Spin imbalance effect on the Larkin-Ovchinnikov-Fulde-Ferrel state

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We study spin imbalance effects on the Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) state relevant for superconductors under a strong magnetic field and spin polarized ultracold Fermi gas. We obtain the exact solution for the condensates with arbitrary spin imbalance and the fermion spectrum perturbatively in the presence of small spin imbalance. We also obtain fermion zero mode exactly without perturbation theory.

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I. INTRODUCTION

The exotic superconducting state called Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) state has been proposed to arise in superconductors under a strong magnetic field.[1,2] The LOFF state has a spatially varying order parameter associated with Cooper pairs with finite center-of-mass momentum. If a strong magnetic field induces spin polarization, such Cooper pairs are considered to form between electrons with different Fermi momenta. The LOFF state is also relevant for the physics of color superconductivity where quarks with different masses form pairs.[3] This state was not observed for over 40 years since its proposal, in spite of tremendous efforts. In the last couple of years, there have been several claims of its observation in heavy fermion materials[4] and organic superconductors,[5] but direct confirmation is yet to be given (see Ref. 6 for a review).

Recent developments of research in cold atomic Fermi gases have renewed interest in the LOFF state (see Ref. 7 for a review). In two-component Fermi gases consisting of atoms in two different hyperfine states, (pseudo)spin polarization can be controlled by changing the populations of the two components.[7] Furthermore, atomic interaction can be tuned in this system by using the Feshbach resonance which allows one to explore the interesting BCS-BEC crossover physics. Thus, a spin polarized Fermi gas is an ideal system for realizing and exploring the LOFF state. In Fermi gases in a toroidal trap, it has been shown that a new state called angular LOFF state is possible in which the rotational symmetry is spontaneously broken,[8] instead of the translational symmetry for the usual LOFF state. Recently, observation of spin polarized superfluid state was reported[9] and it is expected that the LOFF state has been achieved in this experiment. However, direct observation of its oscillating order parameter is still lacking.

The Bogoliubov–de Gennes (BdG) equation has been widely employed to study the LOFF state. Machida and Nakanishi[10] derived the self-consistent LOFF state solution of the one-dimensional (1D) BdG equation making use of the analytical solutions of the 1D Peierls problem.[11–15] However, they assume that electrons with up and down spins have the same Fermi velocities ($v_{F\uparrow} = v_{F\downarrow}$), so that their LOFF solutions are valid only when spin polarization is small. This assumption is appropriate to superconducting states, because in ordinary superconductors the splitting of the Fermi surfaces is of an order of the pair potential at the Pauli limit, which is much smaller than the radius of the Fermi surfaces.[10] On the other hand, the Fermi surface mismatch is in general not small for cold atomic Fermi gases[7] and therefore we have to take into account large spin polarization. The spin imbalance effect was previously studied in the Peierls problem.[16]

Recently a new approach for solving the BdG equation has been proposed by Başar and Dunne.[17] They derived the nonlinear Schrödinger equation (NLSE) for the order parameter Δ with a suitable ansatz for Gor'kov Green's function. Since the derived NLSE is a closed equation for the order parameter $\Delta(x)$, this enables one to avoid the self-consistent calculation of the coupled equations of the BdG equation and the gap equation. Using this approach, they found a self-consistent solution for a complex kink crystal, which includes all previously known solutions as special cases, such as the solutions of the LOFF state (real kink crystal)[10] and Shei's complex (twisted) kink.[18] This new approach and the complex kink crystal solution have been further developed[17] for the massless Gross-Neveu model[19] and the Nambu-Jona-Lasinio model in $1+1$ dimension.[20] However, this approach has not been extended to spin polarized system.

In this paper, we investigate the self-consistent solutions of the BdG equation for spin *imbalanced* Fermi condensates. We extend the approach developed by Başar and Dunne to obtain the analytic solutions for the LOFF state in which the order parameter exhibits spatial oscillations. In contrast to the solutions of Machida and Nakanishi, we take into account the difference in the Fermi velocities ($v_{F\uparrow} \neq v_{F\downarrow}$) and derive the exact solutions for the condensate $\Delta(x)$ which are valid for *any* spin polarizations. We show how the effect of large spin polarization changes the form of the nonlinear Schrödinger equation for the order parameter. We also develop the perturbation theory for the BdG equation in the presence of small spin polarization, and obtain the fermion zero mode which is exact for arbitrary spin polarization.

