Polarization dependence of Landau parameters for normal Fermi liquids in two dimensions

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We examine the polarization dependence of the Landau parameters for normal Fermi liquids in two dimensions in the limit of low densities. We obtain the ground-state energy to second order in the s-wave and p-wave low-energy T-matrix components. By functional differentiation, the Landau parameters, $f^{\uparrow\uparrow}, f^{\uparrow\downarrow}, f^{\downarrow\downarrow}$, are obtained exactly and analytically as functions of the occupation of each Fermi sea. We generalize the expressions for the state-dependent effective masses, specific heat, compressibility, spin susceptibility, and zero sound for a two-dimensional Fermi liquid at arbitrary polarization. We then apply the theory to the thin film limit of ^{3}He. The ^{3}He s-wave and p-wave T-matrix components at a density of 0.026 Å^{-2} were determined by fitting zero polarization measurements of the effective mass and spin susceptibility. We find $g_0 = 0.76$ and $g_1 = 1.89$ Å^2. The value for g_0 was tested using scattering theory and a first-principles, microscopic particle-hole interaction. Using these interaction components, we calculate predictions for the polarization dependence of the ^{3}He state-dependent effective masses, specific heat, compressibility, spin susceptibility, and zero sound. We find that the effective masses should be much smaller in the limit of full polarization. We compare our zero polarization first sound speed to that obtained from a small-k analysis of a first-principles structure factor and find reasonable agreement. Using the present technique for analyzing the Landau kinetic equation, we find that zero sound should be stable at all polarizations, and the speed should monotonically increase with increasing polarization. However, spin zero sound is probably not stable at any polarization. We show that the spin susceptibility decreases with increasing polarization but in two-dimensions it does not vanish in the limit of full polarization. Finally, we discuss the forward-scattering sum rule and conclude that it is not applicable to this sort of perturbation expansion.

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I. INTRODUCTION

In the mid-1950s, Landau developed a first-principles theory for the class of quantum fluids called normal Fermi liquids. In particular, he showed[1] that the low-temperature collective excitations and thermodynamic properties could be encoded in a few parameters, the Landau parameters, and that these parameters were related to a certain limiting value of the microscopic scattering function.[2] The theory has been applied with great success to numerous many-fermion systems including ^{3}He [Ref. 3] and atomic nuclei.[4]

From the very beginning, there was strong interest in the application of Landau's theory to polarized Fermi liquids.[5] A rigorous discussion of Fermi liquid theory for polarized systems was given by Bedell and Quader in the mid-1980s.[6] This latter work culminated in the 1989 study of polarized bulk liquid ^{3}He by Sanchez-Castro, Bedell, and Wiegers.[7] In this paper, expressions for the thermodynamic properties and collective excitations for three-dimensional ^{3}He as a function of polarization were derived and were then evaluated using a phenomenological model for the Landau parameters.

In the 1970s, experiments on ^{3}He adsorbed onto exfoliated graphite stimulated work on Fermi liquid theory for two-dimensional systems.[8,9] In particular, Bloom[15] adapted Galitskii's approach[10] for the three-dimensional Fermi gas to the two-dimensional Fermi gas, and evaluated the resulting principal value integrals numerically. In 1990, P. W. Anderson, examining possible models for high-T_c superconductivity, suggested[11] that the ground-state of the two-dimensional Hubbard model may be Luttinger-liquid type rather than a normal Fermi liquid. Anderson's argument would not exclude the low-density Fermi gas in two dimensions from also having a Luttinger-liquid ground state rather than a normal Fermi liquid. Subsequently, Engelbrecht and Randeria[12] re-examined the low-density Fermi gas in two dimensions and argued that their analysis showed no breakdown in Fermi liquid theory. In 1992, Engelbrecht, Randeria, and Zhang[13] (ERZ) obtained an analytic solution for the s-wave contribution to the low-density, unpolarized Fermi gas in two dimensions. The exact solution by ERZ provided some corrections to the previous numerical results of Bloom. As stressed by ERZ the key to obtaining analytic results in two dimensions is the constant density of states.

The perturbation theory approach used by ERZ was first applied to the low-density Fermi gas in three dimensions by Abrikosov and Khalatnikov.[14] In this paper, we generalize the results of ERZ to include the p-wave contributions to the low-density, two-dimensional Fermi gas Landau parameters. This will enable us to calculate the polarization dependence of the thermodynamic parameters and also the collective excitations. In Sec. II we utilize second-order perturbation theory to compute the Landau parameters. The perturbation theory interactions are expanded to include s-wave and p-wave contributions. The details of the integrations are presented in the appendixes. In Sec. III we derive expressions for the polarization-dependent effective masses, heat capacity, compressibility, spin susceptibility, zero-sound and spin-zero-sound collective excitations. We also derive the forward-scattering sum rules at nonzero polarization. Section IV is the application of the polarization-dependent Landau theory to the system of thin ^{3}He films. We utilize existing experimental data to determine the values of the s-wave and p-wave interaction components. Using scattering theory and a first-principles particle-hole effective interaction, we show that the value of the s-wave interaction parameter is consistent with the range of the effective interaction. Finally, we compute the polarization dependence of the thermodynamic quantities and the collective excitations for a ^{3}He film of density 0.026 Å^{-2}.

II. GROUND STATE AND EXCITATIONS

A. Perturbation theory

We examine a system of N spin-$\frac{1}{2}$ fermions in a box of area L^2. The particles have bare mass m and interact with two-body potential $V(r)$ that is assumed to depend only on the scalar distance between the particles. The particles fill two Fermi seas up to Fermi momenta $k_\uparrow$ and $k_\downarrow$, and we introduce the convention that the spin-down Fermi sea will always be the minority Fermi sea in the case of nonzero polarization. In second-order of perturbation theory, the ground-state energy can be written[15]

$$\begin{aligned} E =& \frac{1}{2}\sum_{\mathbf{p},\mathbf{q},\sigma}[V(0)-V(2p)]\left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\right)\\ &+\frac{1}{2}\sum_{\mathbf{p},\mathbf{q}}V(0)\left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}+n_{\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\right)\\ &+\frac{1}{2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q},\sigma}\frac{|V(\mathbf{p}-\mathbf{p}')|^2-V(\mathbf{p}-\mathbf{p}')V(-\mathbf{p}-\mathbf{p}')}{\Delta T}\\ &\times\left[n_{\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\left(1-n_{\mathbf{p}'+\frac{\mathbf{q}}{2},\sigma}\right)\left(1-n_{-\mathbf{p}'+\frac{\mathbf{q}}{2},\sigma}\right)\right]\\ &+\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{|V(\mathbf{p}-\mathbf{p}')|^2}{\Delta T}\left[n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\left(1-n_{\mathbf{p}'+\frac{\mathbf{q}}{2},\uparrow}\right)\right.\\ &\left.\times\left(1-n_{-\mathbf{p}'+\frac{\mathbf{q}}{2},\downarrow}\right)\right], \end{aligned} \quad (2.1)$$

where the kinetic energy denominators are given by

$$\Delta T=\frac{\hbar^2}{m}(p^2-p'^2). \quad (2.2)$$

The spin variable $\sigma=\uparrow,\downarrow$ and the $n_{k,\sigma}$ are the noninteracting distribution functions, equal to 1 for $k<k_\sigma$ and 0 for $k>k_\sigma$. The potential function $V(p)$ is the Fourier transform of some local two-body interaction $V(r)$ as defined by the box normalized form:

$$V(p)=\frac{1}{L^2}\int d\mathbf{r}\,V(r)e^{i\mathbf{p}\cdot\mathbf{r}}. \quad (2.3)$$

As pointed out by Abrikosov and Khalatnikov[14] (AK), when the numerators of the two last terms in Eq. (2.1) are symmetric with respect to interchange of p and p' then the quartic contributions in $n_{p,\sigma}$ must vanish. This is the case here and so in what follows we can neglect the quartic terms. Thus, the energy can be written

$$E=E^{(2)}+E^{(3)}, \quad (2.4)$$

where the quadratic and cubic components are defined by

$$\begin{aligned} E^{(2)} =& \frac{1}{2}\sum_{\mathbf{p},\mathbf{q},\sigma}[V(0)-V(2p)]\left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\right)\\ &+\frac{1}{2}\sum_{\mathbf{p},\mathbf{q}}V(0)\left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}+n_{\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\right)\\ &+\frac{1}{2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q},\sigma}\frac{|V(\mathbf{p}-\mathbf{p}')|^2-V(\mathbf{p}-\mathbf{p}')V(-\mathbf{p}-\mathbf{p}')}{\Delta T}\\ &\times\left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\right)+\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{|V(\mathbf{p}-\mathbf{p}')|^2}{\Delta T}\\ &\times\left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\right), \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} E^{(3)} =& -\frac{1}{2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q},\sigma}\frac{|V(\mathbf{p}-\mathbf{p}')|^2-V(\mathbf{p}-\mathbf{p}')V(-\mathbf{p}-\mathbf{p}')}{\Delta T}\\ &\times\left[n_{\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\left(n_{\mathbf{p}'+\frac{\mathbf{q}}{2},\sigma}+n_{-\mathbf{p}'+\frac{\mathbf{q}}{2},\sigma}\right)\right]\\ &-\frac{1}{2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{|V(\mathbf{p}-\mathbf{p}')|^2}{\Delta T}\left[n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\left(n_{\mathbf{p}'+\frac{\mathbf{q}}{2},\uparrow}\right.\right.\\ &\left.\left.+n_{-\mathbf{p}'+\frac{\mathbf{q}}{2},\downarrow}\right)\right]. \end{aligned} \quad (2.6)$$

The low-density expansion of the energy can now be carried out. Using the notation of Randeria, Duan, and Shieh,[16] the momentum interaction of Eq. (2.3) can be written

$$V(\mathbf{k}-\mathbf{m})=\langle\mathbf{k}|V|\mathbf{m}\rangle\equiv\frac{1}{L^2}\int d\mathbf{r}e^{i\mathbf{k}\cdot\mathbf{r}}V(r)e^{-i\mathbf{m}\cdot\mathbf{r}}. \quad (2.7)$$

The plane waves can be expanded using

$$e^{ix\cos\theta}=\sum_{\ell=-\infty}^{\infty}i^\ell e^{i\ell\theta}J_\ell(x), \quad (2.8)$$

where the J_ℓ's are integer order Bessel functions. Substituting into (2.7) and carrying out the angular integration yields

$$\langle\mathbf{k}|V|\mathbf{m}\rangle=\sum_{\ell=0}^{\infty}\alpha_\ell\cos(\ell\theta_{km})V_{km}^{(\ell)}, \quad (2.9)$$

where we have defined

$$V_{km}^{(\ell)}=\frac{2\pi}{L^2}\int_0^\infty dr\,r\,J_\ell(kr)V(r)J_\ell(mr), \quad (2.10a)$$

$$\theta_{km}=\theta_{\mathbf{k}}-\theta_{\mathbf{m}}, \quad (2.10b)$$

and

$$\alpha_\ell=\begin{cases}1 & \text{if} \quad \ell=0,\\ 2 & \text{if} \quad \ell\geqslant 1.\end{cases} \quad (2.10c)$$

The terms $T_\ell(\cos\theta)\equiv\cos(\ell\theta)$ are Chebyshev polynomials of the first kind and will be discussed in more detail in Sec. III.

The low-density theory is obtained by truncating the series Eq. (2.9) after the $\ell=1$ term and then taking the low-density limits $kr\ll 1$ in the Bessel functions:

$$V(\mathbf{k}-\mathbf{m})\approx V_0+2\cos(\theta_{km})kmV_1, \quad (2.11)$$

where the s- and p-wave low-density potential parameters are defined by

$$\lim_{k,m\to 0}V_{km}^{(0)}\equiv V_0=\frac{2\pi}{L^2}\int_0^\infty dr\,r\,V(r), \quad (2.12a)$$

$$\lim_{k,m\to 0}V_{km}^{(1)}\equiv kmV_1=\frac{km}{4}\frac{2\pi}{L^2}\int_0^\infty dr\,r^3V(r). \quad (2.12b)$$

Thus, we find

$$V(\pm\mathbf{p}-\mathbf{p}') = V_0 \pm 2\cos(\theta_{pp'})pp'V_1. \tag{2.13}$$

If one substitutes Eq. (2.13) into $E^{(2)}$ one immediately finds divergences in the quadratic terms. This was first noticed by AK in three dimensions and was discussed in detail by ERZ for two dimensions. The divergences can be removed by replacing the bare interaction by an effective interaction, the two-particle T matrix.

B. Removing the divergence

The Lippman-Schwinger equation for the T matrix is

$$T_{pp'}(E) = V_{pp'} + \sum_{\mathbf{k}} V_{pk}G_0(E)T_{kp'}(E), \tag{2.14}$$

where $T_{pp'}(E) = \langle \mathbf{p},\sigma|T(E)|\mathbf{p}',\sigma'\rangle$ with the spin indices suppressed, $G_0(E)$ is the single-particle propagator

$$G_0(E) = \left(\frac{1}{E-2\epsilon_k^0+i\eta}\right), \tag{2.15}$$

and $\epsilon_k^0 \equiv \hbar^2k^2/2m$. Introducing an angular decomposition of the T matrix equivalent to Eq. (2.9) yields

$$T_{pp'}^{(\ell)} = V_{pp'}^{(\ell)} + L^2\int_0^\infty \frac{dk}{2\pi}kV_{pk}^{(\ell)}\left(\frac{1}{E-2\epsilon_k^0+i\eta}\right)T_{kp'}^{(\ell)}. \tag{2.16}$$

Inverting the low-density $\ell = 0,1$ equations and then truncating to second-order in the T-matrix components yields

$$V_0 = \tau_0(1-\mathbb{P}_0\tau_0), \tag{2.17a}$$
$$V_1 = \tau_1(1-\mathbb{P}_1\tau_1), \tag{2.17b}$$

where, basically following the notation of ERZ, we have defined the low-energy limits of the T matrices,

$$\lim_{E\to 0} T_{pp'}^{(0)} = \tau_0, \tag{2.18a}$$
$$\lim_{E\to 0} T_{pp'}^{(1)} = pp'\tau_1, \tag{2.18b}$$

and the integrals over the propagators are the formally infinite quantities

$$\mathbb{P}_0 = \frac{L^2}{2\pi}\int_0^\infty dk\frac{k}{E-2\epsilon_k^0+i\eta}, \tag{2.19a}$$
$$\mathbb{P}_1 = \frac{L^2}{2\pi}\int_0^\infty dk\frac{k^3}{E-2\epsilon_k^0+i\eta}. \tag{2.19b}$$

Substituting Eqs. (2.17) into $E^{(2)}$ yields

$$E^{(2)} = \frac{1}{2}\sum_{\mathbf{p},\mathbf{q},\sigma}\left[4p^2\tau_1 + 4p^2\tau_1^2\left(\sum_{\mathbf{p}'}\frac{(p')^2}{\Delta T}-\mathbb{P}_1\right)\right] \times \left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\right) + \sum_{\mathbf{p},\mathbf{q}}\left[(\tau_0+2p^2\tau_1) + \tau_0^2\left(\sum_{\mathbf{p}'}\frac{1}{\Delta T}-\mathbb{P}_0\right) + 2p^2\tau_1^2\left(\sum_{\mathbf{p}'}\frac{(p')^2}{\Delta T}-\mathbb{P}_1\right)\right] \times \left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\right). \tag{2.20}$$

The real parts of $\mathbb{P}_0$ and $\mathbb{P}_1$ are defined as Cauchy principal values. The contribution of $E^{(2)}$ to the Landau parameters is

$$\delta f_{kk'}^{(2)\sigma\sigma'} = \frac{\delta^2E^{(2)}}{\delta n_{k,\sigma}\delta n_{k',\sigma'}}, \tag{2.21}$$

where in the following we shall treat as associated pairs of variables $(\mathbf{k},\sigma)$ and $(\mathbf{k}',\sigma')$. The δ functions generated by the functional derivatives fix $\mathbf{p} = \frac{1}{2}(\mathbf{k}-\mathbf{k}')$ and $\mathbf{q} = \mathbf{k}+\mathbf{k}'$. Using (2.2), the denominators can be written: $\Delta T_{\sigma,\sigma'} = \frac{1}{2}\epsilon^0_{\mathbf{k}-\mathbf{k}'} - 2\epsilon^0_{p'}$, where we have defined $\epsilon^0_{\mathbf{k}} = \hbar^2k^2/2m$. By inspection of Eq. (2.20), choosing state-dependent energy parameters

$$E_{\sigma,\sigma'} = \frac{1}{2}\epsilon^0_{\mathbf{k}-\mathbf{k}'} = \frac{1}{2}\left(\epsilon^0_{\mathbf{k}\sigma}+\epsilon^0_{\mathbf{k}'\sigma'}\right) - \frac{\hbar^2}{2m}\mathbf{k}\cdot\mathbf{k}', \tag{2.22}$$

causes the divergent terms to cancel exactly. In the unpolarized limit, we have $E = \epsilon^0_{\mathrm{F}}(1-\cos\theta_{kk'})$. In the fully polarized limit, we have $E = \frac{1}{2}\epsilon^0_{\mathrm{F}\uparrow}$. We note that the former is different from the choice of energy parameter made by ERZ. However, we point out that for the s-state the divergence is removed for any nonzero value of E; however, for a choice other than that above there remains a residual contribution to the Landau parameters that is logarithmic in E. For the p state this flexibility disappears because of the coefficients of the log terms generated by the divergent integrals and the only choice that one has to remove the divergence is the above $E_{\sigma,\sigma'}$. We note that the divergence of concern is at the upper limit of the integrals. The pole along the axis of integration has zero principal value.

