Gaudin models solver based on the correspondence between Bethe ansatz and ordinary differential equations

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We present a numerical approach which allows the solving of Bethe equations whose solutions define the eigenstates of Gaudin models. By focusing on a different set of variables, the canceling divergences which occur for certain values of the coupling strength no longer appear explicitly. The problem is thus reduced to a set of quadratic algebraic equations. The required inverse transformation can then be realized using only linear operations and a standard polynomial root-finding algorithm. The method is applied to Richardson's fermionic pairing model, the central spin model, and the generalized Dicke model.

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I. INTRODUCTION

The correspondence between the Bethe ansatz and an ordinary differential equation (ODE) was found some time ago[1] on the basis of the similarity between the T-Q system (Transfer Matrix/Auxiliary Matrix) of the Bethe ansatz and functional relations including spectral determinants on the ODE side. For extensive review we refer to Ref. 2. The correspondence is related to the Langland correspondence where the ODE part has to do with so-called Miura opers.[3] It has many interesting links with conformal field theory, quasiexact solvability, supersymmetry, and PT-symmetric quantum mechanics. It was also used in the context of physics of cold atoms and the quantum impurity problem.[4] Here we develop further this remarkable correspondence to facilitate the solution of otherwise difficult-to-solve Gaudin systems and implement it in several physically interesting situations.

Indeed, numerous integrable models derived from a generalized Gaudin algebra have been used to describe the properties of physical systems. The Richardson fermionic pairing Hamiltonian[5] has been used to explain most properties of superconducting nanograins,[6,7] numerous studies of the decoherence of a single electron spin trapped in a quantum dot rely on the central spin model,[8] while the inhomogeneous Dicke models[9] have been used to study light-matter interaction in many cavity-based systems.

The quantum integrability of such systems brings major simplifications to the structure of their eigenstates. In a system containing N degrees of freedom, they can be fully described by a number M of complex parameters that we call rapidities. M is here a number of the same order as N despite the exponentially large Hilbert space. The possible values of this set of parameters can be obtained by finding solutions to a set of M coupled nonlinear algebraic equation: the Bethe equations. However, analytical solution of the Bethe equations remains impossible except in very specific cases. The problem is still numerically approachable but solution of nonlinear equations remains a challenge that only iterative methods can tackle.

While previous efforts in designing algorithms have shown promising results,[10–14] methods which are sufficiently fast and stable for systematically finding a large number of eigenstates remained elusive. As was shown by one of the authors, efficient computation of only a small fraction of the complete Hilbert space can be sufficient to access static,[15] dynamical,[16] and even nonequilibrium dynamical properties of these systems,[17,18] making such a method highly desirable.

This paper presents an algorithm which finds solutions to the system of equations with unprecedented speed and stability. The first section discusses general considerations related to the quantum systems discussed here. Section III describes in detail the method used to solve for a set of intermediate variables, and we show in Sec. IV how one can recover the rapidities from these variables. Explicit numerical applications to the Richardson model, the generalized Dicke model, and the central spin model are presented in Sec. V and we conclude in the remaining section.

II. THE MODELS

Let us first introduce the generalized Gaudin algebra defined by the operators $S^x(\lambda_i), S^y(\lambda_i), S^z(\lambda_i)$ with λ_i any complex number, and the commutation rules they obey:[19,20]

$$\begin{aligned}
[S^x(\lambda_i), S^y(\lambda_j)] &= i[Y(\lambda_i,\lambda_j)S^z(\lambda_i) - X(\lambda_i,\lambda_j)S^z(\lambda_j)],\\
[S^y(\lambda_i), S^z(\lambda_j)] &= i[Z(\lambda_i,\lambda_j)S^x(\lambda_i) - Y(\lambda_i,\lambda_j)S^x(\lambda_j)],\\
[S^z(\lambda_i), S^x(\lambda_j)] &= i[X(\lambda_i,\lambda_j)S^y(\lambda_i) - Z(\lambda_i,\lambda_j)S^y(\lambda_j)],\\
[S^\kappa(\lambda_i), S^\kappa(\lambda_j)] &= 0, \kappa = x,y,z.
\end{aligned} \tag{1}$$

In this paper, we deal with the rational family of Gaudin models for which

$$X(\lambda_i,\lambda_j) = Y(\lambda_i,\lambda_j) = Z(\lambda_i,\lambda_j) = \frac{g}{\lambda_i - \lambda_j}. \tag{2}$$

The generalized Dicke model that we also consider is, however, obtained from a trigonometric Gaudin model[21] defined by

$$\begin{aligned}
X(\lambda_i,\lambda_j) = Y(\lambda_i,\lambda_j) &= \frac{g}{\sin(\lambda_i - \lambda_j)},\\
Z(\lambda_i,\lambda_j) &= g\cot(\lambda_i - \lambda_j).
\end{aligned} \tag{3}$$

The model is then derived by taking the large-spin limit of a Holstein-Primakoff transformation for one of the degrees of freedom. The resulting limit shares more resemblance with the rational family of Gaudin models and simply constitutes its extension to include a bosonic degree of freedom. We will therefore indiscriminately use the term "rational" to also include models derived from this particular limit.

For any given realization of the Gaudin algebra, one can define a set of N commuting operators R_i allowing one to build exactly solvable Hamiltonians

$$H = \sum_{i=1}^{N} \eta_i R_i, \tag{4}$$

for which the R_i are evidently constants of motion. Eigenstates of these models are, for a given number of excitations M, all defined by the construction

$$|\{\lambda_1, \ldots, \lambda_M\}\rangle \propto \prod_i \mathrm{S}^+(\lambda_i)|0\rangle. \tag{5}$$

Here $\mathrm{S}^+(u) = \mathrm{S}^x(u) + i\mathrm{S}^y(u)$ is a family of generalized creation operators parametrized by the complex parameter u. Its explicit expression in terms of the fundamental operators defining a particular realization will be model dependent. The pseudovacuum $|0>$ is defined as the lowest-weight vector, i.e., $\mathrm{S}^-(u)|0>=0$, and can also differ in distinct realizations of the Gaudin algebra.

