Magnetic field driven instability of a charged center in graphene

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It is shown that a magnetic field dramatically affects the problem of supercritical charge in graphene, making any charge in gapless theory supercritical. The cases of a radially symmetric potential well and a Coulomb center in a homogeneous magnetic field are considered. The local density of states and polarization charge density are calculated in the first order of perturbation theory. It is argued that the magnetically induced instability of the supercritical Coulomb center can be considered as a quantum-mechanical counterpart of the magnetic catalysis phenomenon in graphene.

I. INTRODUCTION

Recently, it was shown[1–4] that atomic collapse in a strong Coulomb field,[5,6] a fundamental quantum relativistic phenomenon still inaccessible in high-energy experiments, can be readily investigated in graphene. In quantum electrodynamics (QED), taking into account the finite size of a nucleus,[7] theoretical works on the Dirac-Kepler problem showed that for atoms with nuclear charge in excess of $Z > 170$, the electron states dive into the lower continuum leading to positron emission.[5,6]

In graphene, the effective Coulomb coupling constant is given by $\beta = Ze^2/\kappa$, where $\alpha = e^2/\hbar v_F \simeq 2.19$ is the "fine-structure" coupling constant, $v_F \approx 10^6$ m/s is the velocity of Dirac quasiparticles, and κ is a dielectric constant. The Hamiltonian of the system is not self-adjoint when the coupling β exceeds the critical value $\beta_c = 1/2$.[1–4] Similar to the Dirac equation in QED, one should replace the singular $1/r$ potential by a regularized potential that takes into account the finite size of the charged impurity, R: $V(r) = -\frac{Ze^2}{\kappa r}\theta(r - R) - \frac{Ze^2}{\kappa R}\theta(R - r)$. For gapped quasiparticles in such a regularized potential, the critical coupling is determined by $\beta_c = 1/2 + \pi^2/\log^2(c\Delta R/\hbar v_F)$,[8] where Δ is a quasiparticle gap and the constant $c \approx 0.21$, and β_c tends to $1/2$ for $\Delta \to 0$ or $R \to 0$.

Since the electrons and holes strongly interact by means of the Coulomb interaction, one may expect[8–10] an excitonic instability in graphene with subsequent phase transition to a phase with gapped quasiparticles that may turn graphene into an insulator. This semimetal-insulator transition in graphene is actively studied in the literature, where numerical simulations give the critical coupling constant $\alpha_c \approx 1.19$.[11,12]

In a many-body system or quantum field theory, the supercritical coupling leads to more dramatic consequences compared to the case of the Dirac equation for the electron in the Coulomb potential. Unlike the case of the Coulomb center, the many-body supercritical coupling instability cannot be resolved through a spontaneous creation of a finite number of electron-positron pairs. Like the Cooper instability in the theory of superconductivity, the QED supercritical coupling instability is resolved only through the formation of a condensate of electron-positron pairs generating a mass gap in the spectrum.[13]

The presence of a magnetic field makes the situation even more interesting. It was shown in Ref. 14 that a magnetic field catalyzes the gap generation for gapless fermions in relativistic-like systems, and even the weakest attraction leads to the formation of a symmetry-breaking condensate. Therefore, the system is always in the supercritical regime once there is an attractive interaction. The magnetic catalysis plays an important role in quantum Hall effect studies in graphene,[15–20] where it is responsible for lifting the degeneracy of the Landau levels.

The magnetic catalysis phenomenon suggests that the Coulomb impurity in a magnetic field in graphene should be supercritical for any Z. The Dirac equation for quasiparticles in graphene in the Coulomb potential in a magnetic field was considered in Ref. 21, where exact solutions were found for certain values of the magnetic field; however, no instability or resonance was found.

In QED in $(3 + 1)$ dimensions, the Coulomb center problem in a magnetic field was studied in Ref. 22. There it was found that magnetic field B confines the transverse electronic motion, and the electron in the magnetic field is closer to the nucleus than in the free atom. Thus, it is situated in a stronger Coulomb field. Therefore, $Z_c\alpha$ decreases with B. This result is consistent with the magnetic catalysis phenomenon,[14] according to which the magnetic field catalyzes gap generation and leads to a zero critical coupling constant in both $(3 + 1)$- and $(2 + 1)$-dimensional theories.

We would like to stress that the presence of a homogeneous magnetic field changes qualitatively the supercritical Coulomb center problem. Indeed, if a magnetic field is absent, then the supercritical Coulomb center instability leads to a resonance that describes an outgoing positron propagating freely to infinity. However, since charged particles confined to a plane do not propagate freely to infinity in a magnetic field, such a behavior is impossible for the in-plane Coulomb center problem in a magnetic field. Therefore, *a priori* it is not clear how the instability suggested by the magnetic catalysis manifests itself in the Coulomb center problem in a magnetic field. The main goal of this paper is to answer this question.

In Secs. II and III, we consider the Dirac equation for the electron in the potential well and Coulomb center in a magnetic field. We study the local density of states (LDOS) and induced charge density for both cases in Sec. IV, where similarities and differences between the cases of gapped and gapless quasiparticles as well as the potential well and Coulomb interactions are discussed. In Sec. V, we give a brief summary

of our results. Finally, we provide the details of our calculations of the LDOS and polarization charge density in Appendix A.

II. POTENTIAL WELL

The electron quasiparticle states in the vicinity of the $K_\pm$ points of graphene in the field of Coulomb impurity and in a homogeneous magnetic field perpendicular to the plane of graphene are described by the Dirac Hamiltonian in $(2+1)$ dimensions,

$$H = \hbar v_F \boldsymbol{\tau} \boldsymbol{p} + \xi \Delta \tau_3 + V(r), \qquad (2.1)$$

where the canonical momentum $\boldsymbol{p} = -i\nabla + e\boldsymbol{A}/c$ includes the vector potential $\boldsymbol{A}$ corresponding to the external magnetic field $\boldsymbol{B}$, τ_i are the Pauli matrices, and Δ is a quasiparticle gap. The two-component spinor $\Psi_{\xi s}$ carries the valley ($\xi = \pm$) and spin ($s = \pm$) indices. We will use the standard convention $\Psi^T_{+s} = (\psi_A, \psi_B)_{K_+s}$, whereas $\Psi^T_{-s} = (\psi_B, \psi_A)_{K_-s}$, and A, B refer to two sublattices of a hexagonal graphene lattice. Since the interaction $V(r)$ does not depend on spin, in what follows we will omit the spin index s.

It is instructive to consider first the Dirac equation for the electron in a potential well $V(r) = -V_0\theta(r_0 - r)$ with $V_0 > 0$ in a magnetic field perpendicular to the plane of graphene. We have

$$\begin{pmatrix} \xi\Delta & \hbar v_F(-iD_x - D_y) \\ \hbar v_F(-iD_x + D_y) & -\xi\Delta \end{pmatrix} \Psi(\mathbf{r})$$
$$= [E - V(r)]\,\Psi(\mathbf{r}), \qquad (2.2)$$

where $D_i = \partial_i + (ie/\hbar c)A_i$ with $i = x, y$ is the covariant derivative, and the symmetric gauge $(A_x, A_y) = (B/2)(-y, x)$ is used for the magnetic field. It is clear that the solution at the K_- point is obtained from the solution at the K_+ point changing $\Delta \to -\Delta$ and exchanging the spinor components $\psi_A \leftrightarrow \psi_B$.

