Fermi-liquid theory of Fermi–Bose mixtures

E. V. Thuneberg and T. H. Virtanen
Department of Physics, P.O. Box 3000, FI-90014 University of Oulu, Finland
(Received 18 January 2011; published 29 June 2011)

We write the basic equations of Fermi-liquid theory for mixtures of fermions and bosons, an example being ^{3}He-^{4}He mixtures at low temperatures. Basically the theory is identical to the one derived by Khalatnikov, but it is derived in a different way, and includes more discussion. A simplifying transformation of the equations is found where the coupling of the normal and superfluid components appears in a simple form. The boundary conditions are discussed.

DOI: 10.1103/PhysRevB.83.245137 PACS number(s): 67.10.Db, 62.10.+s, 67.60.G−

I. INTRODUCTION

The Fermi-liquid theory formulated by Landau has become a paradigm of what can be the effect of interactions in a Fermi system.[1] Landau formulated the theory originally for liquid ^{3}He. Khalatnikov generalized this theory to mixtures of fermions and bosons, and applied it to mixtures of ^{3}He and ^{4}He (Ref. 2). The purpose of this paper is to reformulate Khalatnikov's theory. Although the theory is basically the same, our approach is different. We avoid several complications by concentrating on the laboratory frame, by directly starting with the linearized theory, and by using the osmotic energy. We construct a transformation that eliminates the coupling between the superfluid velocity and the quasiparticle momentum. We discuss boundary conditions. We also try to interpret the theory by discussing how the momentum of a quasiparticle can be divided into three contributions. The purpose is to formulate general Fermi–Bose-liquid equations that are needed to calculate the response on a vibrating wire.[3,4]

A general introduction to Fermi-liquid theory can be found in Refs. 5–8. Although our emphasis is on subjects not found in these reviews, we try to be self-contained. Note that Khalatnikov's theory[2] as well as the present theory makes no assumption about the diluteness of the fermion component in the mixture, which is assumed in several other treatments.[8–10] Thus the Fermi-liquid theory of a pure fermion system can be obtained as a limiting case of the theory.

We start in Sec. II with some basic definitions and introduce the noninteracting system. The basic assumptions of the Fermi–Bose-liquid theory are given in Sec. III. In Sec. IV we define the phenomenological parameters that enter the theory. The equations of motion are formulated in Sec. V. The conserved currents are identified in Sec. VI and some discussion is given. In Sec. VII we make a change to new variables where the coupling between the normal and superfluid components appears only through their densities. The scattering of quasiparticles in the bulk and at surfaces is discussed in Secs. VIII and IX. The hydrodynamic limit of the theory is discussed in Sec. X.

II. BASIC DEFINITIONS

An important thing to realize is that the *mass current density* $\boldsymbol{J}$ is the same as the *momentum density* $\boldsymbol{P}$. The equivalence $\boldsymbol{J} \equiv \boldsymbol{P}$ follows because the momentum of a particle is mass times the velocity, $\boldsymbol{p} = m\boldsymbol{v}$, i.e., the mass m is transported at velocity $\boldsymbol{v}$. This relation is valid in condensed matter under standard (nonrelativistic) conditions since the changes of momentum and mass associated with the interaction field are negligible. This is valid also in crystalline material, where even in the idealized limit of infinitely rigid lattice, one should allow part of the momentum or mass current carried by the lattice. The equivalence $\boldsymbol{J} \equiv \boldsymbol{P}$ is essential in the connection of Eqs. (19)–(21) below.

We consider a system of particles having one type of fermion and one type of boson, with masses $m_{\rm F}$ and $m_{\rm B}$. In this section, we describe the system *in the absence of interactions*. The free fermions have momenta $\boldsymbol{p}$ and energies $\epsilon_p = p^2/2m_{\rm F}$. The state of the fermion system is described by distribution function $n_{\boldsymbol{p}}$ that takes values 0 and 1 for each momentum state. The number density of fermions $n_{\rm F}$ is given by

$$n_{\rm F} = \int n_{\boldsymbol{p}} d\tau, \tag{1}$$

where $d\tau = 2d^3p/(2\pi\hbar)^3$. The fermions have a spin but since we are not considering spin-depedent phenomena, it appears only as a factor of 2 in $d\tau$. The ground state consists of a Fermi sphere, which has filled states, $n_{\boldsymbol{p}} = 1$, for momenta $p < p_{\rm F}$ and empty states, $n_{\boldsymbol{p}} = 0$, for momenta $p > p_{\rm F}$. The Fermi momentum $p_{\rm F}$ is determined by the number density $n_{\rm F} = p_{\rm F}^3/3\pi^2\hbar^3$. The momentum density of fermions is $\boldsymbol{J}_{\rm F} = \int \boldsymbol{p} n_{\boldsymbol{p}} d\tau$. The bosons are assumed to be condensed to a state with velocity $\boldsymbol{v}_{\rm B}$, and all excited states of the boson system are neglected. The number density of bosons is denoted by $n_{\rm B}$ and the momentum density $\boldsymbol{J}_{\rm B} = m_{\rm B} n_{\rm B} \boldsymbol{v}_{\rm B}$. The total momentum density of both fermions and bosons $\boldsymbol{J} = \boldsymbol{J}_{\rm F} + \boldsymbol{J}_{\rm B}$ is thus given by

$$\boldsymbol{J} = m_{\rm B} n_{\rm B} \boldsymbol{v}_{\rm B} + \int \boldsymbol{p}\, n_{\boldsymbol{p}} d\tau. \tag{2}$$

III. BASIC ASSUMPTIONS

We now turn to the discussion of the interacting system. Landau's Fermi-liquid theory applies to low-energy excitations of the system. The basic assumption is that, when interactions are turned on, the low-energy part of the excitation spectrum remains qualitatively the same as it is in the noninteracting system. More precisely, one assumes that the *momenta of the excitations remain the same* when the interactions are turned on at constant densities of both fermions and bosons. The energies of the excitations can be shifted, but *the equilibrium Fermi surface is assumed to remain unchanged*. Also *the quasiparticle energy is assumed to be linear in* p close to the Fermi surface. Thus the excited states can still be specified by a quasiparticle distribution function $n_{\boldsymbol{p}}$, and the ground state corresponds to the filled Fermi sphere.

A particular consequence of the assumptions is that Eq. (1) remains valid in the interacting system. For the total momentum of an arbitrary interacting state we write, following Khalatnikov,[2]

$$\boldsymbol{J} = m_{\rm B} n_{\rm B} \boldsymbol{v}_s + \int \boldsymbol{p}\, n_{\boldsymbol{p}} d\tau. \tag{3}$$

This is the same as Eq. (2) except that instead of the velocity of bosons it defines *the superfluid velocity* $\boldsymbol{v}_s$.