States of the form (5) become eigenstates of the system provided the M rapidities λ_i are solutions of a set of coupled nonlinear algebraic equations: the Bethe equations. For rational models, these equations can be written, in general, as

$$F(\lambda_i) = \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j}, \tag{6}$$

with $S^z(\lambda_i)|0>= F(\lambda_i)|0>$ defining the lowest-weight function F.

One of the major difficulties in solving these equations numerically is the divergences which occur whenever two rapidities coincide. While they are canceled by similarly diverging terms on the left-hand side of the equation, they still have an important impact on numerical stability and computational speed. To circumvent these potential pitfalls, we introduce the function

$$\Lambda(z) \equiv \sum_{k=1}^{M} \frac{1}{z - \lambda_k} = \frac{P'(z)}{P(z)}, \tag{7}$$

where

$$P(z) = \prod_{k=1}^{M} (z - \lambda_k) \tag{8}$$

is the polynomial of degree M whose M roots correspond to the values of λ_k.

Since $\Lambda(z)$ obeys the following Riccati-type differential equation:[22,23]

$$\frac{\partial \Lambda(z)}{\partial z} + \Lambda^2(z) = -\sum_{\alpha} \frac{1}{(z-\lambda_\alpha)^2} + \sum_{\alpha,\beta} \frac{1}{(z-\lambda_\alpha)(z-\lambda_\beta)} = \sum_{\alpha\neq\beta} \frac{2}{(z-\lambda_\alpha)(\lambda_\alpha - \lambda_\beta)}, \tag{9}$$

it is easy to show that when the set $\{\lambda_i\}$ is a solution of the Bethe equations we have

$$\Lambda'(z) + \Lambda^2(z) - \sum_{\alpha} \frac{2F(\lambda_\alpha)}{(z-\lambda_\alpha)} = 0. \tag{10}$$

One can derive this last equation with respect to z any number of times to write additional equalities

$$\begin{aligned} &\Lambda''(z) + 2\Lambda(z)\Lambda'(z) + \sum_{\alpha} \frac{2F(\lambda_\alpha)}{(z-\lambda_\alpha)^2} = 0, \\ &\Lambda'''(z) + 2\Lambda(z)\Lambda''(z) + 2\Lambda'(z)^2 - \sum_{\alpha} \frac{4F(\lambda_\alpha)}{(z-\lambda_\alpha)^3} = 0, \\ &\vdots \end{aligned} \tag{11}$$

From this point on, we will assume a particular form for the F function which, while not general, encompasses a wide variety[20] of physically relevant realizations of the Gaudin algebra. We restrict ourselves to the following:

$$F(\lambda_\alpha) = -\sum_{i=1}^{N} \frac{A_i}{(\epsilon_i - \lambda_\alpha)} + \frac{B}{2g}\lambda_\alpha + \frac{C}{2g}. \tag{12}$$

The exact physical nature of the parameters g and ϵ_i is highly model dependent, but in the cases treated in this paper it will be made explicit in Sec. V when discussing their specifics. In (pseudo)spin models, $A_i = |s_i|\Omega_i$ with $|s_i|$ the norm of the local spin degree of freedom, while Ω_i is an integer relatable to the degeneracy, i.e., the number of elements of the set $\{\epsilon_j\}$ equal to ϵ_i. Consequently, every A_i can then take any integer or half-integer value.

III. SOLVING THE SYSTEM

From the previously found set of differential equations, we can write a new set of algebraic equations by simply taking the limits $z \to \epsilon_j$ of Eqs. (13) and (11). Using the previous form of $F(\lambda_\alpha)$ [Eq. (12)] we find

$$(1-2A_j)\Lambda'(\epsilon_j) + \Lambda^2(\epsilon_j) + \frac{B}{g}M - \frac{B\epsilon_j + C}{g}\Lambda(\epsilon_j) + \sum_{i\neq j} 2A_i \frac{\Lambda(\epsilon_j) - \Lambda(\epsilon_i)}{\epsilon_i - \epsilon_j} = 0, \tag{13}$$

$$\begin{aligned} &(1-A_j)\Lambda''(\epsilon_j) + 2\Lambda(\epsilon_j)\Lambda'(\epsilon_j) - \frac{B}{g}\Lambda(\epsilon_j) \\ &\quad - \frac{B\epsilon_j + C}{g}\Lambda'(\epsilon_j) + \sum_{i\neq j} 2A_i \frac{\Lambda(\epsilon_j) - \Lambda(\epsilon_i)}{(\epsilon_i - \epsilon_j)^2} \\ &\quad + \Lambda'(\epsilon_j) \sum_{i\neq j} \frac{2A_i}{\epsilon_i - \epsilon_j} = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &\left(1 - \frac{2}{3}A_j\right)\Lambda'''(\epsilon_j) + 2\Lambda(\epsilon_j)\Lambda''(\epsilon_j) + 2\Lambda'(\epsilon_j)^2 \\ &\quad - 2\frac{B}{g}\Lambda'(\epsilon_j) - \frac{B\epsilon_j + C}{g}\Lambda''(\epsilon_j) \\ &\quad + \sum_{i\neq j} 4A_i \frac{\Lambda(\epsilon_j) - \Lambda(\epsilon_i)}{(\epsilon_i - \epsilon_j)^3} + \sum_{i\neq j} \frac{4A_i\Lambda'(\epsilon_j)}{(\epsilon_i - \epsilon_j)^2} \\ &\quad + \sum_{i\neq j} \frac{2A_i\Lambda''(\epsilon_j)}{\epsilon_i - \epsilon_j} = 0, \\ &\vdots \end{aligned} \tag{15}$$

One can immediately see that for any nondegenerate spin-1/2 degree of freedom ($A_j = \frac{1}{2}$), the first differential equation reduces to a quadratic algebraic one which depends only on the set of variables $\{\Lambda(\epsilon_j)\}$. This is due to the canceling of the first term ($1 - 2A_j = 0$).