In polar coordinates,

$$iD_x + D_y = e^{-i\phi}\left(i\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial\phi} + \frac{ieBr}{2\hbar c}\right),$$
$$iD_x - D_y = e^{i\phi}\left(i\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial\phi} - \frac{ieBr}{2\hbar c}\right). \qquad (2.3)$$

We can represent $\Psi(\mathbf{r})$ in terms of the eigenfunctions of the conserved angular momentum $J_z = L_z + \sigma_z/2 = -i\partial/\partial\phi + \sigma_z/2$ as follows:

$$\Psi(\mathbf{r}) = \frac{1}{r}\begin{pmatrix} e^{i(j-\frac{1}{2})\phi}f(r) \\ ie^{i(j+\frac{1}{2})\phi}g(r) \end{pmatrix}, \qquad (2.4)$$

with $j = \pm 1/2, \pm 3/2, \dots$. For functions $f(r), g(r)$, we get the following equations:

$$f' - \frac{j+1/2}{r}f - \frac{r}{2l^2}f + \frac{E+\xi\Delta-V(r)}{\hbar v_F}g = 0, \quad (2.5)$$

$$g' + \frac{j-1/2}{r}g + \frac{r}{2l^2}g - \frac{E-\xi\Delta-V(r)}{\hbar v_F}f = 0, \quad (2.6)$$

where $l = \sqrt{\hbar c/|eB|}$ is the magnetic length. These equations are easily solved for the potential well in two regions $r < r_0$ and $r > r_0$ in terms of confluent hypergeometric functions. In

the region $r < r_0$, eliminating the function $g(r)$, we obtain the second-order differential equation for the function $f(r)$:

$$f'' - \frac{1}{\rho}f' + \left[2p_V^2 - j - \frac{1}{2} - \frac{j^2 - j - 3/4}{\rho^2} - \frac{\rho^2}{4}\right]f = 0, \qquad (2.7)$$

and in the region $r > r_0$ we have the same equation but with $V_0 = 0$. Here we introduced the following dimensionless quantities:

$$p_V^2 = \frac{l^2[(E+V_0)^2 - \Delta^2]}{2(\hbar v_F)^2}, \quad p^2 = \frac{l^2(E^2 - \Delta^2)}{2(\hbar v_F)^2}, \qquad (2.8)$$

and $\rho = r/l$. We get the following solutions, which are regular at $r = 0$:

$$f_1(\rho) = \rho^{j+\frac{1}{2}}e^{-\rho^2/4}\frac{C_1}{\Gamma(j+1/2)}$$
$$\times \Phi\left(j + \frac{1}{2} - p_V^2, j + \frac{1}{2}; \frac{\rho^2}{2}\right), \qquad (2.9)$$

$$g_1(\rho) = \frac{l(E+V_0-\xi\Delta)}{\sqrt{2}\hbar v_F}\rho^{j+\frac{3}{2}}e^{-\rho^2/4}\frac{C_1}{\Gamma(j+3/2)}$$
$$\times \Phi\left(j + \frac{1}{2} - p_V^2, j + \frac{3}{2}; \frac{\rho^2}{2}\right), \qquad (2.10)$$

and decrease at infinity,

$$f_2(\rho) = C_2\rho^{j+\frac{1}{2}}e^{-\rho^2/4}$$
$$\times \Psi\left(j + \frac{1}{2} - p^2, j + \frac{1}{2}; \frac{\rho^2}{2}\right), \qquad (2.11)$$

$$g_2(\rho) = \frac{\sqrt{2}\hbar v_F C_2}{l(E+\xi\Delta)}\rho^{j+\frac{3}{2}}e^{-\rho^2/4}$$
$$\times \Psi\left(j + \frac{1}{2} - p^2, j + \frac{3}{2}; \frac{\rho^2}{2}\right), \qquad (2.12)$$

respectively (note that these expressions are valid at all $j = \pm 1/2, \pm 3/2, \dots$).

Then sewing solutions at $r = r_0$,

$$\frac{f_1(\rho)}{f_2(\rho)}\bigg|_{\rho=\rho_0} = \frac{g_1(\rho)}{g_2(\rho)}\bigg|_{\rho=\rho_0}, \quad \rho_0 = \frac{r_0}{l}, \qquad (2.13)$$

we obtain the following transcendental equation for energies of solutions with the total angular momentum j:

$$\frac{2(\hbar v_F)^2\left(j+\frac{1}{2}\right)\Phi\left(j + \frac{1}{2} - p_V^2, j + \frac{1}{2}; \frac{\rho_0^2}{2}\right)}{l^2(E+V_0-\xi\Delta)\Phi\left(j + \frac{1}{2} - p_V^2, j + \frac{3}{2}; \frac{\rho_0^2}{2}\right)}$$
$$= (E+\xi\Delta)\frac{\Psi\left(j + \frac{1}{2} - p^2, j + \frac{1}{2}; \frac{\rho_0^2}{2}\right)}{\Psi\left(j + \frac{1}{2} - p^2, j + \frac{3}{2}; \frac{\rho_0^2}{2}\right)}. \qquad (2.14)$$

Below we analyze this equation analytically and numerically.

A. Instability in the absence of a magnetic field

In this subsection, we consider the problem of the potential well instability in the absence of a magnetic field ($B = 0$). This will serve us as a useful reference point in the next section, where we study instability in a magnetic field. For $B = 0$, the

energy spectrum may be obtained either by solving the Dirac equation from the very beginning, or by taking the limit of zero field ($l \to \infty$) in Eq. (2.14). In the latter case, one needs to use the following formulas:[23]

$$\Phi(a,c;z) = e^z \Phi(c-a,c;-z), \qquad (2.15)$$

$$\lim_{a \to \infty} \Phi(a,c;-z/a) = \Gamma(c) z^{\frac{1-c}{2}} J_{c-1}(2\sqrt{z}), \qquad (2.16)$$

$$\lim_{a \to \infty} [\Gamma(1+a-c)\Psi(a,c;-z/a)]$$
$$= -i\pi e^{i\pi c} z^{\frac{1-c}{2}} H_{c-1}^{(1)}(2\sqrt{z}), \quad \mathrm{Im}\, z > 0, \qquad (2.17)$$

and $|\arg a| < \pi$ for the last two equations.