To stress the nontriviality of Eq. (3) we mention that it is *incorrect* to deduce that the first term would be the momentum density of bosons and the latter that of fermions. The correct decomposition will be given later in Eqs. (22) and (24). Note that an interacting system does not have a single boson velocity $\boldsymbol{v}_{\rm B}$, and therefore it cannot appear in Eq. (3).

In addition we use in the following sections the spherical symmetry of the system, the conservation laws of particle number, momentum, and energy, and some estimates of orders of magnitude.

IV. PARAMETRIZATION

Above we used variables $n_{\boldsymbol{p}}$, $n_{\rm B}$, and $\boldsymbol{v}_s$ to specify the state of the system. In particular, the energy density could be written as $\tilde{E}(\{n_{\boldsymbol{p}}\}, n_{\rm B}, \boldsymbol{v}_s)$. Here the curly brackets indicate that $\tilde{E}$ is a functional of $n_{\boldsymbol{p}}$, i.e., it depends on $n_{\boldsymbol{p}}$ at all values of $\boldsymbol{p}$. With respect to the variable $n_{\rm B}$, it is more convenient to change to the "osmotic energy" defined by

$$E(\{n_{\boldsymbol{p}}\},\mu_{\rm B},\boldsymbol{v}_s) = \tilde{E}(\{n_{\boldsymbol{p}}\},n_{\rm B},\boldsymbol{v}_s) - n_{\rm B}\mu_{\rm B} - n_{\rm F}\mu_{\rm F}^{(0)}. \tag{4}$$

Here the term $-n_{\rm B}\mu_{\rm B}$ effects the standard Legendre transformation from the density $n_{\rm B}$ to the chemical potential $\mu_{\rm B} = \partial \tilde{E}(\{n_{\boldsymbol{p}}\},n_{\rm B},\boldsymbol{v}_s)/\partial n_{\rm B}$. The term $-n_{\rm F}\mu_{\rm F}^{(0)}$ looks formally similar for the fermions, but is different because the chemical potential of fermions in a nonequilibrium state is not defined. Instead, we define a reference state. It has the equilibrium distribution $n_p^{(0)} = \Theta(p_{\rm F} - p)$, where $\Theta(x)$ is the step function. This equilibrium is taken to correspond chemical potentials $\mu_{\rm B}^{(0)}$ and $\mu_{\rm F}^{(0)}$, superfluid velocity $\boldsymbol{v}_s = 0$, and temperature $T = 0$. Thus the osmotic energy of Eq. (4) still depends on the distribution function $n_{\boldsymbol{p}}$, and the effect of the $-n_{\rm F}\mu_{\rm F}^{(0)}$ term is merely a shift of energy so that quasiparticle energies are counted from $\mu_{\rm F}^{(0)}$.

The quasiparticle energy is defined as

$$\epsilon_{\boldsymbol{p}} = \frac{\delta E(\{n_{\boldsymbol{p}}\},\mu_{\rm B},\boldsymbol{v}_s)}{\delta n_{\boldsymbol{p}}}, \tag{5}$$

where the functional derivative is interpreted as $\delta E = \int \epsilon_{\boldsymbol{p}} \delta n_{\boldsymbol{p}} d\tau$. The linearity of the quasiparticle energy near the Fermi surface is satisfied by the choice

$$\epsilon_{\boldsymbol{p}} = v_{\rm F}(p - p_{\rm F}) + \delta\epsilon_{\boldsymbol{p}}. \tag{6}$$

The coefficient of $p - p_{\rm F}$ defines the Fermi velocity $v_{\rm F}$. Writing $v_{\rm F} = p_{\rm F}/m^*$, it defines the effective mass m^*. The second term $\delta\epsilon_{\boldsymbol{p}}$ in Eq. (6) can be written as

$$\delta\epsilon_{\boldsymbol{p}} = (1+\alpha)\delta\mu_{\rm B} + D\boldsymbol{p}\cdot\boldsymbol{v}_s + \int f(\boldsymbol{p},\boldsymbol{p}_1)\big(n_{\boldsymbol{p}_1} - n_{p_1}^{(0)}\big)d\tau_1, \tag{7}$$

where $\delta\mu_{\rm B} = \mu_{\rm B} - \mu_{\rm B}^{(0)}$ is the deviation of the boson chemical potential from its equilibrium value. We see that $\delta\epsilon_{\boldsymbol{p}}$ is linear in the deviation from equilibrium, and thus appears less important than the first term in Eq. (6). It was an important observation of Landau that this term still has to be kept in order to make a consistent theory. Being linear in the deviations of $\mu_{\rm B}$, $\boldsymbol{v}_s$, and $n_{\boldsymbol{p}}$ from equilibrium, we see that $\delta\epsilon_{\boldsymbol{p}}$ in Eq. (7) has the most general form that is allowed by symmetry. For example the most general function of $\boldsymbol{p}$ that is linear in $\boldsymbol{v}_s$ and does not depend on any other direction has the form $D(p)\boldsymbol{p}\cdot\boldsymbol{v}_s$ with some function $D(p)$. For the same reason $f(\boldsymbol{p},\boldsymbol{p}_1)$ cannot depend on the directions of $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{p}}_1$ separately but only on their scalar product $\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}_1$.

We can further simplify $\delta\epsilon_{\boldsymbol{p}}$ in Eq. (7). Since we are considering only excitations near the Fermi surface, we can approximate $\boldsymbol{p} \approx p_{\rm F}\hat{\boldsymbol{p}}$ in the $D\boldsymbol{p}\cdot\boldsymbol{v}_s$ term. Also, we estimate that $f(\boldsymbol{p},\boldsymbol{p}_1)$ changes essentially only on the momentum scale of $p_{\rm F}$. Since $n_{\boldsymbol{p}} - n_p^{(0)}$ is nonzero only in a thin shell around the Fermi surface, we can neglect the dependence of $f(\boldsymbol{p},\boldsymbol{p}_1)$ on the magnitudes p and p_1. Similarly, we can assume α and D to be constants not depending on p. Thus $\delta\epsilon_{\boldsymbol{p}}$ depends only on the momentum direction $\hat{\boldsymbol{p}}$, not on the magnitude p. We can now define an energy integrated distribution function

$$\phi_{\hat{\boldsymbol{p}}} = \int \big(n_{p\hat{\boldsymbol{p}}} - n_p^{(0)}\big)v_{\rm F}dp, \tag{8}$$