For $A_j = 1$, which can occur either due to a doubly degenerate spin 1/2 or a single spin 1, the two first equations form a quadratic system of equations depending on $\{\Lambda(\epsilon_j)\}$ but also on the first derivative $\Lambda'(\epsilon_j)$ evaluated at ϵ_j. Larger values of A_j require additional equations, but it is always possible to write a closed coupled system of quadratic equations. It will depend on $\Lambda(\epsilon_j)$ for variables with $A_j = \frac{1}{2}$, on both $\Lambda(\epsilon_j), \Lambda'(\epsilon_j)$ for variables with $A_j = 1$, on $\Lambda(\epsilon_j), \Lambda'(\epsilon_j), \Lambda''(\epsilon_j)$ for variables with $A_j = \frac{3}{2}$, etc. The resulting system of equations is built by using the n first equations above for any variables with $A_j = \frac{n}{2}$ and its solutions give a one-to-one correspondence with the solutions of the Bethe equations and therefore the eigenstates of the system.

There is one caveat that readers should be aware of. While a spin $S > \frac{1}{2}$ and a set of $2S$ degenerate spins $\frac{1}{2}$ lead to the same set of Bethe equations, only in the former case will the solutions to the Bethe equations give us the full Hilbert space. This is particularly simple to understand for a system containing only one spin $S = 1$ or two degenerate spins $\frac{1}{2}$. In the first case, the Hilbert space dimension is 3 while in the second it is 4 and the resulting Bethe equations have only three distinct solutions. In the degenerate case, the Bethe equations will give us only the highest-weight states (one can think of this as the $J = 1$ triplet for two spins $\frac{1}{2}$) and the remainder of the Hilbert space will need to be reconstructed by building the appropriate set of orthogonal states.

While in principle the system of quadratic equations can be solved for any spins or degeneracies, for the remainder of this paper we will focus on the simplest case of nondegenerate spin-1/2 systems ($A_j = \frac{1}{2}\ \forall\ j$). In this case, the closed set of algebraic equations is given by the N following equations:

$$\Lambda^2(\epsilon_j) = \sum_{i\neq j}^{N} \frac{\Lambda(\epsilon_j) - \Lambda(\epsilon_i)}{\epsilon_j - \epsilon_i} - M\frac{B}{g} + \left(\frac{B}{g}\epsilon_j + \frac{C}{g}\right)\Lambda(\epsilon_j). \tag{16}$$

In the subspace $M < N$ this system of equations is larger than the original one [Eq. (6)]. However, because the problem is quadratic and does not have canceling divergences, it is much easier to tackle numerically.

We also note in passing that the values of $\Lambda(\epsilon_j)$ determine the eigenvalues r_j of the commuting Hamiltonians R_j. The eigenvalues of the transfer matrix $\tau(\lambda) = \frac{1}{2}\mathrm{Tr}(\mathrm{S}^2)$ are given in terms of $F(\lambda)$ [see Eq. (12)] as follows:

$$\tau(\lambda) = F^2(\lambda) + F'(\lambda) + 2\sum_i [F(\lambda) - F(\lambda_i)]/(\lambda - \lambda_i), \tag{17}$$

with the last term reducing to $(BM/2g) + \sum_j A_j\Lambda(\epsilon_j)/(\lambda - \epsilon_j)$. Since the eigenvalue r_j of the conserved operator R_j is given by the pole of $\tau(\lambda)$ at $\lambda = \epsilon_j$ we can read off this eigenvalue by looking at the residue of $\tau(\lambda)$ at this pole:

$$r_j = 2A_j\Lambda(\epsilon_j) - A_j(C + B)/g + 2\sum_{i\neq j} A_iA_j/(\epsilon_j - \epsilon_i). \tag{18}$$

As with any nonlinear system, solution of Eqs. (16) necessitates an iterative method such as the well-known Newton-Raphson approach. Provided we have access to an initial approximate solution that is good enough to be in its basin of attraction, convergence of the method to a specific solution will be quadratic. Finding eigenstates therefore requires a sufficiently good guess which we obtain by slightly deforming known solutions. In the cases treated here, we exclusively know the exact eigenstates of the system at $g = 0$. We therefore approach the problem by slowly deforming these $g = 0$ solutions. Provided g is raised in small enough steps, this guarantees that the previously found solutions can be used to generate a good approximation at the next point.

Remarkably, it is numerically simple to compute the n first derivatives of the variables $\Lambda_j = g\Lambda(\epsilon_j)$ with respect to g. Indeed, one can show that every order needs only the solution of the same linear system with an updated right-hand side that is straightforwardly computed from the previously computed information. Defining $\Lambda_j^{(n)} = \frac{d^n\Lambda_j}{dg^n}$, we obtain in matrix form the following linear system:

$$\mathrm{K}\vec{\Lambda}^{(n)} = \vec{R}_n \tag{19}$$

for some right-hand side that depends only on the lower derivatives of $\Lambda(\epsilon_j)$ and a constant matrix

$$\mathrm{K}_{ij} = \begin{cases} \frac{g}{\epsilon_i - \epsilon_j}, & i \neq j, \\ \sum_{k\neq j}\frac{-g}{\epsilon_k - \epsilon_j} + B\epsilon_j + C - 2\Lambda_j, & i = j. \end{cases} \tag{20}$$