II. NONLINEAR SCHRÖDINGER EQUATION FOR ORDER PARAMETER

In this section, we derive the nonlinear Schrödinger equation for the order parameter Δ in the presence of spin polarization.

A. The Bogolibov–de Gennes equation

We consider a gas of fermions with spin up and down in quasi-one-dimension under a magnetic field. If fermions with different spins interact attractively, the system undergoes a superconducting (superfluid) transition at low temperature. Although the mean field approximation is not valid in strict one dimension, since we assume a quasi-one-dimensional system relevant for experiments, the system can be described by the mean-field BdG equation[21] (we set $\hbar = 1$)

$$\begin{bmatrix} H_\uparrow(x) & \Delta_0(x) \\ \Delta_0^*(x) & -H_\downarrow(x) \end{bmatrix} \begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix} = E \begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix}, \tag{1}$$

$$H_\sigma(x) = -\frac{1}{2M}\frac{\partial^2}{\partial x^2} - \mu_\sigma, \tag{2}$$

where $\sigma(=\uparrow,\downarrow)$ stands for the spin and M is the mass of the fermion. The energy difference due to the Zeeman splitting is included in the chemical potential for each spin state μ_σ ($\sigma = \uparrow,\downarrow$). This model is indeed applicable to an imbalanced cold Fermi gas.[7] Throughout this paper, we restrict ourselves at $T = 0$. In this case, the order parameter $\Delta_0(x)$ satisfies the gap equation

$$\Delta_0(x) = -2g^2 \sum_{E_n<0} u_n(x)v_n(x)^*, \tag{3}$$

where g is the attractive interaction between fermions with different spins and n is the index for eigenstates.

If the attractive interaction is small compared with the Fermi energy $\varepsilon_{F\sigma} = \mu_\sigma$, fermions near the Fermi surfaces form Cooper pairs. If we assume $u_0(x) = e^{ik_{F\uparrow}x}u(x)$ and $v_0(x) = e^{-ik_{F\downarrow}x}v(x)$ ($k_{F\sigma}$ is the Fermi momentum $k_{F\sigma} = \sqrt{2M\varepsilon_{F\sigma}}$), $u(x)$ and $v(x)$ vary much slower than $1/k_{F\sigma}$. Neglecting the second derivative term of $u(x)$ and $v(x)$ (the Andreev approximation[22]), the BdG equation reduces to

$$\begin{bmatrix} -iv_{F\uparrow}\frac{\partial}{\partial x} & \Delta(x) \\ \Delta^*(x) & iv_{F\downarrow}\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = E \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}, \tag{4}$$

where $v_{F\sigma} = k_{F\sigma}/m$ is the Fermi velocity and $\Delta = e^{-i(k_{F\uparrow}+k_{F\downarrow})x}\Delta_0$. When $v_{F\uparrow} = v_{F\downarrow}$, the LOFF state solution of Eq. (4) has been derived in Refs. 10,17.

B. The Başar-Dunne formalism

In Ref. 17, the so-called nonlinear Schrödinger equation (NLSE) for $\Delta(x)$ has been derived for the case of $v_{F\uparrow} = v_{F\downarrow}$, through the analysis of Gor'kov Green's function. This is a convenient way to solve the BdG equation. Since the derived NLSE is a closed equation for $\Delta(x)$, this enables one to avoid the self-consistent calculation of the coupled equations of the BdG equation and the gap equation. We extend this analysis to the case of $v_{F\uparrow} \neq v_{F\downarrow}$.

