful trial state. In Sec. VI, we take a general viewpoint on the product rule symmetry from conformal field theory. We show that the product rule manifests itself as a generic property of all FQH states, which can be written as an expectation value of parafermionic operators, hence, including a large set of both spin-polarized and spin-unpolarized FQH states. Finally, we conjecture in Sec. VII that the product rule symmetry is a structural property of the majority of FQH trial states, including fermionic or bosonic states and spin-polarized or spin-unpolarized states, and may serve as an ingredient to future numerical approaches for FQH systems.

II. BOSONIC STATES

FQH states are analytic functions of the positions of electrons in a magnetic field. The single-particle orbitals in the Landau level are given by $\phi_m(z) = (2\pi m!2^m)^{-1/2} z^m \exp(-|z|^2/4)$ with angular momentum $L_z = m\hbar$, although from now on we will neglect the trivial Gaussian multiplication factors. A noninteracting N-particle basis state can be indexed by a partition λ (an ordered list of the L_z angular momentum of the occupied orbitals). The corresponding occupation number configuration is $n(\lambda) = \{n_m(\lambda), m = 0,1,2,\ldots\}$,[15,16] where m labels the individual single-particle orbitals and $n_m(\lambda)$ is the multiplicity of orbital m in λ. We consider FQH states decomposed in this many-body basis, either of bosons (permanents) or fermions (Slaters) with expansion coefficients c_λ. One central task of this paper is to present methods to compute these expansion coefficients.

We now define a two-body operation on the many-body basis that is important for the purpose of the paper: For a pair of particles in the orbitals m_1 and m_2, with $m_1 < m_2 - 1$, the elementary *squeezing* operation consists of the two particles shifted to different momentum orbitals as $n_{m_{1,2}} \to n_{m_{1,2}} - 1$, $n_{m_{1,2}\pm 1} \to n_{m_{1,2}\pm 1} + 1$. This means that both particles in the m_1, m_2 orbitals are shifted "inward" the partition (as shown in Fig. 1).

The squeezing defines a partial ordering relation between two partitions $\lambda > \mu$ when μ is generated by squeezing operations acting on λ.[17,18] This ordering yields a tree hierarchy, a complete example of which is shown in Appendix B. By contrast, when λ and μ do not relate by squeezing, no ordering relation is set between these partitions.

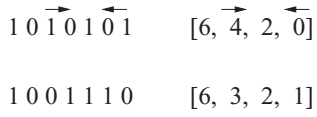

FIG. 1. Pictorial example of the squeezing operation in occupation language (left) and partition language (right). The squeezing operation takes the first row into the second.

The trial FQH states we consider are all squeezed polynomials. They possess a unique partition, called the root partition, dominating all other partitions. This means that all partitions with possible (but not guaranteed) nonzero weight are generated by subsequent squeezing operations acting on the root partition. In many cases, this already allows us to omit a significant (more than half) part of the Hilbert space (see Table I).

In this section, we focus on the bosonic FQH states. The noninteracting basis is given by monomials $\mathcal{M}_\lambda(z_1,\ldots,z_N) = \mathrm{Per}(z_i^{\lambda_j})/\prod_m n_m(\lambda)!$, where λ_j is the momentum index of the jth particle in the partition λ and Per is the permanent. It was shown[16] that the N-particle bosonic RR k series of states (which includes the Laughlin and MR states) are a special class of symmetric polynomials. Specifically, this class is called the $r = 2$ single *Jack* polynomials $J^\alpha_\lambda(z_1,\ldots,z_N)$ of parameter $\alpha = -\frac{k+1}{r-1}$ and root partition $\lambda = [k0^{r-1}k\cdots k0^{r-1}k]$. The Jack wave functions can be related to WA_{k-1} conformal field theories and can be classified in terms of symmetric polynomial categories.[19–25] Moreover, the quasiparticle excitations of the trial state systems can also be written as coherent state superpositions of Jacks.[26,27] This provides a complementary view to that of other approaches for FQH quasiparticle excitation states.[27–46]

TABLE I. Size of the monomial basis for the bosonic Laughlin state $\nu = 1/2$ up to $N = 17$ particles. The second column denotes the dimension of the full Hilbert space. The third column shows the reduced Hilbert space built out of partitions allowed by squeezing operations from the root partition $1010101\ldots0101$.

# Particles	Full dimension	Squeezed dimension
4	18	16
5	73	59
6	338	247
7	1656	1111
8	8512	5302
9	45207	26376
10	246448	135670
11	1371535	716542
12	7764392	3868142
13	44585180	21265884
14	259140928	118741369
15	1521967986	671906876
16	9020077206	3846342253
17	53885028921	22243294360

Jacks are eigenstates of the Laplace-Beltrami (LB) operator[17]

$$\mathcal{H}_{\mathrm{LB}} = \sum_i \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \frac{1}{\alpha} \sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right). \tag{1}$$

Until recently, the spectrum of the LB operator had been studied in detail for only positive α in the context of the Calogero-Sutherland model.[18] Recent progress has shown that the LB operator has well-defined polynomial solutions for certain negative α.[47] In particular, as stated above, some of them are found to correspond to bosonic FQH trial states for the ground-state, quasielectron, and quasihole excitations.[48]

We expand the Jacks into the monomial basis

$$J^{\alpha}_{\lambda} = \sum_{\kappa \leqslant \lambda} c_{\lambda\kappa}(\alpha) \mathcal{M}_{\kappa}, \tag{2}$$

where κ runs over all monomial partitions squeezed from the root partition λ. There is a known recurrence relation for the expansion coefficients $c_{\lambda\kappa}(\alpha)$ (Refs. 49 and 50):

$$c_{\lambda\kappa}(\alpha) = \frac{2/\alpha}{\rho_{\lambda}(\alpha) - \rho_{\kappa}(\alpha)} \sum_{\kappa < \mu \leqslant \lambda} [(l_i + t) - (l_j - t)] c_{\mu\kappa}(\alpha), \tag{3}$$

where $\kappa = [l_1, \ldots, l_i, \ldots, l_j, \ldots]$ and $\mu = [l_1, \ldots, l_i + t, \ldots, l_j - t, \ldots]$ denote partitions. We arrange the momentum orbitals denoted above in decreasing order from left to right, i.e., $l_1 \geqslant l_2 \geqslant l_i \geqslant l_j \cdots$ in κ, and a possible rearrangement occurs in μ depending on t. All partitions μ are understood to be reordered in this way. The sum in (3) extends over all partitions μ strictly dominating κ but being dominated (squeezed from) or equal to λ, which are generated by unsqueezing (i.e., the inverse operation to squeezing). The ρ's are defined as

$$\rho_{\lambda}(\alpha) = \sum_i \lambda_i \left(\lambda_i - 1 - \frac{2}{\alpha}(i-1) \right). \tag{4}$$

We now explain an easily implementable computer algorithm that allows one to obtain a large number of bosonic FQH states with high precision. From Eq. (4), we can uniquely index every partition by $\sum_i 2^{\lambda_i + N - i}$. For any numerical implementation, we order the basis according to this index so that the look-up of a partition in the basis list becomes logarithmic in effort. By recurrence, we can always compute the coefficient of a partition from those of its dominating partitions. The number of dominating partitions is generally small. In the worst case, it scales with the number of fluxes (∼number of orbitals) times the square of the number of particles, i.e., $N^2 N_{\phi}$. Thus, to compute all expansion coefficients, the procedure gives linear effort in the monomial basis dimension. The algorithm to generate the Jack state is sketched in Table II.

In contrast to positive α for which Eq. (3) was originally derived, there are minor caveats for certain negative α. Situations occur in which the denominator in (3) vanishes for certain partitions. In these cases, one finds that two different partitions μ_1 and μ_2 can obey $\rho_{\mu_1}(\alpha) = \rho_{\mu_2}(\alpha)$, a situation that is proved to never arise for positive α. An elementary example would be the six-particle ($k = 3, r = 4$) partitions $\mu_1(-4/3) = [5,5,4,1,1,0]$ and $\mu_2(-4/3) = [3,3,3,3,2,2]$. However, this denominator divergence is always regularized by a vanishing numerator. Under a limiting prescription $\lim_{\epsilon \to 0}[\alpha \to \alpha - \epsilon]$, the quotient either gives 0 or a rational value. Numerically, we let α slightly deviate from its exact value, vary it, and find a plateau value, which is then identified as the resulting expansion coefficient for the partition. This type of ρ degeneracy happens neither for the $r = 2$ RR series nor for the Gaffnian state.[40,51,52] It only occurs for Jacks that are not uniquely defined by a single clustering condition.[48]

TABLE II. Sketched how-to for using Eq. (3) to generate bosonic FQH states in monomial basis and Jacks of arbitrary parameter α.

Algorithmic steps to generate the Jack state	
(i)	Generate squeezed monomial basis $\mathcal{M}_{\kappa}$; $\mathcal{M}_1 = \mathcal{M}_{\lambda}$ is the root partition; order basis states by the integers $\sum_i 2^{\lambda_i + N - i}$
(ii)	Compute all $\rho_{\kappa}(\alpha)$ [Eq. (4)]
(iii)	Loop over all κ: loop over all pairs of elements $l_i < l_j \in \kappa$; for each pair, unsqueeze to upper dominant partitions $\mu > \kappa$ and read off $c_{\lambda\mu}$
(iv)	Compute contribution to $c_{\lambda\kappa}$ by Eq. (3); if $\rho_{\kappa}(\alpha) = \rho_{\lambda}(\alpha)$, compute the limit prescription $\lim_{\epsilon \to 0}[\alpha \to \alpha - \epsilon]$

III. POLARIZED FERMIONIC STATES

Similar to the bosonic case in Sec. II, we start with single-particle orbitals of the Landau level defined before, i.e., $\phi_m(z) = (2\pi m! 2^m)^{-1/2} z^m \exp(-|z|^2/4)$ with angular momentum $L_z = m\hbar$ and m the labeling index for all single-particle orbitals. However, for the many-body state, we now assume that the particles described by the first quantized wave functions obey fermionic statistics. As a consequence, the noninteracting free fermion basis is given by Slater determinant states: $\mathrm{sl}_{\lambda} = \mathcal{A}_z(z_1^{\lambda_1} z_2^{\lambda_2}, \ldots, z_N^{\lambda_N}) = \mathrm{Det}(z_i^{\lambda_j})$, where sl_{λ} is the unnormalized orthogonal Slater determinant and $\mathcal{A}$ denotes the antisymmetrization over all z coordinates. Different normalizations can be applied to put the polynomial wave function on different manifolds such as the plane or the sphere. As in the bosonic case, we describe the free many-body states by partitions (or occupation numbers). We arrange the partition $\lambda = [\lambda_1, \ldots, \lambda_N]$ by decreasing order in angular momentum λ_i of the ith particle. As before, the squeezing operation shifts two particles inward (toward each other) in the partition. For fermions, multiple occupancy is forbidden due to the Pauli principle.

In first quantized notation, bosonic and fermionic trial states can be transformed into each other by multiplication with a Vandermonde determinant. In terms of single-particle coordinates, this polynomial is the Jastrow factor, which is the antisymmetric homogeneous polynomial of degree 1. Starting from a Jack polynomial J^{α}_{λ}, the transformation reads as $J^{\alpha}_{\lambda} \to S^{\alpha}_{\lambda} := J^{\alpha}_{\lambda} \prod_{i<j}(z_i - z_j)$. The S^{α}_{λ} polynomials are the exact fermionic analog of the bosonic (Jack) trial state J^{α}_{λ}. For example, the $\nu = 1/2$ bosonic Laughlin state [Jack of $(k,r) = (1,2)$] becomes the $\nu = 1/3$ fermionic Laughlin state. The filling always changes from bosonic filling $\nu = p/q$ to fermionic

filling $\nu = p/(p+q)$. However, in second quantized notation, multiplication by the Vandermonde determinant does not transform a single monomial to a single Slater. To obtain a one-to-one correspondence between a bosonic monomial basis and fermionic Slaters, one first has to transform the monomials to Schur functions.[53] However, this involves knowledge of all the Kostka numbers, a long-standing unsolved mathematical problem with no known efficient algorithm.[54] There are different ways to remedy this problem. As one path, we can use the knowledge that the transformation from monomials to Schur functions is exactly given by the $J^{\alpha=1}$ Jack polynomial coefficients, which we can compute from Eq. (3).