The components of the right-hand side $R_{n,j}$ can be computed iteratively as

$$\begin{aligned} R_{0,j} &= -\Lambda_j^2, \\ R_{n,j} &= \frac{n}{g}\big(-R_{n-1,j} + \Lambda_j^{(n-1)}[(B\epsilon_j + C) - 2\Lambda_j]\big) \\ &\quad + \sum_{k=1}^{n-1}\binom{n}{k}\Lambda_j^{(k)}\Lambda_j^{(n-k)}. \end{aligned} \tag{21}$$

Since this requires a single matrix inversion (LU or QR decomposition to be exact) which can be reused at every order, it is a numerically fast process to compute the n first derivatives. Using the resulting derivatives, a Taylor expansion gives an excellent initial approximation to $\Lambda(\epsilon_j)$ at $g + \Delta g$ even for fairly large Δg:

$$\tilde{\Lambda}(\epsilon_j)\big|_{g+\Delta g} = \Lambda(\epsilon_j)\big|_g + \sum_{k=1}^{n}\frac{1}{k!}\left.\frac{\partial^k\Lambda(\epsilon_j)}{(\partial g)^k}\right|_g (\Delta g)^k. \tag{22}$$

In principle, the radius of convergence of the Taylor expansion around the current solution would set an upper limit on the Δg step we can take. Nonetheless, one should keep in mind that adding terms to the Taylor series offers only linear convergence while Newton's method converges

quadratically. While computing fewer derivatives and relying on more Newton steps could speed up the calculation, it also comes with the risk that the initial guess falls outside the basin of attraction of the desired solution. Optimization of the computation speed without compromising stability is then a question of balance between the number of computed derivatives and the size of the steps one takes in g.

IV. INVERTING $\Lambda(z)$

Our capacity to compute physically relevant quantities relies on the use of Slavnov's determinant formula[24] which gives expressions for the expectation values and correlation functions in terms of simple determinants[25] built out using the set of rapidities $\{\lambda_i\}$. It is therefore necessary to be able to extract the rapidities from a given solution obtained in terms of the variables $\{\Lambda(\epsilon_i)\}$.

Going back to the definition of $\Lambda(z)$ [Eq. (7)], Newton's identities allow us to write it explicitly in terms of elementary symmetric polynomials. We have

$$\Lambda(\epsilon_j) = \frac{P'(\epsilon_j)}{P(\epsilon_j)} = \frac{\sum_{m=0}^{M} m\epsilon_j^{m-1} P_{M-m}}{\sum_{m=0}^{M} \epsilon_j^m P_{M-m}}, \tag{23}$$

with

$$P_k = (-1)^k \sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant M} \lambda_{j_1}\lambda_{j_2},\ldots,\lambda_{j_k},$$
$$P_0 = 1. \tag{24}$$

Having the N values $\Lambda(\epsilon_j)$ at hand, it is then a trivial matter to write a linear problem for the elementary symmetric polynomials:

$$\sum_{m=0}^{M-1} \left[m\epsilon_j^{m-1} - \Lambda(\epsilon_j)\epsilon_j^m\right] P_{M-m} = \Lambda(\epsilon_j)\epsilon_j^M - M\epsilon_j^{M-1}. \tag{25}$$

A. Case $M < N$

When the number of excitations M is smaller than N, one simply needs to pick M values of (ϵ_j) to extract the polynomials P_{M-m}. The complete set of M elementary symmetric polynomials give the real coefficients of the single-variable polynomial $P(z)$ at every order. The corresponding ensemble of parameters $\{\lambda_i\}$ which defines the eigenstate is then obtained by finding every root of $P(z)$.

This is a well-studied problem for which many methods exists. Here we choose to use Laguerre's method with polynomial deflation. Laguerre's method is an almost "sure-fire" method for finding one root r_1. We then proceed by finding a root r_2 of the deflated polynomial $\frac{P(z)}{z-r_1}$ at the next step, then a root of $\frac{P(z)}{(z-r_1)(z-r_2)}$, etc. Repetition M times allows the extraction of every root of our initial polynomial, i.e., every rapidity λ_i.

This procedure for inverting the $\Lambda(z)$ function is purely local in g in the sense that it only needs $\Lambda(\epsilon_j)$ at a given g. Therefore it is not necessary to perform the inversion at values of g we are not interested in. Solving for $\Lambda(\epsilon_j)$ still requires this scan in g but if we are, for example, only interested in the large-g regime, not having to perform the inversion at intermediate steps can markedly reduce computation time.

B. Case $M > N$

For a Hamiltonian realized in terms of only spin-1/2 degrees of freedom the number of excitations is always $M \leqslant N$. However, in more general cases there exists a subspace where the excitation number is larger than the number of degrees of freedom ($M > N$). In such a case, one cannot simply use the set of N parameters $\Lambda(\epsilon_j)$ to find the $M > N$ elementary symmetric polynomials.

However, for any solutions with $M \leqslant \sum_{i=1}^{N} 2A_i$, one can still extract them by solving only linear problems, since the supplementary information has been obtained from the necessity to solve for certain derivatives $\Lambda'(\epsilon_i)$. The unbounded number of excitations in models containing bosonic degrees of freedom, such as the Dicke models presented here, would give a subset of states with $M > \sum_{i=1}^{N} 2A_i$. These would not allow such a simple inversion. However, since any degree of freedom with A_i can only accommodate up to $2A_i$ excitations, for any model based exclusively on spins (of any magnitude and degeneracy) the total number of excitations is always bounded and every solution has $M \leqslant \sum_{i=1}^{N} 2A_i$.