First, we derive the Gor'kov Green's function that satisfies

$$(H - E)G(x,y;E) = \delta(x - y), \tag{5}$$

where

$$H = \begin{bmatrix} -iv_{F\uparrow}\frac{\partial}{\partial x} & \Delta(x) \\ \Delta^*(x) & iv_{F\downarrow}\frac{\partial}{\partial x} \end{bmatrix}. \tag{6}$$

The Gor'kov Green's function can be constructed from two independent solutions $\psi(x)$ and $\phi(x)$ of Eq. (4) as[23]

$$G(x,y;E) = \begin{pmatrix} 0 & v_{F\uparrow}^{-1} \\ v_{F\downarrow}^{-1} & 0 \end{pmatrix} F^*(x,y;E), \tag{7}$$

$$F(x,y;E) = \frac{1}{iW(x)}[\theta(y - x)\psi(x)\phi^T(y) + \theta(x - y)\phi(x)\psi^T(y)], \tag{8}$$

where $W \equiv i\psi^T\sigma_2\phi$ is a Wronskian. It is easy to show that the Eq. (7) satisfies (5).

On the other hand, the diagonal resolvent is defined by

$$R(x;E) = \langle x|\frac{1}{H - E}|x\rangle. \tag{9}$$

Indeed, Eq. (9) includes all spectral information for fermions in the presence of $\Delta(x)$, such as the single-particle spectral function

$$\rho(E) = \frac{1}{\pi}\mathrm{Im}\int dx \mathrm{Tr}R(x;E + i\delta). \tag{10}$$

From Eq. (7), $R(x;E)$ can be obtained as the coincident limit of Gor'kov Green's function:

$$R(x;E) = \lim_{\delta\to 0^+}\tfrac{1}{2}[G(x,x + \delta;E) + G(x + \delta,x;E)]. \tag{11}$$

We can show that $R(x)$ satisfies the following conditions:

$$\mathrm{Tr}\tilde{R}(x;E)\sigma_3 = 0, \tag{12}$$

$$\det\tilde{R}(x;E) = -\tfrac{1}{4}, \tag{13}$$

where

$$\tilde{R}(x;E) \equiv \begin{pmatrix} v_{F\uparrow} & 0 \\ 0 & v_{F\downarrow} \end{pmatrix} R(x;E). \tag{14}$$

In addition to the above conditions, from the definition Eq. (9), the resolvent must satisfy the Hermiticity condition:

$$R^\dagger(x;E) = R(x;E). \tag{15}$$

By a straightforward calculation, we obtain

$$\partial_x\tilde{R}(x;E)\sigma_3 = i\left[\begin{pmatrix} v_{F\downarrow}^{-1} & 0 \\ 0 & v_{F\uparrow}^{-1} \end{pmatrix}\begin{pmatrix} E & -\Delta(x) \\ \Delta^*(x) & -E \end{pmatrix}, \tilde{R}(x;E)\sigma_3\right]. \tag{16}$$

Equation (16) is known as the Dikii-Eilenberger equation.[23]

Note that we have derived the above equation from the BdG equation only with the Andreev approximation. On the other hand, it is well known that the Eilenberger equation can be derived from the BdG equation or equivalently Gor'kov equation using the quasiclassical approximation in addition to the Andreev approximation in 3D.[24] In the present case,

since we assume the system is (quasi)1D where the Andreev approximation and the quasiclassical approximation are equivalent, the Dikii-Eilenberger equation can be derived without explicitly using the quasiclassical approximation. Therefore, the solutions of Eq. (16) are the exact self-consistent solutions of the BdG equation.

Next step is to make an ansatz for the form of the Gor'kov Green's function. From the gap equation, $\Delta(x)$ must satisfy

$$\Delta(x) \propto \frac{\delta}{\delta\Delta(x)^*} \int dE \rho(E) \ln(1 + e^{-\beta(E-\mu)}). \tag{17}$$

From Eq. (10), the simplest ansatz for $R(x;E)$ to satisfy Eq. (17) is that the diagonal entries are set to be proportional to $|\Delta(x)|^2$. The gap equation can be derived by the functional derivative

$$\Delta(x) \propto \mathrm{Tr}_{D,E}\left[\sigma_1(1+\sigma_3)R(x,E)\right]. \tag{18}$$