This leaves

$$E^{(2)} = \sum_{\mathbf{p},\mathbf{q},\sigma} 2p^2\tau_1\left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\sigma}\right) + \frac{1}{2}\sum_{\mathbf{p},\mathbf{q}}(\tau_0+2p^2\tau_1) \times \left(n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow} + n_{\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\, n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\right). \tag{2.23}$$

The contributions to the Landau parameters from quadratic order are therefore

$$\begin{aligned}
\delta f_{kk'}^{(2)\uparrow\uparrow} &= 4\sin^2(\theta_{k,k'}/2)k_\uparrow^2\tau_1,\\
\delta f_{kk'}^{(2)\downarrow\downarrow} &= 4\sin^2(\theta_{k,k'}/2)k_\downarrow^2\tau_1,\\
\delta f_{kk'}^{(2)\uparrow\downarrow} &= \tau_0 + \tfrac{1}{2}(\mathbf{k}-\mathbf{k}')^2\tau_1.
\end{aligned} \tag{2.24}$$

C. Landau parameters

The Landau parameters $f_{kk'}^{\sigma\sigma'}$ are generated from Eq. (2.4):

$$f_{kk'}^{\sigma\sigma'} = \frac{\delta^2E}{\delta n_{k,\sigma}\delta n_{k',\sigma'}}. \tag{2.25}$$

Taking the functional derivatives and using symmetry arguments to simplify the expressions, we find

$$f_{kk'}^{\uparrow\uparrow} = 4\sin^2(\theta_{k,k'}/2)k_\uparrow^2\tau_1 - 16\frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{[pp'\cos(\theta_{pp'})]^2\tau_1^2}{p^2-p'^2}\left[\delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}n_{\mathbf{p}'+\frac{\mathbf{q}}{2},\uparrow} + 2\delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}\delta_{\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}}n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}\right]$$
$$-\frac{2m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{\left[\tau_0^2+4pp'\cos(\theta_{pp'})\tau_0\tau_1+4(pp'\cos(\theta_{pp'}))\right]^2\tau_1^2}{p^2-p'^2}\left[\delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}\delta_{\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}}n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\right]. \tag{2.26}$$

The Landau parameter $f_{kk'}^{\downarrow\downarrow}$ is obtained from Eq. (2.26) by reversing all the spins. We note again that in the following, as a matter of convention, we consider the spin-down Fermi sea as the minority Fermi sea when the polarization is nonzero. Finally, we have

$$f_{kk'}^{\uparrow\downarrow} = \tau_0 + \frac{1}{2}(\mathbf{k}-\mathbf{k}')^2\tau_1 - \frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{\tau_0^2+4pp'\cos(\theta_{pp'})\tau_0\tau_1+4[pp'\cos(\theta_{pp'})]^2\tau_1^2}{p^2-p'^2}$$
$$\times\left[\delta_{\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow} + \delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{\mathbf{p}'+\frac{\mathbf{q}}{2},\uparrow} + \delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{-\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow} + \delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{-\mathbf{p}'+\frac{\mathbf{q}}{2},\downarrow}\right]. \tag{2.27}$$

The integrations needed to evaluate Eqs. (2.26) and (2.27) are rather involved and so have been moved to Appendices A–D . It is usual to use dimensionless forms for the parameters by multiplying the $f_{kk'}^{\sigma\sigma'}$'s by a density of states.[1] In the following we denote these dimensionless parameters by $\tilde{F}$, that is, $\tilde{F}_{kk'}^{\sigma\sigma'} = \tilde{N}_0 f_{kk'}^{\sigma\sigma'}$, where $\tilde{N}_0 = mL^2/2\pi\hbar^2$ is the *bare* single-spin-state density of states. We introduce this tilde notation because it is usual to use an actual density of states in this definition. Our dimensionless Landau parameters thus differ from those introduced in Ref. 13 by the use of the bare mass instead of the effective mass and also by a factor of two. This choice is simply a matter of notational convenience and no physics depends on it. We also use the $\tilde{N}_0$'s to redefine the T-matrix parameters τ_0 and τ_1:

$$g_0 = \tilde{N}_0\tau_0, \tag{2.28a}$$
$$g_1 = \tilde{N}_0\tau_1. \tag{2.28b}$$

We note that with this definition, g_0 is dimensionless, whereas g_1 has the dimensions of length squared.

The final results for the Landau parameters are

$$\tilde{F}_{kk'}^{\uparrow\uparrow} = 4k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)g_1 + \left[1-\sqrt{1-\frac{k_\downarrow^2}{k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)}}\,\Theta\left(k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)\geqslant k_\downarrow^2\right)\right]g_0^2 + \left[\left(\left(1-\frac{16}{3}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^2+k_\downarrow^2\right)\right.$$
$$-\sqrt{1-\frac{k_\downarrow^2}{k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)}}\left.\left(\left(1-\frac{20}{3}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^2-\frac{1}{3}k_\downarrow^2\right)\Theta\left(k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)\geqslant k_\downarrow^2\right)\right]g_0g_1$$
$$+\left[\left(25-108\sin^2\left(\frac{\theta_{kk'}}{2}\right)+\frac{2240}{15}\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right)\frac{k_\uparrow^4}{4}+\frac{k_\downarrow^4}{4}+\left(1-3\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^2k_\downarrow^2\right.$$
$$-\sqrt{1-\frac{k_\downarrow^2}{k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)}}\left[\left(1-12\sin^2\left(\frac{\theta_{kk'}}{2}\right)+\frac{448}{15}\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right)\frac{k_\uparrow^4}{4}+\frac{k_\downarrow^4}{20}+\left(\frac{1}{2}-\frac{19}{15}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^2k_\downarrow^2\right]$$
$$\left.\times\Theta\left(k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)\geqslant k_\downarrow^2\right)+k_\uparrow^4\tan^2\left(\frac{\theta_{kk'}}{2}\right)\left\{(1+2\cos\theta_{kk'})-\left[\cos\theta_{kk'}+\sin^2\theta_{kk'}\ln\left(\tan\left(\frac{\theta_{kk'}}{2}\right)\right)\right]\right\}\right]g_1^2. \tag{2.29}$$

The quantities $\Theta(x)$ are generalized step functions and are introduced in Appendix A. They are defined such that $\Theta(x) = 1$ if x is true and 0 if x is not true. In the limit of zero polarization, we simply have $\tilde{F}_{kk'}^{\downarrow\downarrow} = \tilde{F}_{kk'}^{\uparrow\uparrow}$ and $\tilde{F}_{kk'}^{\downarrow\downarrow}$ is obtained from (2.29) by reversing all the spins. In the following, we assume finite polarization and $k_\downarrow < k_\uparrow$:

$$\tilde{F}_{kk'}^{\downarrow\downarrow} = 4k_\downarrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)g_1 + g_0^2 + \left[\left(1-\frac{16}{3}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\downarrow^2+k_\uparrow^2\right]g_0g_1 + \left[\left(25-108\sin^2\left(\frac{\theta_{kk'}}{2}\right)+\frac{2240}{15}\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right)\frac{k_\downarrow^4}{4}\right.$$
$$\left.+\frac{k_\uparrow^4}{4}+\left(1-3\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\downarrow^2k_\uparrow^2+k_\downarrow^4\tan^2\left(\frac{\theta_{kk'}}{2}\right)\left\{(1+2\cos\theta_{kk'})-\left[\cos\theta_{kk'}+\sin^2\theta_{kk'}\ln\left(\tan\left(\frac{\theta_{kk'}}{2}\right)\right)\right]\right\}\right]g_1^2. \tag{2.30}$$

$$
\begin{aligned}
\tilde{F}^{\uparrow\downarrow}_{kk'} &= g_0 + \frac{1}{2}|\vec{k} - \vec{k}'|^2 g_1 + \left[\left(\frac{k_\uparrow^2 - k_\uparrow k_\downarrow \cos\theta_{kk'}}{|\mathbf{k}-\mathbf{k}'|^2}\right) + \left(\frac{k_\downarrow^2 - k_\downarrow k_\uparrow \cos\theta_{kk'}}{|\mathbf{k}-\mathbf{k}'|^2}\right) - \ln\left(\frac{2p}{q}\right)\right] g_0^2 \\
&+ \left[-\left(\frac{2p}{q}\right)\cos\theta_{pq}(k_\uparrow^2 - k_\downarrow^2) + (k_\uparrow^2 - k_\uparrow k_\downarrow \cos\theta_{kk'})\left[1 - \frac{8}{3}\frac{(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})^2}{|\vec{k} - \vec{k}'|^4}\right]\right. \\
&+ \left.\frac{(k_\downarrow^2 - k_\downarrow k_\uparrow \cos\theta_{kk'})}{|\vec{k} - \vec{k}'|^2}\left(k_\downarrow^2 + (k_\uparrow^2 - 2k_\uparrow k_\downarrow \cos\theta_{kk'}) - \frac{8}{3}\frac{(k_\downarrow^2 - k_\downarrow k_\uparrow\cos\theta_{kk'})^2}{|\vec{k} - \vec{k}'|^2}\right)\right] g_0 g_1 \\
&+ \left[\frac{(k_\uparrow^2 - k_\uparrow k_\downarrow \cos\theta_{kk'})}{|\vec{k} - \vec{k}'|^2}\frac{k_\uparrow^2}{2}\left[(k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'}) - \frac{k_\uparrow^2}{|\vec{k} - \vec{k}'|^2}(k_\uparrow^2 - 2k_\uparrow k_\downarrow \cos\theta_{kk'} + k_\downarrow^2\cos 2\theta_{kk'}) - \frac{k_\uparrow^2}{2}\right]\right. \\
&+ \frac{(k_\downarrow^2 - k_\uparrow k_\downarrow \cos\theta_{kk'})}{|\vec{k} - \vec{k}'|^2}\frac{k_\downarrow^2}{2}\left[(k_\uparrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'}) - \frac{k_\downarrow^2}{|\vec{k} - \vec{k}'|^2}(k_\downarrow^2 - 2k_\uparrow k_\downarrow \cos\theta_{kk'} + k_\uparrow^2\cos 2\theta_{kk'}) - \frac{k_\downarrow^2}{2}\right] \\
&+ \frac{\frac{1}{4}[(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})(k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'})^2] + \frac{1}{4}[(k_\downarrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})(k_\uparrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'})^2]}{|\vec{k} - \vec{k}'|^2} \\
&+ \left[(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})^3\left[-\frac{4}{3}(k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'}) + \frac{k_\uparrow^2}{|\vec{k} - \vec{k}'|^2}\left(k_\uparrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'} + k_\downarrow^2\left(\cos 2\theta_{kk'} + \frac{2}{3}\sin^2\theta_{kk'}\right)\right)\right]\right. \\
&+ \left.(k_\downarrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})^3\left[-\frac{4}{3}(k_\uparrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'}) + \frac{k_\downarrow^2}{|\vec{k} - \vec{k}'|^2}\left(k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'} + k_\uparrow^2\left(\cos 2\theta_{kk'} + \frac{2}{3}\sin^2\theta_{kk'}\right)\right)\right]\right] \\
&\times \frac{1}{|\vec{k} - \vec{k}'|^4} + \frac{4}{5}\frac{(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})^5 + (k_\downarrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})^5}{|\vec{k} - \vec{k}'|^6} \\
&- \left(\frac{4p^2}{q^2}\right)\cos(2\theta_{\mathrm{p,q}})\left(\frac{1}{4}(k_\uparrow^4 + k_\downarrow^4) + \left(-p^2 + \frac{q^2}{4}\right)\frac{1}{2}(k_\uparrow^2 + k_\downarrow^2)\right) \\
&+ \left.\left[4p^2\cos^2(\theta_{pq})\frac{1}{2}(k_\uparrow^2 + k_\downarrow^2) + \left(-p^2 + \frac{q^2}{4}\right)\frac{p^2}{2}\left(1 + \frac{4p^2}{q^2}\right)\cos(2\theta_{pq}) - (2p^4)\ln\left(\frac{2p}{q}\right)\right]\right] g_1^2, \qquad (2.31)
\end{aligned}
$$

where $|\vec{k} - \vec{k}'|^2 = (k_\uparrow^2 + k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'})$, $\quad 2\vec{p} = \vec{k} - \vec{k}'$, and $\vec{q} = \vec{k} + \vec{k}'$.

D. Unpolarized and fully polarized limits

We can now specialize these results to the unpolarized limit $\tilde{F}^{\sigma\sigma'}_{kk'}(0)$ with the magnetization $M/N = (N_\uparrow - N_\downarrow)/N = 0$, and the fully polarized limit $\tilde{F}^{\sigma\sigma'}_{kk'}(1)$ with $M/N = 1$. For the unpolarized limit, we set $k = k' = k_{\mathrm{F}}$:

$$
\begin{aligned}
\tilde{F}^{\uparrow\uparrow}_{kk'}(M/N = 0) = \tilde{F}^{\downarrow\downarrow}_{kk'}(M/N = 0) &= 4\sin^2\left(\frac{\theta_{kk'}}{2}\right)k_F^2 g_1 + g_0^2 + 2\left(1 - \frac{8}{3}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_{\mathrm{F}}^2 g_0 g_1 + \left[\left(30 - 120\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right.\right. \\
&+ \left.\frac{448}{3}\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right) + 4\tan^2\left(\frac{\theta_{kk'}}{2}\right)\left[(1 + 2\cos\theta_{kk'}) - \left(\cos\theta_{kk'} + \sin^2\theta_{kk'}\ln\left(\tan\left(\frac{\theta_{kk'}}{2}\right)\right)\right)\right]\left.\vphantom{\frac{1}{1}}\right]\frac{k_{\mathrm{F}}^4}{4}g_1^2, \qquad (2.32)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}^{\uparrow\downarrow}_{kk'}(M/N = 0) &= g_0 + 2\sin^2\left(\frac{\theta_{kk'}}{2}\right)k_F^2 g_1 + \left(1 - \ln\left(\frac{2p}{q}\right)\right)g_0^2 + \frac{4}{3}\sin^2\left(\frac{\theta_{kk'}}{2}\right)k_{\mathrm{F}}^2 g_0 g_1 + \left[8\left(\frac{2}{5} - \ln\left(\frac{2p}{q}\right)\right)\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right. \\
&+ \left. 2\tan^2\left(\frac{\theta_{kk'}}{2}\right)(1 + 2\cos\theta_{kk'}) - 2\cos\theta_{kk'}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\left(1 + \tan^2\left(\frac{\theta_{kk'}}{2}\right)\right)\right]\frac{k_{\mathrm{F}}^4}{4}g_1^2. \qquad (2.33)
\end{aligned}
$$

For the fully polarized limit, we set $k_\uparrow = k_{\mathrm{F}}, k_\downarrow = 0$:

$$
\begin{aligned}
\tilde{F}^{\uparrow\uparrow}_{kk'}(M/N = 1) &= 4\sin^2\left(\frac{\theta_{kk'}}{2}\right)k_{\mathrm{F}}^2 g_1 + \frac{4}{3}\sin^2\left(\frac{\theta_{kk'}}{2}\right)k_{\mathrm{F}}^2 g_0 g_1 + \left[\left(6 - 24\sin^2\left(\frac{\theta_{kk'}}{2}\right) + \frac{448}{15}\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right)\right. \\
&+ \left.\tan^2\left(\frac{\theta_{kk'}}{2}\right)(1 + 2\cos\theta_{kk'}) - \tan^2\left(\frac{\theta_{kk'}}{2}\right)\left(\cos\theta_{kk'} + \sin^2\theta_{kk'}\ln\left(\tan\left(\frac{\theta_{kk'}}{2}\right)\right)\right)\right]k_{\mathrm{F}}^4 g_1^2, \quad (2.34)
\end{aligned}
$$

$$\tilde{F}^{\downarrow\downarrow}_{kk'}(M/N=1) = g_0^2 + k_F^2 g_0 g_1 + \frac{1}{4}k_F^4 g_1^2, \tag{2.35}$$

$$\tilde{F}^{\uparrow\downarrow}_{kk'}(M/N=1) = g_0 + \frac{1}{2}k_F^2 g_1 + g_0^2 - \frac{8}{3}k_F^2 g_0 g_1 + \frac{26}{20}k_F^4 g_1^2. \tag{2.36}$$

III. THERMODYNAMICS AND COLLECTIVE EXCITATIONS

In this section, we derive those relations for the two-dimensional arbitrarily polarized Fermi system that connect the angular moments of the Landau parameters to measurable thermodynamic properties and collective excitations. As shown by Landau[2] and Nozières and Luttinger[17] the Landau parameter $\tilde{F}^{\sigma\sigma'}_{\mathbf{pp'}}$ is determined by the singular behavior of the scattering function with the momenta $\mathbf{p}$ and $\mathbf{p}'$ fixed at the Fermi momentum. Thus, the only degree of freedom is the angle between the momenta. In two dimensions the angular decomposition can be written

$$\tilde{F}^{\sigma\sigma'}_{\mathbf{pp'}} = \sum_{\ell=0}^{\infty} \alpha_\ell \tilde{F}^{\sigma\sigma'}_\ell T_\ell(\cos\theta_{\mathbf{pp'}}), \tag{3.1}$$

where α_ℓ is the parameter defined in Eq. (2.10c), and $T_\ell(\cos\theta_{\mathbf{pp'}}) = \cos(\ell\theta_{\mathbf{pp'}})$ are Chebyshev polynomials of the first kind.[18] They play the same role in two dimensions that Legendre polynomials play in three dimensions. This form for the angular decomposition requires that $\tilde{F}(\theta)$ is periodic in θ with period 2π, that $\tilde{F}(\theta)$ is real, and that $\tilde{F}(\theta)$ is even in θ. Inverting the decomposition,

$$\tilde{F}^{\sigma\sigma'}_m = \frac{1}{\pi}\int_{-1}^{+1} dx\, w(x) \tilde{F}^{\sigma\sigma'}_{\mathbf{pp'}} T_m(x), \tag{3.2}$$

where $w(x) = 1/\sqrt{1-x^2}$ is the integration weight. In obtaining Eq. (3.2) we have used the general result for the inner product of two Chebyshev polynomials [Eq. (3.3a)]. This integral over Chebyshev polynomials and others that are used in this section have been gathered together in the following:

$$(T_m, T_\ell) \equiv \int_{-1}^{+1} dx\, w(x) T_m(x) T_\ell(x) = \frac{\pi}{\alpha_m}\delta_{m,\ell}, \tag{3.3a}$$

$$\int_0^{2\pi} d\theta' T_m\big(\cos(\theta-\theta')\big) T_n(\cos\theta') = T_m(\cos\theta)\frac{2\pi}{\alpha_m}\delta_{m,n}, \tag{3.3b}$$

$$(T_m\cos\theta, T_\ell) \equiv \int_{-1}^{+1} dx\, w(x) T_m(x)\, x\, T_\ell(x) = \begin{cases} \frac{\pi}{2} & \text{if } \ell=0,\, m=1 \quad \text{or} \quad \ell=1,\, m=0, \\ \frac{\pi}{4} & \text{if } \ell = m\pm 1 \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3c}$$

In general, when an integrand consists of powers of $\cos(\theta)$ times Chebyshev polynomials then integrals over θ from 0 to 2π can be changed to integrals over $\cos(\theta)$ from -1 to $+1$ multiplied by a factor of 2.