In solving the quadratic equations (13), (14), etc. we are provided with $\Lambda(\epsilon_i)$ as well as its $2A_i - 1$ first derivatives. Having these values at hand, to write a linear problem for the elementary symmetric polynomials is simply a matter of defining rational functions which are linear in P_n in both the numerator and denominator. Naturally, we already have from the definition

$$\Lambda(z) = \frac{P'(z)}{P(z)} = \frac{\sum_{m=0}^{N} m z^{m-1} P_{N-m}}{\sum_{m=0}^{N} z^m P_{N-m}}, \tag{26}$$

but we can also write

$$V_1(z) \equiv \Lambda'(z) + \Lambda(z)^2 = \frac{P''(z)}{P(z)},$$
$$V_2(z) \equiv \Lambda''(z) + 3\Lambda(z)\Lambda'(z) + \Lambda(z)^3 = \frac{P'''(z)}{P(z)}, \tag{27}$$
$$\vdots$$

By evaluating the M necessary functions $[\Lambda(z), V_1(z), V_2(z), \ldots]$ at $z = \epsilon_j$, we can write M equations linear in the polynomials P_n. Solving them and using a root-finding algorithm for $P(z)$ would, once again, give us the set of rapidities $\{\lambda_i\}$.

V. APPLICATIONS

We now turn to three specific models derived from a generalized Gaudin algebra in order to demonstrate the efficiency of this approach. First we treat the discrete reduced BCS model[26] (Richardson model). One of the authors has already been involved in solving the Bethe equations in this context.[15–18] However, in this series of papers, the solution was performed without the currently discussed method. This led to serious stability and computation time issues, which are now extremely well controlled in the current approach.

A. Richardson model

The Richardson model,[5] while having a rich history in nuclear physics, has also recently found renewed interest in condensed matter physics in light of tunneling experiments performed on superconducting nanograins.[6] It is nothing but a discrete version of the reduced BCS model which limits the interaction to a uniform s-wave pairing term between time-reversed states. Using the Anderson pseudospin-$\frac{1}{2}$ representation in terms of fermionic operators,

$$\begin{aligned} S_i^z &= c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger c_{i\downarrow} c_{i\uparrow} - \tfrac{1}{2}, \\ S_i^+ &= c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger, \\ S_i^- &= c_{i\downarrow} c_{i\uparrow}, \end{aligned} \tag{28}$$

the Hamiltonian is given by

$$H = \sum_{i=1}^{N} \epsilon_i S_i^z - g \sum_{i,j=1}^{N} S_i^+ S_j^-. \tag{29}$$

Here ϵ_j corresponds to the discrete set of unblocked single-fermion energies which can accommodate a Cooper pair, while g is the pairing strength between time-reversed fermionic states. While its integrability in a strict sense was proven much later,[27] the exact solution of the model was first proposed by Richardson himself.[5] The eigenstates of the system are of the form given in Eq. (5) with a pseudovacuum $|0\rangle = |\downarrow, \downarrow, \ldots, \downarrow\rangle$ that has fully down-polarized pseudospins (Fock vacuum of Cooper pairs) and the Gaudin creation operators given by

$$\mathrm{S}^+(\lambda_\alpha) = \sum_{i=1}^{N} \frac{S_i^+}{\lambda_\alpha - \epsilon_i}. \tag{30}$$

The Bethe equations for this particular realization are given by[28,29]

$$\sum_{\alpha=1}^{N} \frac{1/2}{\lambda_j - \epsilon_\alpha} + \frac{1}{2g} = \sum_{k\neq j}^{M} \frac{1}{\lambda_j - \lambda_k}, \quad j = 1, \ldots, M. \tag{31}$$

This obviously corresponds to an F function of the type given in Eq. (12) with $A_i = \frac{1}{2}$, $B = 0$, and $C = 1$. Solution of the model is straightforwardly carried out using the procedure outlined in this paper. For any set of rapidities which is a solution of the Bethe equations, the eigenenergy of the state is then given by

$$E = \sum_{j=1}^{N} \lambda_j. \tag{32}$$

At zero coupling, the M-pair eigenstates are simply obtained by occupying any M energy levels with a Cooper pair (pseudospin up) while the remaining $N - M$ are empty. This can be represented by the set of rapidities $\lambda_\alpha = \epsilon_{i_\alpha}$ which according to Eq. (30) lead to an up-pseudospin at every energy ϵ_{i_α} present in the set of rapidities. This state obviously leads to diverging $\Lambda(\epsilon_{i_\alpha}) = \sum_\alpha \frac{1}{\epsilon_{i_\alpha} - \lambda_\alpha}$ for the occupied levels, but linearizing the Bethe equations (31) tells us that every rapidity can be written as $\lambda_i = \epsilon_{\alpha_i} - g$ at weak coupling. We consequently solve Eqs. (16) using $\Lambda_j = g\Lambda(\epsilon_j)$, which are given at zero coupling by

$$\Lambda_j = \begin{cases} 1, & \text{occupied } \epsilon_j, \\ 0, & \text{empty } \epsilon_j. \end{cases} \tag{33}$$