where the argument of $n_{\boldsymbol{p}}$ is $\boldsymbol{p} = p\hat{\boldsymbol{p}}$. We get that $f_{\boldsymbol{p},\boldsymbol{p}_1}$ depends only on the scalar product $\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}_1$. Thus we can expand $f_{\boldsymbol{p},\boldsymbol{p}_1}$ using Legendre polynomials $P_l(x)$:

$$F(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') \equiv \frac{m^* p_{\rm F}}{\pi^2\hbar^3} f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') = \sum_{l=0}^{\infty} F_l P_l(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}'), \tag{9}$$

where $P_0(x) = 1$, $P_1(x) = x$, etc., and $m^* p_{\rm F}/\pi^2\hbar^3$ is the quasiparticle density of states at the Fermi surface. As a result we can write Eq. (7) in the form

$$\delta\epsilon_{\hat{\boldsymbol{p}}} = (1+\alpha)\delta\mu_{\rm B} + Dp_{\rm F}\hat{\boldsymbol{p}}\cdot\boldsymbol{v}_s + \sum_{l=0}^{\infty} F_l \langle P_l(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')\phi_{\hat{\boldsymbol{p}}'}\rangle_{\hat{\boldsymbol{p}}'}, \tag{10}$$

where $\langle\ldots\rangle_{\hat{\boldsymbol{p}}}$ denotes the average over the unit sphere of $\hat{\boldsymbol{p}}$.

We have defined parameters m^*, α, D, and F_l's in Eqs. (6) and (10). There is one constraint between these required by translational invariance. The relation is conveniently stated by expressing D in terms of other parameters,

$$D = 1 - \frac{m_{\rm F}}{m^*}\left(1 + \frac{1}{3}F_1\right). \tag{11}$$

This relation will be justified below in connection with Eq. (23). Note that because of using the osmotic energy of

Eq. (4) right from the start, we need to consider only one set of F_l parameters, in contrast to the three sets used in Ref. 2.

We also need to consider changes in the boson density. We parametrize

$$\delta n_{\rm B} \equiv n_{\rm B} - n_{\rm B}^{(0)} = -(1+\alpha)\delta n_{\rm F} + \frac{n_{\rm B}}{m_{\rm B} s^2}\delta\mu_{\rm B}, \qquad (12)$$

where $\delta n_{\rm F} = n_{\rm F} - n_{\rm F}^{(0)}$. The equality of the coefficient $1+\alpha$ with the one in Eqs. (7) and (10) follows because they are second partial derivatives of E in Eq. (4):

$$\frac{\partial^2 E}{\partial n_{\rm F}\partial\mu_{\rm B}} = \frac{\partial\mu_{\rm F}}{\partial\mu_{\rm B}} = -\frac{\partial n_{\rm B}}{\partial n_{\rm F}} = 1+\alpha. \qquad (13)$$

In the limit of vanishing concentration of fermions, α reduces to the parameter α used in Refs. 9 and 8. In the same limit, the parameter s in Eq. (12) reduces to the velocity of sound [as will be evident from Eqs. (18) and (36) below].

V. EQUATIONS OF MOTION

In general the distribution function $n_{\boldsymbol p}$ depends on location and time, $n_{\boldsymbol p}(\boldsymbol r,t)$, and similarly for other variables $\mu_{\rm B}(\boldsymbol r,t)$ and $\boldsymbol v_s(\boldsymbol r,t)$. The quasiparticle distribution obeys the kinetic equation

$$\frac{\partial n_{\boldsymbol p}}{\partial t} + \nabla n_{\boldsymbol p}\cdot\frac{\partial\epsilon_{\boldsymbol p}}{\partial\boldsymbol p} - \frac{\partial n_{\boldsymbol p}}{\partial\boldsymbol p}\cdot\nabla\epsilon_{\boldsymbol p} = I_{\boldsymbol p}, \qquad (14)$$

where $I_{\boldsymbol p}$ is the collision term. The fermion number and momentum conservation in collisions requires that

$$\int I_{\boldsymbol p} d\tau = 0, \quad \int \boldsymbol p I_{\boldsymbol p} d\tau = 0. \qquad (15)$$

Assuming small deviation from equilibrium, we linearize kinetic equation (14) and get

$$\frac{\partial n_{\boldsymbol p}}{\partial t} + v_{\rm F}\hat{\boldsymbol p}\cdot\nabla\left(n_{\boldsymbol p} - \frac{dn_p^{(0)}}{d\epsilon_p}\delta\epsilon_{\hat{\boldsymbol p}}\right) = I_{\boldsymbol p}, \qquad (16)$$

where $dn_p^{(0)}/d\epsilon_p$ denotes the derivative of the equilibrium distribution function with respect to the unperturbed energy $\epsilon_p^{(0)} = v_{\rm F}(p-p_{\rm F})$.

The superfluid velocity $\boldsymbol v_s$ appearing in Eq. (3) is assumed to be curl free:[11]

$$\nabla\times\boldsymbol v_s = 0. \qquad (17)$$

It is assumed to obey the ideal-fluid equation of motion,

$$\frac{\partial\boldsymbol v_s}{\partial t} + \frac{1}{m_{\rm B}}\nabla\mu_{\rm B} = 0. \qquad (18)$$

VI. CURRENTS

From the equations above one should be able to derive conservation laws. In particular, one should be able to derive separate continuity equations for the two components,

$$m_{\rm F}\frac{\partial n_{\rm F}}{\partial t} + \nabla\cdot\boldsymbol J_{\rm F} = 0, \qquad (19)$$

$$m_{\rm B}\frac{\partial n_{\rm B}}{\partial t} + \nabla\cdot\boldsymbol J_{\rm B} = 0, \qquad (20)$$

and the momentum conservation

$$\frac{\partial\boldsymbol J}{\partial t} + \nabla\cdot\overset{\leftrightarrow}{\Pi} = 0, \qquad (21)$$

where $\boldsymbol J = \boldsymbol J_{\rm F} + \boldsymbol J_{\rm B}$. In the following we derive linearized expressions for the fermion mass current $\boldsymbol J_{\rm F}$, for the boson mass current $\boldsymbol J_{\rm B}$, and for the momentum flux tensor $\overset{\leftrightarrow}{\Pi}$.