However, we try to tackle the fermionic trial states from a different angle in the following. We define the decomposition of the S^{α}_{λ} polynomials into Slaters:

$$S^{\alpha}_{\lambda}(z_1,\ldots,z_N) = J^{\alpha}_{\lambda_{\mathrm{B}}} \prod_{i<j}^{N}(z_i - z_j) = \sum_{\mu \leqslant \lambda} b_{\lambda\mu}\,\mathrm{sl}_{\mu}. \tag{5}$$

To avoid confusion, λ_{B} denotes the bosonic root partition and λ the associated fermionic root partition. All partitions μ are squeezed from the fermionic partition λ that is related to the bosonic partition by $\lambda_i = \lambda_i^B + (N-i)$. We now use that the Jack part of S^{α}_{λ} is an eigenstate of the LB operator, i.e., $\mathcal{H}_{\mathrm{LB}} J^{\alpha}_{\lambda_{\mathrm{B}}} = E_{\lambda_{\mathrm{B}}}(\alpha) J^{\alpha}_{\lambda_{\mathrm{B}}}$. We then relate the derivatives acting on $J^{\alpha}_{\lambda_{\mathrm{B}}}$ to derivatives on S^{α}_{λ} (details are given in Appendix C):

$$\begin{aligned}
E_{\lambda_{\mathrm{B}}}(\alpha) S^{\alpha}_{\lambda} &= \prod_{k<l}(z_k - z_l)\left[\sum_i \left(z_i \frac{\partial}{\partial z_i}\right)^2 + \frac{1}{\alpha}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}\left(z_i\frac{\partial}{\partial z_i} - z_j\frac{\partial}{\partial z_j}\right)\right] J^{\alpha}_{\lambda_{\mathrm{B}}} \\
&= \left[\sum_i \left(z_i \frac{\partial}{\partial z_i}\right)^2 - 2\sum_{\substack{i,m \\ m\neq i}} \frac{z_i}{z_i - z_m} z_i \frac{\partial}{\partial z_i} + \frac{1}{\alpha}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}\left(z_i\frac{\partial}{\partial z_i} - z_j\frac{\partial}{\partial z_j}\right)\right] S^{\alpha}_{\lambda} \\
&\quad + \left[\sum_{\substack{i,m \\ i\neq m}} \frac{z_i(z_i+z_m)}{(z_i-z_m)^2} + \sum_{\substack{i,m,n \\ i\neq m\neq n}} \frac{z_i^2}{(z_i-z_m)(z_i-z_n)} + \frac{1}{\alpha}\sum_{i<j}\frac{z_i+z_j}{z_i-z_j}\left(\sum_{m\neq i}\frac{z_i}{z_i-z_m} - \sum_{m\neq j}\frac{z_j}{z_j-z_m}\right)\right] S^{\alpha}_{\lambda}.
\end{aligned} \tag{6}$$

By simplifying several polynomial sums that yield constants as shown in Appendix A, we can define a fermionic Laplace-Beltrami operator that diagonalizes S^{α}_{λ}, i.e., $\mathcal{H}^{\mathrm{F}}_{\mathrm{LB}}(\alpha) S^{\alpha}_{\lambda}(z_1,\ldots,z_N) = E_{\lambda}(\alpha) S^{\alpha}_{\lambda}(z_1,\ldots,z_N)$, with

$$E_{\lambda}(\alpha) = \sum_i \lambda_i \left[\lambda_i - 2\left(\frac{1}{\alpha} - 1\right) i\right] + \left(\frac{1}{\alpha} - 1\right)\left((N+1)\sum_i \lambda_i - N(N-1)\right), \tag{7}$$

$$\mathcal{H}^{\mathrm{F}}_{\mathrm{LB}}(\alpha) = H_K + \frac{1}{2}\left(\frac{1}{\alpha} - 1\right) H_I = \sum_i \left(z_i\frac{\partial}{\partial z_i}\right)^2 + \frac{1}{2}\left(\frac{1}{\alpha} - 1\right)\left[\sum_{i\neq j}\frac{z_i+z_j}{z_i-z_j}\left(z_i\frac{\partial}{\partial z_i} - z_j\frac{\partial}{\partial z_j}\right) - 2\frac{z_i^2+z_j^2}{(z_i-z_j)^2}\right]. \tag{8}$$

We now diagonalize the above operator in the basis of Slater determinants. The action of the kinetic part yields $\sum_i H_K \mathrm{sl}_{\mu} = (\sum_i \mu_i^2)\mathrm{sl}_{\mu}$, where the μ_i denotes the polynomial power of the ith particle in the partition. The action of the interaction part H_I is nondiagonal in Slater determinant basis and demands further calculation. First, we realize that, due to its two-body nature, the action of H_I on any Slater determinant decomposes into the sum of two-particle interaction terms. It is thus sufficient to look at the action on the two-particle Slater determinant $\mathrm{sl}_{\mu=(\mu_1,\mu_2)} = z_1^{\mu_1} z_2^{\mu_2} - z_2^{\mu_1} z_1^{\mu_2}$. Assume $\mu_1 > \mu_2$, and define $p = \mu_1 - \mu_2$:

$$\begin{aligned}
\frac{H_I \mathrm{sl}_{(\mu_1,\mu_2)}}{z_1^{\mu_2} z_2^{\mu_2}} &= p\frac{z_1+z_2}{z_1-z_2}\left(z_1^p + z_2^p\right) - 2\frac{z_1^2+z_2^2}{(z_1-z_2)^2}\left(z_1^p - z_2^p\right) \\
&= \frac{1}{z_1-z_2}\left[p\left(z_1^{p+1} + z_2^{p+1} + z_1^p z_2 + z_2^p z_1\right) - 2\sum_{s=1}^{p/2}\left(z_1^{p-s}z_2^{s+1} + z_2^{p-2}z_1^{s+1} + z_1^{p-s+2}z_2^{s-1} + z_2^{p-s+2}z_1^{s-1}\right)\right] \\
&= \frac{1}{z_1-z_2} 2\sum_{s=1}^{p/2}\left[z_1^{p-s+2}\left(z_1^{s-1} - z_2^{s-1}\right) + z_2^{p-s+2}\left(z_2^{s-1} - z_1^{s-1}\right) + z_1^{p-s} z_2\left(z_1^s - z_2^s\right) + z_2^{p-s} z_1\left(z_2^s - z_1^s\right)\right] \\
&= 2\sum_{s=1}^{p/2}\left(z_1^{p-s+2} - z_2^{p-s+2}\right)\sum_{t=1}^{(s-1)/2}\left(z_1^{s-t-1}z_2^{t-1} + z_2^{s-t-1}z_1^{t-1}\right) + 2\sum_{s=1}^{p/2}\left(z_1^{p-s}z_2 - z_2^{p-s}z_1\right)\sum_{t=1}^{s/2}\left(z_1^{s-t}z_2^{t-1} + z_2^{s-t}z_1^{t-1}\right).
\end{aligned} \tag{9}$$

The two terms are already grouped to yield two-particle Slater determinants. Collecting all prefactors, this gives

$$H_I \mathrm{sl}_{(\mu_1,\mu_2)} = (\mu_1 - \mu_2 - 2)\mathrm{sl}_{(\mu_1,\mu_2)} + 2\sum_{s=1}^{(\mu_1-\mu_2)/2} (\mu_1 - \mu_2 - 2s)\mathrm{sl}_{(\mu_1-s,\mu_2+s)}. \quad (10)$$

Equation (10) has a particular form: it only scatters "inward" the two-particle basis of Slater determinants, i.e., toward decreasing relative momentum of the particles, and thus to a squeezed partition. Let us now look at the total action of $H_{\mathrm{LB}}^{\mathrm{F}}$ on S_λ^α expanded in Slaters. The above scattering Hamiltonian and the linear independence of Slater determinants provide a recurrence relation for the coefficients $b_{\lambda\mu}$ in (5). We collect all diagonal terms and invert the sum over s in Eq. (10) to a sum over all dominating partitions:

$$b_{\lambda\mu} = \frac{2\left(\frac{1}{\alpha} - 1\right)}{\rho_\lambda^{\mathrm{F}}(\alpha) - \rho_\mu^{\mathrm{F}}(\lambda)} \sum_{\theta;\ \mu<\theta\leqslant\lambda} (\mu_i - \mu_j) b_{\lambda\theta} (-1)^{N_{\mathrm{SW}}}, \quad (11)$$

where $\rho_\lambda^{\mathrm{F}}(\alpha) = \sum_i \lambda_i[\lambda_i + 2i(1 - 1/\alpha)]$. Similar to the bosonic recurrence formula in Eq. (3), the sum in (11) extends over all partitions $\theta = [\mu_1, \ldots, \mu_i + s, \ldots, \mu_j - s, \ldots, \mu_N]$ that dominate the partition $\mu = [\mu_1, \ldots, \mu_N]$ and are squeezed from the root partition λ. A new factor $(-1)^{N^{\mathrm{SW}}}$ appears according to the parity of the number of transpositions (swaps) of particles from a given dominating partition θ back to μ. This term appears since the reordering of the partition in Slater determinant language may cause a minus sign due to fermionic anticommutation relations. N_{SW} starts from zero for partition μ and advances by one every time the momentum of the unsqueezed electron passes through the value of the momentum for another electron. As a further difference from the bosonic Jack recurrence relation, the terms summed in Eq. (11) do not explicitly depend on the partition θ. This is because the rescaling of the s in (10) exactly cancels the term's dependence on s. For $\alpha = -(k+1)$, Eq. (11) gives the coefficients of the fermionic Read-Rezayi states (an example of the partition coefficient for the 4 particle $\nu = 1/3$ Laughlin state is shown in Fig. 2). The complete decomposition of the $N = 4$ particle $\nu = 1/3$ Laughlin state is given in Appendix B. As in the bosonic case, this method constitutes an advance in the numerical computation of the coefficients: the computational effort required to generate the state scales linearly with basis size. This approach has already been applied to increase the maximally reachable system in finite-size studies.[55]

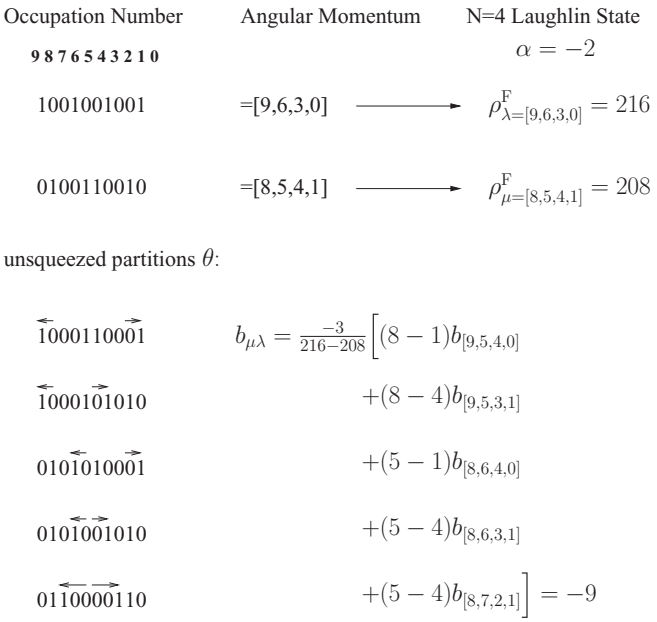

FIG. 2. The recurrence relation (10) for the $N = 4$ particle $\nu = 1/3$ Laughlin state. The partitions are written in decreasing order of orbital angular momentum, which ranges from 9 to 0 in the case considered. The coefficient of the partition $\mu = 0100110010$ is computed with the knowledge of the coefficient for the partitions dominating μ.

IV. PRODUCT RULES

A. Introduction

The coefficients of the monomials (Slaters) in the bosonic (fermionic) FQH states exhibit a hidden symmetry found in Ref. 14 and named product rule. The product rule is valid in *any quantum-mechanical normalization*, be it on the plane, sphere, cylinder, disk, or any other genus-0 geometry. However, its explanation is easier in the basis of Eqs. (3) and (11), for which we have already developed the formalism in the previous sections. Once a state is obtained in this basis, it is only a matter of specific change in the normalization of the free many-body wave functions to switch between different genus-0 geometries. The product rule found in Ref. 14 and explained in detail in this section is valid not only for FQH Jacks, but for *all* Jacks at *any*α, of any partition λ. Furthermore, as further elaborated on in Sec. VI below, the product rule is a property of quantum Hall trial states even beyond Jack polynomials.[56–58]

We consider a Jack state generated from Eq. (3) [or (11)] that can serve as a suitable example to demonstrate the product rule. We discuss the fermionic MR state of $N = 10$ particles. This state can be written as a linear superposition of Slater determinants squeezed from the root partition $n(\lambda) = 110011001100110011$. We pick a configuration squeezed from λ that has the special property that two parts of the partition can be identified as squeezed from individual root partitions for smaller systems sizes (see Fig. 2). Let us consider 101101000101111010. We observe that the first seven orbitals from the left, i.e., 1011010, can be squeezed from the $N = 4$ partition 1100110. The remainder right part, i.e., 00101111010, can be squeezed from the $N = 6$ partition 01100110011. We find that the product of the two coefficients obtained from the $N = 6$ part and the disconnected $N = 4$ part (-70 and -2, respectively) gives the coefficient $(-70) \times (-2) = 140$ of the $N = 10$ partition. The product rule (symmetry) allows the computation of a certain set of coefficients of an N-particle state from the knowledge of the state for $N - 1$ particles. This hints at similarities with Feynman disconnected diagram summation in interacting systems, where the total contribution is given by the product of the disconnected components (Fig. 3). As such, the product rule is an exact property of most FQH trial state polynomials. As we show below, it can be used to construct an accurate

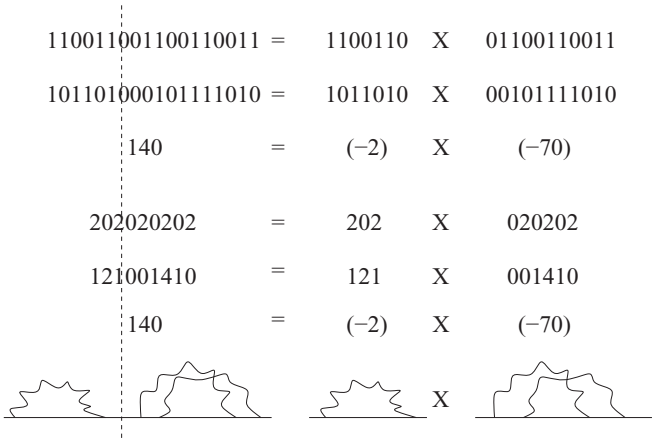

FIG. 3. Product rule applied for a partition of the $N = 10$ MR state in fermionic and bosonic notations. The given partition can be related to the expansion coefficients of products of smaller systems, in this case the $N = 4$ and the 6 MR state partitions.

approximation of the model states, keeping the essential part of the correlations of the FQH state. We prove the product rule for fermionic Jacks by induction principle; a similar proof can be obtained for the bosonic Jacks.