A. Effective mass and heat capacity

State-dependent effective masses have been discussed within the Landau Fermi-liquid picture many times in the literature (see, for example, Refs. 5, 19, and 20). In the absence of a preferred frame of reference the effective mass can be derived by arguments based on the Galilean invariance of the quasiparticle excitation energy as observed from a fixed laboratory frame of reference and a frame moving with velocity $\mathbf{u}$ with respect to the laboratory frame. The derivation proceeds in two dimensions exactly the same as in three dimensions and can be found in Baym and Pethick.[3] One finds that the effective mass obeys the expression

$$\left(\frac{m}{m^*_\sigma} - 1\right)\mathbf{p}_\sigma\cdot\mathbf{u} = -\sum_{\mathbf{p}',\sigma'} f^{\sigma\sigma'}_{\mathbf{pp'}}\,\delta(\epsilon_{p',\sigma'} - \epsilon_{F\sigma'})\frac{p'}{m^*_{\sigma'}} m\, u\cos\theta_{up'}, \tag{3.4}$$

where $\epsilon_{p,\sigma} = p^2/2m^*_\sigma$ is the quasiparticle energy. Using the angular decomposition (3.1) one immediately finds

$$\frac{m^*_\sigma}{m} = \frac{1}{1 - \left(\tilde{F}^{\sigma\sigma}_1 + \frac{k_{F-\sigma}}{k_{F\sigma}}\tilde{F}^{\sigma-\sigma}_1\right)}. \tag{3.5}$$

In this expression the notation $-\sigma$ simply denotes the opposite of σ; thus, if $\sigma = \uparrow$ then $-\sigma = \downarrow$.

We can now obtain the effective masses in the unpolarized and fully polarized limits. For zero polarization, in agreement with ERZ, we obtain

$$\frac{m^*}{m} = \frac{1}{1-2\tilde{F}^s_1} = 1 + 2F^s_1, \tag{3.6}$$

where $F^s_1 = (m^*/m)\tilde{F}^s_1 \equiv N_0 f^s_1$ is a dimensionless Landau parameter written in terms of a single-spin-state density of states that itself contains the effective mass. [Here and henceforth, N_0 will be used to denote the single-spin-state density of states in the zero polarization case.] The symmetric and antisymmetric Landau parameters used in the zero polarization limit are defined as usual by

$$\tilde{F}^{\sigma\sigma'}_\ell = \tilde{F}^s_\ell + \sigma\sigma'\tilde{F}^a_\ell, \tag{3.7}$$

where for this definition we associate $\sigma(\uparrow) = +1$ and $\sigma(\downarrow) = -1$. In the fully polarized limit ($k_\uparrow = k_F,\quad k_\downarrow = 0$),

$$\frac{m^*_\uparrow}{m} = \frac{1}{1-\tilde{F}^{\uparrow\uparrow}_1}, \tag{3.8a}$$

$$\frac{m^*_\downarrow}{m} = \frac{1}{1 - k_\uparrow \lim_{k_\downarrow\to 0}\left(\frac{\tilde{F}^{\uparrow\downarrow}_1}{k_\downarrow}\right)}. \tag{3.8b}$$

These results are similar to those in three dimensions where they were discussed by Bedell.[20] Bedell argued from scattering theory that

$\lim_{k_\downarrow\to 0}\tilde{F}_\ell^{\uparrow\downarrow}\to 0$ and $\lim_{k_\downarrow\to 0}\tilde{F}_\ell^{\downarrow\downarrow}\to 0$ for $\ell\geqslant 1$. He also noted that Eq. (3.8b) does not vanish. This is simply the effective mass for a single down-spin impurity in an up-spin Fermi liquid environment. Thus, he concludes that $\lim_{k_\downarrow\to 0}\tilde{F}_\ell^{\uparrow\downarrow}=O(k_\downarrow)$. We note in passing that, in agreement with Bedell's argument, our Eqs. (2.35) and (2.36) have only $\ell=0$ terms.

The low-temperature heat capacity is simply proportional to the number of available states. Thus, in a derivation that follows that in three-dimensions,[3] we find

$$C=\frac{\pi^2}{3}\tilde{N}_0\left(\frac{m_\uparrow^*}{m}+\frac{m_\downarrow^*}{m}\right)k_B^2T=\frac{\pi k_B^2L^2}{6\hbar^2}(m_\uparrow^*+m_\downarrow^*)T. \tag{3.9}$$

B. Compressibility and spin susceptibility

The compressibility and spin-susceptibility for a three-dimensional Fermi liquid with arbitrary polarization was obtained by Sanchez-Castro, Bedell, and Wiegers.[7] In this section, we obtain equivalent results for two dimensions by treating the Fermi liquid as a thermodynamic mixture. The Gibbs free energy for the system can be written

$$G=\mu_\uparrow N_\uparrow+\mu_\downarrow N_\downarrow=\tfrac{1}{2}N(\mu_\uparrow+\mu_\downarrow)+\tfrac{1}{2}M(\mu_\uparrow-\mu_\downarrow), \tag{3.10}$$

where $N=N_\uparrow+N_\downarrow$ and $M=N_\uparrow-N_\downarrow$. The species chemical potentials are given by $\mu_\sigma=(\partial G/\partial N_\sigma)_{N_{-\sigma}}$. From Eq. (3.10) we can associate a field $h\equiv\frac{1}{2}(\mu_\uparrow-\mu_\downarrow)$ with the difference in the species chemical potentials, and also the chemical potential is $\mu=\frac{1}{2}(\mu_\uparrow+\mu_\downarrow)$. Using (3.10), the Gibbs-Duhem relation at fixed temperature can be written

$$dP=\overline{n}_\uparrow d\mu_\uparrow+\overline{n}_\downarrow d\mu_\downarrow, \tag{3.11}$$

where $\overline{n}_\sigma\equiv N_\sigma/A$ are areal densities, and P is the spreading pressure. The inverse isothermal compressibility is defined by

$$\kappa_T^{-1}=-A\left(\frac{\partial P}{\partial A}\right)_{T,N_\uparrow,N_\downarrow}. \tag{3.12}$$

Then using (3.11) we find the well-known result:

$$\kappa_T^{-1}=\overline{n}_\uparrow^2\frac{\partial\mu_\uparrow}{\partial\overline{n}_\uparrow}+\overline{n}_\uparrow\overline{n}_\downarrow\left(\frac{\partial\mu_\uparrow}{\partial\overline{n}_\downarrow}+\frac{\partial\mu_\downarrow}{\partial\overline{n}_\uparrow}\right)+\overline{n}_\downarrow^2\frac{\partial\mu_\downarrow}{\partial\overline{n}_\downarrow}. \tag{3.13}$$

The isothermal spin susceptibility is defined by

$$\chi=\frac{1}{A}\left(\frac{\partial M}{\partial h}\right)_{T,N}. \tag{3.14}$$

Thus, both the compressibility and the spin susceptibility depend on the same sort of partial derivatives of the species chemical potentials with respect to the species densities. These quantities are simple to obtain from the usual Fermi liquid methods.

The fundamental starting place is the relation between the quasiparticle energy fluctuations and the species distributions fluctuations:

$$\delta\epsilon_{\mathbf{k},\sigma}=\sum_{\mathbf{k}',\sigma'}f_{\mathbf{kk}'}^{\sigma\sigma'}\delta n_{\mathbf{k}'\sigma'}. \tag{3.15}$$

We now use the angular decomposition (3.1) for the Landau parameter and the argument that for an isotropic system we expect that the response to a small fluctuation as measured by the compressibility or spin susceptibility should be isotropic and therefore only the $\ell=0$ term can survive:

$$\delta\epsilon_{\mathbf{k},\sigma}=f_0^{\sigma\uparrow}\delta n_{\mathbf{k}\uparrow}+f_0^{\sigma\downarrow}\delta n_{\mathbf{k}\downarrow}, \tag{3.16}$$

where we used $\alpha_0=1$ and $T_0(x)=1$. From the Fermi function form for the distribution function, we can also write

$$\delta n_\sigma=-\frac{L^2}{(2\pi)^2}\int d\mathbf{k}\,\delta(\epsilon_{\mathbf{k},\sigma}-\mu_\sigma)(\delta\epsilon_{\mathbf{k},\sigma}-\delta\mu_\sigma). \tag{3.17}$$

If we substitute (3.16) into (3.17) we obtain the basic relations

$$\left(\frac{1}{N_0^\uparrow}+f_0^{\uparrow\uparrow}\right)\delta n_\uparrow+f_0^{\uparrow\downarrow}\delta n_\downarrow=\delta\mu_\uparrow, \tag{3.18a}$$

$$\left(\frac{1}{N_0^\downarrow}+f_0^{\downarrow\downarrow}\right)\delta n_\downarrow+f_0^{\downarrow\uparrow}\delta n_\uparrow=\delta\mu_\downarrow, \tag{3.18b}$$

where $N_0^\sigma=m_\sigma^*L^2/2\pi\hbar^2$ is the density of states at the Fermi surface for the σ spin state. The partial derivatives needed for κ_T^{-1} (3.13) can be obtained by inspection and we find

$$\kappa_T^{-1}=\frac{2\pi\hbar^2}{m}\left[\left(\frac{m}{m_\uparrow^*}\overline{n}_\uparrow^2+\frac{m}{m_\downarrow^*}\overline{n}_\downarrow^2\right)+\tilde{F}_0^{\uparrow\uparrow}\overline{n}_\uparrow^2+2\tilde{F}_0^{\uparrow\downarrow}\overline{n}_\uparrow\overline{n}_\downarrow+\tilde{F}_0^{\downarrow\downarrow}\overline{n}_\downarrow^2\right]. \tag{3.19}$$

The compressibility is related to the first sound speed c_1 by $mc_1^2=\kappa^{-1}/\overline{n}$. As pointed out by Landau, first sound does not propagate in fermion systems at absolute zero. The propagating mode, zero sound, is discussed in the next section.

For the spin susceptibility we subtract (3.18b) from (3.18a):

$$\delta\mu_\uparrow-\delta\mu_\downarrow=\left(\frac{1}{N_0^\uparrow}+f_0^{\uparrow\uparrow}-f_0^{\uparrow\downarrow}\right)\delta n_\uparrow-\left(\frac{1}{N_0^\downarrow}+f_0^{\downarrow\downarrow}-f_0^{\uparrow\downarrow}\right)\delta n_\downarrow. \tag{3.20}$$

We now use $\delta h=\frac{1}{2}(\delta\mu_\uparrow-\delta\mu_\downarrow)$, $\delta\overline{n}_\uparrow=\frac{1}{2}(\delta\overline{n}+\delta\overline{m})$, and $\delta\overline{n}_\downarrow=\frac{1}{2}(\delta\overline{n}-\delta\overline{m})$, where $\overline{n}\equiv N/A,\overline{m}\equiv M/A$:

$$\delta h=\left[\left(\frac{1}{N_0^\uparrow}+f_0^{\uparrow\uparrow}\right)+\left(\frac{1}{N_0^\downarrow}+f_0^{\downarrow\downarrow}\right)-2f_0^{\uparrow\downarrow}\right]\frac{A}{4}\delta\overline{m}+\left[\left(\frac{1}{N_0^\uparrow}+f_0^{\uparrow\uparrow}\right)-\left(\frac{1}{N_0^\downarrow}+f_0^{\downarrow\downarrow}\right)\right]\frac{A}{4}\delta\overline{n}. \tag{3.21}$$

The spin susceptibility is therefore

$$\chi^{-1}=\frac{\pi\hbar^2}{2m}\left[\left(\frac{m}{m_\uparrow^*}+\frac{m}{m_\downarrow^*}\right)+\tilde{F}_0^{\uparrow\uparrow}-2\tilde{F}_0^{\uparrow\downarrow}+\tilde{F}_0^{\downarrow\downarrow}\right]. \tag{3.22}$$

Equation (3.19) and (3.22) are in agreement with the equivalent results in three-dimensions.[7] For alternative derivations of the spin susceptibility, see Ref. 21.

In the limit of zero polarization $\overline{n}_\uparrow = \overline{n}_\downarrow = \overline{n}/2$, we find

$$\kappa_T^{-1}(0) = \frac{\pi\hbar^2}{m^*}\overline{n}^2\left[1 + 2\frac{m^*}{m}\tilde{F}_0^s\right] = \frac{\pi\hbar^2}{m^*}\overline{n}^2[1 + 2F_0^s], \tag{3.23a}$$

$$\chi^{-1}(0) = \frac{\pi\hbar^2}{m^*}\left[1 + 2\frac{m^*}{m}\tilde{F}_0^a\right] = \frac{\pi\hbar^2}{m^*}[1 + 2F_0^a], \tag{3.23b}$$

and in the limit of full polarization, $\overline{n}_\uparrow = \overline{n}, \overline{n}_\downarrow = 0$:

$$\kappa_T^{-1}(1) = \frac{2\pi\hbar^2}{m_\uparrow^*}\overline{n}^2\left[1 + \frac{m_\uparrow^*}{m}\tilde{F}_0^{\uparrow\uparrow}\right], \tag{3.24a}$$

$$\chi^{-1}(1) = \frac{\pi\hbar^2}{2m}\left[\left(\frac{m}{m_\uparrow^*} + \frac{m}{m_\downarrow^*}\right) + \tilde{F}_0^{\uparrow\uparrow} - 2\tilde{F}_0^{\uparrow\downarrow} + \tilde{F}_0^{\downarrow\downarrow}\right]. \tag{3.24b}$$

C. Zero sound

The derivation of the zero-sound dispersion relations proceeds exactly as in three dimensions, beginning with Landau's linearized kinetic equation:[3]

$$(\mathbf{q}\cdot\mathbf{v_k} - \omega)\delta n_{\mathbf{k},\sigma} - (\mathbf{q}\cdot\mathbf{v_k})\frac{\partial n^0_{\mathbf{k}\sigma}}{\partial\epsilon_{\mathbf{k}\sigma}}\sum_{\mathbf{k}'\sigma'} f_{kk'}^{\sigma\sigma'}\delta n_{\mathbf{k}',\sigma'} = 0. \tag{3.25}$$

Introducing a Fermi-surface displacement function,

$$\delta n_{\mathbf{k},\sigma} = \delta(\epsilon_{\mathbf{k}\sigma} - \mu_\sigma)u_{\mathbf{k}\sigma}(\theta), \tag{3.26}$$

where we note that in two dimensions $u = u_{\mathbf{k}\sigma}(\theta)$ is not an approximation, we find

$$(s_\sigma - \cos\theta)u_{\mathbf{k}\sigma}(\theta) - \cos\theta\sum_{\sigma'}\frac{m_{\sigma'}^*L^2}{4\pi^2\hbar^2}\int_0^{2\pi} d\theta' f_{kk'}^{\sigma\sigma'}u_{\mathbf{k}'\sigma'}(\theta') = 0, \tag{3.27}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{q}$, θ' is the angle between $\mathbf{k}'$ and $\mathbf{q}$, and $s_\sigma \equiv \omega/qv_{F\sigma}$ is the usual dimensionless frequency. Introducing angular decompositions for the Landau parameters (3.1) and the displacement function $u_{\mathbf{k}\sigma}(\theta) = \sum_{\ell=0}^{\infty}\alpha_\ell u_{\ell\sigma}T_\ell(\cos\theta)$:

$$(s_\sigma - \cos\theta)\sum_{\ell=0}^{\infty}\alpha_\ell u_{\ell\sigma}T_\ell(\cos\theta) - \cos\theta\sum_{\sigma'}\frac{1}{2\pi}N_0^{\sigma'}\sum_{m,n=0}^{\infty}\alpha_m\alpha_n f_m^{\sigma\sigma'}u_{n\sigma'}\int_0^{2\pi}d\theta' T_m \times[\cos(\theta-\theta')]T_n(\cos\theta') = 0. \tag{3.28}$$

The convolution integral is given in Eq. (3.3b), and this yields

$$(s_\sigma - \cos\theta)\sum_{\ell=0}^{\infty}\alpha_\ell u_{\ell\sigma}T_\ell(\cos\theta) - \cos\theta\sum_{\sigma'}N_0^{\sigma'}\sum_{\ell=0}^{\infty}\alpha_\ell f_\ell^{\sigma\sigma'}u_{\ell\sigma'}T_\ell(\cos\theta) = 0. \tag{3.29}$$

We now multiply by $T_m(\cos\theta)$ and integrate over θ for the first three coefficients u_m with $m = 0,1,2$. This procedure was introduced in Ref. 7 for the three-dimensional polarized Fermi system. The matrix elements $(T_m\cos\theta, T_\ell)$ are given in Eq. (3.3c). We find

$$m = 0, \quad s_\sigma u_{0\sigma} = u_{1\sigma} + \sum_{\sigma'}N_0^{\sigma'}f_1^{\sigma\sigma'}u_{1\sigma'}, \tag{3.30a}$$

$$m = 1, \quad s_\sigma u_{1\sigma} = \frac{1}{2}(u_{0\sigma} + u_{2\sigma}) + \sum_{\sigma'}N_0^{\sigma'} \times\frac{1}{2}\left(f_0^{\sigma\sigma'}u_{0\sigma'} + f_2^{\sigma\sigma'}u_{2\sigma'}\right), \tag{3.30b}$$

$$m = 2, \quad s_\sigma u_{2\sigma} = \frac{1}{2}\left(u_{1\sigma} + \sum_{\sigma'}N_0^{\sigma'}f_1^{\sigma\sigma'}u_{1\sigma'}\right) = \frac{1}{2}s_\sigma u_{0\sigma}. \tag{3.30c}$$

The second equality in Eq. (3.30c) is by inspection. We can use this to eliminate $u_{2\sigma}$ in Eq. (3.30b) and solve for $u_{1\sigma}$ in terms of $u_{0\sigma}$. We then substitute that into Eq. (3.30a) and obtain a quadratic equation for $u_{0\sigma}$. The roots of this equation are the zero-sound and spin-zero-sound frequencies.