We integrate kinetic equation (16) over all momenta. Using Eqs. (1) and (10) and comparing with Eq. (19), we get

$$\boldsymbol J_{\rm F} = Dm_{\rm F}n_{\rm F}\boldsymbol v_s + \frac{m_{\rm F}}{m^*}\left(1+\frac{1}{3}F_1\right)\int\boldsymbol p\left(n_{\boldsymbol p} - n_p^{(0)}\right)d\tau. \qquad (22)$$

From Eqs. (3) and (22) we can eliminate the integral term and get

$$\boldsymbol J = \left(m_{\rm B}n_{\rm B} - \frac{Dm^*n_{\rm F}}{1+\frac{1}{3}F_1}\right)\boldsymbol v_s + \frac{m^*}{m_{\rm F}\left(1+\frac{1}{3}F_1\right)}\boldsymbol J_{\rm F}. \qquad (23)$$

This can be used to justify relation (11). Consider the whole system moving with constant velocity $\boldsymbol v$, but being otherwise in equilibrium. Because it has to be that $\boldsymbol J = (m_{\rm F}n_{\rm F} + m_{\rm B}n_{\rm B})\boldsymbol v$, $\boldsymbol v_s = \boldsymbol v$ and $\boldsymbol J_{\rm F} = m_{\rm F}n_{\rm F}\boldsymbol v$, we get Eq. (11).

Using Eqs. (3) and (22) we calculate $\boldsymbol J_{\rm B} = \boldsymbol J - \boldsymbol J_{\rm F}$ and get

$$\boldsymbol J_{\rm B} = (m_{\rm B}n_{\rm B} - Dm_{\rm F}n_{\rm F})\boldsymbol v_s + D\int\boldsymbol p\left(n_{\boldsymbol p} - n_p^{(0)}\right)d\tau. \qquad (24)$$

Let us consider Eqs. (22) and (24) in the frame where $\boldsymbol v_s = 0$. From Eq. (24) we see that a single quasiparticle of momentum $\boldsymbol p$ has fraction D of the momentum carried by bosons. As the principal fermion forming the quasiparticle travels with velocity $v = p/m^*$, it carries the fraction $m_{\rm F}/m^*$ of the quasiparticle momentum. The rest fraction of the momentum $m_{\rm F}F_1/3m^*$ is carried by other fermions. The total fermion contribution is visible in the second term of $\boldsymbol J_{\rm F}$ in Eq. (22).

To clarify the nature of the quasiparticle, we consider a simple model. Assume that the principal fermion forms a sphere of radius a and the bosons and other fermions are modeled as an incompressible ideal fluid of density ρ. In this model one can calculate $m^* = m_{\rm F} + 2\pi a^3\rho/3$, i.e., the effective mass is the fermion mass plus the fluid mass corresponding to half of the sphere. This result is obtained by calculating the momentum $\boldsymbol p = m^*\boldsymbol v$ from Newton's second law when a force is applied to accelerate the sphere, or, equivalently, by calculating the kinetic energy E_k and writing $E_k = \frac{1}{2}m^*v^2$. This model gives a simple picture of how the effective mass is increased due to the medium dragged along by the principal fermion. This is sometimes called "backflow" but this may be misleading since the momentum is in the "forward" direction (as $m^* > m_{\rm F}$).

We note that, when considering a single excited quasiparticle in a finite container, we must also allow a compensating flow. Using the simple quasiparticle model as discussed above, the total momentum of an incompressible fluid in a finite stationary container must vanish. Thus the momentum $\boldsymbol p = m^*\boldsymbol v$ of a quasiparticle must be compensated by an opposite flow of the medium. This compensating momentum arises from force applied by the walls of the container, and thus is separate from the momentum of the quasiparticle.

For further insight, consider the case that the fermions are in equilibrium but moving with velocity $\boldsymbol v$, i.e., $\boldsymbol J_{\rm F} = m_{\rm F}n_{\rm F}\boldsymbol v$. We assume $\boldsymbol v_s = 0$. The mass density dragged in such a normal current is called *normal fluid density* ρ_n, i.e., $\boldsymbol{J} = \rho_n \boldsymbol{v}$. The complementary fraction of the total density $\rho = m_{\rm F} n_{\rm F} + m_{\rm B} n_{\rm B}$ that remains at rest is called the *superfluid density* $\rho_s = \rho - \rho_n$. From Eq. (23) we get

$$\rho_s = m_{\rm B} n_{\rm B} - \frac{D m^* n_{\rm F}}{1+\frac{1}{3}F_1}, \quad \rho_n = \frac{m^* n_{\rm F}}{1+\frac{1}{3}F_1}. \qquad (25)$$

These expressions have a straightforward interpretation in terms of the momentum division introduced above; namely $\boldsymbol{J}_{\rm F} = m_{\rm F}(1+\frac{1}{3}F_1)\boldsymbol{J}/m^*$ and $\boldsymbol{J}_{\rm B} = D\boldsymbol{J}$. Solving $\boldsymbol{J}$ from the former and using $\boldsymbol{J}_{\rm F} = m_{\rm F} n_{\rm F} \boldsymbol{v}$ one gets the normal fluid density in Eqs. (25). One also sees that the mass density $Dm^* n_{\rm F}/(1+F_1/3)$ subtracted from the boson density in ρ_s in Eqs. (25) just corresponds to the boson fraction that is bound to the quasiparticles.

Next we check the momentum conservation in Eq. (21). We take the time derivative of $\boldsymbol{J}$ from Eq. (3) and use kinetic equation of quasiparticles (16) and the equation of motion of the Bose component in Eq. (18). Using Eq. (15) we find the momentum conservation with the momentum flux tensor:

$$\overleftrightarrow{\Pi} = P^{(0)} \overleftrightarrow{1} + n_{\rm B}\delta\mu_{\rm B} \overleftrightarrow{1} + \frac{1}{m^*} \times \int \boldsymbol{p}\,\boldsymbol{p} \left(n_{\boldsymbol{p}} - n_p^{(0)} - \frac{dn_p^{(0)}}{d\epsilon_p}\delta\epsilon_{\hat{\boldsymbol{p}}} \right) d\tau. \qquad (26)$$

Here the constant term arising from the equilibrium pressure $P^{(0)}$ is added.