B. Derivation

Basic induction case. Assume we start with any fermionic polynomial $S^{\alpha}_{\lambda}(z_1,\ldots,z_N)$ of a configuration $\mu \leqslant \lambda$ that can be divided in two disconnected sets: $\mu = (\mu_A,\mu_B)$, with N_A particles in the first subpart A and $N_B = N - N_A$ particles in the second subpart B. This means that μ_A is squeezed from an N_A-particle root partition $\lambda_A = [\lambda_1,\ldots,\lambda_{N_A}]$, and μ_B is squeezed from the N_B-particle root partition $\lambda_B = [\lambda_{N_A+1},\ldots,\lambda_N]$. The basic induction case is given by any partition of the monomials in $S^{\alpha}_{\lambda}(z_1,\ldots,z_N)$ for which the product rule holds. Trivially, for this purpose we can choose the root partition λ. By definition, it has coefficient 1, and we can think of it as being separated into any product of two subpart root partitions. Again, all these have coefficient 1 by definition, so that the product rule holds for the root partition itself.

Induction hypothesis. We now assume the product rule is valid for *all* partitions θ sharing the separable form $\theta = (\theta_A,\theta_B)$, where $\mu_A < \theta_A \leqslant \lambda_A$ and $\mu_B < \theta_B \leqslant \lambda_B$. As shown in Eq. (10), the coefficients of S^{α}_{λ} are given as a recursion from partitions that dominate μ. By construction, any partition dominating μ and entering (10) is also separable according to $\theta = (\theta_A,\theta_B)$. As the unsqueezed operation is a two-body operation, the sum over all dominating partitions θ can be decomposed into $\sum_{\theta;\,\mu<\theta\leqslant\lambda} = \sum_{\mu_A<\theta_A\leqslant\lambda_A} + \sum_{\mu_B<\theta_B\leqslant\lambda_B}$. In particular, the summation over the individual partition entries μ_i only mixes μ_i,μ_j of the left-hand side A and right-hand side B separately, while the remainder right (left) part remains unchanged. Partition-wise, the first sum reads as (θ_A,μ_B), while the second reads as (μ_A,θ_B). Finally, we assume that all partitions dominating μ satisfy the product rule

$$b_{\lambda(\theta_A,\mu_B)}(\alpha) = b_{\lambda(\theta_A,\lambda_B)}(\alpha) b_{\lambda(\lambda_A,\mu_B)}(\alpha), \tag{12}$$

where (θ_A,λ_B) denotes the partition formed by θ_A and the remainder root state partition of part B, i.e., λ_B. This holds vice versa for (λ_A,θ_B).

Induction proof. Let us consider the coefficient $b_{\lambda,\mu}$. By induction hypothesis, we can rewrite Eq. (10) as (we skip the argument α in the notation for the coefficients b)

$$\begin{aligned} b_{\lambda\mu} &= \frac{2\left(\frac{1}{\alpha}-1\right)}{\rho^{\mathrm{F}}_{\lambda}-\rho^{\mathrm{F}}_{\mu}}\left(\sum_{\theta_A;\,\mu_A<\theta_A\leqslant\lambda_A}\left(\mu^A_i-\mu^A_j\right)b_{\lambda(\theta_A,\mu_B)}(-1)^{N_{\mathrm{SW}}}+\sum_{\theta_B;\,\mu_B<\theta_B\leqslant\lambda_B}\left(\mu^B_i-\mu^B_j\right)b_{\lambda(\mu_A,\theta_B)}(-1)^{N_{\mathrm{SW}}}\right)\\ &= \frac{2\left(\frac{1}{\alpha}-1\right)}{\rho^{\mathrm{F}}_{\lambda}-\rho^{\mathrm{F}}_{\mu}}\left(b_{\lambda(\lambda_A,\mu_B)}\sum_{\theta_A;\mu_A<\theta_A\leqslant\lambda_A}\left(\mu^A_i-\mu^A_j\right)b_{\lambda(\theta_A,\lambda_B)}(-1)^{N_{\mathrm{SW}}}+b_{\lambda(\mu_A,\lambda_B)}\sum_{\theta_B;\,\mu_B<\theta_B\leqslant\lambda_B}\left(\mu^B_i-\mu^B_j\right)b_{\lambda(\lambda_A,\theta_B)}(-1)^{N_{\mathrm{SW}}}\right). \end{aligned} \tag{13}$$

Writing out the coefficients $b_{\lambda(\lambda_A,\mu_B)}$ and $b_{\lambda(\mu_A,\lambda_B)}$ according to Eq. (10), we then use $\rho^{\mathrm{F}}_{\lambda}(\alpha)-\rho^{\mathrm{F}}_{\mu}(\alpha) = \rho^{\mathrm{F}}_{\lambda}(\alpha)-\rho^{\mathrm{F}}_{\mu_A\lambda_B}(\alpha)+\rho^{\mathrm{F}}_{\lambda}(\alpha)-\rho^{\mathrm{F}}_{\lambda_A\mu_B}(\alpha)$ to get

$$\begin{aligned} b_{\lambda\mu} = {} & \frac{2\left(\frac{1}{\alpha}-1\right)}{\rho^{\mathrm{F}}_{\lambda}(\alpha)-\rho^{\mathrm{F}}_{\mu_A\lambda_B}(\alpha)+\rho^{\mathrm{F}}_{\lambda}(\alpha)-\rho^{\mathrm{F}}_{\lambda_A\mu_B}(\alpha)}\left(\frac{2\left(\frac{1}{\alpha}-1\right)}{\rho^{\mathrm{F}}_{\lambda}(\alpha)-\rho^{\mathrm{F}}_{\mu_A\lambda_B}(\alpha)}+\frac{2\left(\frac{1}{\alpha}-1\right)}{\rho^{\mathrm{F}}_{\lambda}(\alpha)-\rho^{\mathrm{F}}_{\lambda_A\mu_B}(\alpha)}\right)\\ &\times\sum_{\mu_A<\theta_A\leqslant\lambda_A}\sum_{\mu_B<\theta_B\leqslant\lambda_B}\left(\mu^A_i-\mu^A_j\right)\left(\mu^B_i-\mu^B_j\right)b_{\lambda(\lambda_A,\theta_B)}b_{\lambda(\theta_B,\lambda_A)}(-1)^{N_{\mathrm{SW,A}}+N_{\mathrm{SW,B}}} = b_{\lambda(\mu_A,\lambda_B)}\cdot b_{\lambda(\lambda_A,\mu_B)} \quad \text{q.e.d.} \end{aligned} \tag{14}$$

A similar line of reasoning applies to the bosonic case. We have thus proved the product rule symmetry for this type of piece-separable configurations. This is valid for all bosonic and fermionic Jack polynomials, and hence for all Read-Rezayi states. However, the product rule even applies to a much larger range of polynomials (Sec. VI). Following similar steps as above, the product rule can also be explicitly derived for spin-unpolarized states such as the Haldane-Rezayi state discussed in Sec. V.

For bosonic states, there are certain classes of partitions where it is not immediately clear how to apply the product rule symmetry. Let us look again at the $N = 10$ particle MR state in bosonic notation; this state is squeezed from the root partition $\lambda = 202020202$.

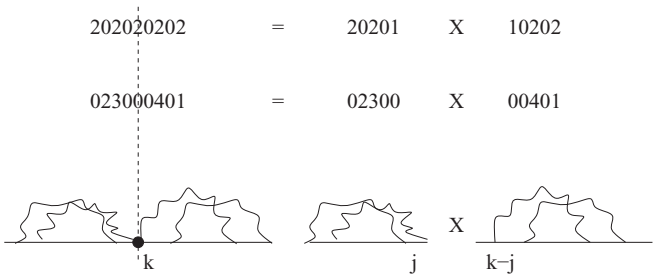

FIG. 4. Generalized product rule for certain partitions of Z_k-parafermion non-Abelian states in bosonic notation (shown: a partition of the $N = 10$ Moore-Read state). In terms of diagrammatic representation of interaction vertices, the orbital along the partition cut is assigned a label k for the occupancy of the associated root partition orbital. The diagrammatic decomposition proceeds by splitting the k particle into j to the left and $k - j$ to the right. This way, the product rule holds for this class of configurations.

Partitions type I. An easy application of the product rule exists for configurations such as $p_1 = 040000600$. We identify the first four orbitals to be squeezed from $\lambda_A = 2020$, while the remainder orbitals are squeezed from $\lambda_B = 020202$; we find the product rule to hold (see also Fig. 3).

Partitions type II. Let us analyze the configuration $p_2 = 023000401$. We can split the configuration in a five-particle separation to the left and a five-particle separation to the right of the cut in Fig. 4. Both parts are disconnected in terms of squeezing operations on the particles. However, what are the root partitions from which we generate the subparts? We have to split one doubly occupied orbital of the associated root partition 202020202 as in Fig. 4. We double copy this orbital and distribute the particles in both subparts. We consider 2020**1** and **1**0202 as the root partitions for subpart A and B: together they make up 202020202, but the orbital where the bold particle **1** is placed is taken to belong to both parts A and B. At the same time, we double copy the fifth orbital of p_2, i.e., 0230**0** and **0**0401 (see Fig. 4). Following this recipe, we find that the product rule holds for these type-II configurations, i.e., we can trace back separable configurations of type p_2 to product rule compositions of smaller system size, which individually correspond to single-hole partitions (Fig. 4). Generalizing this approach to multiple-hole state decompositions is not readily applicable within our orbital basis notation of single partitions.[59]

C. Application

As it stands, the product rules can be used to build an approximate N-particle state given the knowledge of the exact states up to $N - 1$ particles. Let us investigate the accuracy of such an approximated state. To begin with, an important quantity is the number of monomials (or Slater determinants), the configurations of which obey the product rule. Tables III and IV show the ratio between this number and the total size of the squeezed Hilbert space for the Laughlin state ($\nu = 1/2$ and $1/3$) and the MR state, respectively. As a rule of thumb, the product rule allows us to construct more than a third and less than a half of the total Hilbert space. We observe that this ratio decreases with increasing system size.

The important question is how much of the exact state is kept in this part of the Hilbert space generated by the product rule. We have computed the overlap between the exact state for N particles and the state constructed only from the product rule. The overlap is taken using the scalar product of the sphere geometry. We find that the state approximated by the product rule has $>99.9\%$ overlap with the exact state, involving only type-I partitions for the Laughlin state (Table III) or type-I and type-II for the MR state (Table IV). This tells us that the monomials generated by the product rule contain almost all of the exact state by overlap despite comprising only a fraction of its Hilbert space. For all the $(k,r = 2)$ states we have considered, the overlap has the peculiar feature that it *increases* with system size (by contrast, any comparison between a model state and the ground state of some realistic interaction would exhibit the opposite behavior). As such, this provides indication that the product rule symmetry of quantum Hall trial states may become exact in the thermodynamic limit. Fermionic states show a very similar behavior (see Table III for the Laughlin state at $\nu = 1/3$). The overlaps are likewise high but not as good as for their bosonic counterparts.

Up until now, we have only considered partitions subject to the product rule construction that can be decomposed into a pair of ground-state partitions of smaller system size. We can, however, further refine the procedure for Laughlin states. The product rule can be applied not only when considering disconnected squeezing sequences from the root partition, but

TABLE III. Shown is the percentage of Hilbert space that is constructed from the product rule and overlap of the exact state and the state built from the product rule for sphere geometry. N is the number of particles and the overlap is defined as the absolute value of the scalar product. The first two rows of data are the results for the $\nu = 1/2$ bosonic Laughlin state for different numbers of particles. The third and fourth rows of data are the results for the $\nu = 1/3$ fermionic Laughlin state relying only on the knowledge of the Slaters determined by ground-state partitions, while the fifth and sixth rows additionally take into account information stemming from the single quasihole excitations.

N	8	9	10	11	12	13	14	15	16
$\nu = 1/2$ product rules	43.76%	42.01%	40.76%	39.93%	39.52%	39.32%	39.24%	39.19%	39.16%
$\nu = 1/2$ overlap	0.9933	0.9947	0.9947	0.9956	0.9963	0.9968	0.9972	0.9977	0.9979
$\nu = 1/3$ product rules	47.68%	46.41%	45.33%	44.45%	43.73%	43.11%	42.56%	42.08%	41.65%
$\nu = 1/3$ overlap	0.9502	0.9534	0.9557	0.9573	0.9585	0.9593	0.9599	0.9603	0.9605
$\nu = 1/3$ product rules + qh	73.92%	72.52%	71.35%	70.40%	69.61%	68.90%	68.27%	67.70%	67.18%
$\nu = 1/3$ overlap + qh	0.9938	0.9944	0.9949	0.9952	0.9955	0.9956	0.9957	0.9958	0.9958

TABLE IV. Shown is the percentage of Hilbert space that is constructed from the product rule and overlap of the complete MR state and the state built from product rule on the sphere geometry. N is the number of particles and the overlap is defined as the absolute value of the scalar product. The first row of data is the total dimension of the squeezed Hilbert space for different system sizes. The second and third rows are the overlap results obtained when only type-I partitions are taken into account, while the fourth and fifth rows show the results obtained involving both type-I and -II partitions.