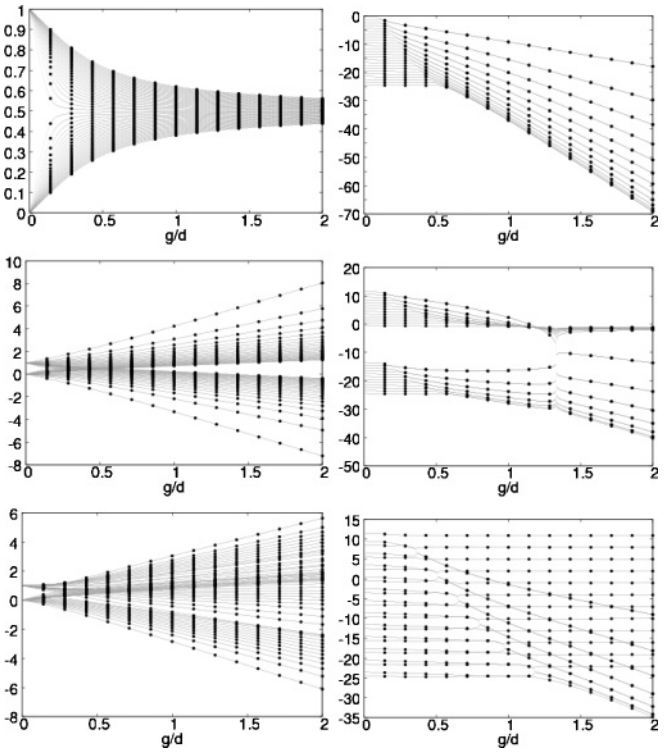

FIG. 1. $g\Lambda(\epsilon_j)$ (left) and the real part of the corresponding rapidities (right) for the ground state (top) and two excited states of the equally spaced Richardson model ($N = 50, M = 25$). The circles are the only points needed when the six first derivatives are used for the regression.

While in other applications, such as nuclear physics,[30,31] a different distribution of the levels could be more suitable, we present results for equally spaced single-particle energy levels within a bandwidth D, i.e., $\epsilon_j = -\frac{D}{2} + \frac{D}{N} j$. This Richardson model is the one typically used in condensed matter systems.[7] Both $\Lambda_j = g\Lambda(\epsilon_j)$ and the corresponding rapidities for the ground state and two excited states are plotted in Fig. 1.

The black circles show the points at which we actually solve the equations, the step size in g between two points being $\Delta_g/d = \frac{1}{7}$. These results were computed using the six first derivatives for the regression (see Sec. III). For reference we present in gray the numerical solution as a continuous function of g/d. The figure therefore shows how this approach allows steps in the coupling constant which are very large compared to the actual scale on which the rapidities themselves vary.

The middle panel shows a state specifically chosen to generate a region of rapid variation of the rapidities that one can see around $g/d = 1.3$. When solving directly for the rapidities themselves using Eqs. (31), one would need an extremely small step size in this region in order to maintain stability. In the current approach, however, this complex structure causes no problem since the variables Λ_j remain smoothly varying

functions. This constitutes a prime example of the remarkable capacities of this approach, which allows the scan in g to be performed orders of magnitude faster than by use of the standard Bethe equations (31).

B. Generalized Dicke model

We now turn to a second related model, the generalized Dicke model. It describes a collection of N multilevel systems coupled uniformly to a single bosonic mode and has been shown to be Bethe ansatz solvable.[21,32] Here, we again consider the case of two-level systems (represented by a spin 1/2), which is evidently the relevant case for cavity-based quantum computing proposals.[33] The Hamiltonian takes the form

$$H = \omega b^\dagger b + \sum_{j=1}^{N} \epsilon_j S_j^z + V \sum_{j=1}^{N} (b^\dagger S_j^- + S_j^+ b), \qquad (34)$$

where $\{S_j^z, S_j^+, S_j^-\}$ are the usual generators of spin-1/2 representations of SU(2) at each site j. The boson frequency is ω, ϵ_j sets the splitting between the two levels of every subsystem, while V controls the strength of the interaction. This Hamiltonian conserves the total number of excitations $b^\dagger b + \sum_j (S_j^z + \frac{1}{2})$. The construction of M-excitation eigenstates is achieved using the Gaudin creation operators

$$\mathrm{S}^+(\lambda_\alpha) = b^\dagger + \sum_j \frac{V}{\lambda_\alpha - \epsilon_j} S_j^+ \qquad (35)$$

acting repeatedly on the pseudovacuum state $|0\rangle = |0; \downarrow, \downarrow, \ldots, \downarrow\rangle$ which contains no boson and is fully down-polarized. The appropriate sets of rapidities $\{\lambda_\alpha\}_{\alpha=1}^M$ must fulfill the Bethe equations[34]

$$\frac{\omega}{2V^2} - \frac{\lambda_\alpha}{2V^2} - \sum_{i=1}^{N} \frac{1/2}{\epsilon_i - \lambda_\alpha} = \sum_{\beta \neq \alpha}^{M} \frac{1}{\lambda_\alpha - \lambda_\beta}. \qquad (36)$$

Here, we can identify the general form [Eq. (12)] with $g = V^2$, $A_i \equiv 1/2$, $B = -1$, and $C = \omega$. The energy eigenvalues corresponding to each set of rapidities are

$$E(\{\lambda_\alpha\}) = \sum_\alpha \lambda_\alpha - \sum_j \epsilon_j. \qquad (37)$$

Now we turn to the solution of (36) using the method proposed in the previous section. The equivalent equations for $\Lambda_j = V^2 \Lambda(\epsilon_j)$ read

$$V^2 \sum_{i \neq j} \frac{\Lambda_i - \Lambda_j}{\epsilon_i - \epsilon_j} = \Lambda_j[(\epsilon_j - \omega) + \Lambda_j] - V^2 M. \qquad (38)$$

By linearizing the Bethe equations (36), one can show that $\lambda_\alpha = \epsilon_j - \frac{V^2}{\omega - \epsilon_j}$ give correct solutions in the limit $V^2 \to 0$. These values lead to $\Lambda_j = \omega - \epsilon_j$ at $V = 0$. As in the Richardson model, $\lambda_\alpha = \epsilon_{j_\alpha}$ leads to an up-spin at level ϵ_{j_α}. Another possible solution is obtained for $\lambda_\alpha = \omega$, which leads to $\Lambda_j = 0$. In this case, every $\lambda_\alpha = \omega$ corresponds to an additional bosonic excitation at $V^2 = 0$, since eigenstates are constructed using the operator (35). Any of the $d = \sum_{i=1}^{M} \binom{N}{i}$ combinations of M excited spins and bosons gives us a possible eigenstate of H at $V^2 = 0$, which we deform numerically to find eigenstates and eigenvalues at nonzero coupling $V^2 > 0$.