VII. ELIMINATION OF SUPERFLUID-VELOCITY COUPLING

Our purpose in this section is to rewrite the theory so that the coupling between the superfluid and normal fluid components appears in a minimal way. For that purpose we define new quantities $\delta\bar{n}_{\boldsymbol{p}}$ and $\delta\bar{\epsilon}_{\boldsymbol{p}}$, and write the kinetic equation in the form

$$\frac{\partial}{\partial t}\left(\delta\bar{n}_{\boldsymbol{p}} + \frac{dn_p^{(0)}}{d\epsilon_p}\delta\bar{\epsilon}_{\hat{\boldsymbol{p}}}\right) + v_{\rm F}\hat{\boldsymbol{p}}\cdot\nabla\delta\bar{n}_{\boldsymbol{p}} = I_{\boldsymbol{p}}, \qquad (27)$$

so that $\delta\bar{\epsilon}_{\boldsymbol{p}}$ will depend on $\delta\mu_{\rm B}$ and on $\delta\bar{n}_{\boldsymbol{p}}$ but not on $\boldsymbol{v}_s$. To achieve this, we scalar multiply superfluid equation (18) by $Av_{\rm F}\hat{\boldsymbol{p}}\, dn_p^{(0)}/d\epsilon_p$ and add it to kinetic equation (16). A is a constant that is fixed below. Comparing this with Eq. (27) we find

$$\delta\bar{n}_{\boldsymbol{p}} = n_{\boldsymbol{p}} - n_p^{(0)} - \frac{dn_p^{(0)}}{d\epsilon_p}\left(\delta\epsilon_{\hat{\boldsymbol{p}}} + \frac{A}{m_{\rm B}}\delta\mu_{\rm B}\right), \qquad (28)$$

$$\delta\bar{\epsilon}_{\hat{\boldsymbol{p}}} = \delta\epsilon_{\hat{\boldsymbol{p}}} + \frac{A}{m_{\rm B}}\delta\mu_{\rm B} - Av_{\rm F}\hat{\boldsymbol{p}}\cdot\boldsymbol{v}_s. \qquad (29)$$

Similar to $\phi_{\hat{\boldsymbol{p}}}$ in Eq. (8), it is convenient to define

$$\psi_{\hat{\boldsymbol{p}}} = \int \delta\bar{n}_{p\hat{\boldsymbol{p}}} v_{\rm F} dp. \qquad (30)$$

The next task is to express $\delta\bar{\epsilon}_{\hat{\boldsymbol{p}}}$ in terms of $\psi_{\hat{\boldsymbol{p}}}$. Doing this one notices that the dependence on $\boldsymbol{v}_s$ drops out by choosing $A = Dm^*/(1+F_1/3)$, and we get

$$\delta\bar{\epsilon}_{\hat{\boldsymbol{p}}} = \frac{K}{1+F_0}\delta\mu_{\rm B} + \sum_{l=0}^{\infty}\frac{F_l}{1+\frac{1}{2l+1}F_l}\langle P_l(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')\psi_{\hat{\boldsymbol{p}}'}\rangle_{\hat{\boldsymbol{p}}'}, \qquad (31)$$

where we have defined

$$K = \frac{m^* D}{m_{\rm B}\left(1+\frac{1}{3}F_1\right)} + 1 + \alpha = \frac{m^*}{m_{\rm B}\left(1+\frac{1}{3}F_1\right)} - \frac{m_{\rm F}}{m_{\rm B}} + 1 + \alpha. \qquad (32)$$

It is useful to express the densities and currents in terms of $\delta\bar{n}_{\boldsymbol{p}}$ from Eq. (28) or $\psi_{\hat{\boldsymbol{p}}}$ from Eq. (30). Here we give the fermion density (1) as

$$\delta n_{\rm F} \equiv n_{\rm F} - n_{\rm F}^{(0)} = \frac{1}{1+F_0}\left(\int\delta\bar{n}_{\boldsymbol{p}}d\tau - \frac{m^* p_{\rm F} K}{\pi^2\hbar^3}\delta\mu_{\rm B}\right) = \frac{m^* p_{\rm F}}{\pi^2\hbar^3(1+F_0)}(\langle\psi_{\hat{\boldsymbol{p}}}\rangle_{\hat{\boldsymbol{p}}} - K\delta\mu_{\rm B}), \qquad (33)$$

the fermion momentum of Eq. (22) as

$$\boldsymbol{J}_{\rm F} = \frac{m_{\rm F}}{m^*}\int \boldsymbol{p}\,\delta\bar{n}_{\boldsymbol{p}} d\tau = \frac{m_{\rm F} p_{\rm F}^2}{\pi^2\hbar^3}\langle\hat{\boldsymbol{p}}\psi_{\hat{\boldsymbol{p}}}\rangle_{\hat{\boldsymbol{p}}}, \qquad (34)$$

and the momentum flux tensor of Eq. (26) as

$$\overleftrightarrow{\Pi} = P^{(0)}\overleftrightarrow{1} + \frac{\rho_s}{m_{\rm B}}\delta\mu_{\rm B}\overleftrightarrow{1} + \frac{1}{m^*}\int\boldsymbol{p}\,\boldsymbol{p}\,\delta\bar{n}_{\boldsymbol{p}}d\tau = P^{(0)}\overleftrightarrow{1} + \frac{\rho_s}{m_{\rm B}}\delta\mu_{\rm B}\overleftrightarrow{1} + 3n_{\rm F}\langle\hat{\boldsymbol{p}}\hat{\boldsymbol{p}}\psi_{\hat{\boldsymbol{p}}}\rangle_{\hat{\boldsymbol{p}}}. \qquad (35)$$

In the momentum flux of Eq. (35) the second and third terms arise from the superfluid and the normal components, respectively. We also rewrite the boson conservation of Eq. (20) using Eq. (12) as

$$\frac{n_{\rm B}}{s^2}\frac{\partial\delta\mu_{\rm B}}{\partial t} + \rho_s\nabla\cdot\boldsymbol{v}_s - m_{\rm B}K\frac{\partial n_{\rm F}}{\partial t} = 0. \qquad (36)$$

Equations (27), (36), and (18) constitute equations of motion in variables $\delta\bar{n}_{\boldsymbol{p}}$, $\delta\mu_{\rm B}$, and $\boldsymbol{v}_s$.

We see that in the system of equations (18), (27), (31), and (36), the coupling of the superfluid and normal components takes place through their densities via the terms proportional to K. Let us estimate this effect considering a problem of length scale a, frequency ω, and relatively long mean free path $\ell \gtrsim a$. We first consider fermion flow with velocity scale u, for which we estimate $\delta n_{\rm F} \sim n_{\rm F} u/v_{\rm F}$. Assuming $\omega a/s \ll 1$, the first term in Eq. (36) can be neglected, and it is important to balance the last two terms. We get that a fermion flow with velocity u generates a superfluid velocity of the order of $K(n_{\rm F}/n_{\rm B})(\omega a/v_{\rm F})u$. The opposite effect of $v_s \sim u$ on the quasiparticles can be estimated from the fermion conservation of Eq. (19) and applying Eqs. (33) and (34). We get that for $\omega a/v_{\rm F} \lesssim 1$ the induced fermion velocity is of the order of $K(m_{\rm B}/m_{\rm F})(\omega a/v_{\rm F})^2 u$. For small frequencies $\omega a/v_{\rm F} \ll 1$ the velocities induced in the other component are small, and the superfluid and quasiparticle components can be treated independently of each other.