N	8	10	12	14	16	18	20	22
Squeezed dimension	119	1070	10751	116287	1326581	15756587	193181910	2429921124
Product rules (type I)	27.73%	25.23%	23.58%	22.48%	21.63%	20.95%	20.40%	19.95%
Overlap (type I)	0.8858	0.9070	0.9188	0.9262	0.9311	0.9344	0.9367	0.9383
Product rules (types I + II)	72.27%	70.09%	68.48%	66.98%	65.79%	64.82%	64.01%	63.34%
Overlap (types I + II)	0.9875	0.9895	0.9919	0.9930	0.9936	0.9941	0.9944	0.9946

also from a partition such as 10010001‖10001 with a cut between the two consecutive particles in the eighth and ninth orbitals. In this case, to reconstruct the Slater determinant weight, one needs to merge two Laughlin states with one quasihole excitation each (which are also Jack polynomials with the same α parameter as the ground state). For our example, this corresponds to the Jacks of smaller system size with root partitions 10010001 and 10001. To apply this rule, the only missing information we additionally need is the weight of a Slater determinant in a Laughlin state, which is obtained from the root configuration by a single squeezing of two neighboring particles $\ldots 1001 \ldots$ into two consecutive occupied orbitals $\ldots 0110 \ldots$. It can be shown that this weight is always equal to 3 in the basis described in Eq. (10). The improvement of the overlap including this additional rule is shown in Table III.

D. Entanglement spectra

While high overlaps of two states as discussed above provide an indication of correspondence between these states, cases are known where the overlap measure can be ambiguous or even misleading. For example, the Jain state at $\nu = 2/5$ and the Gaffnian state have large overlaps while describing two physically different states.[38,51] A more adequate measure is provided by quantities such as the entanglement entropy or the entanglement spectrum. The latter has been recently used to identify the topological fingerprint of quantum Hall states,[60–71] superconductors,[72] and topological insulators,[73–75] and to detect nonlocal order in spin systems[74,76–80] and Bose-Einstein condensates.[81] The procedure to obtain the entanglement entropy and the entanglement spectrum is as follows. We cut the quantum Hall sphere into two parts and separate the orbitals into regions A (with l_A orbitals) and B (with $N_\phi + 1 - l_A$ orbitals). From there, we compute the reduced density matrix of region A as the partial trace of ρ over the degrees of freedom in region B, i.e., $\rho_A = \mathrm{Tr}_B \rho$. The entanglement entropy is the von Neumann entropy associated with ρ_A. We define the entanglement spectral levels ξ as the spectrum of an entanglement Hamiltonian H_A, which relates to ρ_A via $\rho_A = e^{-H_A}$. Depending on the basis for which the partial trace is defined, the entanglement levels possess certain quantum numbers associated with commuting operators $[\rho_A, Q_A] = 0$. For spinless quantum Hall states on the sphere, such quantum numbers are the number of particles N_A and the angular momentum L_z^A in A.

The entanglement entropy allows us to resolve certain deviations of the approximated states built from the product rule as compared to the exact states. For example, using the product rule construction only obtained from the $N/2$-particle exact states would lead to zero entanglement entropy for the approximated state when cutting the system into two equal parts, contrary to the exact state. As discussed in Ref. 14, the typical error for the entanglement entropy of the product rule state generally is around 5%.

A deeper comparison can be achieved by considering the entanglement spectrum. It can be evaluated on any geometry (such as the sphere[60] or the torus[65]). For many generic Coulomb ground states, the thin annulus geometry, the so-called conformal limit,[62] exhibits a clear gap in the entanglement spectrum. Figure 5(a) shows such a spectrum for the $\nu = 1/2$ Laughlin trial state. It only consists of one branch of levels that matches the low-lying levels of the associated Coulomb state below the entanglement gap (not shown). The approximated state generated using the product rules exhibits a different spectrum [see Fig. 5(b)]: it shows an entanglement gap separating a "low-energy" sector and a "high-energy" sector (the size and clear definition of such an entanglement gap in this case may depend on the trial product state considered). Contrary to the Coulomb spectrum at the same filling factor,[62] however, the counting of the eigenvalues in the low-energy sector does not match exactly the one of the Laughlin state. It only matches above some given value L_z^{A*}, which depends on the type of trial state and system size. Below L_z^{A*}, some or even all eigenvalues are missing in the product rule state. In particular, the L_z^A region that has the same counting as the exact Laughlin state is the one where the counting has already reached the thermodynamic value for a given system size.[71] This indicates that the complete counting of the exact trial state may be recovered in the product rule state in the thermodynamical limit. It also suggests the way the approximated state may converge toward the exact state, eventually recovering the full counting correspondence.

V. HALDANE-REZAYI STATE

A. Basic properties

We now turn to the generalization of the bosonic and fermionic states involving the spin degree of freedom of the constituent particles. In the following, we discuss the Haldane-Rezayi (HR) spin-singlet state.[4] The HR state was originally proposed as a trial state for the incompressible plateau state at $\nu = 5/2$.[4] As opposed to the spin-polarized Moore-Read (Pfaffian) state at identical filling,[2,82] it is a spin singlet. The

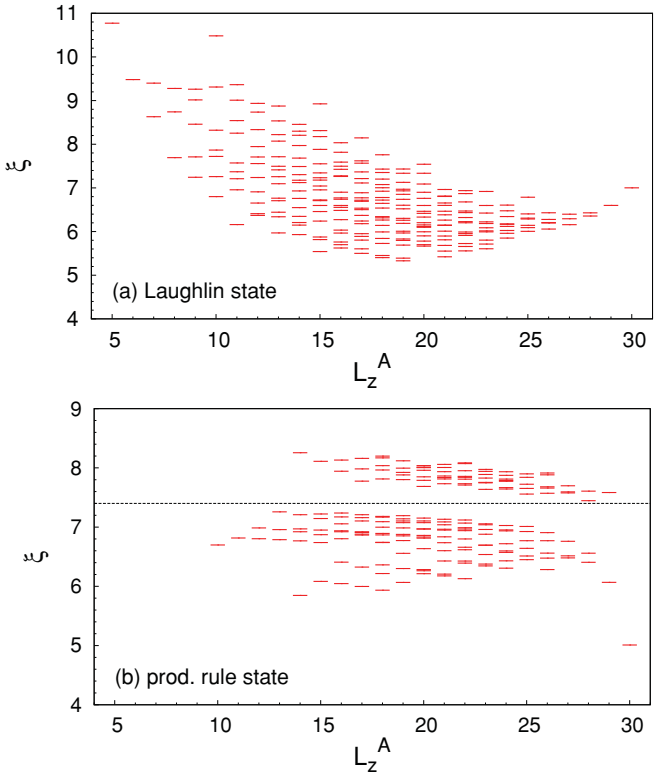

FIG. 5. (Color online) Upper panel: Entanglement spectrum for the bosonic $\nu = 1/2$ Laughlin state for $N = 11$ particles, $l_A = 10$ and $N_A = 5$ in the conformal limit. Lower panel: Entanglement spectrum for the approximation of the bosonic $\nu = 1/2$ Laughlin state using the product rules with $N = 11$, $l_A = 10$, and $N_A = 5$ in the conformal limit (Ref. 62). The spectrum separates into two parts below and above the dashed line. Below the dashed line, the eigenvalue counting matches the one of the exact state from $L_z^A = 26$ to 30, while there are missing eigenvalues compared to the exact state for $L_z^A < 26$. In particular, no eigenvalues are present in the sector $5 \leqslant L_z^A \leqslant 9$. This absence of eigenvalues can be understood by looking at the missing configurations (in particular, the most squeezed ones) within the approximated state.

degree of spin polarization is still an experimental issue that is not yet settled in the $\nu = 5/2$ state. The general belief, supported by numerical evidence from exact diagonalization studies, is that the Moore-Read state is the promising candidate to explain the $\nu = 5/2$ state.[83–86] The HR state attained considerable attention since it can be interpreted as describing the transition point between the strong and weak pairing phases of a spin-singlet d-wave BCS superconductor. This physical interpretation is supported by the fact that the bulk CFT is a nonunitary $c = -2$ theory, which is expected to have gapless bulk excitations.[87]

The HR state is given by

$$\Phi_{\mathrm{HR}} = \mathcal{A}_{z,w}\left(\frac{1}{(z_1-w_1)^2}\frac{1}{(z_2-w_2)^2}\cdots\right) \times \prod_{i<j}^{N}(z_i-z_j)^2 \prod_{i<j}^{N}(w_i-w_j)^2 \prod_{i,j}^{N}(z_i-w_j)^2, \quad (15)$$

where N is the number of spin-up ($\uparrow$) and spin-down ($\downarrow$) particles with positions denoted by z and w, $\mathcal{A}_{z,w}$ is the antisymmetrizer over the z particles and w particles separately. Due to the antisymmetric non-$U(1)$ prefactor in the first line of Eq. (15) and the evenness of the Jastrow factors, the state in (15) is fermionic although a bosonic variant can also be formulated. Let us look at the clustering conditions at the level of first quantization. The wave function dies as the second power of the difference between two equal spin coordinates. However, the prefactor of the Jastrow factors removes the second-order interspin zeros induced by the last Jastrow factor, allowing for configurations where one up and one down spin sit on the same site.

In order to span the Hilbert space of spinful states, we again start from the single-particle Landau orbitals. However, in this case, the single-particle orbitals possess an additional spin quantum number σ taking on the values $\uparrow$ or $\downarrow$. The free many-body basis is given by spinful Slater determinants $\mathrm{sl}_{\lambda_{A\uparrow},\lambda_{B\downarrow}} = \mathcal{A}(z_1^{\lambda_{1,A\uparrow}},\ldots,z_N^{\lambda_{N,A\uparrow}},w_1^{\lambda_{1,B\downarrow}},\ldots,w_N^{\lambda_{N,B\downarrow}})$, where $\lambda_{i,A\uparrow}$ and $\lambda_{i,B\downarrow}$ label the momenta of the ith up-spin and ith down-spin particles, respectively.

The squeezing operation is defined as before, and applies equally to squeezing between $\uparrow$ or $\downarrow$ particles. The root partition of the HR state is given by $X000X\ldots0X$, where $X = \uparrow\downarrow$, i.e., X denotes an orbital occupied by both spins. This is consistent with the first quantized clustering condition and with the filling $\nu = 1/2$ in the highest partially populated Landau level. We decompose the HR state in terms of Slater determinants:

$$\Phi_{\mathrm{HR}} = \sum_{\lambda_{A\uparrow},\lambda_{B\downarrow}\leqslant X000X000X\ldots} c_{\lambda_{A\uparrow},\lambda_{B\downarrow}}\mathrm{sl}_{\lambda_{A\uparrow},\lambda_{B\downarrow}}, \quad (16)$$

where the sum extends over all spinful partitions obtained by squeezing operations on the root partition $X000X\ldots0X$. Let us consider the four-particle HR root configuration $X000X$. It can be expressed in partition language as $[4\uparrow,4\downarrow,0\uparrow,0\downarrow]$ or $[4\uparrow,0\uparrow]\times[4\downarrow,0\downarrow]$. The latter is a factorization in $\uparrow$-spin and $\downarrow$-spin partitions. It differs from the former by a minus sign. Thus, to avoid ambiguities of global minus signs due to different orderings of fermionic operators, we order all partition entries (i.e., $\uparrow$ and $\downarrow$ spin) first by momenta in decreasing order. For a given momentum, we write the $\uparrow$-spin entry before the $\downarrow$-spin entry. Whenever a factorized partition notation of $\uparrow$ spin and $\downarrow$ spin momenta appears in the following text, it is only for reasons of presentation. The ordering of fermions should always be interpreted as explained above. Moreover, Slater determinants differing by an overall spin rotation have equal coefficients. This is so since the HR state is a spin singlet and thus spin rotationally invariant.

B. Differential action for the HR state

From conformal field theory considerations, $\Phi_0(z,w) = \mathcal{A}_{z,w}(\frac{1}{(z_1-w_1)^2}\frac{1}{(z_2-w_2)^2\ldots})$ satisfies the following differential equation[88]:

$$\left[\frac{1}{2}\frac{\partial^2}{\partial z_i^2} - \sum_{j\neq i}\left(\frac{1}{(z_i-w_j)^2} + \frac{1}{z_i-w_j}\frac{\partial}{\partial w_j}\right)\right]\Phi_0 = 0, \quad (17)$$

with the same equation for $z \leftrightarrow w$. Following Sec. III, we use this equation to obtain an operator for which Φ_{HR} is an eigenstate. We rewrite the derivatives acting on Φ_0 as derivatives acting on Φ_{HR}. This amounts to taking into account the additional derivative acting on Jastrow factors for both spin species. By using the intermediate steps explained in Appendix D, we derive the following differential equation:

$$\left\{\frac{1}{2}\sum_i^N\left(z_i\frac{\partial}{\partial z_i}\right)^2+\left(w_i\frac{\partial}{\partial w_i}\right)^2-(3N-2)\sum_i^N\left(z_i\frac{\partial}{\partial z_i}+w_i\frac{\partial}{\partial w_i}\right)-\frac{1}{2}\sum_{i,k}\frac{z_i+w_k}{z_i-w_k}\left(z_i\frac{\partial}{\partial z_i}-w_k\frac{\partial}{\partial w_k}\right)\right.$$
$$\left.-\frac{1}{4}\sum_{i\neq j}\left[\frac{z_i+z_j}{z_i-z_j}\left(z_i\frac{\partial}{\partial z_i}-z_j\frac{\partial}{\partial z_j}\right)+\frac{w_i+w_j}{w_i-w_J}\left(w_i\frac{\partial}{\partial w_i}-w_j\frac{\partial}{\partial w_j}\right)\right]+4N(2N^2-3N+1)\right\}\Phi_{\mathrm{HR}}=0. \quad (18)$$

The above equation contains only two-body interactions. The interaction terms are symmetric with respect to ↑-spin and ↓-spin variables. Both the interspin and intraspin interactions are of Laplace-Beltrami type and are familiar from our previous calculations of the polarized fermionic states.