Before proceeding, there is an important detail that needs to be discussed concerning this approach. For the following discussion we can restrict ourselves to the unpolarized limit with no loss of generality. We note that in the weak coupling limit ($\lim F_0^s \to 0$) we expect that $c_0 \to v_F$. This limit is evidently not obeyed by this approach. By inspection of Eqs. (3.30) one finds instead that $c_0 \approx \sqrt{3/4}v_F$. It is simple to locate the problem. If we solve the kinetic equation (3.29) in the limit $f_\ell^{\sigma\sigma'} = 0$ then we find the recurrence relation

$$u_{1\sigma} = s_\sigma u_{0\sigma}, \tag{3.31a}$$

$$u_{m+1\sigma} = 2s_\sigma u_{m\sigma} - u_{m-1\sigma} \text{ for } m \geqslant 1. \tag{3.31b}$$

Thus, setting $s_\sigma = 1$, the noninteracting limit, we find $u_{m\sigma} = u_{0\sigma}$ for $m \geqslant 1$. From Eq. (3.30c) we see that the truncation instead yields $u_{2\sigma} = (1/2)u_{0\sigma}$.

We still wish to use Eqs. (3.30) since they allow us to include both $\ell = 1$ contributions at arbitrary polarization and solve for the *two* lowest zero-sound modes in a very elegant way. Thus, instead of Eq. (3.30c) we instead use the truncation $u_{2\sigma} = u_{0\sigma}$. We show below that this change gives the correct weak coupling limit. Using the convenient notation of Ref. 7, we find

$$c_{0\pm}^2 = \frac{1}{2}\left(A_{\uparrow\uparrow}v_{F\uparrow}^2 + A_{\downarrow\downarrow}v_{F\downarrow}^2\right) \pm \frac{1}{2}\sqrt{\left[A_{\uparrow\uparrow}v_{F\uparrow}^2 + A_{\downarrow\downarrow}v_{F\downarrow}^2\right]^2 - 4(A_{\uparrow\uparrow}A_{\downarrow\downarrow} - A_{\uparrow\downarrow}A_{\downarrow\uparrow})v_{F\uparrow}^2v_{F\downarrow}^2}. \tag{3.32}$$

The parameters are defined by

$$A_{\uparrow\uparrow} = \left(1 + \frac{1}{2}\frac{m_\uparrow^*}{m}\tilde{F}_0^{\uparrow\uparrow}\right)\left(1 + \frac{m_\uparrow^*}{m}\tilde{F}_1^{\uparrow\uparrow}\right) + \frac{1}{2}\frac{m_\uparrow^* m_\downarrow^*}{m^2}\tilde{F}_1^{\uparrow\downarrow}\tilde{F}_0^{\downarrow\uparrow}\frac{v_{F\downarrow}}{v_{F\uparrow}}, \tag{3.33a}$$

$$A_{\downarrow\downarrow} = \left(1 + \frac{1}{2}\frac{m^*_\downarrow}{m}\tilde{F}_0^{\downarrow\downarrow}\right)\left(1 + \frac{m^*_\downarrow}{m}\tilde{F}_1^{\downarrow\downarrow}\right) + \frac{1}{2}\frac{m^*_\downarrow m^*_\uparrow}{m^2}\tilde{F}_1^{\downarrow\uparrow}\tilde{F}_0^{\uparrow\downarrow}\frac{v_{F\uparrow}}{v_{F\downarrow}}, \tag{3.33b}$$

$$A_{\uparrow\downarrow} = \left(1 + \frac{1}{2}\frac{m^*_\downarrow}{m}\tilde{F}_0^{\downarrow\downarrow}\right)\frac{m^*_\downarrow}{m}\tilde{F}_1^{\uparrow\downarrow} + \frac{1}{2}\frac{m^*_\downarrow}{m}\tilde{F}_0^{\uparrow\downarrow}\left(1 + \frac{m^*_\uparrow}{m}\tilde{F}_1^{\uparrow\uparrow}\right)\frac{v_{F\uparrow}}{v_{F\downarrow}}, \tag{3.33c}$$

$$A_{\downarrow\uparrow} = \left(1 + \frac{1}{2}\frac{m^*_\uparrow}{m}\tilde{F}_0^{\uparrow\uparrow}\right)\frac{m^*_\uparrow}{m}\tilde{F}_1^{\downarrow\uparrow} + \frac{1}{2}\frac{m^*_\uparrow}{m}\tilde{F}_0^{\downarrow\uparrow}\left(1 + \frac{m^*_\downarrow}{m}\tilde{F}_1^{\downarrow\downarrow}\right)\frac{v_{F\downarrow}}{v_{F\uparrow}}. \tag{3.33d}$$

In the weak coupling limit $A_{\sigma\sigma'} = 1$ and thus $c_{0\sigma} = v_{F\sigma}$. The altered truncation of Eq. (3.30c) gives us the correct weak coupling limit at all polarizations.

In the zero-polarization limit ($A_{\uparrow\uparrow} = A_{\downarrow\downarrow}, A_{\uparrow\downarrow} = A_{\downarrow\uparrow}, v_F = v_{F\uparrow} = v_{F\downarrow}$), and $c_{0\pm}^2 = (A_{\uparrow\uparrow} \pm |A_{\uparrow\downarrow}|)v_F^2$. In terms of the Landau parameters,

$$\frac{c_{0s,a}^2}{v_F^2} = \left(1 + 2F_1^{s,a}\right) + \frac{1}{2}\left(2F_0^{s,a} + 4F_0^{s,a}F_1^{s,a}\right), \tag{3.34}$$

where we changed $+$ and $-$ to symmetric and antisymmetric, respectively. The symmetric mode corresponds to zero sound and the antisymmetric mode corresponds to spin zero sound. The coefficients of 2 that appear before the Landau parameters are present because we are using the single-state density of states instead of the usual two-state density of states to write the dimensionless parameters. In the "simplest approximation"[22] we set $F_1^{s,a} = 0$ and find

$$c_{0s,a}^2 \approx \left(1 + F_0^{s,a}\right)v_F^2. \tag{3.35}$$

Thus, in the zero-polarization, strong-coupling limit we find $\lim_{F_0^{s,a}\to\infty} c_{0s,a} \approx \sqrt{F_0^{s,a}}v_F$. In the limit of full polarization ($v_F = v_{F\uparrow}$ and $v_{F\downarrow} = 0$), we immediately find $c_{0+}^2 = A_{\uparrow\uparrow}v_F^2$ and $c_{0-}^2 = 0$. Thus, in terms of the Landau parameters

$$c_{0+}^2 = \left(1 + \frac{1}{2}\frac{m^*_\uparrow}{m}\tilde{F}_0^{\uparrow\uparrow}\right)\left(1 + \frac{m^*_\uparrow}{m}\tilde{F}_1^{\uparrow\uparrow}\right)v_F^2. \tag{3.36}$$

Finally, we need to show that Eqs. (3.32) and (3.33) give the correct strong-coupling limit. We shall proceed by calculating the zero-sound speed in a slightly different way that does not suffer from truncation induced problems. We follow Khalatnikov and Abrikosov[23] (see also Refs. 1 and 3) and rewrite the kinetic equation:

$$u_{m\sigma} - \sum_{\sigma'} N_0^{\sigma'} \sum_{\ell=0}^{\infty} \alpha_\ell f_\ell^{\sigma\sigma'} u_{\ell\sigma'} \Omega_{m,\ell} = 0, \tag{3.37}$$

where the angular integrals are defined by

$$\Omega_{m,\ell} = \frac{1}{\pi}\int_{-1}^{+1} dx\, w(x) T_m(x)\left(\frac{x}{s_\sigma - x}\right)T_\ell(x). \tag{3.38}$$

In the "simplest approximation" we ignore all terms except $m = 0$. Then the zero-sound velocity, in the zero-polarization limit for simplicity, is determined by the solution of $1 - 2F_0^s\Omega_{0,0} = 0$, where

$$\Omega_{0,0} = \frac{s}{\sqrt{s^2 - 1}} - 1. \tag{3.39}$$

Solving for the zero-sound velocity,

$$\frac{c_{0s}^2}{v_F^2} = \frac{\left(1 + 2F_0^s\right)^2}{1 + 4F_0^s}. \tag{3.40}$$

The Khalatnikov/Abrikosov approach has the correct weak coupling limit. We note in particular that the *strong*-coupling limit $c_s^2/v_F^2 \sim F_0^s$ does agree with our previous approach Eq. (3.35). Unfortunately, the Khalatnikov/Abrikosov approach leads to equations that are much more complicated than Eqs. (3.32) and (3.33) when including $\ell = 1$ terms and allowing arbitrary polarization. Thus, in this paper we present results using the simpler truncation approach and leave the Khalatnikov/Abrikosov approach for a subsequent publication. We note that in the important range of values of $F_0^s \approx 1$, the zero-sound speeds calculated with Eq. (3.35) tend to be slightly higher than those calculated with Eq. (3.40).

The polarization dependence of zero sound for a two-dimensional Fermi liquid was considered by Béal-Monod, Valls, and Daniel,[24] who derived an expression for c_{0s}/v_F to $\ell = 0$ order in the Landau parameters and to quadratic order in the applied field. In the limit of zero field their results agree with Eq. (3.40).

D. Forward-scattering sum rule

The forward-scattering sum rule is a constraint placed on the triplet scattering function due to the Pauli principle. The sum rule is important because the scattering function can be written in terms of the Landau parameters and thus the sum rule is, in fact, a constraint that must be obeyed by the Landau parameters. This result was first proved by Landau[2] in 1958. Landau showed that the so-called k limit of the scattering function (basically the forward-scattering amplitude, $a_{pp'}^{\sigma\sigma'}$) and the ω limit of the scattering function (basically the Landau parameter) are related by the following integral equation:[3]

$$a_{\mathbf{pp}'}^{\sigma\sigma'} = f_{\mathbf{pp}'}^{\sigma\sigma'} + \sum_{\mathbf{p}'',\sigma''} f_{\mathbf{pp}''}^{\sigma\sigma''}\frac{\partial n^0_{p''\sigma''}}{\partial\epsilon_{p''\sigma''}} a_{\mathbf{p}''\mathbf{p}'}^{\sigma''\sigma'}. \tag{3.41}$$

We now use

$$\frac{\partial n^0_{p''\sigma''}}{\partial\epsilon_{p''\sigma''}} = -\delta(\epsilon_{p''} - \mu_{\sigma''}), \tag{3.42}$$

$$a_{\mathbf{pp}'}^{\sigma\sigma'} = \sum_{\ell=0}^{\infty}\alpha_\ell a_\ell^{\sigma\sigma'} T_\ell(\cos\theta), \tag{3.43}$$

and also the angular decomposition for the Landau parameter (3.1) to find

$$a_\ell^{\sigma\sigma'} = f_\ell^{\sigma\sigma'} - \sum_{\sigma''} N_0^{\sigma''} f_\ell^{\sigma\sigma''} a_\ell^{\sigma''\sigma'}. \tag{3.44}$$

The convolution integral Eq. (3.3b) was used to obtain this result. We now use (3.44) to find the $a_\ell^{\sigma\sigma'}$ in terms of the $f_\ell^{\sigma\sigma'}$. We first write $a_\ell^{\uparrow\uparrow}$ and $a_\ell^{\uparrow\downarrow}$ explicitly:

$$a_\ell^{\uparrow\uparrow} = f_\ell^{\uparrow\uparrow} - N_0^{\uparrow} f_\ell^{\uparrow\uparrow} a_\ell^{\uparrow\uparrow} - N_0^{\downarrow} f_\ell^{\uparrow\downarrow} a_\ell^{\downarrow\uparrow}, \tag{3.45}$$

$$a_\ell^{\downarrow\uparrow} = f_\ell^{\downarrow\uparrow} - N_0^{\uparrow} f_\ell^{\downarrow\uparrow} a_\ell^{\uparrow\uparrow} - N_0^{\downarrow} f_\ell^{\downarrow\downarrow} a_\ell^{\downarrow\uparrow}. \tag{3.46}$$

Solving these simultaneously yields

$$a_\ell^{\uparrow\uparrow} = \frac{f_\ell^{\uparrow\uparrow}(1 + N_0^{\downarrow} f_\ell^{\downarrow\downarrow}) - N_0^{\downarrow}(f_\ell^{\uparrow\downarrow})^2}{(1 + N_0^{\uparrow} f_\ell^{\uparrow\uparrow})(1 + N_0^{\downarrow} f_\ell^{\downarrow\downarrow}) - N_0^{\uparrow} N_0^{\downarrow}(f_\ell^{\uparrow\downarrow})^2}, \tag{3.47a}$$

$$a_\ell^{\uparrow\downarrow} = \frac{f_\ell^{\uparrow\downarrow}}{(1 + N_0^{\uparrow} f_\ell^{\uparrow\uparrow})(1 + N_0^{\downarrow} f_\ell^{\downarrow\downarrow}) - N_0^{\uparrow} N_0^{\downarrow}(f_\ell^{\uparrow\downarrow})^2}, \tag{3.47b}$$

$$a_\ell^{\downarrow\downarrow} = \frac{f_\ell^{\downarrow\downarrow}(1 + N_0^{\uparrow} f_\ell^{\uparrow\uparrow}) - N_0^{\uparrow}(f_\ell^{\downarrow\uparrow})^2}{(1 + N_0^{\uparrow} f_\ell^{\uparrow\uparrow})(1 + N_0^{\downarrow} f_\ell^{\downarrow\downarrow}) - N_0^{\uparrow} N_0^{\downarrow}(f_\ell^{\uparrow\downarrow})^2}, \tag{3.47c}$$

where $a_\ell^{\downarrow\downarrow}$ was obtained by flipping the spins in $a_\ell^{\uparrow\uparrow}$, and $a_\ell^{\downarrow\uparrow} = a_\ell^{\uparrow\downarrow}$. The analogous results for a three-dimensional fermion system were first obtained by Bedell.[20] For a polarized Fermi liquid, Landau's argument leads to two sum rules:

$$\sum_{\ell=0}^{\infty} \alpha_\ell a_\ell^{\sigma\sigma} = 0, \text{ for } \sigma = \uparrow, \downarrow. \tag{3.48}$$

In the zero-polarization limit, we define $A_\ell^{\sigma\sigma'} = N_0 a_\ell^{\sigma\sigma'}$ and find immediately

$$\sum_{\ell=0}^{\infty} \alpha_\ell A_\ell^{\uparrow\uparrow} = \sum_{\ell=0}^{\infty} \alpha_\ell \left[\frac{F_\ell^s}{1 + 2F_\ell^s} + \frac{F_\ell^a}{1 + 2F_\ell^a}\right] = 0 \tag{3.49}$$

and

$$A_\ell^{\uparrow\downarrow} = \frac{F_\ell^s}{1 + 2F_\ell^s} - \frac{F_\ell^a}{1 + 2F_\ell^a}. \tag{3.50}$$

In the limit of full polarization, remembering Bedell's argument that $\lim_{k_\downarrow \to 0} \tilde{f}_\ell^{\uparrow\downarrow}, \tilde{f}_\ell^{\downarrow\downarrow} \to 0$ for $\ell \geqslant 1$, we find

$$\frac{f_0^{\uparrow\uparrow}(1 + N_0^{\downarrow} f_0^{\downarrow\downarrow}) - N_0^{\downarrow}(f_0^{\uparrow\downarrow})^2}{(1 + N_0^{\uparrow} f_0^{\uparrow\uparrow})(1 + N_0^{\downarrow} f_0^{\downarrow\downarrow}) - N_0^{\uparrow} N_0^{\downarrow}(f_0^{\uparrow\downarrow})^2} + 2\sum_{\ell=1}^{\infty} \frac{f_\ell^{\uparrow\uparrow}}{(1 + N_0^{\uparrow} f_\ell^{\uparrow\uparrow})} = 0, \tag{3.51a}$$

$$f_0^{\downarrow\downarrow} - \frac{N_0^{\uparrow}(f_0^{\downarrow\uparrow})^2}{(1 + N_0^{\uparrow} f_0^{\uparrow\uparrow})} = 0. \tag{3.51b}$$

The second relation is in exactly the same form as the three-dimensional results. The first equation is more complicated than the analogous three-dimensional result because in two dimensions the density of states $N_0^{\downarrow}$ is a constant and so does not vanish in the limit $k_\downarrow \to 0$ as is the case in three dimensions.

IV. RESULTS: APPLICATION TO ^{3}He THIN FILMS

In this section, we try to obtain estimates of the values of the interaction parameters g_0 and g_1 introduced in Sec. II and then calculate predictions for the polarization dependence of thermodynamic quantities and collective excitations for ^{3}He in thin films using the expressions derived in Sec. III.