Assuming $|E| < \Delta$, we obtain

$$\frac{\sqrt{(E+V_0)^2 - \Delta^2}}{E + V_0 - \xi\Delta} \frac{J_{j-1/2}(\beta r_0)}{J_{j+1/2}(\beta r_0)}$$
$$= \frac{\sqrt{E^2 - \Delta^2}}{E - \xi\Delta} \frac{H_{j-1/2}^{(1)}(\beta' r_0)}{H_{j+1/2}^{(1)}(\beta' r_0)}, \qquad (2.18)$$

where $J_\nu(z)$, $H_\nu^{(1)}(z)$ are the Bessel and Hankel functions, respectively, $\beta = \sqrt{(E+V_0)^2 - \Delta^2}/\hbar v_F$, $\beta' = \sqrt{E^2 - \Delta^2}/\hbar v_F$, and the square roots are defined as $\mathrm{Im}\,\beta, \mathrm{Im}\,\beta' > 0$. In the regions with $\mathrm{Im}\,\beta, \mathrm{Im}\,\beta' \neq 0$, one can use the relations $H_\nu^{(1)}(iz) = (2/\pi i)e^{-i\pi\nu/2}K_\nu(z), J_\nu(iz) = e^{i\pi\nu/2}I_\nu(z)$. Equation (2.18) is invariant under the change $j \to -j, \xi \to -\xi$.[24]

Taking for the definiteness the K_- point ($\xi = -$), one can see from Eq. (2.18) that the energy spectrum is continuous for $|E| > \Delta$ and a discrete one for $|E| < \Delta$. The first bound state $E \lesssim \Delta$ appears at an arbitrary small interaction V_0. Indeed, taking $j = -1/2$, which corresponds to the smallest centrifugal barrier, we find

$$E \simeq \Delta \left[1 - 2\left(\frac{\hbar v_F}{\Delta r_0}\right)^2 \exp\left(-\frac{2(\hbar v_F)^2}{V_0 \Delta r_0} - 2\gamma\right)\right], \quad (2.19)$$

where γ is the Euler constant. Note that there is no solution with the energy $E \lesssim \Delta$ at the K_- point with angular momentum $j = 1/2$, but such a solution exists at the K_+ point similarly to the case of the Coulomb potential.[8]

As V_0 grows, at the critical strength of interaction,

$$V_{0\mathrm{cr}} = \Delta \left[1 + \sqrt{1 + \left(\frac{\hbar v_F}{\Delta r_0}\right)^2 j_{0,1}^2}\right] \qquad (2.20)$$

[$j_{0,1} \approx 2.41$ is the first zero of the Bessel function $J_0(x)$], the lowest-energy bound state dives into the lower continuum ($E = -\Delta$). We note that for the zero-gap case ($\Delta = 0$) there are no bound states at all. In the supercritical regime for $V_0 > V_{0\mathrm{cr}} = \hbar v_F j_{0,1}/r_0$ [which follows from Eq. (2.20) at $\Delta = 0$], resonances with complex energies appear, leading to instability of the system similar to the case of the supercritical Coulomb center.[1,8] The occurrence of resonant states synchronously with diving into the lower continuum of the lowest-energy bound state is the standard characteristic of QED systems.[1,5,6] We will see in the next subsection that the presence of a homogeneous magnetic field changes this conclusion.

Near the critical value of coupling, the energy of the resonant state is given by

$$E = -\frac{V_0 - V_{0\mathrm{cr}}}{\ln(1/\delta)} \exp\left(\frac{i\pi}{2\ln(1/\delta)}\right),$$
$$\delta = \frac{(V_0 - V_{0\mathrm{cr}})r_0 e^{\gamma - 1}}{2\hbar v_F}, \quad 0 < \delta \ll 1. \qquad (2.21)$$

The dependence of energy on the $V_0 - V_{0\mathrm{cr}}$ (deviation from the critical value) for the potential well interaction is nonanalytic but differs from the essential singularity that takes place in the Coulomb center problem. This, of course, is related to the absence of scale invariance for the potential well $V(r)$.

B. Magnetically driven instability

Before we consider the instability of the potential well problem in a homogeneous magnetic field, it is useful to recall the Landau energy levels for the electron states in graphene in a magnetic field. If the interaction vanishes ($V_0 = 0, r_0 \to 0$), Eq. (2.14) gives the well-known spectrum of Landau levels:

$$E = -\xi\Delta, \quad j \leqslant -\tfrac{1}{2}, \qquad (2.22)$$

$$E = \pm\sqrt{\Delta^2 + 2n\left(\frac{\hbar v_F}{l}\right)^2}, \quad n = 1, 2, \dots, j + \frac{1}{2} \leqslant n. \qquad (2.23)$$

Note that the level $E = \Delta$ ($E = -\Delta$) is present only at the K_- (K_+) point.

For nonzero V_0, the Landau energy levels are no longer degenerate. Using the sewing equation (2.14), we can determine the evolution of degenerate solutions with V_0. For solutions of the Landau level $E = \Delta$ with different j, their energies as a function of V_0/Δ (at fixed magnetic field B) are plotted in Fig. 1 for $l\Delta/(\sqrt{2}\hbar v_F) = 0.1$ and $\rho_0 = r_0/l = 0.02$. We see that as V_0 increases, more and more solutions with different j *cross the energy level $E = -\Delta$*. As usual,[5,6] this means that vacuum of the second quantized theory is unstable with respect to the creation of electron-hole pairs. However, as we discussed in the Introduction, there are *no resonance solutions* in the
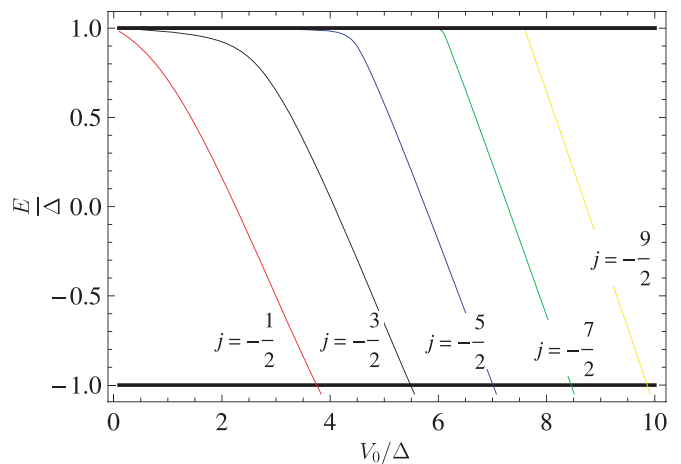

FIG. 1. (Color online) The evolution of degenerate solutions of the lowest Landau level at the K_- point as a function of the dimensionless ratio V_0/Δ.

presence of a constant magnetic field. The reason for that is the presence of the positive $r^2/4l^2$ term due to the magnetic field in the effective Schrödinger-like equation for one component of the spinor function [see Eq. (3.4) and Fig. 3 in the next section], which qualitatively changes the asymptotics of the effective potential: in nonzero field it grows at infinity instead of decreasing, as in the case $B = 0$. Therefore, quasiparticles are confined in such a potential and cannot escape to infinity forming only discrete levels.

We would like to note that the situation under consideration is analogous to that for a deep level vacancy in a many-electron atom. There electron states as solutions of the Dirac equation in the Coulomb potential of the nucleus are stable. However, taking into account the interaction with the second quantized electromagnetic field, the electrons on higher energy levels are unstable with respect to the transition to the vacant state with emission of photons.