VIII. COLLISION TERM

For detailed discussion of the collision term $I_{\boldsymbol{p}}$ we refer to Sykes and Brooker.[12] In the collision of two quasiparticles, their total energy is conserved. It follows that the collision integral can be written as a function $\delta\bar{n}_{\boldsymbol{p}}$ in Eq. (28). The collision term can in a few cases be analyzed exactly.[12] A simpler approach is to use a relaxation-time approximation. In its simplest version one assumes one relaxation time τ and writes[8]

$$I_{\boldsymbol{p}} = -\frac{1}{\tau}\left[n_{\boldsymbol{p}} - n^{(0)}\left(\epsilon_p^{(0)} + \delta\epsilon_{\boldsymbol{p}} - c - \boldsymbol{b}\cdot\boldsymbol{p}\right)\right]. \tag{37}$$

Here $n^{(0)}$ is the equilibrium distribution evaluated at the energy that consist of the exact energy of the quasiparticle $\epsilon_p^{(0)} + \delta\epsilon_{\boldsymbol{p}}$, and the coefficients c and $\boldsymbol{b}$ are chosen so that conditions (15) are satisfied. The relaxation time can also be specified by the mean free path $\ell = v_{\mathrm{F}}\tau$. The relaxation-time approximation leads to essential simplification because, instead of using $\delta n_{\boldsymbol{p}}$ or $\delta\bar{n}_{\boldsymbol{p}}$, one can construct equations for $\psi_{\hat{\boldsymbol{p}}}$ in Eq. (30), which does not depend on the magnitude of the momentum. In the relaxation-time approximation kinetic equation (27) takes the form

$$\begin{aligned}\frac{\partial}{\partial t}(\psi_{\hat{\boldsymbol{p}}} - \delta\bar{\epsilon}_{\hat{\boldsymbol{p}}}) + v_{\mathrm{F}}\hat{\boldsymbol{p}}\cdot\nabla\psi_{\hat{\boldsymbol{p}}} \\ = -\frac{1}{\tau}(\psi_{\hat{\boldsymbol{p}}} - \langle\psi_{\hat{\boldsymbol{p}}'}\rangle_{\hat{\boldsymbol{p}}'} - 3\langle\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}'\psi_{\hat{\boldsymbol{p}}'}\rangle_{\hat{\boldsymbol{p}}'}).\end{aligned} \tag{38}$$

It is also possible to introduce different relaxation times τ_l (with $l = 2,3,\ldots$) corresponding to the spherical harmonic decomposition of $\psi_{\hat{\boldsymbol{p}}}$ in Eq. (54).

IX. BOUNDARY CONDITIONS

The boundary conditions at walls depend on the scattering properties of the wall. Two common models for surface scattering are specular and diffuse scattering. We generalize the analysis of Ref. 13 to Fermi–Bose liquids and include general interactions of Eq. (9). We first consider the boundary conditions in the rest frame of a wall, and after that we generalize to the case of a moving wall.

We assume a wall with normal $\hat{\boldsymbol{n}}$ that is inpenetrable to both bosons and fermions. From Eq. (23) we see that the vanishing of the particle fluxes $\boldsymbol{J}$ and $\boldsymbol{J}_{\mathrm{F}}$ normal to the wall imply vanishing of the normal component of the superfluid velocity,

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{v}_s = 0. \tag{39}$$

Specular scattering means that the quasiparticle momentum changes from $\boldsymbol{p}$ to $\boldsymbol{p}_R = \boldsymbol{p} - 2\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\boldsymbol{p})$, and thus

$$n_{\boldsymbol{p}} = n_{\boldsymbol{p}-2\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\boldsymbol{p})}. \tag{40}$$

Before accepting this one should check that it conserves the quasiparticle energy of Eq. (6), i.e., we calculate

$$\begin{aligned}\delta\epsilon_{\hat{\boldsymbol{p}}} - \delta\epsilon_{\hat{\boldsymbol{p}}_R} &= Dp_{\mathrm{F}}(\hat{\boldsymbol{p}} - \hat{\boldsymbol{p}}_R)\cdot\boldsymbol{v}_s \\ &\quad + \langle F(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')\phi_{\hat{\boldsymbol{p}}'} - F(\hat{\boldsymbol{p}}_R\cdot\hat{\boldsymbol{p}}')\phi_{\hat{\boldsymbol{p}}'}\rangle_{\hat{\boldsymbol{p}}'} \\ &= 2Dp_{\mathrm{F}}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}})(\hat{\boldsymbol{n}}\cdot\boldsymbol{v}_s) \\ &\quad + \left\langle F(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')\phi_{\hat{\boldsymbol{p}}'} - F(\hat{\boldsymbol{p}}_R\cdot\hat{\boldsymbol{p}}'_R)\phi_{\hat{\boldsymbol{p}}'_R}\right\rangle_{\hat{\boldsymbol{p}}'} \\ &= 2Dp_{\mathrm{F}}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}})(\hat{\boldsymbol{n}}\cdot\boldsymbol{v}_s) \\ &\quad + \left\langle F(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')\left[\phi_{\hat{\boldsymbol{p}}'} - \phi_{\hat{\boldsymbol{p}}'_R}\right]\right\rangle_{\hat{\boldsymbol{p}}'} = 0,\end{aligned} \tag{41}$$

where we have used Eq. (10), $\hat{\boldsymbol{p}}_R\cdot\hat{\boldsymbol{p}}'_R = \hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}'$, Eq. (39), and $\phi_{\hat{\boldsymbol{p}}'} - \phi_{\hat{\boldsymbol{p}}'_R} = 0$, which follows from Eq. (40).

Diffuse scattering means that the quasiparticles reflected from the wall are in equilibrium evaluated at the exact quasiparticle energy of Eq. (6). Thus the distribution of the reflected quasiparticles is

$$\begin{aligned}n_{\boldsymbol{p}} &= n^{(0)}\left(\epsilon_p^{(0)} + \delta\epsilon_{\boldsymbol{p}} - c\right) \\ &= n_p^{(0)} + \frac{dn_p^{(0)}}{d\epsilon_p}(\delta\epsilon_{\boldsymbol{p}} - c) \quad \text{for } \hat{\boldsymbol{n}}\cdot\boldsymbol{p} > 0,\end{aligned} \tag{42}$$

where the constant c is determined by the condition that the particle flux of Eq. (34) to the wall vanishes, $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}_{\mathrm{F}} = 0$. More conveniently, the boundary is expressed as

$$\delta\bar{n}_{\boldsymbol{p}} = -\frac{dn_p^{(0)}}{d\epsilon_p}\bar{c} \quad \text{for } \hat{\boldsymbol{n}}\cdot\boldsymbol{p} > 0, \tag{43}$$

with a new constant $\bar{c}$. The constant can be determined by evaluating the condition $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}_{\mathrm{F}} = 0$ for reflected quasiparticles, which allows us to write the diffuse boundary condition as

$$\delta\bar{n}_{\boldsymbol{p}} = \frac{4\pi^2\hbar^3}{m^* p_{\mathrm{F}}}\frac{dn_p^{(0)}}{d\epsilon_p}\int_{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}'<0}\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}'\,\delta\bar{n}_{\boldsymbol{p}'}\,d\tau' \quad \text{for } \hat{\boldsymbol{n}}\cdot\boldsymbol{p} > 0. \tag{44}$$