A. Interaction parameters: Experiment and scattering theory

Quasi-two-dimensional ^{3}He has been studied for many years in the form of adsorbed films on various solid surfaces.[25] By quasi-two-dimensional we mean that the film is studied at temperatures small relative to the characteristic transverse excitation energy. Thus, in this sense, the third dimension is "frozen out." There have been a number of high-precision measurements of the thermodynamic properties of these films. All of these measurements are on an unpolarized system. We point especially to heat capacity measurements[26–28] that enabled accurate estimates of the heat-capacity effective mass and also measurements of the spin susceptibility.[29]

The spin susceptibility can be obtained from Fig. 1 of Ref. 29. The authors write the spin susceptibility in the form

$$\frac{\chi}{\chi_0} = \frac{m^*/m}{1 + 2(m^*/m)\tilde{F}_0^a}, \tag{4.1a}$$

$$\text{where} \quad m^*/m = \frac{1}{1 - 2\tilde{F}_1^s}. \tag{4.1b}$$

We have used the single-state density of states from Eq. (3.23b) and $\chi_0 = 2\tilde{N}_0 = N/\epsilon_F$. Then we estimate $\chi/\chi_0 \approx 5$ at $\overline{n} \approx 0.026$ Å^{-2}.

The effective mass will be obtained from the data shown in Fig. 12 of Ref. 27. At $\overline{n} = 0.026$ Å^{-2} we estimate that $m^*/m \approx 1.8$. At this density, these results are slightly larger than those of Ref. 26. This discrepancy is accounted for by Greywall as due to a higher minimum temperature at which the data of Van Sciver and Vilches were taken.

We can now use the two experimental numbers in Eqs. (4.1) to obtain approximate values of the interaction parameters at $\overline{n} = 0.026$ Å^{-2}. We note that, like the situation for bulk ^{3}He, we also seem to have at our disposal the constraint of the forward-scattering sum rule. The possible application of the forward-scattering sum rule to a low-density expansion is discussed in the Conclusion, Sec. V. We find that the following values of g_0 and g_1 fit the experimental data:

$$g_0 = 0.76, \tag{4.2a}$$

$$g_1 = 1.89\ \text{Å}^2. \tag{4.2b}$$

In order to compare approximately the magnitudes of the s-wave and p-wave interaction components, we can, for example, multiply g_1 by k_F^2 to create a dimensionless quantity; then $g_1 k_F^2 \approx 0.31$. In this sense, the p-wave component is roughly 40% of the size of the s-wave interaction component.

These interaction parameters are related to the effective two-body interaction in the low-density limit by scattering theory. For quantum systems in two dimensions the relevant scattering theory has been derived by Adhikari.[30] The theory was applied to this problem by Randeria, Duan, and Shieh

(RDS) in Ref. 16. The approach begins with an angular decomposition of the T matrix as in Eq. (2.16):

$$\begin{aligned} T_{kk'} &= \sum_{\ell=0}^{\infty} \alpha_\ell T^{(\ell)}_{kk'} T_\ell(\cos\theta_{kk'}), \\ T^{(\ell)}_{kk'} &= \frac{4\hbar^2}{m}(-\cot\delta_\ell + i)^{-1}, \end{aligned} \tag{4.3}$$

where δ_ℓ is the ℓ-wave scattering phase shift. For a finite range interaction $V_{\rm eff}(r)$, say, where $V_{\rm eff}(r) = 0$ for $r \geqslant R$, the phase shifts are obtained in the usual way by requiring continuity of the logarithmic derivative of the wave function at $r = R$. The general result for two dimensions can be found in Appendix A Ref. 16. For s waves in the low-density limit, one can show that $\pi \cot\delta_0 = \ln(E/E_{\rm a}) + O[(kr)^2]$, where $E_{\rm a}$ is a parameter with the dimensions of energy. The real part of the low-density T matrix element can therefore be written

$$\tau_0 = \frac{4\pi\hbar^2}{m}[\ln(E_{\rm a}/E)]^{-1}, \tag{4.4}$$

and from Eq. (2.28) we have

$$g_0 = 2[\ln(E_{\rm a}/E)]^{-1}. \tag{4.5}$$

At this point we note that RDS make the important observation that from Eq. (4.4) there is a low-density pole in the T matrix located at $E = -E_{\rm a}$. Then, using the analytic properties of the T matrix,[30] RDS are able to identify $E_{\rm a}$ as a two-body bound state. A simple expression for the bound-state energy in two dimensions can be found in Landau and Lifshitz:[31]

$$E_{\rm a} = \left(\frac{\hbar^2}{2mR^2}\right)\exp\left[\frac{2\pi\hbar^2}{m}\frac{1}{\overline{V}_{\rm R}}\right], \tag{4.6a}$$

$$\text{where } \overline{V}_{\rm R} = 2\pi\int_0^R dr\, r\, V_{\rm eff}(r). \tag{4.6b}$$

We note that this expression is derived under the assumptions that $V_{\rm eff}(r)$ is external, and that $E_{\rm a} \ll \hbar^2/2mR^2$. For the energy parameter, we choose $E = \hbar^2k^2/2m = \epsilon_{\rm F}^0 = \hbar^2\pi\overline{n}/m$ (where $\overline{n} = N/A$). This choice is consistent with the $\ell = 0$ part of the energy parameter discussed in Eq. (2.22). Thus, following RDS, we have a scattering theory result for the low-density s-wave interaction component:

$$g_0 = 2\left[\ln\left(\frac{1}{2\pi\overline{n}R^2}\exp\left[\frac{2\pi\hbar^2}{m}\frac{1}{\overline{V}_{\rm R}}\right]\right)\right]^{-1}. \tag{4.7}$$

There exist a number of alternative formulations of an effective interaction for ^{3}He (Refs. 32–34). These are all local, static two-body interactions. For our purposes, we utilize the so-called correlated RPA as described in Ref. 33. In Ref. 35, this potential was used to generate the coverage dependence of effective masses of ^{3}He in thin ^{3}He films. The good agreement that was found with existing experimental results suggests that this potential would be an excellent choice for our analysis. The effective interaction in the correlated RPA approximation is given by

$$\overline{V}_{\rm eff}(k) = \frac{\hbar^2k^2}{4m}\left(\frac{1}{S^2(k)} - \frac{1}{S_{\rm F}^2(k)}\right), \tag{4.8}$$

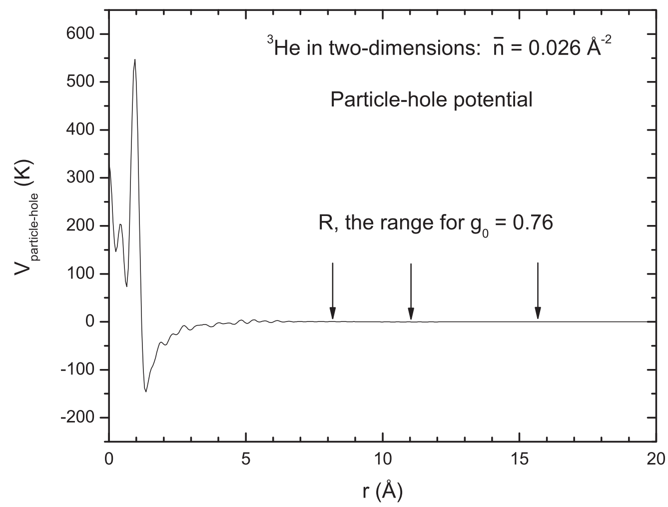

FIG. 1. The particle-hole effective interaction for ^{3}He in two dimensions at a density of $\overline{n} = 0.026$ Å^{-2}. The vertical arrows show the positions of the range parameter R for which Eq. (4.7) agrees with the s-wave t-matrix component $g_0 = 0.76$. There are multiple positions because of the oscillations of the effective interaction.

where $S(k)$ is the ^{3}He static structure function[36] and $S_{\rm F}(k)$ is the static structure function for a two-dimensional ideal Fermi gas. $\overline{V}_{\rm eff}(k)$ is the Fourier transform of $V_{\rm eff}(r)$:

$$V_{\rm eff}(r) = \frac{1}{2\pi\overline{n}}\int_0^\infty dk\, k\, J_0(kr)\overline{V}_{\rm eff}(k), \tag{4.9}$$

and $J_0(kr)$ is a Bessel function. Then, using Eq. (4.9) in Eq. (4.6b) we can compute $\overline{V}_{\rm R}$ as a function of the range R and then Eq. (4.7) yields g_0 as a function of the range. We note that the functions $\overline{V}_{\rm eff}(k)$ and $S(k)$ that we use below have been supplied to us by the authors of Ref. 35.

In Fig. 1, we show the results of this analysis. The vertical arrows are drawn at the values of R that correspond to the value of g_0 in Eq. (4.2a). The strongly oscillatory nature of the effective potential also causes the integral $\overline{V}_{\rm R}$ to oscillate. This behavior yields multiple solutions as indicated by the three vertical arrows. Although the leading arrow in particular is located at a position that one might reasonably associate as an approximate "range" of this potential, in fact these solutions correspond to the domain in which $E_{\rm a} \gg \hbar^2/2mR^2$. Thus, we must conclude that this simple analytic approach is not applicable to a strongly correlated system like ^{3}He.

B. Effective mass, thermodynamics, and collective excitations

With values for the s-wave and p-wave interaction components as obtained in Sec. IV A, we can follow the analysis in Sec. II and compute the Landau parameters, $\{\tilde{F}^{\uparrow\uparrow}, \tilde{F}^{\uparrow\downarrow}, \tilde{F}^{\downarrow\downarrow}\}$, for a ^{3}He thin film at a density of $\overline{n} = 0.026$ Å^{-2}. We then compute the $\ell = 0$ and $\ell = 1$ components of an angular momentum decomposition of the Landau parameters which from Sec. III yield various measurable quantities.

Figures 2 and 3 show the $\ell = 0,1$ components of the Landau parameters, respectively, as a function of polarization. The Landau parameter $\tilde{F}_0^{\uparrow\uparrow}$ shows a monotonic increase as a function of polarization. We note that 100% polarization corresponds to a completely filled up-spin Fermi sea. At

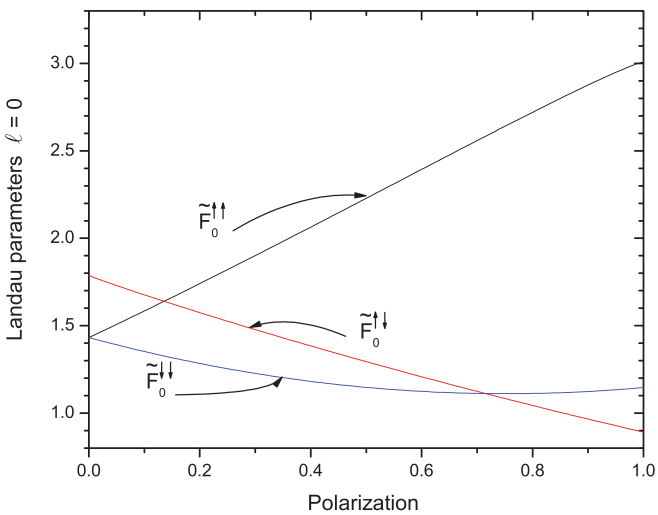

FIG. 2. (Color online) The $\ell = 0$ Landau parameters $\{\tilde{F}_0^{\uparrow\uparrow}, \tilde{F}_0^{\uparrow\downarrow}, \tilde{F}_0^{\downarrow\downarrow}\}$ for ^{3}He in two dimensions at a density of $\bar{n} = 0.026\,\text{Å}^{-2}$ as a function of polarization.

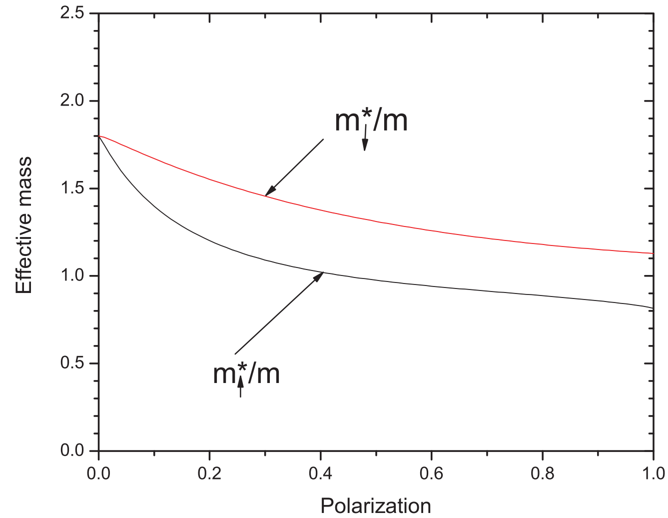

FIG. 4. (Color online) The spin-up and spin-down effective masses for ^{3}He in two dimensions at a density of $\bar{n} = 0.026\,\text{Å}^{-2}$, as a function of polarization from Eq. (3.5). The zero polarization limit is from Ref. 27 as discussed in the text.

zero polarization, $\tilde{F}_0^{\uparrow\downarrow}$ is larger than the parallel-spin Landau parameters. However, it decreases monotonically with increasing polarization and eventually crosses $\tilde{F}_0^{\downarrow\downarrow}$ at around 70% polarization. This behavior has an important consequence, as is discussed below.

Figure 3 shows that the $\ell = 1$ parameters are monotonically decreasing functions of polarization for this particular problem. At zero polarization the $\tilde{F}_1^{\uparrow\downarrow}$ parameter dominates, whereas at complete polarization the $\tilde{F}_1^{\uparrow\uparrow}$ has the greatest magnitude, as is to be expected. Further, at complete polarization $\tilde{F}_1^{\uparrow\downarrow} = \tilde{F}_1^{\downarrow\downarrow} = 0$ which ensures that the spin-down effective mass in that limit Eq. (3.8b) is well defined.

1. *Effective mass and heat capacity*

The effective mass as a function of polarization is given in Eq. (3.5). Using the $\ell = 1$ Landau parameters shown in Fig. 3 we calculate the results shown in Fig. 4. Both the spin-up and spin-down effective masses are a maximum at zero polarization and decrease monotonically with increasing polarization. At all polarizations, $m_\downarrow^* \geqslant m_\uparrow^*$. This behavior is in qualitative agreement with arguments presented by Bedell[6] for the three-dimensional system. The zero-polarization value for the effective masses is one of the experimental numbers (from Ref. 27) used to determine the interaction components, g_0 and g_1. Thus, the finite polarization values are the predictions for this model. We find a dramatic decrease in both the spin-up and spin-down effective masses as a function of polarization. This behavior was also predicted for the three-dimensional system.

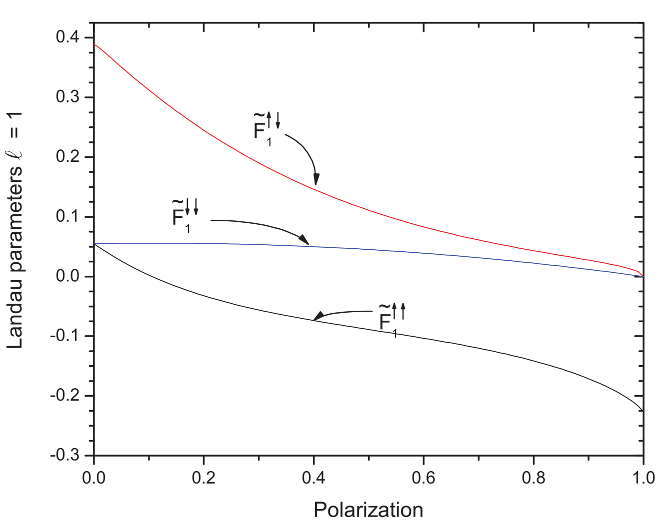

FIG. 3. (Color online) The $\ell = 1$ Landau parameters $\{\tilde{F}_1^{\uparrow\uparrow}, \tilde{F}_1^{\uparrow\downarrow}, \tilde{F}_1^{\downarrow\downarrow}\}$ for ^{3}He in two dimensions at a density of $\bar{n} = 0.026\,\text{Å}^{-2}$ as a function of polarization.

The slope of the low-temperature heat capacity as a function of polarization Eq. (3.9) is shown in Fig. 5. Since the number of accessible states gets smaller with increasing polarization, we expect that at a given temperature the heat capacity would decrease monotonically as a function of polarization and this is clearly shown in this figure.

2. *Compressibility and first sound*

In Fig. 6 we show the first sound speed mc_1^2, which from Eq. (3.19) is essentially the inverse compressibility, as a function of polarization. On the right-hand ordinate we show c_1 in units of m/s. As pointed out by Landau[1] first sound cannot propagate in a Fermi liquid at absolute zero. At any finite temperature, however, hydrodynamic response will dominate in the limit of small frequencies.[37] The magnitude of the first sound speed as a function of polarization is a prediction of this model. In particular at zero polarization, $c_1 = 117$ m/s. We first point out that this value is large compared to what would be expected from an ideal Fermi gas. For an ideal Fermi gas (with an effective mass) the first sound speed is given by $c_1^0 = \sqrt{\frac{m^*}{2m}}v_{\text{F}}$. The Fermi velocity at zero polarization for our system is $v_{\text{F}} = 47$ m/s. Thus, the ideal gas first sound speed is $c_1^0 \approx 45$ m / s $\ll c_1$. This certainly

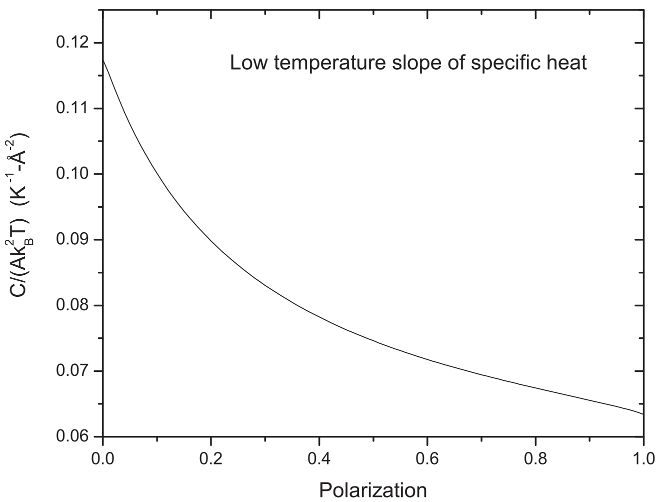

FIG. 5. The slope of the low-temperature heat capacity for ^{3}He in two dimensions at a density of $\overline{n} = 0.026$ Å^{-2}, as a function of polarization from Eq. (3.9). The slope decreases monotonically with polarization as expected.

emphasizes the importance of interactions even in this fairly dilute system. We can use the small-k limit of the static structure function $S(k)$ discussed in Sec. IV A above to obtain an independent estimate of the first sound speed. From the f-sum rule and the compressibility sum rule Feenberg[36] shows that $\lim_{k\to 0} S(k) \leqslant \Delta \frac{k}{k_F}$, where $\Delta \equiv \frac{\hbar k_F}{2mc_1}$. By inspection we find the slope $\Delta \approx 0.4$. This returns a first sound speed estimate of $c_1 \approx 106$ m/s which is in reasonable agreement with the zero-polarization first sound speed prediction shown in Fig. 6.