The critical potential V_{0cr} is defined as the potential for which the first crossing occurs. According to Fig. 1, such a crossing is first realized for the state with $j = -1/2$ (the potential well interaction lifts the degeneracy of the Landau levels in quantum number j). Let us analyze in detail how this state evolves with V_0. For the state with $j = -1/2$, Eq. (2.14) becomes

$$(E + V_0 - \Delta)\frac{\rho_0^2}{2}\frac{\Phi\left(1 - p_V^2, 2; \frac{\rho_0^2}{2}\right)}{\Phi\left(-p_V^2, 1; \frac{\rho_0^2}{2}\right)}$$
$$= -(E - \Delta)\frac{\Psi\left(-p^2, 0; \frac{\rho_0^2}{2}\right)}{\Psi\left(-p^2, 1; \frac{\rho_0^2}{2}\right)}, \qquad (2.24)$$

where we used the relation

$$\lim_{c \to -m}\frac{\Phi(a, c; x)}{\Gamma(c)}$$
$$= \frac{\Gamma(a + m + 1)}{\Gamma(a)(m + 1)!}x^{m+1}\Phi(a + m + 1, m + 2; x),$$
$$m = 0, 1, \ldots \qquad (2.25)$$

For $V_0 \to 0$, Eq. (2.24) implies the following bound state at the K_- point:

$$E = \Delta - V_0(1 - e^{-r_0^2/2l^2}), \qquad (2.26)$$

which is in contrast with nonanalytic behavior in the coupling V_0 in the absence of magnetic field described by Eq. (2.19). (At the K_+ point, a similar bound state exists but with angular momentum $j = +1/2$.)

As the coupling V_0 grows, the energy of this bound state decreases and finally crosses the level $E = -\Delta$ at some critical value V_{0cr}. For $E = -\Delta$, $p^2 = 0$, and $p_V^2 = l^2(V_0^2 - 2\Delta V_0)/2(\hbar v_F)^2$, using $\Psi(0, 0; z) = \Psi(0, 1; z) = 1$ we find that Eq. (2.24) defines the following equation for V_{0cr}:

$$V_{0cr} = 2\Delta\left[1 + \frac{2\Phi\left(a, 1, \frac{\rho_0^2}{2}\right)}{\rho_0^2\Phi\left(1 + a, 2, \frac{\rho_0^2}{2}\right)}\right], \qquad (2.27)$$

where $a = -l^2V_{0cr}(V_{0cr} - 2\Delta)/2(\hbar v_F)^2$. [Note that at zero magnetic field ($l = \infty$), Eq. (2.27) reduces to Eq. (2.20) for V_{0cr}, which tends to a finite value in the gapless limit.] The

critical potential strength V_{0cr} as a function of Δ is plotted in Fig. 2 for different values of the parameter ρ_0, which defines the ratio of the potential well width to the magnetic length.

Analytically, it is not difficult to find that for $\rho_0 \ll 1$, Eq. (2.27) implies

$$V_{0cr} = 2\Delta\left(1 + 2l^2/r_0^2\right). \qquad (2.28)$$

It is clearly seen from Fig. 2 and from Eq. (2.28) that the critical potential strength V_{0cr} decreases with the growth of a magnetic field (or with the decrease of l) at fixed r_0 and Δ. The physical reason for that is that the magnetic field forces electron orbits to become closer to the charge center, making attraction stronger and, thus, effectively lowering the critical coupling.

What is surprising here is that V_{0cr} tends to zero as $\Delta \to 0$. Thus, the presence of a homogeneous magnetic field leads to the instability of gapless quasiparticles in the second quantized theory for *any* value of the potential strength V_0. This result suggests that the Coulomb center in gapless graphene in a magnetic field may also be unstable for any value Ze, which is the problem we study in the next section.

Finally, we will analyze states with energies near $\pm\Delta$ and large by modulus negative momenta j. We find that there exists an infinite series of levels approaching the energies $\pm\Delta$ asymptotically at large $|j + 1/2|$ (i.e., for sufficiently large j, the effect of the potential interaction V_0 can be neglected and the Landau levels are recovered). For $V_0 \to 0$, they behave as

$$E \simeq -\xi\Delta - \frac{V_0 e^{-\rho_0^2/2}}{\Gamma(k + 1)}\left(\frac{\rho_0^2}{2}\right)^{k+1}, \quad k = -\left(j + \frac{1}{2}\right) \gg 1. \qquad (2.29)$$

This can be found directly by solving Eq. (2.14), first taking there the limit $j + 1/2 \to -k$ by means of Eq. (2.25) and then analyzing the equation at weak coupling and large
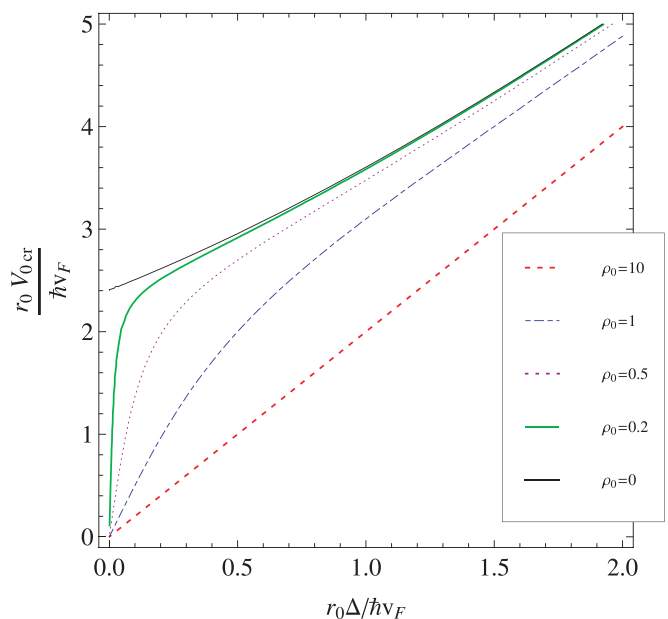

FIG. 2. (Color online) The critical potential V_{0cr} as a function of a gap for different values of ρ_0. The case of zero magnetic field corresponds to $\rho_0 = 0$.

k. Alternatively, Eq. (2.29) is obtained as the first-order correction in the interaction to the levels $\pm\Delta$ at $K_\pm$ points in a magnetic field. Note that the levels (2.29) lie below Δ for the K_- point and below $-\Delta$ for the K_+ point, respectively.