The simplest solution for Eq. (18) is for off-diagonal entries of the $R(x;E)$ to be proportional to $\Delta(x)$ [or $\Delta^*(x)$]. However, the consistency between (16) and (18) requires that the derivative term $\Delta'(x)$ [or $\Delta^{*\prime}(x)$] should be in the off-diagonal entries. The last condition comes from Eq. (12). If we assume the following form for the resolvent:

$$R(x;E) = \mathcal{N}\begin{pmatrix} v_{F\downarrow}(a+|\Delta(x)|^2) & b\Delta(x) - ic\Delta'(x) \\ b\Delta^*(x) + ic\Delta^{*\prime}(x) & v_{F\uparrow}(a+|\Delta(x)|^2) \end{pmatrix}, \tag{19}$$

by substituting Eq. (19) into the right hand side of Eq. (16), we obtain

$$\partial_x \tilde{R}(x;E)\sigma_3 = \mathcal{N}(E)\begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix}, \tag{20}$$

where

$$A = c(|\Delta|^2)', \tag{21}$$

$$B = iv_{F\downarrow}\left(v_{F\uparrow}^{-1} + v_{F\downarrow}^{-1}\right)E[b(E)\Delta^* + ic\Delta^{*\prime}] - 2iv_{F\downarrow}\Delta^*[a(E) + |\Delta|^2]. \tag{22}$$

Then, we find

$$c = v_{F\uparrow}v_{F\downarrow}, \tag{23}$$

from the diagonal part, and

$$\tilde{\Delta}'' + i[\tilde{b} - 2\tilde{E}]\tilde{\Delta}' - 2[\tilde{a} - \tilde{E}\tilde{b}]\tilde{\Delta} - 2\tilde{\Delta}|\tilde{\Delta}|^2 = 0, \tag{24}$$

from the off-diagonal part. Here, we have defined

$$\tilde{a} = \alpha^{-2}a, \quad \tilde{b} = \alpha^{-2}b, \quad \tilde{E} = \alpha^{-2}E, \quad \tilde{\Delta} = \alpha^{-1}\Delta, \tag{25}$$

where α is the imbalance parameter, that is

$$\alpha \equiv \frac{2\sqrt{v_{F\uparrow}v_{F\downarrow}}}{v_{F\uparrow} + v_{F\downarrow}} = \frac{\sqrt{v_{F\uparrow}v_{F\downarrow}}}{v_F}. \tag{26}$$

$0 \leqslant \alpha \leqslant 1$. We introduce the mean Fermi velocity by $v_F = (v_{F\uparrow} + v_{F\downarrow})/2$, and set $v_F = 1$. In the balanced case ($v_{F\uparrow} = v_{F\downarrow}$), it has been shown that this equation reproduce the well-known solutions, e.g., homogeneous condensate (BCS theory), single kink condensate,[14] and real kink crystal.[17]

To confirm the consistency condition Eq. (13), we calculate the determinant of the resolvent:

$$\det\tilde{R} = \alpha^8\mathcal{N}^2[|\tilde{\Delta}|^4 - |\tilde{\Delta}'|^2 + (2\tilde{a} - \tilde{b}^2)|\tilde{\Delta}|^2 + i\tilde{b}(\tilde{\Delta}'\tilde{\Delta}^* - \tilde{\Delta}\tilde{\Delta}^{*\prime}) + \tilde{a}^2]. \tag{27}$$

The NLSE implies that the right hand side of the above equation is constant as follows:

$$\frac{d}{dx}\left(\frac{\det\tilde{R}}{\alpha^8\mathcal{N}^2}\right) = (2|\tilde{\Delta}|^2 + 2\tilde{a} - \tilde{b}^2)(|\tilde{\Delta}|^2)' - (\tilde{\Delta}''\tilde{\Delta}^{*\prime} + \tilde{\Delta}'\tilde{\Delta}^{*\prime\prime}) + i\tilde{b}(\tilde{\Delta}''\tilde{\Delta}^* - \tilde{\Delta}^{*\prime\prime}\tilde{\Delta}) = 0. \tag{28}$$

It is remarkable that by the scalings in Eq. (25), the NLSE (24) for finite spin polarization takes exactly the same form as the one for zero spin polarization. This means that there exist solutions of Eq. (24) which correspond to each of the solutions of the NLSE for the balanced case. Thus, the exact solutions of Eq. (24) can be easily derived by scaling the solutions of the NLSE for the balanced case. The solution corresponding to a complex kink crystal is the most general one which includes other solutions in some limits. We derive the solutions of Eq. (24) in Sec. IV including that of the LOFF state.