3. Spin susceptibility

The spin susceptibility Eq. (3.22) as a function of polarization is shown in Fig. 7. Unlike the situation in three dimensions,[7] the two-dimensional spin susceptibility is a

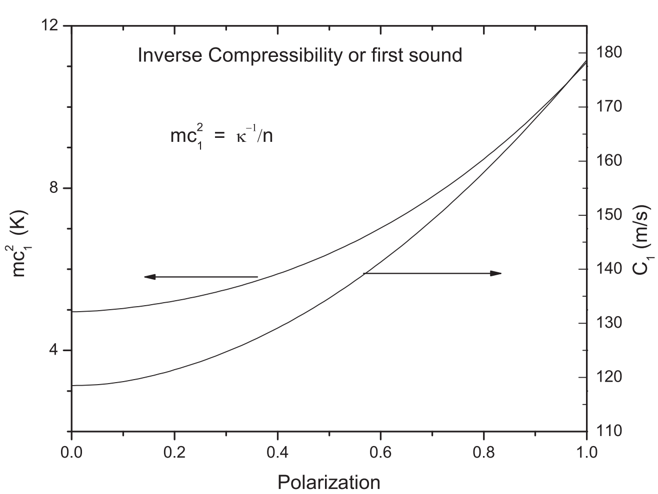

FIG. 6. The incompressibility mc_1^2 from Eq. (3.19) for ^{3}He in two dimensions at a density of $\overline{n} = 0.026$ Å^{-2} as a function of polarization. The right-hand ordinate shows the first sound speed c_1 in units of m/s. The system becomes less compressible with increasing polarization, as is to be expected.

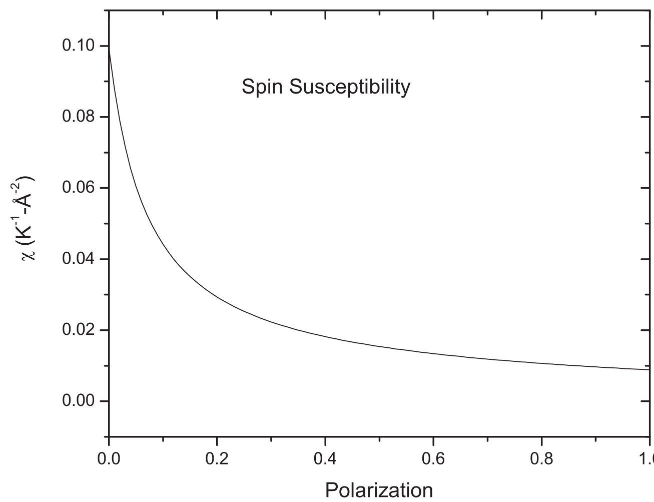

FIG. 7. The spin susceptibility for ^{3}He in two dimensions at a density of $\overline{n} = 0.026$ Å^{-2} as a function of polarization. We note that as a special feature of two dimensions the susceptibility does not vanish in the limit of complete polarization. The zero-polarization limit is from Ref. 29, as discussed in the text.

monotonically decreasing function of the polarization (at least at this density). Another important difference between two and three dimensions is the behavior of the susceptibility in the limit of complete polarization. Because the minority spin density of states vanishes in the limit of complete polarization in three dimensions, the spin susceptibility also vanishes in that limit. However, in two dimensions the density of states is a constant and so, as can be seen in Fig. 7, the susceptibility is small but not zero in the complete polarization limit.

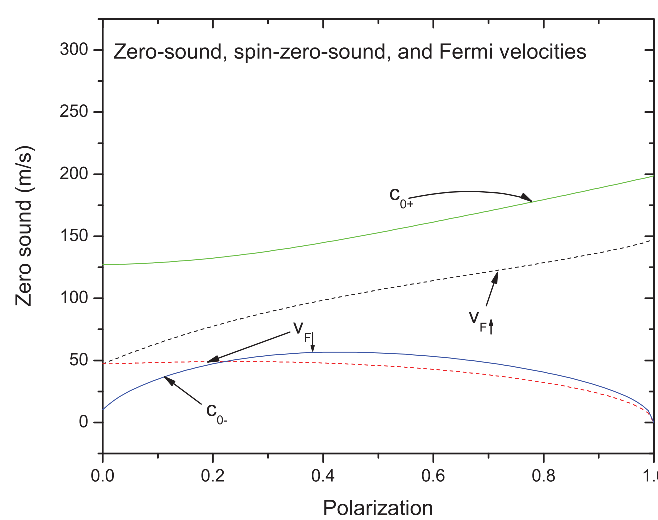

FIG. 8. (Color online) The zero-sound and spin-zero-sound speeds for ^{3}He in two dimensions at a density of $\overline{n} = 0.026$ Å^{-2} as a function of polarization. The solid lines are zero and spin zero sound, and the dashed lines are the up- and down-spin Fermi velocities. Zero sound is stable when the speed is much greater than $v_{F\uparrow}$ and spin zero sound is stable when the speed is much greater than $v_{F\downarrow}$. Thus, the results shown here imply that zero sound should be stable over the whole range of polarizations, whereas spin zero sound may be marginally stable at polarizations $\gtrsim 0.5$.

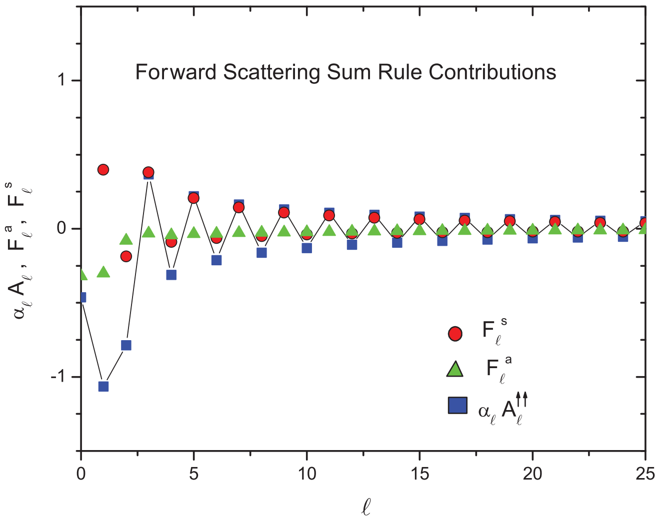

FIG. 9. (Color online) The components of the forward-scattering sum rule from $\ell = 0$ to 25 at zero polarization for ^{3}He in two dimensions at a density of $\overline{n} = 0.026$ Å^{-2} [see Eq. (3.49)]. The circles and triangles are F^s_ℓ and F^a_ℓ, respectively. The squares connected by a line as a guide for the eye are the components of the forward-scattering amplitudes $A^{\uparrow\uparrow}_\ell$ multiplied by the parameter α_ℓ. The oscillation in sign of the scattering amplitude is driven by the symmetric Landau parameter. The asymptotic value of the sum of the $\alpha_\ell A^{\uparrow\uparrow}_\ell$ components is ≈ -2.24. The magnitude of the final term shown in this figure is 0.026, two orders of magnitude smaller than the sum.

4. Zero sound

The allowed collective modes at absolute zero in fermion systems correspond to oscillations of the Fermi seas and are known as zero sound. In Sec. III C we derived the expressions for the zero-sound and spin-zero-sound speeds as a function of polarization Eq. (3.32). The results for our ^{3}He system are shown in Fig. 8. We pointed out in Sec. III C that we had to be careful to not apply this method if the Landau parameters $F^{s,a}_0$ were small relative to unity. In Table I we show the calculated values of the (unpolarized limit) Landau parameters $F^{s,a}_0$ and $F^{s,a}_1$ for our ^{3}He film.

Figure 8 shows that zero sound is a stable collective excitation for the entire polarization range. At zero polarization the stability of the zero-sound mode is due to the fairly large value of F^s_0. Likewise, the instability of the spin-zero-sound mode is due to the small and negative value of F^a_0. We note that the increase in the speed of the zero-sound mode as a function of polarization in the small polarization region is in agreement with the results of Ref. 24.

TABLE I. The symmetric and antisymmetric Landau parameters for $\ell = 0,1$ for ^{3}He in two dimensions at a density of $\overline{n} = 0.026$ Å^{-2} at zero polarization. Note that these are defined with the single-spin-state density of states, as discussed in the text.

Landau parameter	Value
F^s_0	2.89
F^a_0	−0.32
F^s_1	0.40
F^a_1	−0.30

V. CONCLUSION

We have extended the theory of the low-density, two-dimensional Fermi liquid to include the contributions of p-wave interactions. We have calculated exact, analytic expressions for the Landau parameters, $f^{\sigma\sigma'}_{\mathbf{pp'}}$, to quadratic order in the s- and p-wave interaction parameters for arbitrary polarization. A systematic procedure for performing the integrations that relies on the analytic behavior of the angular integral is discussed in the Appendixes. We have generalized to finite polarization the expressions for the effective mass, thermodynamic response, collective excitations, and the forward-scattering sum rule for a two-dimensional Fermi liquid. We discussed the application of the theory to a ^{3}He film.

The density of the ^{3}He film 0.026 Å^{-2} was chosen for convenience. At this density, it is simple to ascertain the experimental numbers that were used to fit the s-wave and p-wave parameters, and in addition we had available a microscopic particle-hole potential to obtain additional information. This density translates into an average spacing between particles of approximately 6.2 Å, which is large but perhaps not large enough for this low-density theory. Equivalently, this density corresponds to 40% of a conventional ^{3}He monolayer. We determined that the interaction parameters with values $g_0 = 0.76$ and $g_1 = 1.89$ Å^2 provide a fit to the effective mass and spin susceptibility experimental data in the zero polarization limit. We then calculated the polarization dependence of the state-dependent effective masses, heat capacity, compressibility, spin susceptibility, zero sound, and spin zero sound.

Our results predict a significant drop in the state-dependent effective masses as a function of polarization. Our calculated first sound speed in the zero polarization limit was consistent with a small-k limit of a completely independent microscopic structure factor. The results predict a significant stiffening of the equation of state with increasing polarization. We show explicitly that, unlike in three-dimensions, the spin-susceptibility decreases monotonically with increasing polarization and does not vanish at full polarization. Finally, our results indicate that zero sound will propagate at all polarizations. As discussed in the text, the method we employed to compute the zero sound speeds is not accurate enough to definitively rule out the possibility that spin zero sound may also propagate at higher polarizations and further work will continue on this issue. These results are all essentially predictions. Testing these predictions will be difficult since polarizing the ^{3}He system means ordering a *nuclear* moment. There are additional issues for future investigation especially concerning collective excitations in the polarized system. The question of attenuation as a function of polarization needs to be addressed. In addition, there is the interesting question of whether there is a Mermin's theorem[38] in a Fermi liquid with a finite polarization.

In Fig. 9 we show the components of the forward-scattering sum rule for our ^{3}He system. This is at zero polarization where the interaction components have been fit, and also where the two sum rules collapse into one. It is straightforward to calculate Landau parameters for any value of ℓ. In the figure, we show the symmetric and antisymmetric Landau parameters (circles and triangles, respectively) and also the scattering amplitudes A_ℓ for $\ell = 0$ to 25. For $\ell \gtrsim 3$ the contribution to the scattering amplitude is dominated by the symmetric term. It is the symmetric term that oscillates in sign not the antisymmetric term. The sum does not vanish: The approximate asymptotic value is -2.24. This nonzero result cannot be due to truncating the angular decomposition. The remaining possibilities are that this result is signaling a problem with our s-wave and p-wave interaction parameters or that we need to include d-wave and higher components in order to satisfy the sum rule. However, the interaction parameters are continuous functionals of the effective interaction [see, for example, Eq. (4.7)]. The sum rule clearly cannot be satisfied for arbitrary values of g_0 and g_1. The implication is that we need the higher angular momentum interaction parameters in order to satisfy the sum rules at arbitrary polarization even in the low-energy limit. This issue of the relationship between the low-density perturbation expansion and the forward-scattering sum rule is one to be addressed in future work.

There has been recent work by Akimoto, Cummings, and Hallock[39] measuring the Fermi liquid properties of a related system, ^{3}He–^{4}He thin-film mixtures. Mixture films are not isotropic. The ^{3}He rides on "top" of the ^{4}He film with the ^{4}He film playing the role of a dynamic, external adsorption potential. The fact that the ^{3}He floats on the ^{4}He component is clear from the measured[40] zero ^{3}He-concentration effective mass ≈ 1.38, which is in basic agreement with the classical result for a ball floating on a liquid surface. Most of the results in Sec. III are immediately applicable to this system if the ^{3}He component is treated as a two-dimensional Fermi liquid with a set of discrete single-particle states that model the effects of the adsorption potential.[21] We note that the presence of the discrete single-particle states for the ^{3}He induce important changes in the thermodynamic response.[41] Finally, if we assume that the ^{3}He effective interaction is local in character as in BBP theory,[42] then even the expression for the effective masses Eq. (3.5) can be utilized for this system.

Finally, we note that the ultracold Fermi gases form very-low-density systems that may be amenable to the analysis in this paper.[43] These systems can be prepared in quasi-two-dimensions as an incoherent mixture of hyperfine species that plays the role of a polarization. The ability to manipulate the strength of the interaction may allow the Landau parameters to be tuned to study the various zero-sound modes.

ACKNOWLEDGMENTS

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APPENDIX A: ANGULAR INTEGRATION

The integrations necessary to obtain the analytic results shown in Eqs. (2.29), (2.30), and (2.31) can be generated in three basic steps from the intermediate results shown in Eqs. (2.26) and (2.27). We note that all of these expressions have the same structure: one two-dimensional momentum integration, with each integrand containing a product of two δ functions and a ground-state Fermi function. We first change integration variable to that of the Fermi function momentum so that this function directly cuts off the momentum integration. Next we perform the principal value integration over the angular variable which introduces an important inequality. Finally, the remaining integrals are of known form.

The key result that needs to be demonstrated is that the angular integral

$$I = \int_0^{2\pi} \frac{dx}{a - b\cos x} \tag{A1}$$

is nonzero if and only if $|a| > |b|$. Here a and b are real and nonzero but not necessarily positive. For the case $|a| \leqslant |b|$ the angular integral in (A1) has poles located at $\theta_1 = \cos^{-1}(a/b)$ and $\theta_2 = 2\pi - \cos^{-1}(a/b)$. It is straightforward to show that the Cauchy principal value *vanishes*. Assume that $a < b$ and define the three integration regions by

$$I = \lim_{\epsilon\to 0} \int_0^{\theta_1-\epsilon} (\mathrm{I}) + \int_{\theta_1+\epsilon}^{\theta_2-\epsilon} (\mathrm{II}) + \int_{\theta_2+\epsilon}^{2\pi} (\mathrm{III}) \frac{dx}{a - b\cos x}. \tag{A2}$$

The integrals in the three regions are given by

$$\int \frac{dx}{a - b\cos x} = \begin{cases} -2\tanh^{-1}\left[\zeta\tan\left(\frac{x}{2}\right)\right]/\sqrt{b^2-a^2} & \text{Regions I and III,} \\ -2\coth^{-1}\left[\zeta\tan\left(\frac{x}{2}\right)\right]/\sqrt{b^2-a^2} & \text{Region II,} \end{cases} \tag{A3}$$

where we have defined $\zeta = (a+b)/\sqrt{b^2-a^2}$. By inspection, we note that at the end points $x = 0, 2\pi$ the integrals vanish. At the pole θ_1, we need to consider the following limit:

$$\lim_{\epsilon\to 0}\left\{\tanh^{-1}\left[\zeta\tan\left(\frac{\theta_1-\epsilon}{2}\right)\right] - \coth^{-1}\left[\zeta\tan\left(\frac{\theta_1+\epsilon}{2}\right)\right]\right\}. \tag{A4}$$

Using an identity, we find

$$= \lim_{\epsilon\to 0}\left\{\coth^{-1}\left[\frac{\zeta\left[\tan\left(\frac{\theta_1+\epsilon}{2}\right) - \tan\left(\frac{\theta_1-\epsilon}{2}\right)\right]}{\zeta^2\tan\left(\frac{\theta_1+\epsilon}{2}\right)\tan\left(\frac{\theta_1-\epsilon}{2}\right) - 1}\right]\right\},$$
$$= \lim_{\epsilon\to 0}\left\{\coth^{-1}\left[\frac{\zeta\sin\epsilon}{\zeta^2\sin\left(\frac{\theta_1+\epsilon}{2}\right)\sin\left(\frac{\theta_1-\epsilon}{2}\right) - \cos\left(\frac{\theta_1+\epsilon}{2}\right)\cos\left(\frac{\theta_1-\epsilon}{2}\right)}\right]\right\}. \tag{A5}$$

In the $\lim \epsilon \to 0$ the denominator goes to $\frac{1}{2}(1-\zeta^2)(\epsilon^2/2)$. Thus, the limit can be taken

$$= \lim_{\epsilon\to 0}\coth^{-1}\left[O\left(\frac{1}{\epsilon}\right)\right] \to 0, \tag{A6}$$

and the Cauchy principal value at pole θ_1 is zero. This result also follows for the pole at θ_2. We conclude that the integral Eq. (A1) vanishes in the case $|a| < |b|$ when defined as a Cauchy principal value.