III. THE COULOMB CENTER

The equations for the functions $f(r)$ and $g(r)$ for the Coulomb center problem follow directly from Eqs. (2.5) and (2.6) by setting $V(r) = -Ze^2/r\ \theta(r-R) - Ze^2/R\ \theta(R-r)$ there (we take the dielectric constant $\kappa = 1$). Eliminating, for example, the function $f(r)$, one can get a second-order differential equation for the function $g(r)$. Further, introducing the function $\chi(r)$ by means of the relation

$$[E - \xi\Delta - V(r)]^{1/2}\chi(r) = \frac{g(r)}{\sqrt{r}}, \qquad (3.1)$$

we get the Schrödinger-like equation

$$-\chi''(r) + U(r)\chi(r) = \mathcal{E}\chi(r), \qquad (3.2)$$

where

$$\mathcal{E} = E^2 - \Delta^2, \qquad (3.3)$$

and the effective potential, $U = U_1 + U_2$,

$$U_1 = \frac{V(2E-V)}{(\hbar v_F)^2} + \frac{j(j+1)}{r^2} + \frac{r^2}{4l^4} + \frac{j-1/2}{l^2}, \quad (3.4)$$

$$U_2 = \frac{1}{2}\left[\frac{V''}{E - \xi\Delta - V} + \frac{3}{2}\left(\frac{V'}{E - \xi\Delta - V}\right)^2 \right. $$
$$\left. - \left(\frac{j}{r} + \frac{r}{2l^2}\right)\frac{2V'}{E - \xi\Delta - V}\right]. \qquad (3.5)$$

We plot the effective potential $U(r)$ near the K_- point for $E = -\Delta$ and $j = -1/2$ in Fig. 3. There the energy barrier in the absence of magnetic field is clearly seen, which leads to the appearance of resonances for sufficiently large charge. The presence of nonzero magnetic field *changes* the asymptotics of the effective potential at infinity and, thus, forbids the occurrence of resonance states. This feature distinguishes *qualitatively* the Coulomb center problem (as well as the potential well problem) in a magnetic form from that at $B = 0$.
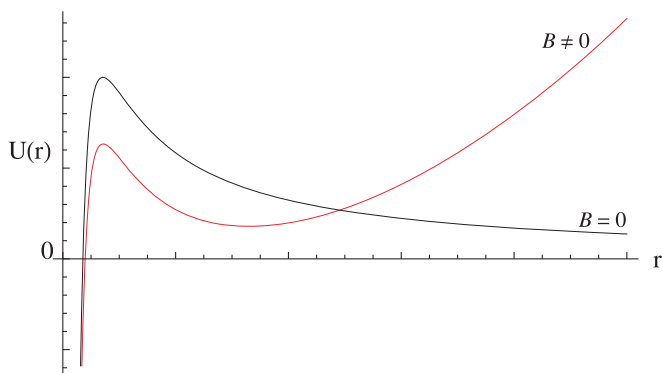

FIG. 3. (Color online) The potential $U(r)$ as a function of the distance from the Coulomb center at zero and nonzero magnetic field for the state with $E = -\Delta$ and $j = -1/2$.

Unfortunately, Eq. (3.2) belongs to the class of equations with two regular and one irregular singular (at $r = \infty$) points, and exact solutions of this equation cannot be expressed in closed form in terms of known special functions.

Since we are interested in solving Eqs. (2.5) and (2.6) with the Coulomb potential in the regime $Z\alpha \to 0$ ($\alpha = e^2/\hbar v_F$), we can find solutions using perturbation theory. For $Z\alpha = 0$, solutions of Eqs. (2.5) and (2.6) are the well known Landau states degenerate in the total angular momentum j. For the level $E^{(0)} = \Delta$, their normalized wave functions (2.4) have the form (at the K_- point)

$$\Psi_k(r,\phi) = \frac{(-1)^k}{l\sqrt{2\pi k!}}e^{-r^2/4l^2}\begin{pmatrix} 0 \\ \left(\frac{r^2}{2l^2}\right)^{k/2}e^{-ik\phi} \end{pmatrix}, \qquad (3.6)$$

where $k = -(j + 1/2) = 0,1,2,\dots$. The Coulomb potential removes degeneracy in j. Energy corrections of perturbed states of the Landau level $E^{(0)} = \Delta$ are found from the secular equation

$$|E^{(1)} - V_{k_1 k_2}| = 0.$$

Since $V_{k_1 k_2}$ is a diagonal matrix, we easily obtain

$$E_k^{(1)} = V_{kk} = -\frac{Ze^2}{k!2^k l}\int_0^\infty d\rho\, \rho^{2k}e^{-\rho^2/2}$$
$$= -\frac{Ze^2\Gamma\left(k + \frac{1}{2}\right)}{l\sqrt{2}\Gamma(k+1)}. \qquad (3.7)$$

Thus at large k, the energy levels accumulate near $E = \Delta$:

$$E_k \simeq \Delta - \frac{Ze^2}{l\sqrt{2k}}. \qquad (3.8)$$

As in the case of the potential well interaction considered in the previous section, the largest correction by modulus $E_0^{(1)} = -Z\alpha\hbar v_F\sqrt{\pi}/l\sqrt{2}$ occurs for the state with $j = -1/2$ ($k = 0$). The critical charge is determined by the condition $E = E^{(0)} + E_0^{(1)} = -\Delta$ when the level E crosses the level of the filled states. This gives

$$Z_c\alpha = \frac{2\sqrt{2}\Delta l}{\sqrt{\pi}\hbar v_F}. \qquad (3.9)$$

As in the case of the potential well in a magnetic field, the critical charge (3.9) tends to zero as $\Delta \to 0$. This means that a magnetic field indeed dramatically affects the Coulomb center problem in graphene, making any charge in gapless theory supercritical.

Equation (3.9) gives the critical Coulomb coupling in the regime $Z\alpha \to 0$ in the first order of perturbation theory. For arbitrary values of $Z\alpha$, we calculated the dependence of the critical coupling on the gap numerically. The corresponding results are presented in Fig. 4, where, for the parameter regularizing the Coulomb potential, we took $R = 10^{-3}l$. The dashed red line in Fig. 4 gives the critical Coulomb coupling $Z_c\alpha = 1/2 + \pi^2/\log^2(c\Delta R/\hbar v_F)$ in the absence of a magnetic field (see Fig. 1 in Ref. 8). Thus, at weak magnetic field ($l \to \infty$), the critical coupling tends to 1/2 while $Z_c\alpha \to 0$ for $l\Delta \to 0$ in the gapless or strong magnetic-field regime.
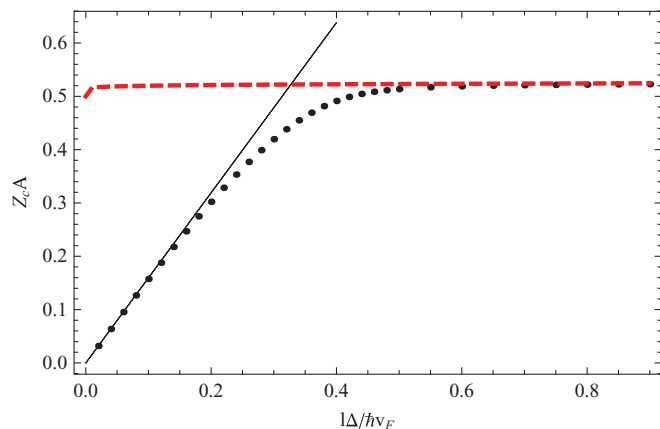
FIG. 4. (Color online) The critical Coulomb coupling $Z_c\alpha$ as a function of gap at zero (dashed red line) and nonzero magnetic field (dotted black line) for the state with $j = -1/2$. The straight black line corresponds to the critical Coulomb coupling in the first order of perturbation theory given by Eq. (3.9).