1. Equal spin action

Let us first compute the action of the terms in Eq. (18) consisting of equal spin interactions. Once solved for one species, e.g., the ↑ spin variables z, this similarly applies for the ↓ spin terms. The corresponding part of the HR operator in (18) is given by

$$\frac{z_i+z_j}{z_i-z_j}\left(z_i\frac{\partial}{\partial z_i}-z_j\frac{\partial}{\partial z_j}\right), \quad (19)$$

which we have previously encountered as a part of the fermionic LB operator. However, the term $\sim 1/(z_i-z_j)^2$ of the LB operator is missing. This is an important difference. It implies that the operator in Eq. (19) does not map a single *general* Slater determinant of arbitrary degree into another Slater determinant. The equivalent of (10) can not be written in the current case by using only the operator in Eq. (19): Acting with the operator in Eq. (19) on a single Slater determinant usually leads to a fraction. However, the special sum of Slater determinants that comprise the HR state *does* map back into a sum of Slater determinants. This is so since the HR state has two Jastrow terms in equal spins; they cancel any fractions that might appear upon the action of (19) on a Slater (see Appendix D). A single Jastrow factor would not be sufficient to ensure this property. However, a mapping of a linear superposition of Slaters constrained in the way above maps back to the space of Slaters under action of (19). Details are given in Appendix D. We force the identity

$$\sum_{i\neq j}\frac{z_i+z_j}{z_i-z_j}\left(z_i\frac{\partial}{\partial z_i}-z_j\frac{\partial}{\partial z_j}\right)\sum_\mu a_\mu \mathrm{sl}_\mu=\sum_\mu b_\mu \mathrm{sl}_\mu, \quad (20)$$

where sl_μ is defined as the Slater determinant of one spin species. The partition of the other spin species is omitted in typing but should be implicated. Since from the expression of the HR state we know that the linear combination of states that form the HR state have a double zero $\sum_\mu a_\mu \mathrm{sl}_\mu(z_1,\ldots,z_M)\sim\prod_{i,j}(z_i-z_j)^2$, the forced identity (20) is clearly true. The coefficients b_μ are then given in terms of the coefficients a_μ. We find

$$\sum_{\mu\leqslant\lambda}b_\mu \mathrm{sl}_\mu=\sum_{\mu\leqslant\lambda}\sum_{i<j}(\mu_i-\mu_j)a_\mu \mathrm{sl}_\mu + 2a_\mu(\mu_i-\mu_j)\sum_{\theta<\mu}\mathrm{sl}_\theta(-1)^{N_{\mathrm{SW}}}, \quad (21)$$

where $\theta=[\mu_1,\ldots,\mu_i-s,\mu_{i+1},\ldots,\mu_j+s,\ldots,\mu_N]$, and N_{SW} again denotes the number of swaps needed to reorder the partition.

2. Different spin action

In order to compute the term in (18) involving action on both spin species, we consider one single spinful Slater determinant $\mathrm{sl}_{\lambda_A\uparrow,\lambda_B\downarrow}$ as defined before. Following the calculation detailed in Appendix D, the spin-rotated partition $\mathrm{sl}_{\lambda_B\uparrow,\lambda_A\downarrow}$ always appears with the same coefficient. In this way, we find that the action of the $\uparrow\leftrightarrow\downarrow$ part of the operator also maps back to Slaters:

$$\sum_{i,k}\frac{z_i+w_k}{z_i-w_k}\left(z_i\frac{\partial}{\partial z_i}-w_k\frac{\partial}{\partial z_k}\right)\mathrm{sl}_{\lambda_A\uparrow,\lambda_B\downarrow}$$
$$=\sum_{i,j,\sigma;\lambda_{i,\sigma}>\lambda_{j,\bar\sigma}}(\lambda_{i,\sigma}-\lambda_{i,\bar\sigma})\mathrm{sl}_{\lambda_A\uparrow,\lambda_B\downarrow}+\sum_{\theta_A,\theta_B;i,j,\sigma;\lambda_{i,\sigma}>\lambda_{j,\bar\sigma}}(\lambda_{i,\sigma}-\lambda_{i,\bar\sigma})\mathrm{sl}_{\theta_A\uparrow,\theta_B\downarrow}(2-\delta_{\theta_A,\theta_B}\delta_{\lambda_{i,\sigma}-s,\lambda_{j,\bar\sigma}+s})\cdot(-1)^{N_{\mathrm{SW}\uparrow}+N_{\mathrm{SW}\downarrow}}, \quad (22)$$

where θ_A,θ_B are the two partitions of spin ↑ and ↓, respectively. They are obtained by squeezing only *opposite* spins in $\lambda_{A\uparrow},\lambda_{B\downarrow}$; s parametrizes the changed partition component obtained from squeezing spin ↑ with spin ↓: $\theta_A=[\lambda_{1,A},\ldots,\lambda_{i,A}+s,\ldots,\lambda_{N,A}]$ and $\theta_B=[\lambda_{1,B},\ldots,\lambda_{j,B}-s,\ldots,\lambda_{N,B}]$. Thus, the summation takes

each possible pairwise combination of spin ↑ and spin ↓ particles, and squeezes them. $N_{\mathrm{SW}\sigma}$ denotes the reordering swaps of spins from species σ upon this interspin squeezing operation.

C. Recurrence relation

We are now ready to compute the full action of the operator in Eq. (18) on the Haldane-Rezayi state. The remainder terms are of noninteracting kinetic type and straightforward to compute. By power counting, we find

$$\sum_i^N \left(z_i \frac{\partial}{\partial z_i} + w_i \frac{\partial}{\partial w_i} \right) \Phi_{\mathrm{HR}} = 4N(N-1)\Phi_{\mathrm{HR}}. \quad (23)$$

The second-order derivative terms give

$$\sum_i \left(z_i \frac{\partial}{\partial z_i} \right)^2 \Phi_{\mathrm{HR}} = \sum_i \lambda_i^2 \Phi_{\mathrm{HR}}. \quad (24)$$

In the intraspin term, the sums over differences of λ's can be computed as

$$\sum_{i<j}^N (\lambda_i - \lambda_j) = \sum_i^N (N+1-2i)\lambda_i, \quad (25)$$

while there is no similar closed form expression for the interspin term. Summing up all terms of (18), we find the following equation:

$$\begin{aligned}
&\sum_{\lambda_A\uparrow,\lambda_B\downarrow \leqslant X000X000X\ldots} c_{\lambda_A\uparrow,\lambda_B\downarrow} \mathrm{sl}_{\lambda_A\uparrow,\lambda_B\downarrow} \left(2N(3N^2-4N+1) - \frac{1}{2}\sum_i \left(\lambda_{Ai}^2 + \lambda_{Bi}^2 + 2i\lambda_{Ai} + 2i\lambda_{Bi}\right) + \frac{1}{2} \sum_{i,j,\sigma;\lambda_{i,\sigma}>\lambda_{j,\bar\sigma}} (\lambda_{i,\sigma} - \lambda_{i,\bar\sigma}) \right) \\
&= -\sum_{\lambda_A\uparrow,\lambda_B\downarrow \leqslant X000X000X\ldots} c_{\lambda_a\uparrow,\lambda_B\downarrow} \left(\sum_{\theta_A<\lambda_A} (\lambda_{A,i} - \lambda_{A,j}) \mathrm{sl}_{\theta_A\uparrow,\lambda_B\downarrow} (-1)^{N_{\mathrm{SW}}} + \sum_{\theta_B<\lambda_B} (\lambda_{B,i} - \lambda_{B,j}) \mathrm{sl}_{\lambda_A\uparrow,\theta_B\downarrow} (-1)^{N_{\mathrm{SW}}} \right. \\
&\left. + \frac{1}{2} \sum_{i,j,\sigma;\lambda_{i,\sigma}>\lambda_{j,\bar\sigma}} (\lambda_{i,\sigma} - \lambda_{i,\bar\sigma}) \mathrm{sl}_{\theta_{A,\uparrow},\theta_{B,\downarrow}} (2 - \delta_{\theta_A,\theta_B}\delta_{\lambda_{i,\sigma}-s,\lambda_{j,\bar\sigma}+s}) (-1)^{N_{\mathrm{SW},\uparrow}+N_{\mathrm{SW},\downarrow}} \right). \quad (26)
\end{aligned}$$

In Eq. (26), we have grouped all diagonal terms to the left and all interactions to the right of the equality. Accordingly, the right-hand side is made of three parts: parts of sums of spin-↑ partitions squeezed from the spin-↑ partition λ_A, parts of sums of spin-↓ partitions squeezed from the spin-↓ partition λ_B, and parts of sums of spin-↑ and spin-↓ partitions squeezed only from particles of different spin from the partition $[\lambda_A\uparrow,\lambda_B\downarrow]$. We now define the quantities

$$\rho_{\lambda_A\uparrow,\lambda_B\downarrow} = \frac{1}{2}\sum_i \left(\lambda_{A,i}^2 + \lambda_{B,i}^2 + 2i\lambda_{A,i} + 2i\lambda_{B,i}\right) - \frac{1}{2} \sum_{i,j,\sigma;\lambda_{i,\sigma}>\lambda_{j,\bar\sigma}} (\lambda_{i,\sigma} - \lambda_{i,\bar\sigma}). \quad (27)$$

We can then immediately show that the root partition obeys $\rho_{X000X\ldots X} = 2N(3N^2-4N+1)$, and thus corresponds to the constant terms on the left side of Eq. (26). By equating the coefficients in (26), we derive a recurrence relation for the coefficients of the decomposition of the HR state into Slater determinants:

$$\begin{aligned}
c_{\mu_A\uparrow,\mu_B\downarrow} = \frac{1}{\rho_{X000X000X\ldots} - \rho_{\mu_A\uparrow,\mu_B\downarrow}} &\left(\sum_{\theta_A;\mu_A<\theta_A\leqslant\uparrow000\uparrow000\uparrow\ldots \;\; \theta_A=[\mu_{A,1},\ldots,\mu_{A,i}+t,\ldots,\mu_{A,j}-t,\ldots,\mu_{A,N}]} (\mu_{A,i} - \mu_{A,j} + 2t) c_{\theta_A\uparrow,\mu_B\downarrow} \cdot (-1)^{N_{\mathrm{SW}}} \right. \\
&+ \sum_{\theta_B;\mu_B<\theta_B\leqslant\downarrow000\downarrow000\downarrow\ldots \;\; \theta_B=[\mu_{B,1},\ldots,\mu_{B,i}+t,\ldots,\mu_{B,j}-t,\ldots,\mu_{B,N}]} (\mu_{B,i} - \mu_{B,j} + 2t) c_{\mu_A\uparrow,\theta_B\downarrow} \cdot (-1)^{N_{\mathrm{SW}}} \\
&\left. + \frac{1}{2} \sum_{\theta_A,\theta_B;\;\; [\mu_A\uparrow,\mu_B\downarrow]<[\theta_A,\theta_B]\leqslant[X000X000X\ldots]} (\mu_\sigma - \mu_{\bar\sigma} + 2t) c_{\theta_A\uparrow,\theta_B\downarrow} (2 - \delta_{\mu_A,\mu_B}\delta_{\mu_i\uparrow,\mu_j\downarrow}) \cdot (-1)^{N_{\mathrm{SW}\uparrow}+N_{\mathrm{SW}\downarrow}} \right), \quad (28)
\end{aligned}$$

where the last sum over the interspin terms extends over all partitions θ_A,θ_B with the property $[\theta_A,\theta_B] = [[\mu_{A,1},\mu_{A,2},\ldots,\mu_{A,i}+t,\ldots,\mu_{A,N}],[\mu_{B,1},\mu_{B,2},\ldots,\mu_{B,j}-t,\ldots,\mu_{B,N}]]$ with $\mu_{A,i} \geqslant \mu_{B,j}$, or $[\theta_A,\theta_B] = [[\mu_{A,1},\mu_{A,2},\ldots,\mu_{A,j}-t,\ldots,\mu_{A,N}],[\mu_{B,1},\mu_{B,2},\ldots,\mu_{B,i}+t,\ldots,\mu_{B,N}]]$ with $\mu_{B,i} \geqslant \mu_{A,j}$. One explicit example of computation is presented in Fig. 6, and a complete four-particle HR Slater decomposition is shown in Appendix E.

In analogy to Eqs. (3) and (10), the Slater decomposition coefficients of the HR state can be read off from (28). The HR state can be generated with linear effort in Hilbert-space dimension.