III. IMBALANCE EFFECT ON SINGLE-PARTICLE STATES

In this section, we examine the effect of spin imbalance on single-particle states. For simplicity, we consider the case when the order parameter is real.

A. Imbalance effect on fermionic zero mode

First, we consider the fermionic zero mode, i.e., the solution of Eq. (4) with $E = 0$. The zero mode plays crucial roles for the LOFF state. The wave function of the zero mode is localized around the nodes of the order parameter and accommodate the excess spin component. For the zero mode solution, we can exactly solve the BdG equation. When $E = 0$, by the scaling transformations

$$\tilde{u} \equiv (1+\epsilon/2)^{-\frac{1}{2}}u, \quad \tilde{v} \equiv (1-\epsilon/2)^{-\frac{1}{2}}v, \tag{29}$$

$$\tilde{x} \equiv (1-\epsilon^2/4)x = \alpha^2 x, \tag{30}$$

Eq. (4) can be rewritten as

$$\begin{bmatrix} -i\frac{\partial}{\partial\tilde{x}} & \tilde{\Delta} \\ \tilde{\Delta} & i\frac{\partial}{\partial\tilde{x}} \end{bmatrix}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix} = 0, \tag{31}$$

where $\epsilon \equiv v_{F\uparrow} - v_{F\downarrow}$. It is clear that the above equation has the same form as the one for $v_{F\uparrow} = v_{F\downarrow}$. This indicates that if the BdG equation has a zero mode solution for balanced case, there exists a corresponding zero mode solution for imbalanced case, and the two solutions are related by the scaling transformations (29) and (30).

Furthermore, the zero mode solution can be explicitly constructed as follows. If one applies the unitary transformation

$$\begin{bmatrix}\tilde{f}_+\\ \tilde{f}_-\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix}\begin{bmatrix}\tilde{u}\\ \tilde{v}\end{bmatrix}, \tag{32}$$

Eq. (31) yields

$$[\partial_{\tilde{x}} \mp \tilde{\Delta}]\tilde{f}_\pm = 0. \tag{33}$$

Hence, the solution of Eq. (33) can be formally written as

$$\tilde{f}_{\pm}(x) \propto \exp\left[\pm \int_0^x dy\, \alpha^2 \tilde{\Delta}(y)\right]. \tag{34}$$

Equation (34) is valid if $\tilde{f}_{\pm}$ is normalizable. That is $\int_{-\infty}^{\infty} dx\, \tilde{f}_{\pm}^2$ is finite. The solution (34) is exactly the same as the one for the balanced case up to the scaling factor for the order parameter.

B. Perturbation theory for massive modes

We develop a perturbation theory for massive modes ($E > 0$). We calculate spin imbalance correction for the solutions of Eq. (4) perturbatively taking ϵ as a small parameter. We expand the solution of Eq. (4) by ϵ as

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} u^{(0)}(x) \\ v^{(0)}(x) \end{bmatrix} + \epsilon \begin{bmatrix} u^{(1)}(x) \\ v^{(1)}(x) \end{bmatrix} + O(\epsilon^2), \tag{35}$$

$$E = E^{(0)} + \epsilon E^{(1)} + O(\epsilon^2). \tag{36}$$

In the zeroth order, we indeed obtain the equation for the balanced case

$$\begin{bmatrix} -i\frac{\partial}{\partial x} & \tilde{\Delta}(x) \\ \tilde{\Delta}(x) & i\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u^{(0)}(x) \\ v^{(0)}(x) \end{bmatrix} = E^{(0)} \begin{bmatrix} u^{(0)}(x) \\ v^{(0)}(x) \end{bmatrix}. \tag{37}$$

Here, we used $\Delta = \alpha\tilde{\Delta} = \sqrt{1-\epsilon^2/4}\tilde{\Delta} \approx (1-\epsilon^2/8)\tilde{\Delta}$. Making use of the unitary transformation

$$\begin{bmatrix} f_+ \\ f_- \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{38}$$

Eq. (37) becomes

$$\left(-\partial_x^2 \mp \tilde{\Delta}' + \tilde{\Delta}^2 - E^{(0)}\right) f_{\pm}^{(0)}(x) = 0. \tag{39}$$

Once $\tilde{\Delta}$ is obtained by solving Eq. (24), the Shrödinger type equation (39) yields a set of unperturbed eigenstates.