IV. THE LOCAL DENSITY OF STATES

It is interesting to see how a magnetic field and the Coulomb center affect the LDOS of quasiparticles that can be directly measured in scanning tunneling microscope (STM) experiments. The crucial difference in the case of gapless quasiparticles from that of gapped ones in a magnetic field is that the critical charge is zero for gapless quasiparticles, therefore energies of all previously degenerate states of the lowest Landau level become negative.

The LDOS at the distance $\mathbf{r}$ from impurity is given by

$$\rho(E;\mathbf{r}) = -\frac{1}{\pi}\mathrm{tr}\,\mathrm{Im}\,G(\mathbf{r},\mathbf{r};E+i\eta), \quad \eta \to 0, \quad (4.1)$$

where the trace includes the summation over the valley, sublattice, and spin degrees of freedom. The retarded Green's function $G(\mathbf{r},\mathbf{r}';E+i\eta)$ in a constant magnetic field can be written in the form

$$G(\mathbf{r},\mathbf{r}';E) = e^{i\Phi(\mathbf{r},\mathbf{r}')}\tilde{G}(\mathbf{r},\mathbf{r}';E), \quad (4.2)$$

$$\Phi(\mathbf{r},\mathbf{r}') = \frac{e}{\hbar c}\int_{\mathbf{r}'}^{\mathbf{r}} A_i^{\mathrm{ext}}(z)dz^i, \quad (4.3)$$

where we separated the gauge-dependent (Schwinger) phase $\Phi(\mathbf{r},\mathbf{r}')$ from a gauge invariant part of the Green's function $\tilde{G}(\mathbf{r},\mathbf{r}';E)$. The last one satisfies the following Lippmann-Schwinger equation:

$$\tilde{G}(\mathbf{r},\mathbf{r}';E) = \tilde{G}_0(\mathbf{r}-\mathbf{r}';E) + \int d\mathbf{r}''\tilde{G}_0(\mathbf{r}-\mathbf{r}'';E)$$
$$\times V(\mathbf{r}'')\tilde{G}(\mathbf{r}'',\mathbf{r}';E)e^{i[\Phi(\mathbf{r},\mathbf{r}'')+\Phi(\mathbf{r}'',\mathbf{r}')+\Phi(\mathbf{r}',\mathbf{r})]}. \quad (4.4)$$

[Note that the Green function $\tilde{G}(\mathbf{r},\mathbf{r}';E)$ is not translation-invariant in the presence of an impurity, unlike the noninteracting function $\tilde{G}_0(\mathbf{r}-\mathbf{r}';E)$.] For weak interaction, we can calculate the LDOS in the first order in the perturbation theory,

$$\rho(E;\mathbf{r}) = \rho_0(E;\mathbf{r}) + \delta\rho(E;\mathbf{r}), \quad (4.5)$$

where $\rho_0(E;\mathbf{r})$ is the LDOS for free quasiparticles in a magnetic field, and

$$\delta\rho(E;\mathbf{r}) = -\frac{1}{\pi}\mathrm{Im}\int d\mathbf{r}'\mathrm{tr}\,[\tilde{G}_0(\mathbf{r}-\mathbf{r}')V(\mathbf{r}')\tilde{G}_0(\mathbf{r}'-\mathbf{r})]. \quad (4.6)$$

First we consider the case of gapless quasiparticles. In this case, $\rho_0(E;\mathbf{r})$ and $\delta\rho(E;\mathbf{r})$ are calculated for gapless quasiparticles in the Appendix. We obtained there that the LDOS decreases at large distances ($r \gg r_0, l$) as

$$\delta\rho(\mathbf{r},E) \simeq \frac{V_0 r_0^2}{2\pi(\hbar v_F)^2}\mathrm{Im}[\lambda\Gamma^2(-\lambda)]e^{-\mathbf{r}^2/2l^2}\left(\frac{\mathbf{r}^2}{2l^2}\right)^{2\lambda} \quad (4.7)$$

for the potential well, while for the Coulomb center we obtained

$$\delta\rho(\mathbf{r},\mathbf{r},E) = \frac{Ze^2}{\kappa}\frac{1}{2\pi(\hbar v_F)^2}\left[\frac{B_0(\lambda)}{r} + \frac{l^2 B_1(\lambda)}{2r^3}\right] \quad (4.8)$$

with the functions $B_i(\lambda)$ given by Eqs. (A14) and (A15), and λ is defined after Eq. (A4). For the induced charge density,

$$n_{\mathrm{ind}}(\mathbf{r}) = -e\int_{-\infty}^{0} dE\,\delta\rho(\mathbf{r},E), \quad (4.9)$$

using Eqs. (4.7) and (4.8), we find that it is positive at large distances and decreases exponentially for the potential well and, due to $\int_{-\infty}^{0} dE\,B_0(\lambda) = 0$, as $1/r^3$ for the Coulomb interaction,

$$n_{\mathrm{ind}}(\mathbf{r}) \simeq a\frac{Ze^3 l}{\kappa\hbar v_F}\frac{1}{r^3}, \quad a = -\frac{3\zeta(-1/2)}{2\pi\sqrt{2}} \approx 0.07. \quad (4.10)$$

We remind the reader that in the absence of a magnetic field, the polarization charge diminishes as $1/r^2$ both in the supercritical $Z\alpha/\kappa > 1/2$ Coulomb center[1] and the potential well $V_0 > V_{0\mathrm{cr}} = \hbar v_F j_{0,1}/r_0$ (Ref. 25) interactions.

The situation is quite different in the case of gapped quasiparticles. Here we will consider the polarization charge density for the most interesting case of the Coulomb center in a homogeneous magnetic field. The polarization charge density (4.9) can be rewritten in the more familiar form

$$n_{\mathrm{ind}}(\mathbf{r}) = -e\sum_{E\leqslant-\Delta}[|\Psi_E(\mathbf{r})|^2 - |\psi_E(\mathbf{r})|^2], \quad (4.11)$$

where ψ_E and Ψ_E are the Landau wave functions and the wave functions of the Coulomb center problem in a magnetic field, respectively. Since we consider the case in which $Z\alpha$ is small, the corrections to the wave functions of negative energy states of the deep Landau levels defined by Eq. (2.23) can be ignored. We will consider only the corrections to the lowest Landau level states given by Eq. (2.22).