D. Zero-weight partitions

The recurrence relations we encountered for the spin-polarized bosonic (fermionic) states in Eq. (3) [Eq. (10)] only produced accidental (about $1/N!$ of the total Hilbert-space dimension) zero weights for squeezed monomial (Slater) configurations. By contrast, we observe an extensive number

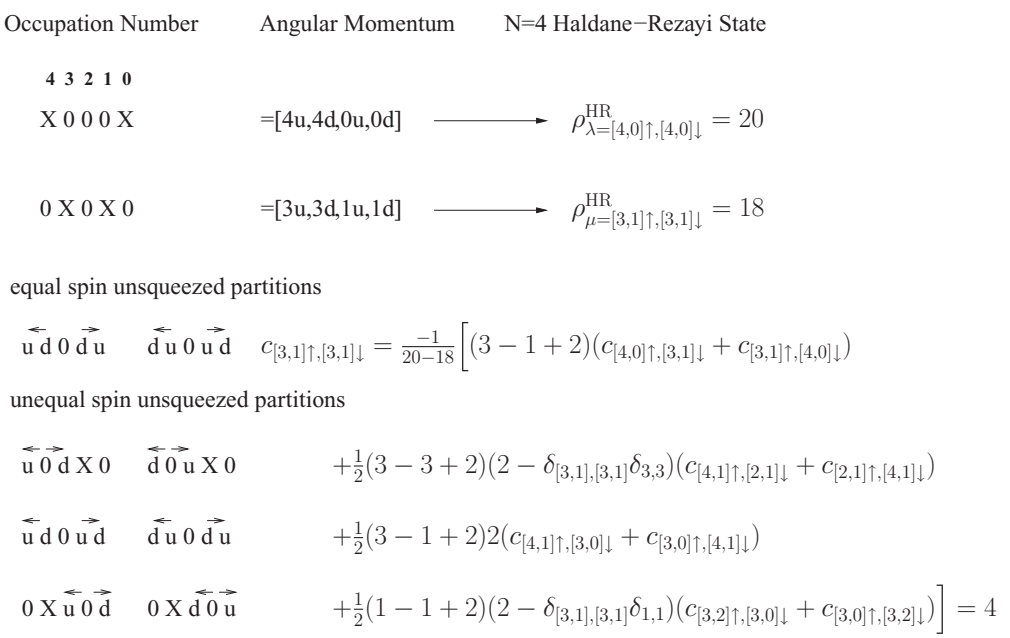

FIG. 6. Explication of the recurrence relation (28) for the $N = 4$ Haldane-Rezayi state. We have chosen to present the explicit computation of the coefficient of partition $\mu = 0X0X0$, which involves all conceptually different terms appearing in (28).

of zero-weight configurations from the HR recurrence relation that hints at further structure in the HR coefficients beyond the spinful squeezed Slater basis. The key observation is that the HR state can be written as[89]

$$\Phi_{\mathrm{HR}} = \prod_{i<j}(z_i - z_j)^2(w_i - w_j)^2 P_1(z_i, w_j), \tag{29}$$

where P_1 is an angular momentum $L = 0$ polynomial squeezed from the root partition $X0X0X0\ldots X0X$.

Interpreted in this way, the spinful $2N$-particle HR partitions are generated from the space of partitions of P_1 squeezed from $X0X0X\ldots X0X$ times partitions from the N-particle bosonic $\nu = 1/2$ Laughlin factor squeezed from $1010101\ldots101$ for $\uparrow$ spins and $\downarrow$ spins, respectively.

We investigate the $2N$-particle HR partition $X000X000X\ldots$, where we count the momentum in terms of polynomial powers beginning from $m = 0$ at the north pole extending to $m = 4(N-1)$ at the south pole. We define the momentum imbalance between total $\uparrow$ spin and $\downarrow$ spin particle momentum within a partition λ: $|\sum_i \lambda_{i,\uparrow} - \sum_i \lambda_{i,\downarrow}|$. The total momentum summed over $\uparrow$ and $\downarrow$ spins, i.e., $\sum_{i,\sigma}\lambda_{i,\sigma}$, is $4N(N-1)$. For $N = 2$, the partitions of maximum momentum imbalance are given by $\downarrow 0\ X\ 0\uparrow$ and $\uparrow 0\ X\ 0\downarrow$ [both with momentum imbalance $4 + 2 - (2 + 0) = 4$]. They are obtained by squeezing only $\uparrow$ spins to the left and only $\downarrow$ spins to the right, and vice versa. If we arrange all $\uparrow$ and $\downarrow$ spins to generate the highest momentum imbalance between the spin species (which means to arrange all $\uparrow$ and $\downarrow$ spins around opposite poles), we find a partition structure $\downarrow 0\downarrow 0\ldots\downarrow 0X0\uparrow\ldots 0\uparrow 0\uparrow$. Thus, we find a maximum imbalance of $2N(N-1)$ for partitions squeezed from the HR root partition.

Let us now construct the HR state starting from P_1. Multiplying a P_1 partition of a given momentum imbalance with a $\nu = 1/2$ Laughlin partition for $\uparrow$ and $\downarrow$ spins does not change the momentum imbalance, as the total momentum added to both spin species is equal. Thus, the available range of possible momentum imbalance in HR partitions is given by the partitions squeezed from $X0X0X0\ldots X0X$. There, arranging the different spin species to opposite poles gives a partition structure $\downarrow\downarrow\ldots\downarrow X\uparrow\ldots\uparrow\uparrow$. The maximum momentum imbalance is $N(N-1)$ and, thus, only one-half of the maximum imbalance for partitions squeezed from the HR root partition. As a consequence, we can remove all squeezed partitions in the HR state with momentum imbalance $> N(N-1)$, as they must have zero weight. For $N = 2$, this applies for the partition $\downarrow 0\ X\ 0\uparrow$ and its spin-rotated counterpart $\uparrow 0\ X\ 0\downarrow$, as the momentum imbalance is 4, while the maximum allowed momentum imbalance is 2 (see Appendix E). While this rule is easily implemented numerically, there are even more zero-weight partitions squeezed from the HR root partition, which can not be written in a similar closed form as the momentum imbalance constraint. Rather, we have to explicitly check whether a partition squeezed from the HR root partition can be generated from P_1 times Laughlin partitions. We have sketched the algorithm in Table V by which partitions squeezed from the HR root partition can be efficiently denied or accepted.

E. Spinful entanglement spectra

We now focus our attention on the entanglement spectrum of the HR state. As discussed in Sec. IV D, the entanglement levels possess certain quantum numbers associated with commuting operators $[\rho_A, Q_A] = 0$. For the HR state, the levels are specified by quantum numbers for the number of particles, angular momentum, and spin z component in A, i.e., $Q_A = N_A$, L_z^A, S_z^A. As the full HR state is a spin singlet, it follows that the reduced density matrix not only commutes with S_z^A, but also $S^{A\,2}$ such that ρ_A decomposes into spin

TABLE V. Sketched algorithm to remove zero-weight partitions squeezed from the HR root partition $X000X000X\ldots$. We apply the structure of the HR state according to Eq. (29). By this procedure, the total Hilbert space is reduced to about 83% of the full squeezed Hilbert space.

	Algorithmic steps to accept or deny a $2N$-particle HR-squeezed partition μ
	For a given partition μ
(i)	Check whether total momentum imbalance $\lvert\sum_i \mu_{i,\uparrow} - \sum_i \mu_{i,\downarrow}\rvert$ is $\leqslant N(N-1)$.
(ii)	Loop over all $\uparrow$ and $\downarrow$ N-particle Laughlin partitions κ
(ii.1)	Divide partition μ by κ: Check whether all momenta are still in allowed range and no equal spin momenta are mapped onto each other (fermionic state)
(ii.2)	Check whether the resulting partition can be squeezed from P_1

multiplets. The $N = 12$ HR spectrum for the half sphere cut with $N_A = 6$ is shown in Fig. 7. In the case of an even number of sites in region A, which gives a decomposition into integer spin multiplets, we always choose to look at the $S_z^A = 0$ spin sector where we can study all total spin multiplets in one sector. Analyzing the single highest angular-momentum entanglement level, which also is a spin singlet (see $L_z^A = 36$ in Fig. 7), we find that the eigenstate of this entanglement level is exactly the $N = 6$ HR state. This demonstrates the product rule property of the HR state: the highest angular-momentum entanglement level of a trial states corresponds to the trial state itself for the number of particles in the remainr regio A. This holds for all trial states where the product rule applies [the product rule can also be obtained from the explicit recurrence relation between the coefficients for the HR state [Eq. (28)] by performing similar steps to the ones used in proving the product rule for spin-polarized states]. The second highest angular-momentum sector shows a single entanglement level that is degenerate in entanglement energy with the highest momentum state. Again, for a fermionic Slater determinant

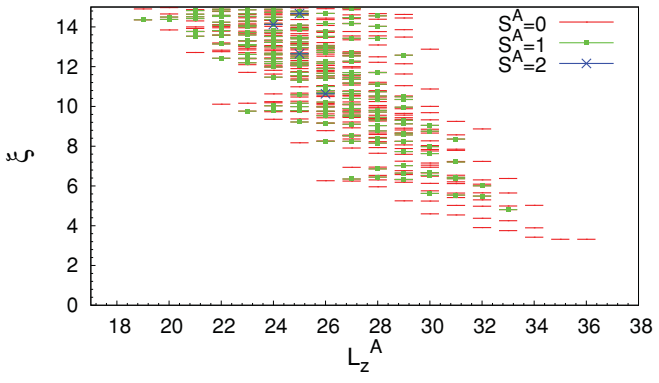

FIG. 7. (Color online) Entanglement spectrum for the $N = 12$ Haldane-Rezayi state, half sphere cut, $N_A = 6$, $S_z^A = 0$ sector, entanglement levels plotted versus the angular momentum L_z^A. Levels relating to different spin multiplets, denoted by S^A, are given in red bars (singlets), square green bars (triplets), and crossed blue bars (quintuplets). A large part of the main entanglement weight resides in the singlet sector.

description of quantum Hall trial states, this degeneracy is constrained by symmetry of how to arrange the Slater with one less fermionic angular momentum,[14] and, hence, can again be understood from the Slater determinant structure of the HR state.

VI. CONFORMAL FIELD THEORY

We investigate the product rule property of FQH states from the perspective of conformal field theory (CFT). For the FQHE, we are concerned with chiral CFTs in two dimensions: all the fields are holomorphic, i.e., they depend on $z = x + iy$, and not on $\bar{z}$. This is an illustration of broken time-reversal symmetry in this context. For an in-depth introduction to CFT, we engage the reader to Ref. 88. In this section, we present the key notions that are needed to derive the product rule of quantum Hall wave functions from CFT.

The conformal dimensions are quantities that appear in the two-point correlation functions of the fields Φ_i present in the CFT. Conformal invariance fixes any two-point function to be of the form

$$\langle \Phi_1(z_1)\Phi_2(z_2)\rangle = \frac{C_{1,2}}{(z_1 - z_2)^{\Delta_1+\Delta_2}} \tag{30}$$

and its scaling properties are described by the so-called conformal dimension Δ_i.

Primary fields are of special importance in a CFT. Among the infinite number of fields of a given CFT, primary fields can be thought of as highest weight fields. All nonprimary fields ("descendants") are created by acting on primary fields with certain types of lowering operators: the (negative) Virasoro modes. The two-point function of primary fields Φ_i and Φ_j reads as

$$\langle \Phi_i(z_1)\Phi_j(z_2)\rangle = \frac{\delta_{i,\bar{j}}}{(z_1 - z_2)^{2\Delta_i}}. \tag{31}$$

It is nonvanishing only if Φ_j and Φ_j are conjugate from one another, namely, if the identity field 1 is produced in their fusion. The identity is a very special field, and for our purposes it will be the only primary field with a vanishing conformal dimension. As a consequence, it is its own conjugate, $\bar{1} = 1$. In some cases, such as the logarithmic $c = -2$ ghost system corresponding to the Haldane-Rezayi state encountered in Sec. V, this is not the case, and one has to treat these cases more carefully as discussed below.

Conformal invariance also fixes the form of the correlation function between three primary fields:

$$\langle \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3)\rangle = \frac{D_{1,2,3}}{(z_1 - z_2)^{\Delta-2\Delta_3}(z_2 - z_3)^{\Delta-2\Delta_1}(z_1 - z_3)^{\Delta-2\Delta_2}}, \tag{32}$$

where $\Delta = \Delta_1 + \Delta_2 + \Delta_3$. When the fusion coefficient $D_{1,2,3}$ is nonzero, the field $\Phi_{\bar{3}}$ is said to be produced in the fusion $\Phi_1 \times \Phi_2$. The fusion rules encode those primary fields that are produced in any fusion. For instance, a fusion rule of the form

$$\Phi_1 \times \Phi_2 = \Phi_a + \Phi_b + \Phi_c \tag{33}$$

tells us that the fusion of Φ_1 and Φ_2 only produces three primaries: Φ_a, Φ_b, and Φ_c. Implicitly, all their descendants will also appear.