Thus, the integral is nonzero only if $|a| > |b|$. In the following, we list the angular integrals that will be needed below:

$$\int_0^{2\pi} \frac{dx}{a - b\cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}\mathrm{sgn}(a)\Theta(|a| > |b|), \tag{A7a}$$

$$\int_0^{2\pi} \frac{\cos x}{a - b\cos x}dx = -\frac{2\pi}{b} + 2\pi\frac{a}{b}\frac{1}{\sqrt{a^2 - b^2}}\mathrm{sgn}(a)\Theta(|a| > |b|), \tag{A7b}$$

$$\int_0^{2\pi} \frac{\cos^2 x}{a - b\cos x}dx = -2\pi\frac{a}{b^2} + 2\pi\frac{a^2}{b^2}\frac{1}{\sqrt{a^2 - b^2}}\mathrm{sgn}(a)\Theta(|a| > |b|), \tag{A7c}$$

where sgn is the sign (signum) function. For convenience in denoting the parametric constraints, we have introduced a *generalized* step function $\Theta(x)$ defined such that

$$\Theta(x) = \begin{cases} 1 & \text{if } x \text{ is true,} \\ 0 & \text{otherwise.} \end{cases} \tag{A8}$$

Finally, we note that from Eq. (A7a) the angular integral is *independent* of the sign of b. This can also be seen by changing integration variables in Eq. (A1):

$$I = \int_0^{\pi} dx \left[\frac{1}{a - b\cos x} + \frac{1}{a + b\cos x}\right]. \tag{A9}$$

APPENDIX B: $f_{kk'}^{\uparrow\uparrow}$

We begin with the first integral in Eq. (2.26):

$$f_{kk'}^{\uparrow\uparrow}(I) = -16\frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} \frac{(pp'\cos(\theta_{pp'})^2\tau_1^2}{p^2 - p'^2} \times \delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}n_{\mathbf{p}'+\frac{\mathbf{q}}{2},\uparrow}. \tag{B1}$$

Using the δ functions to integrate over $\mathbf{p}$ and $\mathbf{q}$ yields $\mathbf{p} = \frac{1}{2}(\mathbf{k}' - \mathbf{k})$ and $\mathbf{q} = (\mathbf{k}' + \mathbf{k})$. We introduce a new integration variable

$$\mathbf{p}'' = \mathbf{p}' + \frac{\mathbf{q}}{2} = \mathbf{p}' + \frac{1}{2}\left(\mathbf{k}' + \mathbf{k}\right). \tag{B2}$$

In terms of these variables the denominator becomes

$$p^2 - p'^2 = -(p''^2 + k_\uparrow^2\cos\theta_{kk'}) + \left(2k_\uparrow\cos\left(\frac{\theta_{kk'}}{2}\right)p''\right)\cos(\theta''),$$

$$\equiv -a + b\cos(\theta''), \tag{B3}$$

where $\theta'' = \theta_{\mathbf{p}''\mathbf{q}}$. Similarly, the numerator can be written

$$[pp'\cos(\theta_{pp'})]^2 = k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)(p'')^2\sin^2(\theta''). \tag{B4}$$

Thus, the integral becomes

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{I}) = -16\frac{m}{\hbar^2}\frac{L^2}{(2\pi)^2}\tau_1^2 k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)\int_0^{k_\uparrow} dp''(p'')^3 \times \int_0^{2\pi} d\theta''\frac{1 - \cos^2(\theta'')}{-a + b\cos(\theta'')}. \tag{B5}$$

Performing the angular integrations yields

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{I}) = -16\frac{m}{\hbar^2}\frac{L^2}{(2\pi)}\tau_1^2 k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)\int_0^{k_\uparrow} dp''(p'')^3 \times \frac{1}{b^2}(\sqrt{a^2 - b^2}\mathrm{sgn}(a)\Theta(|a| > |b|) - a). \tag{B6}$$

The constraint in the first term of (B6), $|(p'')^2 + k_\uparrow^2\cos\theta_{kk'}| > |2k_\uparrow\cos(\frac{\theta_{kk'}}{2})p''|$, restricts the allowed values of p''. It is easily shown that $0 \leqslant p'' \leqslant k_\uparrow\sqrt{1 - \sin\theta_{kk'}}$. The p'' integrations are elementary and we find

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{I}) = \left(\frac{mL^2}{2\pi\hbar^2}\right)\tau_1^2 k_\uparrow^4\tan^2\left(\frac{\theta_{kk'}}{2}\right)\left\{(1 + 2\cos\theta_{kk'}) - \left[\cos\theta_{kk'} + \sin^2(\theta_{kk'})\ln\left(\tan\left(\frac{\theta_{kk'}}{2}\right)\right)\right]\right\}. \tag{B7}$$

The second integral is

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{II}) = -32\frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{[pp'\cos(\theta_{pp'})]^2\tau_1^2}{p^2 - p'^2}\delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}\delta_{\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}}n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}. \tag{B8}$$

The δ functions yield $\mathbf{p} = \mathbf{p}' + (\mathbf{k}' - \mathbf{k})$, $\mathbf{q} = -2\mathbf{p}' + 2\mathbf{k}$, and the new integration variable becomes $\mathbf{p}'' = -\mathbf{p} + \frac{\mathbf{q}}{2}$. The denominator is

$$p^2 - p'^2 = -a + b\cos(\theta''), \tag{B9}$$

where

$$\begin{cases} a = 2k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right), \\ b = 2k_\uparrow|\sin\left(\frac{\theta_{kk'}}{2}\right)|p'', \end{cases} \tag{B10}$$

and $\theta'' = \theta_{\mathbf{p}'',\mathbf{k}-\mathbf{k}'}$. Now consider the numerator:

$$pp'\cos\theta_{pp'} = \frac{(p'')^2}{4} - \frac{k_\uparrow}{2}\left[\cos\theta''|\sin\left(\frac{\theta_{kk'}}{2}\right)| + \sin\theta''\cos\left(\frac{\theta_{kk'}}{2}\right)\right]p'' + \frac{k_\uparrow^2}{4}(2\cos\theta_{kk'} - 1). \tag{B11}$$

We square (B11) and neglect terms that are odd powers of $\sin\theta''$. This leaves

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{II}) = -\frac{2m}{\hbar^2}\frac{L^2}{(2\pi)^2}\tau_1^2\int_0^{k_\uparrow} dp''p''\int_0^{2\pi} d\theta''\frac{\alpha + \beta\cos\theta'' + \gamma\cos^2\theta''}{-a + b\cos\theta''}, \tag{B12}$$

where

$$\begin{cases} \alpha = (p'')^4 + 6k_\uparrow^2 (p'')^2 \cos\theta_{kk'} + k_\uparrow^4 (2\cos\theta_{kk'} - 1)^2, \\ \beta = -\left[4k_\uparrow (p'')^3 + 4k_\uparrow^3 p'' (2\cos\theta_{kk'} - 1)\right] |\sin\left(\frac{\theta_{kk'}}{2}\right)|, \\ \gamma = -4k_\uparrow^2 (p'')^2 \cos\theta_{kk'}. \end{cases} \tag{B13}$$

The angular integrals can be done using Eqs. (A7):

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{II}) = -2\left(\frac{mL^2}{2\pi\hbar^2}\right)\tau_1^2 \int_0^{k_\uparrow} dp'' p'' \left\{ \left[\frac{1}{b}\beta + \frac{a}{b^2}\gamma\right] - \left[\alpha + \frac{a}{b}\beta + \frac{a^2}{b^2}\gamma\right] \frac{\Theta(k_\uparrow |\sin\left(\frac{\theta_{kk'}}{2}\right)| > p'')}{2k_\uparrow |\sin\left(\frac{\theta_{kk'}}{2}\right)| \sqrt{k_\uparrow^2 \sin^2\left(\frac{\theta_{kk'}}{2}\right) - (p'')^2}} \right\}. \tag{B14}$$

The momentum integrations with the square root factor can be done by utilizing the following integrals:

$$\int_0^\xi dx \frac{1}{\sqrt{x_0 - x}} = 2x_0^{\frac{1}{2}} - 2\sqrt{x_0 - \xi}, \tag{B15a}$$

$$\int_0^\xi dx \frac{x}{\sqrt{x_0 - x}} = \frac{4}{3}x_0^{\frac{3}{2}} - \frac{2}{3}\sqrt{x_0 - \xi}\,(2x_0 + \xi), \tag{B15b}$$

$$\int_0^\xi dx \frac{x^2}{\sqrt{x_0 - x}} = \frac{16}{15}x_0^{\frac{5}{2}} - \frac{2}{15}\sqrt{x_0 - \xi}\,\left(8x_0^2 + 4x_0\xi + 3\xi^2\right). \tag{B15c}$$

After some algebra, we obtain

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{II}) = \left(\frac{mL^2}{2\pi\hbar^2}\right)\tau_1^2 2k_\uparrow^4 \left[3 - 12\sin^2\left(\frac{\theta_{kk'}}{2}\right) + \frac{224}{15}\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right]. \tag{B16}$$

The third integral is

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{III}) = -\frac{2m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} \frac{\left\{\tau_0^2 + 4pp'\cos(\theta_{pp'})\tau_0\tau_1 + 4[pp'\cos(\theta_{pp'})]^2\tau_1^2\right\}}{p^2 - p'^2}\left[\delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}\delta_{\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}}n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}\right]. \tag{B17}$$

The δ function constraints are the same as for $f_{kk'}^{\uparrow\uparrow}(\mathrm{II})$ [Eq. (B8)]. The momentum integration variable is also the same except now the integration is over the down-spin Fermi sea. We can now square (B11) and neglect terms that are odd powers of $\sin\theta''$. This leaves

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{III}) = -\frac{2m}{\hbar^2}\frac{L^2}{(2\pi)^2}\int_0^{k_\downarrow} dp'' p'' \int_0^{2\pi} d\theta'' \frac{\mathcal{A} + \mathcal{B}\cos\theta'' + \mathcal{C}\cos^2\theta''}{-a + b\cos\theta''}, \tag{B18}$$

where

$$\begin{cases} \mathcal{A} = \tau_0^2 + [(p'')^2 + k_\uparrow^2(2\cos\theta_{kk'} - 1)]\tau_0\tau_1 + \frac{1}{4}\alpha\tau_1^2, \\ \mathcal{B} = -\left[2k_\uparrow |\sin\left(\frac{\theta_{kk'}}{2}\right)| p''\right]\tau_0\tau_1 + \frac{1}{4}\beta\tau_1^2, \\ \mathcal{C} = \frac{1}{4}\gamma\tau_1^2. \end{cases} \tag{B19}$$

The parameters a,b are defined in Eqs. (B10), and α,β,γ are defined in Eqs. (B13). The angular integrals can be done using Eqs. (A7):

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{III}) = -\frac{2m}{\hbar^2}\frac{L^2}{(2\pi)}\int_0^{k_\downarrow} dp'' p'' \left\{ \left[\frac{1}{b}\mathcal{B} + \frac{a}{b^2}\mathcal{C}\right] - \left[\mathcal{A} + \frac{a}{b}\mathcal{B} + \frac{a^2}{b^2}\mathcal{C}\right] \frac{\Theta(k_\uparrow |\sin\left(\frac{\theta_{kk'}}{2}\right)| > p'')}{2k_\uparrow |\sin\left(\frac{\theta_{kk'}}{2}\right)| \sqrt{k_\uparrow^2 \sin^2\left(\frac{\theta_{kk'}}{2}\right) - (p'')^2}} \right\}. \tag{B20}$$

Due to the constraint introduced by the angular integration in the second term of Eq. (B20), the upper limit of that p'' integration becomes $p''_{\max} = \min\,[k_\downarrow, k_\uparrow |\sin\left(\frac{\theta_{kk'}}{2}\right)|]$. The momentum integrations are handled by Eqs. (B15), and we find the following after some algebra:

$$f_{kk'}^{\uparrow\uparrow}(\mathrm{III}) = \left(\frac{mL^2}{2\pi\hbar^2}\right)\left\{\tau_0^2 + \left[\left(1 - \frac{16}{3}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^2 + k_\downarrow^2\right]\tau_0\tau_1 + \left[\left(\frac{1}{4} - 3\sin^2\left(\frac{\theta_{kk'}}{2}\right) + \frac{112}{15}\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^4\right.\right.$$
$$\left.\left. + \left(1 - 3\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^2 k_\downarrow^2 + \frac{1}{4}k_\downarrow^4\right]\tau_1^2\right\} - \left(\frac{mL^2}{2\pi\hbar^2}\right)\left\{\left(1 - \frac{k_\downarrow^2}{k_\uparrow^2\sin^2\left(\frac{\theta_{kk'}}{2}\right)}\right)^{\frac{1}{2}}\right.$$
$$\times\left[\tau_0^2 + \left[\left(1 - \frac{20}{3}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^2 - \frac{1}{3}k_\downarrow^2\right]\tau_0\tau_1 + \left[\left(\frac{1}{4} - 3\sin^2\left(\frac{\theta_{kk'}}{2}\right) + \frac{112}{15}\sin^4\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^4\right.\right.$$
$$\left.\left.\left. + \left(\frac{1}{2} - \frac{19}{15}\sin^2\left(\frac{\theta_{kk'}}{2}\right)\right)k_\uparrow^2 k_\downarrow^2 + \frac{1}{20}k_\downarrow^4\right]\tau_1^2\right]\right\} \times \Theta\left(k_\uparrow\left|\sin\left(\frac{\theta_{kk'}}{2}\right)\right| > k_\downarrow\right). \tag{B21}$$

APPENDIX C: $f_{kk'}^{\downarrow\downarrow}$

For $f_{kk'}^{\downarrow\downarrow}$, we simply flip the arrows in $f_{kk'}^{\uparrow\uparrow}(\mathrm{I})$ [Eq. (B7)] and $f_{kk'}^{\uparrow\uparrow}(\mathrm{II})$[Eq. (B16)]. For $f_{kk'}^{\downarrow\downarrow}(\mathrm{III})$, we note that because of our convention that $k_\downarrow \leqslant k_\uparrow$, the inequality $k_\downarrow|\sin(\frac{\theta_{kk'}}{2})| > k_\uparrow$ can never be satisfied, and so for $f_{kk'}^{\downarrow\downarrow}(\mathrm{III})$ we flip the arrows in Eq. (B21) and then ignore the terms multiplied by the generalized step function.

APPENDIX D: $f_{kk'}^{\uparrow\downarrow}$

From Eq. (2.27), the first integral is

$$f_{kk'}^{\uparrow\downarrow}(I) = -\frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{\tau_0^2 + 4(pp'\cos\theta_{pp'})\tau_0\tau_1 + 4\left(pp'\cos\theta_{pp'}\right)^2\tau_1^2}{p^2 - p'^2}\delta_{\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{\mathbf{p}+\frac{\mathbf{q}}{2},\uparrow}. \tag{D1}$$

The δ functions yield $-\mathbf{p} + \frac{\mathbf{q}}{2} = \mathbf{k}'$ and $\mathbf{p}' + \frac{\mathbf{q}}{2} = \mathbf{k}$, and we can integrate over $\mathbf{p}$ and $\mathbf{p}'$. We introduce a new integration variable $\mathbf{p}'' = \mathbf{p} + \frac{\mathbf{q}}{2}$. In terms of these variables the denominator becomes

$$\begin{aligned} p^2 - p'^2 &= -k_\uparrow^2 + k_\uparrow k_\downarrow\cos\theta_{kk'} + |\mathbf{k} - \mathbf{k}'|p''\cos\theta'', \\ &\equiv -a + b\cos\theta'', \end{aligned} \tag{D2}$$

where $\theta'' = \theta_{\mathbf{p}'',\mathbf{k}-\mathbf{k}'}$. The basic term in the numerator can be written

$$pp'\cos\theta_{pp'} = A + B\cos\theta'' + C\sin\theta'', \tag{D3}$$

where

$$\begin{cases} A = -\frac{1}{4}(p'')^2 + \frac{1}{4}k_\downarrow^2 - \frac{1}{2}k_\uparrow k_\downarrow\cos\theta_{kk'}, \\ B = -\frac{1}{2}k_\uparrow p''(k_\uparrow - k_\downarrow\cos\theta_{kk'})/|\mathbf{k} - \mathbf{k}'|, \\ C = \frac{1}{2}k_\uparrow p''(k_\downarrow\sin\theta_{kk'})/|\mathbf{k} - \mathbf{k}'|. \end{cases} \tag{D4}$$

Substituting (D2) and (D3) into (D1) the integral becomes

$$f_{kk'}^{\uparrow\downarrow}(\mathrm{I}) = -\frac{m}{\hbar^2}\frac{L^2}{(2\pi)^2}\int_0^{k_\uparrow}dp''(p'')\int_0^{2\pi}d\theta''\left[\frac{\alpha + \beta\cos\theta'' + \gamma\cos^2\theta''}{-a + b\cos\theta''}\right], \tag{D5}$$

where

$$\begin{cases} \alpha = \tau_0^2 + 4A\tau_0\tau_1 + 4(A^2 + C^2)\tau_1^2, \\ \beta = 4B\tau_0\tau_1 + 8AB\tau_1^2, \\ \gamma = 4(B^2 - C^2)\tau_1^2. \end{cases} \tag{D6}$$

Terms linear in $\sin\theta''$ has been omitted. Performing the angular integration yields

$$f_{kk'}^{\uparrow\downarrow}(\mathrm{I}) = -\left(\frac{mL^2}{2\pi\hbar^2}\right)\int_0^{k_\uparrow} dp''p''\left\{\left[\frac{1}{b}\beta + \frac{a}{b^2}\gamma\right] - \left[\alpha + \frac{a}{b}\beta + \frac{a^2}{b^2}\gamma\right]\frac{\Theta(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'} > |\mathbf{k}-\mathbf{k}'|p'')}{\sqrt{k_\uparrow^2(|\mathbf{k}-\mathbf{k}'|^2 - k_\downarrow^2\sin^2\theta_{kk'}) - |\mathbf{k}-\mathbf{k}'|^2(p'')^2}}\right\}. \tag{D7}$$