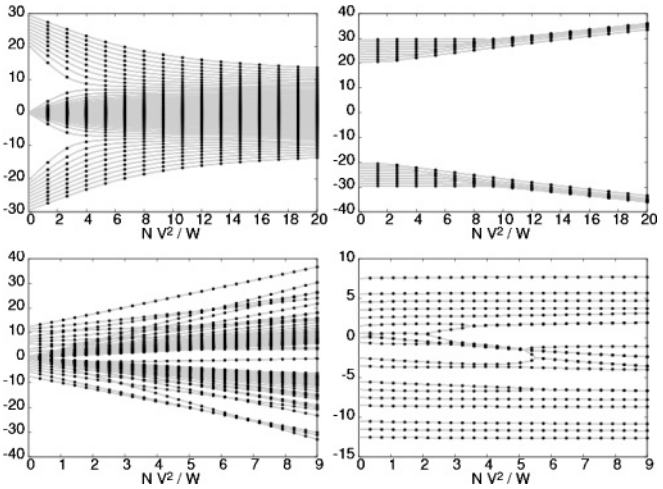

FIG. 2. $V^2\Lambda(\epsilon_j)$ (left) and the real part of the corresponding rapidities (right) for the equally spaced generalized Dicke model ($N = 60, M = 20$). The circles are the only points needed when the five first derivatives are used for the regression.

Figure 2 shows the rapidities and Λ_j for two different states as a function of V^2. We made a specific choice for the inhomogeneity by using equally spaced splittings $\epsilon_j = -\frac{D}{2} + \frac{D}{N} j$. This would ultimately correspond to a coarse-graining of a flat distribution of splittings within a bandwidth D. Moreover, we choose the bosonic frequency to be at the midpoint of the band, i.e., $\omega = 0$.

C. Central spin model

Finally, we apply the method to the central spin (CS) model, which in its most frequently encountered form describes a single spin coupled to an external magnetic field and, via hyperfine interaction, to a spin bath. It is straightforwardly obtained from the commuting conserved quantities of the SU(2) *XXX* Gaudin models given by

$$R_i = S_i^z + \sum_{j \neq i} \frac{g}{\epsilon_i - \epsilon_j} \vec{S}_i \cdot \vec{S}_j \qquad (39)$$

by defining the exactly solvable model as $H = \frac{R_0}{g}$, with the choice $\epsilon_0 = 0$, $h_0 = \frac{1}{g}$, and $A_j = -\frac{1}{\epsilon_j}$, i.e.,

$$H = h_0 S_0^z + \sum_{j \neq 0} A_j \vec{S}_0 \cdot \vec{S}_j. \qquad (40)$$

While the external magnetic field is coupled only to the central spin $i = 0$, the conservation of the total excitation number $S_0^z + \sum_{j \neq 0} S_j^z$ makes the model equivalent to

$$H = h g_e S_0^z + h g_N \sum_{j \neq 0} S_j^z + \sum_{j \neq 0} A_j \vec{S}_0 \cdot \vec{S}_j, \qquad (41)$$

which would also include coupling of the bath spins to the external magnetic field.

In principle, any linear combination of the integrals of motion also leads to an integrable model which commutes with

Hamiltonian (40) and therefore shares the same eigenbasis. Consequently, any Hamiltonian

$$H = \sum_i \eta_i R_i = \sum_i h\eta_i S_i^z + \sum_{i,j>i} \frac{\eta_i - \eta_j}{\epsilon_i - \epsilon_j} \vec{S}_i \cdot \vec{S}_j$$
$$\equiv h_0 S_0^z + \sum_{i\neq 0} h_i S_i^z + \sum_{i\neq 0} A_i \vec{S}_0 \cdot \vec{S}_i + \sum_{i\neq 0, j>i} B_{i,j} \vec{S}_i \cdot \vec{S}_j \tag{42}$$

is also Bethe ansatz solvable. While this allows the treatment of models which include hyperfine interactions between the bath spins $i > 0$, integrability restricts the allowed values of the couplings. For example, if we suppose that bath spins are homogeneously coupled to the external field, i.e., $\eta_j = \eta \ \forall \ j > 0$, integrability imposes the absence of coupling between bath spins. On the other hand, models with nonzero interaction in the bath can be built from any choice of η_j, but this in turn will force the presence of an inhomogeneous magnetic field acting differently on every bath spin.

In the limit $h = 0$, one could study models with bath interactions provided the couplings respect the constraints

$$\frac{B_{jk}}{\eta_j - \eta_k}[A_j(\eta_0 - \eta_k) - A_k(\eta_0 - \eta_j)] = A_j A_k, \tag{43}$$

which has important consequences. For example, when any two nuclear spins are equally coupled ($A_j = A_k = A$) to the central one, the coupling between those two spins also needs to be given by $B_{jk} = A$. For a given set of distinct A_j, it would, however, still be possible and interesting to study central spin systems with uniform bath couplings $B_{i,j} = B$ or any integrable case that emerges from a given choice of distinct η_j.