In order to compute correlation functions, fusion rules are not sufficient. An operator product expansion (OPE) is a refinement of the fusion rules. It is a formal (and exact) expansion of the product of two fields, and is an implicit way to define any correlation functions of a CFT by iteratively reducing the number of fields involved. It has the following generic form:

$$\Phi_1(z_1)\Phi_2(z_2) = \sum_k \frac{C_{1,2}^k}{(z_1 - z_2)^{\Delta_1+\Delta_2-\Delta_k}} \Phi_k(z_2), \quad (34)$$

where the sum is formally over all fields of the CFT. Thanks to conformal invariance, it is sufficient to know only the singular terms, those for which $\Delta_i + \Delta_j - \Delta_k > 0$, and there will be a finite number of them for any rational CFT. N-point OPEs are a straightforward generalization:

$$\Phi_1(z_1), \ldots, \Phi_N(z_N) = \sum_k F_k(z_1, \ldots, z_N)\Phi_k(0). \quad (35)$$

We are now in a position to derive the key property of CFT correlation functions that is responsible for the product rule. Consider a generic $N = N_A + N_B$-point correlation function of primary fields. For simplicity, we denote the complex coordinates of the fields by x_i and y_j:

$$C(x_i|y_j) = \langle \Phi_1(x_1), \ldots, \Phi_{N_A}(x_{N_A})\Phi_{N_A+1}(y_1), \ldots, \Phi_N(y_{N_B})\rangle. \quad (36)$$

A particle cut between N_A and N_B particles is directly related to the asymptotic behavior of the correlation function as we spatially separate the two sets of variables x_i and y_j. We consider the limit $\gamma \to \infty$ of $C(x_i|\gamma y_j)$. By using the N_A-point OPE [Eq. (35)] for $\Phi_1(x_1), \ldots, \Phi_{N_A}(x_{N_A})$ around 0, we obtain

$$C(x_i|\gamma y_j) = \sum_k F_k(x_i)\langle \Phi_k(0)\Phi_{N_A+1}(\gamma y_1), \ldots, \Phi_N(\gamma y_{N_B})\rangle. \quad (37)$$

Thus, conformal invariance tells us that the correlation function $\langle \Phi_k(0)\Phi_{N_A+1}(\gamma y_1), \ldots, \Phi_N(\gamma y_{N_B})\rangle$, if it is nonvanishing, scales as $\gamma^{-\Delta_{N_A+1}-\cdots-\Delta_N-\Delta_k}$ as $\gamma \to \infty$. From this, we can immediately infer that the behavior is dominated by the (primary) field Φ_a (and its conjugate $\Phi_{\bar{a}}$), which is the field with lowest conformal dimension Δ_a appearing in both fusions:

$$\Phi_1 \times \cdots \times \Phi_{N_A} \to \Phi_a, \quad (38)$$

$$\Phi_{N_A+1} \times \cdots \times \Phi_N \to \Phi_{\bar{a}}. \quad (39)$$

In addition, for a primary field Φ_a the function F_a appearing in the N_A-point OPE [Eq. (35)] is given by the correlation function

$$F_a(x_1, \ldots, x_{N_A}) = \left\langle \Phi_1(x_1), \ldots, \Phi_{N_A}\left(x_{N_A}\right)\Phi_{\bar{a}}(\infty)\right\rangle \lim_{x\to\infty} x^{2\Delta_a} \times \left\langle \Phi_1(x_1), \ldots, \Phi_{N_A}\left(x_{N_A}\right)\Phi_{\bar{a}}(x)\right\rangle. \quad (40)$$

Finally, we obtain the asymptotic behavior for $C(x_i|\gamma y_j)$ as $\gamma \to \infty$ given by

$$\gamma^{\Delta_{N_A+1}+\cdots+\Delta_N+\Delta_a} C(x_i|\gamma y_j) \sim \left\langle \Phi_1(x_1), \ldots, \Phi_{N_A}\left(x_{N_A}\right)\Phi_{\bar{a}}(\infty)\right\rangle \times \left\langle \Phi_a(0)\Phi_{N_A+1}(y_1), \ldots, \Phi_N\left(y_{N_B}\right)\right\rangle \quad (41)$$

for any correlation function of primary fields $C(x_i|y_j)$ [Eq. (36)]. Using first principles of CFT, we hence have derived a factorized form of the correlation function spatially separating two sets of variables x_i and y_j. We will see in the following that this property implies both squeezing and the product rule for FQH wave functions. In what follows, we do not discuss Jain states or other hierarchy states that can not be generated through a single CFT correlator. There, both the notion of a product rule itself as well as the description within CFT is a more involved task and we defer this to a later stage.

A. Spinless FQH wave functions and parafermionic CFTs

1. Parafermionic CFTs

Many fully polarized QH wave functions can be written in terms of parafermionic CFTs. This includes Laughlin,[1], Moore-Read,[2] and the Read-Rezayi states[3] (as well as all Jack states[16]), but also any generalized parafermionic states, such as $N = 1$ superconformal or S_3 wave functions.[24,56]

Here, we constrain ourselves to a condensed derivation required to obtain the product rule from CFT. More details about parafermions in the context of FQHE can be found in Ref. 24. These parafermionic CFTs, denoted as $\mathbb{Z}_k^{(r)}$, contain a set of k parafermionic primary fields $\{\Psi_0 = 1, \Psi_1, \ldots \Psi_{k-1}\}$ with the following fusion rules:

$$\Psi_n \times \Psi_m \to \Psi_{n+m} \mod k. \quad (42)$$

The conformal dimension of the field Ψ_n is $\Delta_n = \frac{r}{2}\frac{n(k-n)}{k}$, and its conjugate is Ψ_{k-n}.

Parafermionic FQHE wave functions take the following form:

$$P(z_1, \ldots, z_N) = \langle \Psi(z_1), \ldots, \Psi(z_N)\rangle \prod_{i<j}(z_i - z_j)^{r/k}. \quad (43)$$

Using (41) for $N = N_A + N_B$ in the case in which all the primary fields Φ_i are taken to be parafermionic fields Ψ in a $\mathbb{Z}_k^{(r)}$ theory, we have

$$\gamma^{N_B\Delta_1+\Delta_a} C(x_i|\gamma y_j) \sim \left\langle \Psi(x_1), \ldots, \Psi\left(x_{N_A}\right)\Psi_{-a}(\infty)\right\rangle \times \left\langle \Psi_a(0)\Psi(y_1), \ldots, \Psi\left(y_{N_B}\right)\right\rangle, \quad (44)$$

where $a = N_A \bmod k$ corresponds to the sector in which the N_A particles live after the cut, and $\Delta_a = \frac{r}{2}\frac{a(k-a)}{k}$. Equivalently, for the wave function, this reads as $P(x_i|y_j) = C(x_i|y_j)\prod(x_i - x_j)^{\frac{r}{k}}\prod(y_i - y_j)^{\frac{r}{k}}\prod(x_i - y_j)^{\frac{r}{k}}$:

$$\gamma^{-\frac{r}{2k}[2N_AN_B+N_B(N_B-k)-a(k-a)]} P_N(x_i|\gamma y_j) \sim P_{N_A}^{(a)}(x_i)P_{N_B}^{(\bar{a})}(-1/y_j)\left(\prod_i y_i\right)^{\frac{r}{k}(N-k)}. \quad (45)$$

In particular, in the neutral sector $a = 0$, we have

$$P_N(x_i|\gamma y_j) \sim \gamma^{N_BN_\Phi(N_B)/2}\prod_i(\gamma y_i)^{rN_A/k} P_{N_A}(x)P_{N_B}(y), \quad (46)$$

where we introduced $N_\phi(N) = r(N-k)/k$.

2. *Squeezing and product rule*

From Eq. (45), one can derive the following properties.

Squeezing. Consider the power of γ in (45) for $N_B = 1$:

$$P_N(z_1, z_2, \ldots, z_{N-1}, \gamma z_N) \sim_{\gamma\to\infty} \gamma^{r(N-k)/k}. \quad (47)$$

This shows that for any monomial m_μ entering the decomposition of P_N, we have $\mu_1 \leqslant \lambda_1 = \frac{r}{k}(N-k)$. By iteration on N_B in (45), one finds that (i) the root partition is $\lambda = (k0^{r-1}k0^{r-1}\cdots k)$ and (ii) any other partition μ obeys $\mu_1 + \cdots + \mu_i \leqslant \lambda_1 + \cdots + \lambda_i$, i.e. is obtained by squeezing from λ.

Product rule. In the limit we consider, we send the N_B particles to infinity:

$$\lim_{\gamma\to\infty} \gamma^{-\frac{r}{2k}[2N_AN_B + N_B(N_B-k) - a(k-a)]} P_N(x_i|\gamma y_j) \quad (48)$$

only the monomials m_μ such that

$$\mu_1 + \cdots \mu_{N_B} = \lambda_1 + \cdots \lambda_{N_B} \quad (49)$$

survive. This kills all monomials obtained by squeezing through the cut between N_A and N_B particles, and leaves the others invariant. Now that squeezing has been established, the product rule is simply equivalent to the monomial decomposition of the factorization property:

$$\lim_{\gamma\to\infty} \gamma^{-(\lambda_1+\cdots+\lambda_{N_B})} P_N(x_i|\gamma y_j) = P^{(a)}_{N_A}(x_i)\tilde{P}^{(\bar{a})}_{N_B}(y_j)\left(\prod_{j=1}^{N_B} y_j\right)^{rN_A/k}, \quad (50)$$

where $\tilde{P}_{N_B}$ stands for the north (south) pole reflection of P_{N_B}.

B. Spin-singlet states

The same argument from above applies to spin-unpolarized states, such as the Haldane-Rezayi,[4] Halperin,[5] and NASS states.[6] Their CFT description involves several types of electron operators, typically consisting of a parafermion field and a vertex operator of a set of chiral boson fields. These boson fields Φ_c and Φ_s usually describe charge and spin associated with the particles. The electron creation operators take the generic form

$$V_\uparrow(z) = \Psi_\uparrow(z) : e^{\frac{i}{\sqrt{2}}(\sqrt{\beta+\gamma}\Phi_c + \sqrt{\beta-\gamma}\Phi_s)} :, \quad (51)$$

$$V_\downarrow(w) = \Psi_\downarrow(w) : e^{\frac{i}{\sqrt{2}}(\sqrt{\beta+\gamma}\Phi_c - \sqrt{\beta-\gamma}\Phi_s)} :, \quad (52)$$

where $\Psi_\uparrow$ and $\Psi_\downarrow$ can be trivial fields (Halperin), Gepner parafermions (NASS) (Ref. 90), or ghosts (Haldane-Rezayi).[87] The values of the rational numbers β and γ entering the vertex operators depend on this CFT.

The spin-polarized wave function assumes the form

$$P(z_i, w_j) = \langle \Psi_\uparrow(z_1)\Psi_\downarrow(w_1)\cdots\Psi_\uparrow(z_N)\Psi_\downarrow(w_N)\rangle \times \prod_{i<j}(z_i - z_j)^\beta (w_i - w_j)^\beta \prod_{i,j}(z_i - w_j)^\gamma. \quad (53)$$

Taking the asymptotic factorized behavior as n up-spin and m down-spin electrons are taken to infinity as in (41), we first obtain a *weak form of squeezing*: there exists an integer $N(n,m)$ dominating all the partitions $(\mu^\uparrow, \mu^\downarrow)$ in the sense that

$$\mu^\uparrow_1 + \cdots \mu^\uparrow_n + \mu^\downarrow_1 + \cdots \mu^\downarrow_m \leqslant N(n,m), \quad (54)$$

which is due to the asymptotic behavior $P \sim \gamma^{N(n,m)}$. This integer $N(n,m)$ can be expressed in terms of the CFT data α, β, $\Delta_\uparrow$, $\Delta_\downarrow$, and Δ_a.

Second, we obtain a *product rule* if a partition $\mu = (\mu_\uparrow, \mu_\downarrow)$ is separable, i.e., $\mu = \mu_A + \mu_B$. With μ_A (μ_B) squeezed from λ_A (λ_B), the corresponding monomial (Slater) coefficient is $m_{\lambda\mu} = m_{\lambda_1\mu_1} \times m_{\lambda_2\mu_2}$. In order to obtain a stronger form of squeezing, a more detailed analysis is needed. One would have to specify the OPEs of the operators $\Psi_\uparrow$ and $\Psi_\downarrow$ and, in particular, work out the dimension of the field Φ_a appearing in the fusion of n fields $\Psi_\uparrow$ and m fields $\Psi_\downarrow$, as we did for the spin-polarized case. The detailed analysis of the NASS and Halperin states is beyond the scope of this paper. We treat the HR state in Appendix F.

VII. CONCLUSION

In this paper, we have given an extended derivation of a recurrence formula for the Slater decomposition of fermionic Jack polynomial states. We have given a rigorous and detailed account on the product rule symmetry for spin-polarized quantum Hall trial states first presented in Ref. 14. We generalized the whole approach to spinful states and specifically derived a recurrence relation for the spinful Slater determinant decomposition of the Haldane-Rezayi state for which we have computed its spinful geometric entanglement spectrum. The product rule symmetry is found to be a deep general property of quantum Hall trial states, involving both fermionic and bosonic as well as spinful and spin-polarized states. For states described by parafermionic conformal field theory (which include but transcend the Jack polynomials) we have shown that the product rule comes out of the parafermionic fusion properties. In addition, for many other FQH states, the product holds even though the states can not be described by analog conformal field theory. We showed that the product rule can be used as an increasingly good approximation of the FQH state.

While the product rule symmetry is not exactly valid for ground states built from realistic interactions (such as the Coulomb potential), we would like to investigate if an approximated version of this symmetry also manifests itself in these cases. Such property may help to improve our general understanding and future computational descriptions of fractional quantum Hall systems.

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APPENDIX A: HELPFUL FORMULAS

We present several formulas used to simplify certain polynomial terms appearing in the calculation. They follow from elementary algebra and are easily derived by exploiting the symmetry with respect to the summation indices. In Sec. III, we use

$$\sum_{i\neq m}^{N} \frac{z_i}{z_i - z_m} = \frac{1}{2}N(N-1), \tag{A1}$$

$$\sum_{\substack{i,m,n \\ i\neq j\neq m\neq i}} \frac{z_i^2}{(z_i - z_m)(z_i - z_n)} = \frac{1}{3}N(N-1)(N-2). \tag{A2}$$

In Sec. V and Appendix D, we use

$$\begin{aligned}\sum_{i\ k\neq l} \frac{z_i^2}{(z_i - w_k)(z_i - w_l)} &+ \sum_{j\neq i;k} \frac{z_i z_j - w_k(z_i - z_j)}{(z_i - w_k)(z_j - w_k)} \\ &+ \sum_{i;k\neq l} \frac{w_i^2}{(w_i - z_k)(w_i - z_l)} + \sum_{j\neq i;k} \frac{w_i w_j - z_k(w_i + w_j)}{(w_i - z_k)(w_j - z_k)} \\ &= 2N^2(N-1).\end{aligned} \tag{A3}$$

APPENDIX B: EXAMPLE FOR MONOMIAL DECOMPOSITION

In Fig. 8, we give an example of how to use the squeezing and product rule properties to obtain the $N = 4$ particle $\nu = 1/3$ Laughlin state.