In the first order of perturbation theory, wave functions of the Landau level $E = -\Delta$ are sought as a superposition of all degenerate states with unknown coefficients $\Psi_E = \sum_{j\leqslant1/2} c_j\psi_{-\Delta}^j$, where $\psi_{-\Delta}^j$ are wave functions of the Landau level with $E = -\Delta$ and total momentum j. The unknown c_j are determined by the equation

$$\sum_{j_2\leqslant-1/2}(V_{j_1 j_2} - E^{(1)}\delta_{j_1 j_2})c_{j_2} = 0. \quad (4.12)$$

Since $V_{j_1 j_2} = V_{j_1 j_1} \delta_{j_1 j_2}$ is a diagonal matrix, the secular equation $|V_{j_1 j_2} - E^{(1)}| = 0$ is trivially solved and $E_j^{(1)} = V_{jj}$. Then Eq. (4.12) implies that the corresponding unknown coefficients c_{j_1} are equal to $c_{j_1} = 1$ for $j_1 = j$ and zero for all other j_1. Notice that $c_j = 1$ because the wave functions $\psi^j_{-\Delta}$ are normalized. This means that the wave functions of perturbed states of the Landau level $E = -\Delta$ do not change in the first order of perturbation theory. Consequently, according to Eq. (4.11), they do not contribute to the polarization charge density. Clearly, the polarization charge appears only when the first perturbed state of the Landau level $E = \Delta$ with $j = -1/2$ crosses the threshold of filled states of the lowest Landau level $E = -\Delta$. Using Eqs. (3.6) and (4.11), and the fact that perturbed wave functions of the $E = \Delta$ Landau level states do not change in the first order in the Coulomb potential, similarly to the case of perturbed wave functions of the $E = -\Delta$ Landau level considered above, we conclude that for the critical charge $Z_c \alpha$ given by Eq. (3.9), the polarization charge density equals

$$n_{\mathrm{ind}}(\mathbf{r}) = -\frac{e}{2\pi\, l^2} e^{-r^2/2l^2} . \tag{4.13}$$

Thus, the polarization charge density is concentrated near the impurity where it is negative and quickly decreases at large distances.

V. CONCLUSION

In this paper, we showed that in an external magnetic field, the value of the critical coupling for the onset of instability of a system of planar Dirac gapless quasiparticles interacting with a charged center (charged impurity) reduces to zero. This result serves as a quantum-mechanical single-particle counterpart of the magnetic catalysis phenomenon in graphene. The cases of a radially symmetric potential well and a Coulomb center were analytically considered. The local density of states and induced charge density were calculated in the first order of perturbation theory for gapless quasiparticles.

The crucial ingredient for instability is the existence of a zero-energy level for gapless Dirac fermions in a magnetic field that is infinitely degenerate. In this case, any weak attraction leads to the appearance of empty states in the Dirac sea of negative energy states and to the instability of a system.

The qualitative difference in the phenomenon of instability between gapped and gapless quasiparticles should be stressed. In the case of gapped quasiparticles, there is a *finite* critical value for the strength of interaction when the lowest unfilled level crosses the first filled one forming a hole in the sea of filled states. As the coupling grows, more and more levels cross that level. Clearly, the system tries to rearrange itself, filling in empty states whose presence is a signal of instability. The important difference in the case of gapless quasiparticles from that of gapped ones, besides the critical coupling being zero, is that an infinite number of states of the previously degenerate lowest Landau level become vacant.

Thus, the presence of an external magnetic field changes dramatically the problem of atomic collapse in graphene in a strong Coulomb field.[1] Clearly the problem becomes a many-body one and requires field-theoretical methods to find a true ground state. One should expect that the gap generation for initially gapless quasiparticles will take place already in the weak-coupling regime in the presence of a magnetic field.[16]

APPENDIX

The Green's function of free quasiparticles in a magnetic field is well known (see, for example, Refs. 14 and 16), and in the configuration space it has the form of a series over the Landau levels (we consider the zero gap case),

$$\tilde{G}_0(\mathbf{r}; E) = \frac{1}{2\pi l^2} e^{-\frac{\mathbf{r}^2}{4l^2}} \sum_{n=0}^{\infty} \frac{1}{(E+i\eta)^2 - M_n^2}$$
$$\times \left\{ E\left[P_- L_n\left(\frac{\mathbf{r}^2}{2l^2}\right) + P_+ L_{n-1}\left(\frac{\mathbf{r}^2}{2l^2}\right)\right] \right.$$
$$\left. + i\hbar v_F \frac{\boldsymbol{\tau}\mathbf{r}}{l^2} L_{n-1}^1\left(\frac{\mathbf{r}^2}{2l^2}\right) \right\}, \tag{A1}$$

where $M_n = (\hbar v_F / l)\sqrt{2n}$ are the energies of the Landau levels, $P_\pm = (1 \pm \tau_3)/2$ being the projectors, $L_n^\alpha(z)$ the generalized Laguerre polynomials [by definition $L_n(z) \equiv L_n^0(z)$ and $L_{-1}^\alpha(z) \equiv 0$], and the Pauli matrices $\tau_{1,2}$ act in the sublattice space. The sum over the Landau levels can be explicitly performed by means of the formula

$$\sum_{n=0}^{\infty} \frac{L_n^\alpha(x)}{n+b} = \Gamma(b)\Psi(b; 1+\alpha; x) \tag{A2}$$

[see Eq. (6.12.3) in the Ref. 23], leading to a closed expression for the free Green's function (see Refs. 26 and 27),

$$\tilde{G}_0(\mathbf{r}; E) = -\frac{e^{-\frac{\mathbf{r}^2}{4l^2}}}{4\pi\hbar^2 v_F^2}\left\{ E\left[P_- \Gamma(-\lambda)\Psi\left(-\lambda; 1; \frac{\mathbf{r}^2}{2l^2}\right)\right. \right.$$
$$\left. + P_+ \Gamma(1-\lambda)\Psi\left(1-\lambda; 1; \frac{\mathbf{r}^2}{2l^2}\right)\right]$$
$$\left. + i\hbar v_F \frac{\boldsymbol{\tau}\mathbf{r}}{l^2}\Gamma(1-\lambda)\Psi\left(1-\lambda; 2; \frac{\mathbf{r}^2}{2l^2}\right)\right\}. \tag{A3}$$

Here $\Psi(a; c; x)$ is the confluent hypergeometric function, which is related to the Whittaker function,

$$\Psi(a; c; x) = e^{\frac{x}{2}} x^{-\frac{1}{2}-\mu} W_{\kappa,\mu}(x), \quad \kappa = \frac{c}{2} - a, \quad \mu = \frac{c-1}{2}, \tag{A4}$$

$\Gamma(x)$ is the Euler gamma function, and $\lambda = (E + i\eta)^2 l^2/(2\hbar^2 v_F^2)$.

The LDOS of free quasiparticles in a magnetic field does not depend on **r** and is given by

$$\rho_0(E) = -\frac{1}{\pi} \lim_{\mathbf{r}\to 0} \operatorname{Im} \operatorname{tr}[\tilde{G}_0(\mathbf{r}; E + i\eta)] = \frac{1}{\pi^2 \hbar^2 v_F^2}$$

$$\times \lim_{\mathbf{r}\to 0} \operatorname{Im} \left\{ (E + i\eta) \left[\Gamma(-\lambda)\Psi\left(-\lambda; 1; \frac{\mathbf{r}^2}{2l^2}\right) \right. \right.$$

$$\left. \left. + \Gamma(1-\lambda)\Psi\left(1-\lambda; 1; \frac{\mathbf{r}^2}{2l^2}\right) \right] \right\}. \quad (A5)$$

The hypergeometric function $\Psi(a; c; x)$ at small x behaves as

$$\Psi(a; 1; x) \simeq -\frac{1}{\Gamma(a)}[\ln x + \psi(a) + 2\gamma] + O(x \ln x),$$

$$\Psi(a; 2; x) \simeq \frac{1}{\Gamma(a)x} + \frac{1}{\Gamma(a-1)}[\ln x + \psi(a) + 2\gamma - 1] + O(x \ln x), \quad (A6)$$

where $\psi(z)$ is the digamma function. Therefore,

$$\rho_0(E) = -\frac{1}{(\pi\hbar v_F)^2} \operatorname{Im}\{(E + i\delta)[\psi(-\lambda) + \psi(1-\lambda)]\}, \quad (A7)$$

and the LDOS of free quasiparticles in a magnetic field finally is found to be

$$\rho_0(E) = \frac{2}{\pi l^2} \left[\delta(E) + \sum_{n=1}^{\infty} [\delta(E - M_n) + \delta(E + M_n)] \right] \quad (A8)$$

[compare with Eq. (4.2) in Ref. 28].