We still choose to focus on Hamiltonians of the form (40). Is is straightforward to show that it commutes with the Richardson Hamiltonian [Eq. (29)] defined by parameters $g = -\frac{1}{h}, \epsilon_0 = 0, \epsilon_j = -\frac{1}{A_j}$. They therefore share common eigenstates so that, in principle, solving Eq. (31) for negative values of g would give us the eigenstates of the central spin Hamiltonian. However, we follow the alternative road of inverting the spin quantization axis ($\hat{z} \to -\hat{z}$) and solving Eq. (31) for positive g. Eigenstates are then obtained by using the operators

$$S^+(\lambda) = \sum_{j=0}^{N_b} \frac{S_j^-}{\lambda - \epsilon_j} \tag{44}$$

acting on the fully up-polarized pseudovacuum $|0\rangle = |\uparrow\uparrow \ldots \uparrow\rangle$, and the eigenenergy of a given eigenstate is given by

$$E_0(\{\lambda_k\}_{k=1,\ldots,M}) = \sum_{j=1}^{N_b} \frac{1}{\epsilon_0 - \epsilon_j} + 2\sum_{k=1}^{M} \frac{1}{\lambda_k - \epsilon_0} + \frac{h}{2}. \tag{45}$$

In some of the physically relevant systems that can be described with the central spin model [such as quantum dots or nitrogen vacancies (NVs) in diamond[35]], the proper interactions between the CS and the nuclei do not give rise to equally spaced values of $\epsilon_j = -\frac{1}{A_j}$. In fact, some of the parameters ϵ_j can be very close to one another while others are strongly separated. While the Bethe equations are exactly the same as for the Richardson model, this distribution of ϵ_j has a direct consequence on the application of the current method. Since for equally spaced levels with separation d the Bethe equations can be written only in terms of g/d, the size of the steps in g that one can use is simply controlled by the d parameter. However, when a number of levels are very close together while others are far from one another, these variable spacings lead to variations of $\Lambda(\epsilon_j)$ controlled by different scales. Naturally, use of a step in g that is a given fraction of the smallest distance $\epsilon_j - \epsilon_{j+1}$ would ensure stability, but it would ultimately necessitate a large number of points to reach the strong $g = \frac{1}{h}$ limit (weak magnetic field). It is therefore beneficial to use a variable step size. Starting from small steps at low g, we turn to larger ones when g has increased sufficiently and the behavior of the solutions varies on a much slower scale. Variables steps could also be used in the previously studied models, but in the current case it is more or less necessary to do so in order to achieve fast computation.

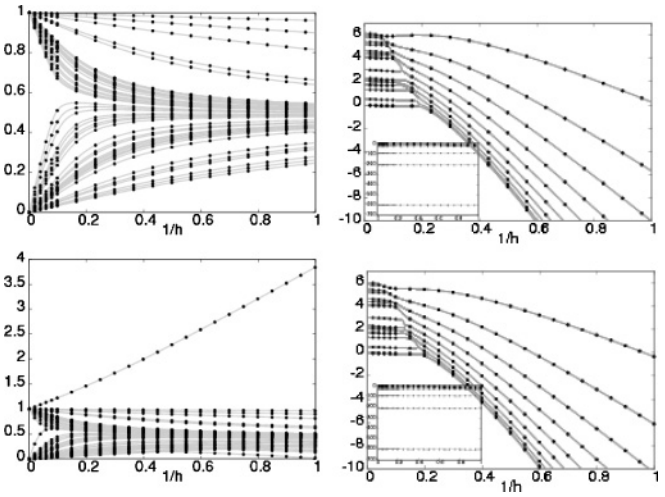

FIG. 3. $g\Lambda(\epsilon_j)$ (left) and the rapidities (right) for the dipole couplings of an NV center (Ref. 36). $\epsilon_j = -4/A_j$ and $A_j = 5.6(a_{jj}/R_j)^3[1 - 3\cos(\theta_j)^2]$ (MHz), where $a_{jj} = 1.54$ Å is the nearest neighbor distance for diamond and R_j is the distance between the jth carbon atom and the defect (center). θ_j is the angle between the magnetic field and $\vec{R}_j$. The vectors $\vec{R}_j$ are corrected by a small amount of randomness to avoid degeneracies. Top panel shows the ground state; bottom one shows an excited state. Insets show the full range of rapidities while the main figures focus on the region with nontrivial structure in the given range of magnetic field.

Figure 3 shows the behavior of the rapidities for an NV center ($N = 50$ and $M = 25$); in the interval g going from 0 to 0.1 the Newton steps were taken for $gS(\epsilon_j)$ with a step size $\Delta g = 1/50$; for $g > 0.1$ the step size was increased to $\Delta g = 1/15$ ($\Delta g = 1/20$ for the excited state). The first six derivatives were used.

Due to the structure of the levels ϵ_j we choose to restrict the $y$axis in the plots of the rapidities to make the intricate structure of the solutions visible. The insets present them in the complete range.

Once again, the black dots represent the points at which the equations are solved. One can see the increase in the step size at large $g \equiv \frac{1}{h}$. In spite of the necessity of using smaller steps, these plots show without a doubt that this approach still

allows us to realize the scan in g using large steps on the scale on which rapidities themselves vary.

A variable step size could naturally be defined by making use of the fact that we compute the truncated Taylor expansion up to a given order. One could fix the appropriate steps simply by defining a desired level of precision for the limited expansion. This would allow one to control stability, while adapting the steps to the behavior of the $\Lambda(\epsilon_j)$ variables at the current g point.

VI. CONCLUSIONS

In this work we have shown how, for rational Gaudin models, the set of nonlinear coupled Bethe equations can be transformed to a different set of equations which is significantly simpler to solve numerically. We presented the complete algorithm and applied it to a variety of models derived from the generalized Gaudin algebra.

This work exclusively presents and demonstrates the capacities of this approach for solution of Bethe equations. In light of previous work,[15–18] it should however be clear that by allowing rapid and systematic solution of a decent fraction of the full Hilbert space this can directly be used for relevant physical calculations. For example, studies of the nonequilibrium dynamics of the discussed models are currently being pursued by the authors. The outcome of these calculations should prove invaluable, considering that this technique gives direct access to regimes which have been hard to describe before. To name only this one, the weak-magnetic-field limit of the central spin models is a prime example.

While we focused on rational Gaudin models, algorithms tailored to the trigonometric or hyperbolic *XXZ* (Ref. 20) or general *XYZ* (Ref. 37) models could possibly be built along similar lines. This is a more fundamental question that is left open for the time being.

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