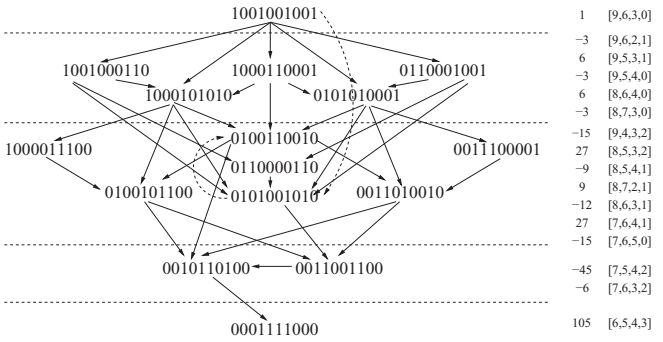

FIG. 8. Monomial decomposition of $N = 4$ particle $\nu = 1/3$ Laughlin state. The particle positions are denoted by 1. The coefficients of the Slater partitions are computed according to Eq. (10), with $\alpha = -2$. The arrows denote a squeezing relation from the upper to the lower partition. In total, there are four squeezing levels until the maximally squeezed partition is reached.

APPENDIX C: FERMIONIC LAPLACE-BELTRAMI OPERATOR

We want to rephrase the derivatives on $J^{\alpha}_{\lambda_B}$ as a derivative action on S^{α}_{λ}. This can be done in a compact form since they only differ by a Jastrow factor. The first derivative yields

$$z_i \frac{\partial}{\partial z_i} S^{\alpha}_{\lambda} = \left[z_i \frac{\partial}{\partial z_i} J^{\alpha}_{\lambda_B} \right] \prod_{k<l} (z_k - z_l) + \sum_{\substack{m \\ i\neq m}} \frac{z_i}{z_i - z_m} S^{\alpha}_{\lambda}. \tag{C1}$$

Similarly, plugging in the previous result for the first derivative, the second derivative action can be rewritten as

$$\begin{aligned}\left(z_i \frac{\partial}{\partial z_i} \right)^2 S^{\alpha}_{\lambda} &= \left[\left(z_i \frac{\partial}{\partial z_i} \right)^2 J^{\alpha}_{\lambda_B} \right] \prod_{k<l} (z_k - z_l) + \sum_{\substack{m \\ i\neq m}} \frac{2z_i^2}{z_i - z_m} \frac{\partial}{\partial z_i} S^{\alpha}_{\lambda} \\ &+ \left[\sum_{\substack{m \\ i\neq m}} \frac{z_i}{z_i - z_m} - \frac{z_i^2}{(z_i - z_m)^2} - \sum_{\substack{m,n \\ m\neq i, n\neq i}} \frac{z_i^2}{(z_i - z_m)(z_i - z_n)} \right] S^{\alpha}_{\lambda}.\end{aligned} \tag{C2}$$

APPENDIX D: HALDANE-REZAYI STATE

A. Derivative action

Starting from the CFT differential equation for the nonunitary term (17), we perform a calculation similar to the previous derivation of the fermionic LB operator. For generality, we can consider the generalized Haldane-Rezayi state

$$\Phi^{m,m,n}_{\mathrm{HR}} = \Phi_0 \prod_{i<j} (z_i - z_j)^m \prod_{i<j} (w_i - w_j)^m \prod_{i,j} (z_i - w_j)^n. \tag{D1}$$

Following a tedious, but straightforward calculation, we find

$$\Bigg[\frac{1}{2}\sum_i\left(z_i\frac{\partial}{\partial z_i}\right)^2-\frac{1}{2}\sum_i z_i\frac{\partial}{\partial z_i}-\sum_{i\neq j}(z_i+z_j)\frac{\partial}{\partial z_j}-(m-1)\sum_{j\neq i}\frac{z_i^2}{z_i-z_j}\frac{\partial}{\partial z_i}-n\sum_{i,k}\frac{z_i^2}{z_i-w_k}\frac{\partial}{\partial z_i}-\sum_{i,k}\frac{z_i^2}{z_i-w_k}\frac{\partial}{\partial w_k}$$
$$+\frac{m^2-m-2}{2}\sum_{i\neq j}\frac{z_i^2}{(z_i-z_j)^2}+\frac{n^2-n-2}{2}\sum_{k,i}\frac{z_i^2}{(z_i-w_k)^2}+\frac{n^2+m}{2}\sum_{i;k\neq l}\frac{z_i^2}{(z_i-w_k)(z_i-w_l)}$$
$$+\frac{n(m+1)}{2}\sum_{j\neq i;k}\frac{z_iz_j-w_k(z_i+z_j)}{(z_i-w_k)(z_j-w_k)}+\frac{m(m+1)}{6}N(N-1)(N-2)\Bigg]\Phi_{\mathrm{HR}}^{m,m,n}=0. \tag{D2}$$

One observes that the $m=2$, $n=2$ case is special: the three body terms, i.e. the last term in the second line and first term in the third line of (D2), have an identical coefficient equal to 3. These three-body terms reduce to a constant once the above equation for z is added to the similar equation for w. By use of (A3), this leads to the expression (18).

B. Recurrence formula

1. *Equal spin action*

From the differential equation (18), we know that under the action of the equal- and different-spin terms, the sum over Slaters contained in the HR state must yield another sum of Slaters. Thus, to deduce a decomposition formula, we enforce this condition. Without loss of generality, we can consider the operator action on a two-particle Slater state, and explicitly demand

$$\frac{z_1+z_2}{z_1-z_2}\left(z_1\frac{\partial}{\partial z_1}-z_2\frac{\partial}{\partial z_2}\right)\sum_{n_1,n_2}a_{n_1,n_2}\mathrm{sl}_{n_1,n_2}=\sum_{n_1,n_2}b_{n_1,n_2}\mathrm{sl}_{n_1,n_2}, \tag{D3}$$

where we use n_1 (n_2) as momentum indices of the two-particle same-spin Slater determinants. Working out the derivative action and expanding the expression inside the equality, we find

$$\sum_{n_1,n_2}\left[a_{n_1,n_2}(n_1-n_2)+a_{n_1+1,n_2-1}(n_1-n_2+2)\right]\left(z_1^{n_1+1}z_2^{n_2}+z_2^{n_1+1}z_1^{n_2}\right)=\sum_{m_1,m_2}\left(b_{m_1,m_2}-b_{m_1+1,m_2-1}\right)\left(z_1^{m_1+1}z_2^{m_2}+z_2^{m_1+1}z_1^{m_2}\right). \tag{D4}$$

This equation can be worked out iteratively, starting from the maximum polynomial degree. In terms of the b coefficients in the final Slater superposition, the coefficients a are given by

$$b_{\mu_1,\mu_2}=(\mu_1-\mu_2)a_{\mu_1,\mu_2}+2\sum_i a_{\mu_1+i,\mu_2-i}(\mu_1-\mu_2+2i). \tag{D5}$$

Since the interaction is pairwise, this can be directly generalized to larger particle numbers of Slaters, and finally yields expression (21).

We now illustrate that the degree of the Jastrow factor in the polynomial determines whether the action of Eq. (D3) causes fractions or not. As an example, we consider the $\mathrm{sl}_{3,0}=z_1^3-z_2^3$ two-particle Slater determinant. The action of the operator in Eq. (D3) gives

$$\frac{z_1+z_2}{z_1-z_2}\left(z_1\frac{\partial}{\partial z_1}-z_2\frac{\partial}{\partial z_2}\right)\mathrm{sl}_{3,0}=3\frac{z_1+z_2}{z_1-z_2}\left(z_1^3+z_2^3\right), \tag{D6}$$

which can not be decomposed into polynomials without fractions. As stated before, this case does not occur for the HR state, since there is an equal-spin Jastrow factor of second power. Let us now consider the polynomial that is constructed from $\mathrm{sl}_{3,0}$ times a Jastrow factor:

$$\frac{z_1+z_2}{z_1-z_2}\left(z_1\frac{\partial}{\partial z_1}-z_2\frac{\partial}{\partial z_2}\right)\mathrm{sl}_{3,0}(z_1-z_2)$$
$$=\frac{z_1+z_2}{z_1-z_2}\left[3\left(z_1^3+z_2^3\right)(z_1-z_2)+z_1z_2\left(z_1^2-z_2^2\right)+z_1^4-z_2^4\right]. \tag{D7}$$

Since all terms from above cancel the z_1-z_2 fraction, the polynomial composed out of a Slater times a Jastrow factor gives no fractions upon the action of the operator in Eq. (D3). This holds for any higher power of Jastrow factors multiplied with the Slater determinant.

2. *Interspin action*

To compute the action on the interspin term, we again constrain ourselves to a two-particle mixed-spin Slater partition $z_1^nw_1^m+z_1^mw_1^n$, and assume $n\geqslant m$ (we remember that the $m\geqslant n$ Slater has the same coefficient):

$$\frac{z_1+w_1}{z_1-w_1}\left(z_1\frac{\partial}{\partial z_1}-w_1\frac{\partial}{\partial w_1}\right)\left(z_1^nw_1^m+z_1^mw_1^n\right)$$
$$=(n-m)\left(z_1^nw_1^m+z_1^mw_1^n\right)$$

$$+2(n-m)\sum_{i=1}^{(m-n)/2-1}\left(z_1^{n-i}w_1^{m+i}+z_1^{m+i}w_1^{n-i}\right)$$
$$+2(n-m)z_1^{(n+m)/2}w_1^{(n+m)/2}. \quad \text{(D8)}$$

For $(n-m)\equiv 1 \bmod 2$, the upper limit of the sum is a half-integer. In this case, the sum is evaluated as an analytical extension from an integer to a half-integer upper limit. To each of these Slaters, we also add the counterpart $n \leftrightarrow m$. To prevent double counting, whenever $m=n$, the counterpart must not be added. In the language of single Slaters, this demands the division of the equal-momentum Slater prefactor by 2 as compared to other Slaters. For the general particle case, this yields the following recipe: Sum over all combinations of pairs (i,k) with one coordinate from the $\uparrow$ spin partition and one from the $\downarrow$ spin partition. If $\lambda_i > \lambda_k$ ($n > m$ as above), take the prefactor as given in (D8). If $\lambda_i < \lambda_k$, change $\uparrow \leftrightarrow \downarrow$ and add it (the spin-rotated Slater must have the same coefficient). Produce all allowed squeezings and add the Slater terms. If one pair (i,k) and some squeezed partition produces an $n=m$ state (i.e., a doubly occupied single-particle state), and the remainder $\uparrow$ spin and $\downarrow$ spin partition are equivalent as well, this factor must be divided by two (or rather, in our case, changed from 2 to 1). We then obtain Eq. (22).

APPENDIX E: EXAMPLE FOR SPINFUL SLATER DECOMPOSITION

In Fig. 9, we give an example of the HR state decomposed into squeezed partition from the root partition. We show the full decomposition of the $N=4$ HR state into spinful partitions squeezed from $X000X$. We observe an example of a partition squeezed from $X000X$ that has zero coefficient, i.e., u0X0d (d $\equiv\downarrow$ and u $\equiv\uparrow$): it can not be constructed from $P_1 = X0X$ times 101 for $\uparrow$ spins and 101 for $\downarrow$ spins.

APPENDIX F: HALDANE-REZAYI STATE, CFT ANALYSIS

As mentioned before in Sec. V, the CFT corresponding to the HR state is a logarithmic theory that contains two fields 1 and $\tilde{1}$ with vanishing conformal dimension.[87] This particularity

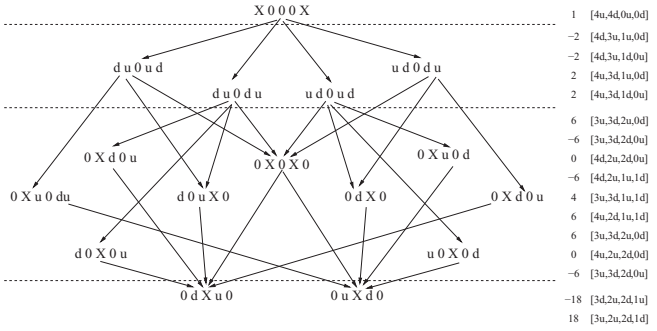

FIG. 9. Spinful Slater decomposition of the $N=4$ Haldane-Rezayi state. The notation corresponds to $X=\uparrow\downarrow$ on the same orbital, and $u\equiv\uparrow$, $d\equiv\downarrow$. The spinful partitions are written in spin-mixed fashion and ordered with respect to momentum. The coefficients are computed corresponding to Eq. (28). The two appearing zero-weight partitions are a manifestation of the zero-weight rule according to Eq. (29). The arrows denote that a given partition can be squeezed from the partition above. There are three squeezing levels until the maximally squeezed partitions are reached.

makes the CFT treatment more complicated. Fortunately, the factorization can already be obtained from the explicit form of the correlation function

$$\langle\partial\theta(z_1)\partial\bar{\theta}(w_1)\cdots\partial\theta(z_N)\partial\bar{\theta}(w_N)\tilde{I}\rangle = \sum_\sigma \epsilon(\sigma)\prod_i \frac{1}{(z_i-w_{\sigma(i)})^2}. \quad \text{(F1)}$$

The right-hand side trivially obeys the following asymptotic behavior as N_B up spins and N_B down spins are taken to infinity:

$$\langle\partial\theta(z_1)\partial\bar{\theta}(w_1)\cdots\partial\bar{\theta}(w_{N_A})\partial\theta(\gamma z_{N_A+1})\cdots\partial\bar{\theta}(\gamma w_{N_A+N_B})\rangle \sim \gamma^{-2N_B}\langle\partial\theta(z_1)\cdots\partial\bar{\theta}(w_{N_A})\rangle\langle\partial\theta(z_{N_A+1})\cdots\partial\bar{\theta}(w_{N_A+N_B})\rangle. \quad \text{(F2)}$$

This yields the product rule (for a cut through an empty orbital) using the same steps as in the spin-polarized case. Cuts through an occupied orbital can be analyzed in a similar way by looking at the asymptotic behavior as N_B up spins and N_B+1 down spins are sent to infinity.

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