From the first order terms, we obtain

$$\left(-\partial_x^2 \mp \tilde{\Delta}' + \tilde{\Delta}^2 - E^{(0)}\right) f_{\pm}^{(1)}(x) + \frac{i}{2}\tilde{\Delta} f_{\mp}^{(0)}(x) + iE^{(0)}\partial_x f_{\pm}^{(0)}(x) - 2E^{(0)}E^{(1)} f_{\pm}^{(0)}(x) = 0. \tag{40}$$

Thus, the first order correction for the energy can be calculated from the nonperturbative eigenvalues $E_n^{(0)}$ and eigenstates $f_{\pm,n}^{(0)}$ as

$$E_n^{(1)} = -\frac{i}{2}\int dx \left[f_{+,n}^{(0)*}(x)\ f_{-,n}^{(0)*}(x)\right]\partial_x \begin{bmatrix} f_{+,n}^{(0)}(x) \\ f_{-,n}^{(0)}(x) \end{bmatrix}. \tag{41}$$

IV. SPIN IMBALANCE CORRECTION FOR VARIOUS CONDENSATES

A. Homogeneous condensate

Here we consider the homogeneous condensate

$$\tilde{\Delta}(x) = m. \tag{42}$$

We can always take m to be real due to the chiral symmetry of the model. The substitution Eq. (42) into Eq. (24) yields[25]

$$\tilde{a} = 2\tilde{E}^2 - m^2, \quad \tilde{b} = 2\tilde{E}. \tag{43}$$

Next, we calculate the energy spectrum for the quasiparticles. In this case, the order parameter is constant and thus the zero mode (34) is not allowed, i.e., it is not normalizable. Then we calculate the massive modes. From Eq. (37), we obtain

$$\begin{bmatrix} -i\frac{\partial}{\partial x} & m \\ m & i\frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u^{(0)}(x) \\ v^{(0)}(x) \end{bmatrix} = E^{(0)} \begin{bmatrix} u^{(0)}(x) \\ v^{(0)}(x) \end{bmatrix}. \tag{44}$$

Then we obtain

$$E_{\pm}^{(0)}(k) = \pm\sqrt{k^2+m^2}, \tag{45}$$

and the eigenspinor is

$$\begin{bmatrix} u_k^{\pm(0)}(x) \\ v_k^{\pm(0)}(x) \end{bmatrix} = e^{ikx} \begin{bmatrix} u_k^{\pm} \\ v_k^{\pm} \end{bmatrix}, \tag{46}$$

where u_k and v_k is independent of x. Substituting Eq. (46) into Eq. (41) yields

$$E_{\pm}^{(1)}(k) = \tfrac{1}{2}k. \tag{47}$$

Then we obtain the energy dispersion

$$E_{\pm}(k) = \pm\sqrt{k^2+m^2} + \frac{\epsilon}{2}k + O(\epsilon^2). \tag{48}$$

In this case, the spectrum of the BdG equation (4) can also be calculated exactly. By substituting Eqs. (25) and (42) into Eq. (4), we obtain

$$E_{\pm}(k) = \pm\sqrt{k^2+\alpha^2 m^2} + \frac{\epsilon}{2}k. \tag{49}$$

This dispersion relation is plotted in Fig. 1. When $\epsilon \ll 1$, $E_{\pm}(k) \simeq \frac{\epsilon}{2}k \pm \sqrt{k^2+m^2}$, which is consistent with the result (48) obtained by the perturbation theory.