The boundary conditions in the case of a moving wall can be found by changing the reference frame. In the laboratory frame the quasiparticle momentum $\boldsymbol{p} = \boldsymbol{p}' + m_{\mathrm{F}}\boldsymbol{u}$, where $\boldsymbol{p}'$ is the momentum in the rest frame of the wall and $\boldsymbol{u}$ is the velocity of the wall. (This relation follows because the momenta are assumed to remain unchanged when the interactions are switched on.) The quasiparticle distributions in the two frames are the same, $n_{\boldsymbol{p}} = n'_{\boldsymbol{p}'}$. A change arises in $\delta n_{\boldsymbol{p}}$:

$$\delta n_{\boldsymbol{p}} = n_{\boldsymbol{p}} - n_p^{(0)} = n'_{\boldsymbol{p}'} - n^{(0)}_{\boldsymbol{p}'+m_{\mathrm{F}}\boldsymbol{u}} = \delta n'_{\boldsymbol{p}'} - \frac{dn_p^{(0)}}{d\epsilon_p}m_{\mathrm{F}}v_{\mathrm{F}}\hat{\boldsymbol{p}}\cdot\boldsymbol{u}. \tag{45}$$

From this we find $\phi_{\hat{\boldsymbol{p}}} = \phi'_{\hat{\boldsymbol{p}}} + m_{\mathrm{F}}v_{\mathrm{F}}\hat{\boldsymbol{p}}\cdot\boldsymbol{u}$, and using $\boldsymbol{v}_s = \boldsymbol{v}'_s + \boldsymbol{u}$ we get $\delta\epsilon_{\hat{\boldsymbol{p}}} = \delta\epsilon'_{\hat{\boldsymbol{p}}} + (m^* - m_{\mathrm{F}})v_{\mathrm{F}}\hat{\boldsymbol{p}}\cdot\boldsymbol{u}$. The transformation for $\delta\bar{n}_{\boldsymbol{p}}$ is then

$$\delta\bar{n}_{\boldsymbol{p}} = \delta\bar{n}'_{\boldsymbol{p}'} - \frac{dn_p^{(0)}}{d\epsilon_p}p_{\mathrm{F}}\hat{\boldsymbol{p}}\cdot\boldsymbol{u}, \tag{46}$$

which implies $\psi_{\hat{\boldsymbol{p}}} = \psi'_{\hat{\boldsymbol{p}}} + p_{\mathrm{F}}\hat{\boldsymbol{p}}\cdot\boldsymbol{u}$.

Applying the transformation rules to Eqs. (39), (40), and (44) gives the superfluid condition

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{v}_s = \hat{\boldsymbol{n}}\cdot\boldsymbol{u}, \tag{47}$$

the specular boundary condition

$$\delta\bar{n}_{\boldsymbol{p}} = \delta\bar{n}_{\boldsymbol{p}-2\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\boldsymbol{p})} - \frac{dn_p^{(0)}}{d\epsilon_p}2p_{\mathrm{F}}\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}\,\hat{\boldsymbol{n}}\cdot\boldsymbol{u}, \tag{48}$$

and the diffusive boundary condition

$$\begin{aligned}\delta\bar{n}_{\boldsymbol{p}} &= 4\frac{dn_p^{(0)}}{d\epsilon_p}\frac{d\epsilon_p}{d\tau}\int_{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}'<0}\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}'\,\delta\bar{n}_{\boldsymbol{p}'}\,d\tau' \\ &\quad - \frac{dn_p^{(0)}}{d\epsilon_p}p_{\mathrm{F}}\left(\hat{\boldsymbol{p}} + \frac{2}{3}\hat{\boldsymbol{n}}\right)\cdot\boldsymbol{u} \quad \text{for } \hat{\boldsymbol{n}}\cdot\boldsymbol{p} > 0.\end{aligned} \tag{49}$$

In terms of $\psi_{\hat{\boldsymbol{p}}}$ the specular and diffuse conditions can be written

$$\psi_{\hat{\boldsymbol{p}}} = \psi_{\hat{\boldsymbol{p}}-2\hat{\boldsymbol{n}}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}})} + 2p_{\rm F}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}})(\hat{\boldsymbol{n}}\cdot\boldsymbol{u}), \tag{50}$$

$$\psi_{\hat{\boldsymbol{p}}_{\rm out}} = -2\langle\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}_{\rm in}\psi_{\hat{\boldsymbol{p}}_{\rm in}}\rangle_{\hat{\boldsymbol{p}}_{\rm in}} + p_{\rm F}\left(\hat{\boldsymbol{p}}_{\rm out} + \tfrac{2}{3}\hat{\boldsymbol{n}}\right)\cdot\boldsymbol{u}, \tag{51}$$

with an average over half of the unit sphere ($\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{p}}_{\rm in} < 0$).

A consistency check for the boundary conditions is that the fermion particle current in Eq. (34) behaves as expected:

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{J}_{\rm F} = m_{\rm F}n_{\rm F}\hat{\boldsymbol{n}}\cdot\boldsymbol{u}. \tag{52}$$

For specular boundary condition (50) we deduce from Eq. (35) the vanishing of the transverse momentum flux,

$$\hat{\boldsymbol{n}}\cdot\overleftrightarrow{\Pi}\times\hat{\boldsymbol{n}} = 0. \tag{53}$$

X. HYDRODYNAMIC LIMIT

The kinetic theory reduces to hydrodynamic theory in the limit of the small mean free path of quasiparticles. We sketch the derivation here starting from the relaxation-time approximation. The quasiparticle distribution function $\psi_{\hat{\boldsymbol{p}}}$ can be expanded in spherical harmonics:

$$\psi_{\hat{\boldsymbol{p}}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_l^m Y_l^m(\hat{\boldsymbol{p}}). \tag{54}$$