The first order correction to the LDOS due to the interaction is given by Eq. (4.6). For the radial well to find the asymptotic at distances $r \gg r_0$, where r_0 is a range of the potential, we can set $r' = 0$ in the arguments of the free Green's functions in Eq. (4.6) and get the following behavior:

$$\delta\rho(\mathbf{r}, \mathbf{r}; E) = V_0 r_0^2 \operatorname{Im} \operatorname{tr}[\tilde{G}_0(\mathbf{r}; E)\tilde{G}_0(-\mathbf{r}; E)]$$

$$\simeq \frac{2V_0 r_0^2}{(\pi\hbar v_F l)^2} \operatorname{Im}[\lambda\psi(-\lambda)] \ln \frac{\mathbf{r}^2}{2l^2}, \quad (A9)$$

$$\frac{V_0 r_0^2}{2(\pi\hbar v_F l)^2} \operatorname{Im}[\lambda\Gamma^2(-\lambda)] e^{-\mathbf{r}^2/2l^2} \left(\frac{\mathbf{r}^2}{2l^2}\right)^{2|\lambda|}, \quad (A10)$$

in the regions $l \gg r \gg r_0$ and $r \gg \max(l, r_0)$, respectively. As is seen, the correction to the free LDOS is an odd function of energy.

To find the first-order correction due to the Coulomb potential, we first calculate the correction to the Green's function, which is given by

$$\delta G(\mathbf{r}, \mathbf{r}; E) = -\frac{Ze^2}{\kappa} \int d\mathbf{r}' \tilde{G}_0(\mathbf{r} - \mathbf{r}'; E) \frac{1}{|\mathbf{r}'|} \tilde{G}_0(\mathbf{r}' - \mathbf{r}; E)$$

$$= -\frac{Ze^2}{\kappa} \int d\mathbf{r}' \tilde{G}_0(\mathbf{r}'; E) \frac{1}{|\mathbf{r}' - \mathbf{r}|} \tilde{G}_0(-\mathbf{r}'; E). \quad (A11)$$

Taking trace over spin and Dirac indices, performing integration over the angle by means of the formula

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \frac{4}{r + r'} K\left(\sqrt{\frac{4rr'}{(r + r')^2}}\right), \quad (A12)$$

where $K(k)$ is the elliptic integral of the first kind $[K(0) = \pi/2]$, and calculating the imaginary part, we obtain

$$\delta\rho(\mathbf{r}, \mathbf{r}, E) = \frac{2Ze^2}{\kappa} \frac{\operatorname{sgn}(E)}{(\pi\hbar v_F l)^2} \int_0^{\infty} \frac{dr'r'}{r + r'} e^{-x} K\left(\sqrt{\frac{4rr'}{(r + r')^2}}\right)$$

$$\times \left\{ \sum_{n=0}^{\infty} \left\{ [\lambda([L_n(x)]^2 + [L_{n-1}(x)]^2) \right. \right.$$

$$\left. + 2x[L_{n-1}^1(x)]^2 \right\} \delta'(\lambda - n)$$

$$- \sum_{n,m=0, n\neq m}^{\infty} \left\{ \lambda [L_n(x)L_m(x) + L_{n-1}(x)L_{m-1}(x)] \right.$$

$$\left. \left. + 2x L_{n-1}^1(x)L_{m-1}^1(x) \right\} \frac{\delta(\lambda - n) - \delta(\lambda - m)}{n - m} \right\}, \quad (A13)$$

where $x = r'^2/2l^2$. The correction to the LDOS at large distances $r \gg l$ is given by Eq. (4.8), where the energy dependence is given by the functions

$$B_0(\lambda) = \operatorname{sgn}(E) \left[\lambda\delta'(\lambda) + 2\sum_{n=1}^{\infty} (\lambda + n)\delta'(\lambda - n) \right], \quad (A14)$$

$$B_1(\lambda) = \operatorname{sgn}(E) \left[\lambda\delta'(\lambda) + 4\sum_{n=1}^{\infty} n(\lambda + n)\delta'(\lambda - n) \right.$$

$$+ 2\delta(\lambda - 1) + 2\sum_{n=1}^{\infty} [\lambda(2n + 1) + 2n(n + 1)]$$

$$\left. \times [\delta(\lambda - n - 1) - \delta(\lambda - n)] \right]. \quad (A15)$$

To calculate the integrals of the Laguerre polynomials, we used the following generating function (see Appendix A in Ref. 26):

$$I_{nm}^{\alpha}(y) = \int_0^{\infty} dt\, e^{-t} t^{\alpha} J_0(2\sqrt{yt}) L_n^{\alpha}(t) L_m^{\alpha}(t)$$

$$= (-1)^{n+m} \frac{(m + \alpha)!}{m!} e^{-y} L_n^{m-n}(y) L_{m+\alpha}^{n-m}(y). \quad (A16)$$

Expanding the left and right sides in y, we find the standard orthogonality relation,

$$\int_0^{\infty} dt\, e^{-t} t^{\alpha} L_n^{\alpha}(t) L_m^{\alpha}(t) = \frac{\Gamma(\alpha + n + 1)}{n!} \delta_{nm} \quad (A17)$$

and the integrals

$$\int_0^{\infty} dt\, e^{-t} t L_n(t) L_m(t)$$

$$= (-1)^{n+m} [(2n + 1)\delta_{nm} + (m + 1)\delta_{n,m+1} + (n + 1)\delta_{m,n+1}], \quad (A18)$$

$$\int_0^\infty dt\, e^{-t} t^2 L_n^1(t) L_m^1(t)$$

$$= (-1)^{n+m} (m+1)(n+1)[2\delta_{nm} + \delta_{n,m+1} + \delta_{m,n+1}].$$

$$\text{(A19)}$$

For the integrals of the functions $B_i(\lambda)$, we get $\int_{-\infty}^0 dE\, B_0(\lambda) = 0$, while

$$\int_{-\infty}^0 dE\, B_1(\lambda) = \frac{\hbar v_F}{l} 6\sqrt{2}\,\zeta(-1/2), \qquad \text{(A20)}$$

where $\zeta(z)$ is the Riemann zeta function.[30]

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