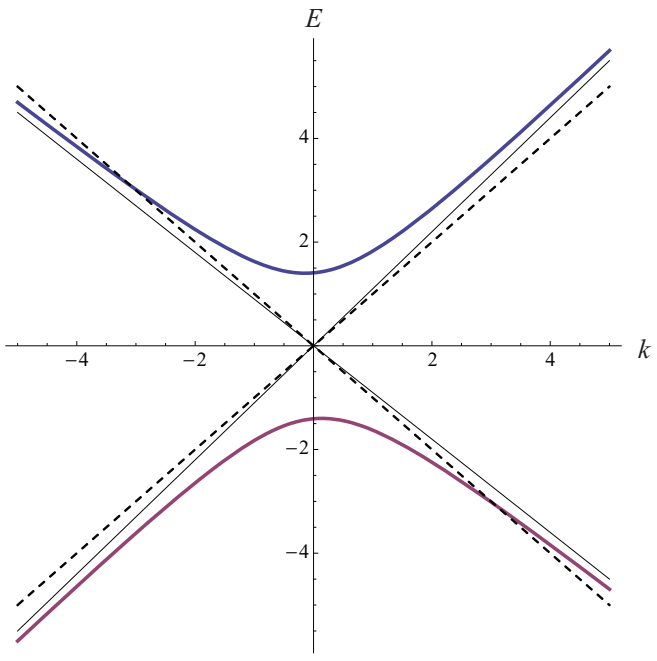

FIG. 1. (Color online) Fermionic spectrum in the case of homogeneous condensate $\Delta = 2$ with $\epsilon = 0.2$. The slope of the asymptotes changes by $\epsilon/2$ as a consequence of the spin imbalance. The dashed line shows the asymptotes for the spectrum when $\epsilon = 0$.

When $|k| \gg \alpha^2 m^2$, $E_{\pm}(k) \simeq \pm|k| + \epsilon k/2$. This indicates that the energy dispersion asymptotically becomes that of the free fermion ($E = v_{F\uparrow}k, -v_{F\downarrow}k$).

Note that the energy gap contracts by a factor α^2 compared to the balanced case, namely the edges of the positive and negative energy bands become $\pm m\alpha^2$ as plotted in Fig. 1.

B. Single real kink condensate

Next solution we consider is the single real kink (antikink) condensate

$$\tilde{\Delta}(x) = \pm m \tanh(mx). \tag{50}$$

This solution can be obtained by setting

$$\tilde{a} = 2\tilde{E}^2 - m^2, \quad \tilde{b} = 2\tilde{E}. \tag{51}$$

Now we analyze the spectrum of the associated BdG equation. The energy spectrum for single kink condensate is obtained in the limit of infinite periodicity of the real kink crystal, which will be discussed in the following section. The only exception is that the normalizable zero mode exists in this case. For the real kink case (antikink case), the eigenstate $\tilde{f}_+(x)$ [$\tilde{f}_-(x)$] in Eq. (34) is normalizable and $\tilde{f}(x)$ [$= \tilde{f}_+(x)$ for kink, $= \tilde{f}_-(x)$ for antikink] becomes

$$\tilde{f}(x) = N\,[\mathrm{sech}(mx)]^{\alpha^2}, \tag{52}$$

where N is the normalization constant.

C. LOFF state

As shown in Ref. 17, the NLSE for balanced case has the LOFF solution (real kink crystal). Then, we can immediately conclude that Eq. (24) has the corresponding solution

$$\tilde{\Delta}(x) = \sqrt{\nu}\frac{2m}{1+\sqrt{\nu}}\mathrm{sn}\left(\frac{2m}{1+\sqrt{\nu}}x;\nu\right), \tag{53}$$

where sn is the Jacobi elliptic function with real elliptic parameter $0 \leqslant \nu \leqslant 1$. Then we can conclude that the spin imbalance results in the dilatation with a factor α^2 of the condensate. The substitution Eq. (53) into Eq. (24) yields

$$\tilde{a}(\tilde{E}) = 2\tilde{E}^2 - 2m^2\frac{1+\nu}{(1+\sqrt{\nu})^2}, \tag{54}$$

$$\tilde{b}(\tilde{E}) = 2\tilde{E}. \tag{55}$$

The eigenstates of the quasiparticles for the real kink crystal order parameter (53) are given as follows[10]:

$$f^{(0)}_{+,n}(x) = \left[\frac{\wp(x+\omega_3) - e}{2L(\bar{\wp} - e)}\right]^{\frac{1}{2}} \times \exp\left[iC(E)\int_0^x \frac{dx'}{\wp(x'+\omega_3) - e}\right], \tag{56}$$