The lowest-order coefficients c_0^0 and c_1^m can be related to the fermion number of Eq. (33) and current of Eq. (34). Due to rapid scattering, all higher-order coefficients are assumed to be small. We define *normal fluid velocity* $\boldsymbol{v}_n$ by writing the current $\boldsymbol{J}_{\rm F} = m_{\rm F}n_{\rm F}\boldsymbol{v}_n$. We multiply kinetic equation (38) by $\hat{\boldsymbol{p}}\hat{\boldsymbol{p}} - \overleftrightarrow{1}/3$ and average over $\hat{\boldsymbol{p}}$. We get

$$\frac{\partial}{\partial t}\langle(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}} - \overleftrightarrow{1}/3)[\psi_{\hat{\boldsymbol{p}}} - \delta\bar{\epsilon}_{\hat{\boldsymbol{p}}}]\rangle + v_{\rm F}\langle(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}} - \overleftrightarrow{1}/3)\hat{\boldsymbol{p}}\cdot\nabla\psi_{\hat{\boldsymbol{p}}}\rangle$$
$$= -\frac{1}{\tau}\langle(\hat{\boldsymbol{p}}\hat{\boldsymbol{p}} - \overleftrightarrow{1}/3)\psi_{\hat{\boldsymbol{p}}}\rangle. \tag{55}$$

In the limit of small gradients and time derivatives, the time-derivative term and the coupling to the third-order term of Eq. (54) in the gradient term of Eq. (55) can be neglected. This allows us to write the momentum flux tensor of Eq. (35) in the form

$$\Pi_{ij} = P_0\delta_{ij} + \left(\frac{\rho_s}{m_{\rm B}}\delta\mu_{\rm B} + \delta P^*\right)\delta_{ij} - \eta\left(\frac{\partial v_{ni}}{\partial x_j} + \frac{\partial v_{nj}}{\partial x_i} - \frac{2}{3}\nabla\cdot\boldsymbol{v}_n\delta_{ij}\right), \tag{56}$$

where the effective pressure of the normal component

$$\delta P^* = Kn_{\rm F}\delta\mu_{\rm B} + \tfrac{1}{3}m^*v_{\rm F}^2(1+F_0)\delta n_{\rm F} \tag{57}$$

and the coefficient of viscosity

$$\eta = \tfrac{1}{5}n_{\rm F}v_{\rm F}p_{\rm F}\tau. \tag{58}$$

Momentum conservation equation (21) combined with flux equation (56) gives an equation of motion. Combined with superfluid equation of motion (18), fermion conservation law (19) and the conservation law (36) give a complete set of linearized hydrodynamic equations. Using variables $(\delta n_{\rm F},\delta\mu_{\rm B},\boldsymbol{v}_n,\boldsymbol{v}_s)$ the set can be written as

$$\frac{\partial\delta n_{\rm F}}{\partial t} + n_{\rm F}\nabla\cdot\boldsymbol{v}_n = 0, \tag{59}$$

$$\frac{n_{\rm B}}{s^2}\frac{\partial\delta\mu_{\rm B}}{\partial t} + \rho_s\nabla\cdot\boldsymbol{v}_s + m_{\rm B}Kn_{\rm F}\nabla\cdot\boldsymbol{v}_n = 0, \tag{60}$$

$$\rho_n\frac{\partial\boldsymbol{v}_n}{\partial t} + \nabla\delta P^* = \eta\nabla^2\boldsymbol{v}_n + \frac{1}{3}\eta\nabla\nabla\cdot\boldsymbol{v}_n, \tag{61}$$

$$m_{\rm B}\frac{\partial\boldsymbol{v}_s}{\partial t} + \nabla\delta\mu_{\rm B} = 0. \tag{62}$$

Here Eq. (61) is the Navier–Stokes equation describing the viscous normal component.

One application of Eqs. (59)–(62) is to determine the velocities of the first and second sound modes. The results are identical to those found by Khalatnikov.[2]

XI. DISCUSSION

For *stationary* phenomena, the Fermi–Bose-liquid theory does not differ from the noninteracting system, when written in terms of proper variables. This follows because, dropping the time-derivative term in kinetic equation (27), the only difference in the noninteracting case is that the distribution $\delta n_{\boldsymbol{p}}$ is replaced by $\delta\bar{n}_{\boldsymbol{p}}$. Boundary conditions (48) and (49) and observables (33) and (34) also are functions of $\delta\bar{n}_{\boldsymbol{p}}$, and they depend on the interaction parameters m^*, α, D, and F_l's in a simple scaling manner, if at all.

The force applied to slowly moving objects in a Fermi liquid in the ballistic limit was calculated in Ref. 14. All the results presented in Ref. 14 [Eqs. (13)–(19)] concern the time-independent case. Therefore their independence of the interactions parameters m^*, α, D, and F_l follows most simply from the general argument of the preceding paragraph.

In time-dependent problems, the interactions have an important effect through the $\delta\bar{\epsilon}$ term in kinetic equation (27). For small frequencies, however, the coupling to the superfluid motion can be neglected, as argued in Sec. VII. Thus at low frequencies the response of Fermi–Bose liquid to external perturbation is the sum of independent superfluid and Fermi-liquid responses.

The theory formulated here is applied to calculate the force on a vibrating cylinder in Refs. 3, 4, 14, and 15.

[1] L. D. Landau, Sov. Phys. JETP **3**, 920 (1957).
[2] I. M. Khalatnikov, Sov. Phys. JETP **28**, 1014 (1969).
[3] T. H. Virtanen and E. V. Thuneberg, Phys. Rev. Lett. **106**, 055301 (2011).
[4] T. H. Virtanen and E. V. Thuneberg, Phys. Rev. B (to be published).
[5] A. A. Abrikosov and I. M. Khalatnikov, Rep. Prog. Phys. **22**, 329 (1959).
[6] P. Nozieres, *Theory of Interacting Fermi systems* (Bejamin, New York, 1964).
[7] E. M. Lifshitz and L. P. Pitaevski, *Statistical Physics*, Part 2 (Pergamon, Oxford, 1980).
[8] G. Baym and C. Pethick, *Landau Fermi-Liquid Theory* (Wiley, New York, 1991).
[9] J. Bardeen, G. Baym, and D. Pines, Phys. Rev. **156**, 207 (1967).

[10]C. Ebner and D. O. Edwards, Phys. Rep. **2**, 77 (1971).

[11]I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).

[12]J. Sykes and G. A. Brooker, Ann. Phys. NY **56**, 1 (1970).

[13]I. L. Bekarevich and I. M. Khalatnikov, Sov. Phys. JETP **12**, 1187 (1961).

[14]T. H. Virtanen and E. V. Thuneberg, AIP Confe. Proc. **850**, 113 (2006). The diffusive boundary condition, Eq. (8) of this reference, should be replaced by Eq. (51) of this paper.

[15]T. H. Virtanen and E. V. Thuneberg, J. Phys. Conf. Ser. **150**, 032115 (2009).