The constraint in the second term states that p'' must be less than the projection of $\mathbf{k}$ onto $\mathbf{k}-\mathbf{k}'$ which for nonzero $\mathbf{k}'$ must be less than $k_\uparrow$. For simplicity, we separate $f_{kk'}^{\uparrow\downarrow}(\mathrm{I})$ [Eq. (D7)] into two parts. Thus,

$$f_{kk'}^{\uparrow\downarrow}(\mathrm{I}) = f_{kk'}^{\uparrow\downarrow}(\text{I-1}) + f_{kk'}^{\uparrow\downarrow}(\text{I-2}), \tag{D8}$$

where the first part corresponds to the first term in square brackets and the second term corresponds to the second term with the constraint on the upper limit of integration. The term in square brackets in $f_{kk'}^{\uparrow\downarrow}(\text{I-1})$ is a simple quadratic in $(p'')^2$ and so we immediately obtain

$$f_{kk'}^{\uparrow\downarrow}(\text{I-1}) = \left(\frac{mL^2}{2\pi\hbar^2}\right)\frac{(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})}{|\mathbf{k}-\mathbf{k}'|^2}\left\{k_\uparrow^2\tau_0\tau_1 + \left[k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'} - \frac{k_\uparrow^2}{|\mathbf{k}-\mathbf{k}'|^2}(k_\uparrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'} + k_\downarrow^2\cos 2\theta_{kk'}) - \frac{k_\uparrow^2}{2}\right]\frac{k_\uparrow^2}{2}\tau_1^2\right\}. \tag{D9}$$

For the second integral, we first reorganize the integrand in powers of p''. The momentum integrals are done by utilizing Eqs. (B15) where the integration upper limit is given by $p''_{\max} = k_\uparrow\cos\theta_{\mathbf{k},\mathbf{k}-\mathbf{k}'} = (k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})/|\mathbf{k}-\mathbf{k}'|$. These yield

$$\begin{aligned} f_{kk'}^{\uparrow\downarrow}(\text{I-2}) = &\left(\frac{mL^2}{2\pi\hbar^2}\right)\left(\frac{k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'}}{|\mathbf{k}-\mathbf{k}'|^2}\right)\left\{\tau_0^2 + \left[(k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'}) - \frac{8}{3}\frac{(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})^2}{|\mathbf{k}-\mathbf{k}'|^2}\right]\tau_0\tau_1\right. \\ &+ \left\{\frac{1}{4}(k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'})^2 + \left(\frac{(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})^2}{|\mathbf{k}-\mathbf{k}'|^2}\right)\right. \\ &\times\left[-\frac{4}{3}(k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'}) + \frac{k_\uparrow^2(k_\uparrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'} + k_\downarrow^2\cos 2\theta_{kk'})}{|\mathbf{k}-\mathbf{k}'|^2}\right. \\ &\left.\left.\left.+\frac{2}{3}\frac{(k_\uparrow k_\downarrow\sin\theta_{kk'})^2}{|\mathbf{k}-\mathbf{k}'|^2}\right] + \frac{4}{5}\left(\frac{(k_\uparrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})^4}{|\mathbf{k}-\mathbf{k}'|^4}\right)\right\}\tau_1^2\right\}. \end{aligned} \tag{D10}$$

The next term to be considered in Eq. (2.27) is

$$f_{kk'}^{\uparrow\downarrow}(\mathrm{II}) = -\frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{\tau_0^2 + 4(pp'\cos\theta_{pp'})\tau_0\tau_1 + 4(pp'\cos\theta_{pp'})^2\tau_1^2}{p^2 - p'^2}\delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{-\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{-\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}. \tag{D11}$$

We note that under the transformations $\mathbf{p} \to -\mathbf{p}$ and $\mathbf{p}' \to -\mathbf{p}'$ this integral becomes

$$f_{kk'}^{\uparrow\downarrow}(\mathrm{II}) = -\frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{\tau_0^2 + 4(pp'\cos\theta_{pp'})\tau_0\tau_1 + 4(pp'\cos\theta_{pp'})^2\tau_1^2}{p^2 - p'^2}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{\mathbf{p}'+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{\mathbf{p}+\frac{\mathbf{q}}{2},\downarrow}, \tag{D12}$$

which is identical to (D1), except for the spin-down dependence of the Fermi function. We can obtain the integral of Eq. (D12) just by reversing spins in Eqs. (D9) and (D10). Thus, we find

$$\begin{aligned} f_{kk'}^{\uparrow\downarrow}(\text{II-1}) = &\left(\frac{mL^2}{2\pi\hbar^2}\right)\frac{(k_\downarrow^2 - k_\uparrow k_\downarrow\cos\theta_{kk'})}{|\mathbf{k}-\mathbf{k}'|^2}\left\{k_\downarrow^2\tau_0\tau_1 + \left[k_\uparrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'} - \frac{k_\downarrow^2}{|\mathbf{k}-\mathbf{k}'|^2}\right.\right. \\ &\left.\left.\times(k_\downarrow^2 - 2k_\uparrow k_\downarrow\cos\theta_{kk'} + k_\uparrow^2\cos 2\theta_{kk'}) - \frac{k_\downarrow^2}{2}\right]\frac{k_\downarrow^2}{2}\tau_1^2\right\}, \end{aligned} \tag{D13}$$

and similarly

$$f_{kk'}^{\uparrow\downarrow}(\text{II-2}) = \left(\frac{mL^2}{2\pi\hbar^2}\right)\left(\frac{k_\downarrow^2 - k_\downarrow k_\uparrow\cos\theta_{kk'}}{|\mathbf{k}-\mathbf{k}'|^2}\right)\left\{\tau_0^2 + \left[\left(k_\uparrow^2 - 2k_\downarrow k_\uparrow\cos\theta_{kk'}\right) - \frac{8}{3}\frac{(k_\downarrow^2 - k_\downarrow k_\uparrow\cos\theta_{kk'})^2}{|\mathbf{k}-\mathbf{k}'|^2}\right]\tau_0\tau_1\right.$$

$$+\left\{\frac{1}{4}(k_\uparrow^2-2k_\downarrow k_\uparrow\cos\theta_{kk'})^2+\left(\frac{(k_\downarrow^2-k_\downarrow k_\uparrow\cos\theta_{kk'})^2}{|\mathbf{k}-\mathbf{k}'|^2}\right)\right.$$
$$\times\left[-\frac{4}{3}(k_\uparrow^2-2k_\downarrow k_\uparrow\cos\theta_{kk'})+\frac{k_\downarrow^2(k_\downarrow^2-2k_\downarrow k_\uparrow\cos\theta_{kk'}+k_\uparrow^2\cos 2\theta_{kk'})}{|\mathbf{k}-\mathbf{k}'|^2}\right.$$
$$\left.\left.+\frac{2}{3}\frac{(k_\downarrow k_\uparrow\sin\theta_{kk'})^2}{|\mathbf{k}-\mathbf{k}'|^2}\right]+\frac{4}{5}\left(\frac{(k_\downarrow^2-k_\downarrow k_\uparrow\cos\theta_{kk'})^4}{|\mathbf{k}-\mathbf{k}'|^4}\right)\right\}\tau_1^2\Bigg\}. \tag{D14}$$

The third term in Eq. (2.27) is

$$f_{kk'}^{\uparrow\downarrow}(\text{III})=-\frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{\tau_0^2+4(pp'\cos\theta_{pp'})\tau_0\tau_1+4(pp'\cos\theta_{pp'})^2\tau_1^2}{p^2-p'^2}\delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{\mathbf{p}'+\frac{\mathbf{q}}{2},\uparrow}. \tag{D15}$$

The δ functions yield $-\mathbf{p}+\frac{\mathbf{q}}{2}=\mathbf{k}'$ and $\mathbf{p}+\frac{\mathbf{q}}{2}=\mathbf{k}$, and so we can integrate over $\mathbf{p}$ and $\mathbf{p}'$. We can solve these for the parameters $\mathbf{p}$ and $\mathbf{q}$ that play an important role in this section:

$$\mathbf{p}=\tfrac{1}{2}(\mathbf{k}-\mathbf{k}'), \tag{D16a}$$
$$\mathbf{q}=\mathbf{k}+\mathbf{k}'. \tag{D16b}$$

We introduce a new integration variable $\mathbf{p}''=\mathbf{p}'+\frac{\mathbf{q}}{2}$. In terms of this variable the denominator becomes

$$\begin{aligned}p^2-p'^2&=-((p'')^2+\mathbf{k}\cdot\mathbf{k}')+|\mathbf{k}+\mathbf{k}'|p''\cos\theta'',\\&\equiv -a+b\cos\theta'',\end{aligned} \tag{D17}$$

where $\theta''=\theta_{\mathbf{p}'',\mathbf{k}+\mathbf{k}'}$. Similarly, the numerator can be written in terms of these variables by using

$$pp'\cos\theta_{pp'}=-\tfrac{1}{4}(k^2-k'^2)+\tfrac{1}{2}p''|\mathbf{k}-\mathbf{k}'|\cos\theta_{\mathbf{p}'',\mathbf{k}-\mathbf{k}'}. \tag{D18}$$

Substituting (D17) and (D18) into (D15) the integral becomes

$$f_{kk'}^{\uparrow\downarrow}(\text{III})=-\frac{m}{\hbar^2}\frac{L^2}{(2\pi)^2}\int_0^{k_\uparrow}dp''(p'')\int_0^{2\pi}d\theta''\left[\frac{\alpha+\beta\cos\theta''+\gamma\cos^2\theta''}{-a+b\cos\theta''}\right], \tag{D19}$$

where

$$\begin{cases}\alpha=\tau_0^2-(k^2-k'^2)\tau_0\tau_1+\left[\frac{1}{4}(k^2-k'^2)+(p'')^2|\mathbf{k}-\mathbf{k}'|^2\sin^2\theta_{p,q}\right]\tau_1^2,\\ \beta=2p''|\mathbf{k}-\mathbf{k}'|\cos\theta_{pq}\tau_0\tau_1-(k^2-k'^2)|\mathbf{k}-\mathbf{k}'|p''\cos\theta_{pq}\tau_1^2,\\ \gamma=(p'')^2|\mathbf{k}-\mathbf{k}'|^2\cos(2\theta_{p,q})\tau_1^2.\end{cases} \tag{D20}$$

Terms linear in $\sin\theta''$ have been omitted. Performing the angular integration yields

$$f_{kk'}^{\uparrow\downarrow}(\text{III})=-\left(\frac{mL^2}{2\pi\hbar^2}\right)\int_0^{k_\uparrow}dp''p''\left\{\left[\frac{1}{b}\beta+\frac{a}{b^2}\gamma\right]-\left[\alpha+\frac{a}{b}\beta+\frac{a^2}{b^2}\gamma\right]\frac{\text{sgn}(a)\Theta(|a|>|b|)}{\sqrt{a^2-b^2}}\right\}. \tag{D21}$$

The first integrand in (D21) (the term without the constraint) is a simple quadratic in p'' and so the result is immediately

$$f_{kk'}^{\uparrow\downarrow}(\text{III-1})=-\left(\frac{mL^2}{2\pi\hbar^2}\right)\left\{\left(\frac{2p}{q}\right)k_\uparrow^2\cos\theta_{pq}\tau_0\tau_1+\left[-2qp\cos^2\theta_{pq}+\left(\frac{2p}{q}\right)\cos(2\theta_{p,q})\left(\left(-p^2+\frac{q^2}{4}\right)+\frac{k_\uparrow^2}{2}\right)\right]\left(\frac{2p}{q}\right)\frac{k_\uparrow^2}{2}\tau_1^2\right\}. \tag{D22}$$

For the second term in (D21) we must analyze the constraint. From Eq. (D17), the allowed integration range is determined by $|a|>|b|$ which implies $|((p'')^2+\mathbf{k}\cdot\mathbf{k}')|>|\mathbf{k}+\mathbf{k}'|p''$. It can be shown that for both $\mathbf{k}\cdot\mathbf{k}'>0$ and $\mathbf{k}\cdot\mathbf{k}'<0$ the constraint fixes the upper limit of the momentum integration $p''_{\max}$, where $p''_{\max}=\frac{1}{2}q-p$. Thus, after performing the angular integration the second integral can be written,

$$f_{kk'}^{\uparrow\downarrow}(\text{III-2}) = \left(\frac{mL^2}{2\pi\hbar^2}\right)\text{sgn}(\cos\theta_{kk'})\int_0^{p''_{\max}} dp''\,p''\left[\frac{[A + B(p'')^2 + C(p'')^4]}{\sqrt{(p'')^4 + (2\mathbf{k}\cdot\mathbf{k}' - |\mathbf{k}+\mathbf{k}'|^2)(p'')^2 + (\mathbf{k}\cdot\mathbf{k}')^2}}\right], \tag{D23}$$

where

$$\begin{cases} A = \tau_0^2 + \left[-(k^2 - k'^2) + 2\left(\frac{2p}{q}\right)\mathbf{k}\cdot\mathbf{k}'\cos\theta_{pq}\right]\tau_0\tau_1 + \left[(k^2 - k'^2)\left(\frac{1}{4}(k^2 - k'^2) - \left(\frac{2p}{q}\right)(\mathbf{k}\cdot\mathbf{k}')\cos\theta_{pq}\right)\right. \\ \qquad\qquad \left. + \left(\frac{2p}{q}\right)^2(\mathbf{k}\cdot\mathbf{k}')^2\cos(2\theta_{pq})\right]\tau_1^2, \\ B = \left[2\left(\frac{2p}{q}\right)\cos\theta_{pq}\right]\tau_0\tau_1 + \left[|\mathbf{k}-\mathbf{k}'|^2\sin^2\theta_{pq} - \left(\frac{2p}{q}\right)(k^2 - k'^2)\cos\theta_{pq} + 2\left(\frac{2p}{q}\right)^2(\mathbf{k}\cdot\mathbf{k}')\cos(2\theta_{pq})\right]\tau_1^2, \\ C = \left[\left(\frac{2p}{q}\right)^2\cos(2\theta_{pq})\right]\tau_1^2. \end{cases}$$

The momentum integrations with the particular square root factors in Eq. (D23) can be done by utilizing the following integrals:

$$\int_0^{\xi} dx\frac{A}{\sqrt{x^2 + Dx + E}} = A\ln\left(\frac{D + 2\xi}{D + 2\sqrt{E}}\right), \tag{D24a}$$

$$\int_0^{\xi} dx\frac{Bx}{\sqrt{x^2 + Dx + E}} = B\left[-\sqrt{E} - \frac{1}{2}D\ln\left(\frac{D + 2\xi}{D + 2\sqrt{E}}\right)\right], \tag{D24b}$$

$$\int_0^{\xi} dx\frac{Cx^2}{\sqrt{x^2 + Dx + E}} = C\left[\frac{3}{4}D\sqrt{E} + \frac{1}{8}\left(3D^2 - 4E\right)\ln\left(\frac{D + 2\xi}{D + 2\sqrt{E}}\right)\right], \tag{D24c}$$

where we have used the fact that ξ is a root of the denominator of the integrand. After some algebra, we find

$$f_{kk'}^{\uparrow\downarrow}(\text{III-2}) = \left(\frac{mL^2}{2\pi\hbar^2}\right)\frac{1}{2}\left\{-\left[\tau_0^2 + 2p^4\tau_1^2\right]\ln\left(\frac{2p}{q}\right) + \left(\frac{q^2}{4} - p^2\right)\left[-\frac{4p}{q}\cos\theta_{pq}\tau_0\tau_1 + \frac{1}{2}p^2\left(1 + \frac{4p^2}{q^2}\right)\cos(2\theta_{pq})\tau_1^2\right]\right\}. \tag{D25}$$

The final integral from Eq. (2.27) to be considered is

$$f_{kk'}^{\uparrow\downarrow}(\text{IV}) = -\frac{m}{\hbar^2}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}}\frac{\tau_0^2 + 4pp'\cos\theta_{pp'}\tau_0\tau_1 + 4(pp'\cos\theta_{pp'})^2\tau_1^2}{p^2 - p'^2}\delta_{\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}}\delta_{-\mathbf{p}+\frac{\mathbf{q}}{2},\mathbf{k}'}n_{-\mathbf{p}'+\frac{\mathbf{q}}{2},\downarrow}. \tag{D26}$$

If we let $\mathbf{p}' \to -\mathbf{p}'$ and reverse spins then Eq. (D26) becomes identical to (D15), except we must also let $V(\mathbf{p}-\mathbf{p}') \to V(\mathbf{p}+\mathbf{p}')$. From Eq. (2.13) this is accomplished just by flipping the sign of the terms proportional to $\tau_0\tau_1$. Thus, from Eqs. (D22) and (D25),

$$f_{kk'}^{\uparrow\downarrow}(\text{IV}) = \left(\frac{mL^2}{2\pi\hbar^2}\right)\left\{\left(\frac{2p}{q}\right)k_\downarrow^2\cos\theta_{pq}\tau_0\tau_1 + \left[2qp\cos^2\theta_{pq} - \left(\frac{2p}{q}\right)\cos(2\theta_{pq})\left(\left(-p^2 + \frac{q^2}{4}\right) + \frac{k_\downarrow^2}{2}\right)\right]\left(\frac{2p}{q}\right)\frac{k_\downarrow^2}{2}\tau_1^2\right.$$
$$\left. + \frac{1}{2}\left\{-\left[\tau_0^2 + 2p^4\tau_1^2\right]\ln\left(\left(\frac{2p}{q}\right)\right) + \left(\frac{q^2}{4} - p^2\right)\left[\frac{4p}{q}\cos\theta_{pq}\tau_0\tau_1 + \frac{1}{2}p^2\left(1 + \frac{4p^2}{q^2}\right)\cos(2\theta_{pq})\tau_1^2\right]\right\}\right\}. \tag{D27}$$

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