where L is the size of the system and $\wp$ is the Weierstrass function which obeys $\tilde{\Delta}^2(x) - \tilde{\Delta}'(x) = e_1 + 2\wp(x+\omega_3)$ with

$$e_1 = \frac{2m^2}{3(1+\sqrt{\nu})^2}(1+\nu), \tag{57}$$

$$e_2 = -\frac{m^2}{3(1+\sqrt{\nu})^2}(1-6\sqrt{\nu}+\nu), \tag{58}$$

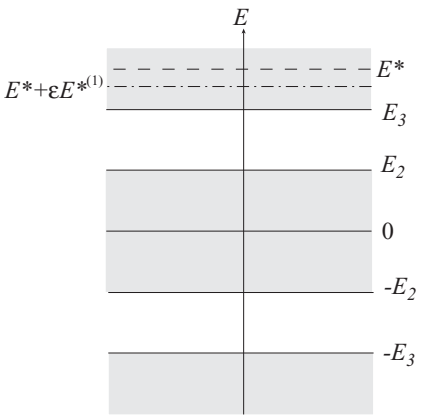

FIG. 2. Schematic illustration of the first order correction to the fermionic energy levels with respect to the balanced case for the real kink crystal condensate.

$$e_3 = -\frac{m^2}{3(1+\sqrt{\nu})^2}(1+6\sqrt{\nu}+\nu), \tag{59}$$

$$e = e_1 - E^2. \tag{60}$$

The amplitude of $f_{+,n}$ has the half periodicity of

$$\omega = K(\nu)/m, \tag{61}$$

where $K(\nu)$ is the complete elliptic integral of the first kind. We define $\bar{\wp}$ as the average of $\wp$

$$\bar{\wp} = \frac{1}{\omega}\int_0^{\omega} \wp(x+\omega_3)dx. \tag{62}$$

The coefficient $C(E)$ in Eq. (56) is defined by

$$C(E) = \pm E\sqrt{(E^2 - E_2^2)(E^2 - E_3^2)}, \tag{63}$$

$$E_i^2 = e_1 - e_i \quad (i = 2,3). \tag{64}$$

Substituting Eq. (56) into Eq. (41), we obtain

$$E^{(1)} = -\frac{C(E)}{2(\bar{\wp} - e)}. \tag{65}$$

This result implies, in particular, that the two gaps shrink as a consequence of nonzero imbalance, as illustrated in Fig. 2.

In the limit of $\nu \to 1$, the periodicity becomes infinite and then the LOFF state (the real kink crystal condensate) reduces to the single kink condensate.

V. CONCLUSIONS

We conclude this paper with few remarks. In this paper, we have investigated the spin imbalance correction for the BdG equation. We have expanded the method in Ref. 17, which is valid for the balanced case, and have obtained the nonlinear Schrödinger equation for the order parameter with spin imbalance. We have shown that the imbalance correction for the order parameter is only included in the reparametrization, however this result is nontrivial without using this method. We have obtained the fermionic zero mode exactly for arbitrary spin imbalance by a scale transformation of the one in the balanced case, which implies the stability of the fermionic zero mode against the spin imbalance. We also

have analyzed the massive fermionic spectrum of the BdG equation with small spin imbalance by perturbation theory. We have applied the method for homogeneous condensate, the single kink condensate and the LOFF state (the real kink crystal condensate). For the homogeneous condensate, we show the consistency between the perturbation theory and the exact solution for fermionic spectrum in first order. For the real kink crystal condensate, we have obtained the imbalance correction for the order parameter and the fermionic spectrum. This result completely generalizes those of Ref. 10.

Finally, we make few remarks, (i) we have analyzed the fermionic problem by the perturbation at the first order in the spin imbalance parameter ϵ, however the higher expansion is straightforward. (ii) As in the case of homogeneous condensate, we may obtain the exact solution. (iii) We have dealt with the real condensate in this paper, however the method used here can be generalized to the complex case, such as the twisted kink[18] and the twisted kink crystal.[17]

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[25] In order to obtain Eq. (43), we put m to be $me^{-\epsilon x}$ and then we take the limit of $\epsilon \to 0$.