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Josephson currents in quantum Hall devices

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We consider a simple model for a superconductor/normal-metal/superconductor Josephson junction in which the "normal metal" is a section of a filling-factor $\nu = 2$ integer quantum Hall edge. We provide analytic expressions for the current/phase relations to all orders in the coupling between the superconductor and the quantum Hall edge modes, and for all temperatures. Our conclusions are consistent with the earlier perturbative study by Ma and Zyuzin [Europhys. Lett. **21**, 941 (1993)]: The Josephson current is independent of the distance between the superconducting leads, and the upper bound on the maximum Josephson current is inversely proportional to the perimeter of the Hall device.

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I. INTRODUCTION

The zero-voltage Josephson current in a superconductor/normal-metal/superconductor (SNS) junction[1] arises from Andreev scattering[2] at the SN and NS interfaces. In the ideal case, an electron incident on one superconductor from the normal metal will be reflected back into the normal metal as a hole, and this hole, on striking the second superconductor, will be reflected back toward the first superconductor as an electron. When the relative phase of the order parameters is such that constructive interference occurs, the back-and-forth process continues *ad infinitum* and transfers two electrons from superconductor to superconductor in each cycle.[3–7] A round trip takes time $2W/v_F$, where v_F is the Fermi velocity and W is the separation between the superconductors. The current will therefore be ev_F/W for each open transverse channel. In practice, the probability of Andreev reflection is less than unity[8,9] and the motion in the metal may be diffusive, but ev_F/W per channel remains an upper bound on the critical current.

An interesting question arises as to what happens when the "normal" metal consists of the chiral fermions at the edge of a quantum Hall (QH) bar.[10] In this case the holes move in the *same* direction as the electrons, so conventional Andreev retroreflection is impossible. A two-electron charge transfer requires a (phase coherent) passage around the entire perimeter of the Hall bar, and this lengthy excursion suggests that the small "W" of the conventional junction be replaced by the much larger perimeter L of the Hall bar. A perturbative study of a S-QH-S system in Ref. 11 supports this conclusion and estimates that the maximum Josephson current will be very small—in the order of 1 nA for millimeter scale devices. In view of ongoing experiments on quantum Hall Josephson junctions, however, it seems worth revisiting the problem to see if devices might be engineered to provide larger critical currents.

In this paper we introduce a model of an S-QH-S junction that is simple enough that it can be studied nonperturbatively. We obtain analytic expressions for the Josephson current/phase relation to all orders in the S-QH coupling and at all temperatures. Despite our greater control over the model, the key conclusions of the perturbative studies in Ref. 11 (see also Ref. 12) are unchanged: At filling fraction $\nu = 2$ an upper bound for the critical Josephson current is given by $2ev_d/L$, where v_d is the edge-mode drift velocity and L is the perimeter of the Hall device. Further, the temperature scale at which the Jospehson current is washed out by thermal effects is set by the edge-mode level spacing $E_{n+1} - E_n = 2\pi\hbar v_d/L$. Thus, if we wish to see Josephson-junction physics in quantum Hall devices, we should construct the junctions by coupling superconducting probes to mesoscale Hall dots.

In Sec. II we introduce the model and solve the associated Bogoliubov–de Gennes equation. In Sec. III we introduce an analytic regularization scheme to handle the otherwise ill-defined sums that appear in the current/phase relation. In Sec. IV we demonstrate that our regularization scheme is consistent with conventional perturbation theory at both zero and nonzero temperatures. We finish with a brief discussion of effects that we have not taken into account and that may or may not be significant.

II. THE MODEL

We consider a $\nu = 2$ quantum Hall edge (two spins therefore) in an interaction with superconducting (SC) leads Fig. 1. We model the system by a linear-dispersion edge-mode Hamiltonian,

$$H = \oint \{-iv_d\psi_\uparrow^\dagger(\partial_x - ieA)\psi_\uparrow - iv_d\psi_\downarrow^\dagger(\partial_x - ieA)\psi_\downarrow + |\Delta(x)|e^{i\theta(x)}\psi_\uparrow^\dagger\psi_\downarrow^\dagger + |\Delta(x)|e^{-i\theta(x)}\psi_\downarrow\psi_\uparrow\}dx. \quad (1)$$

Here v_d is the edge-mode drift velocity that is proportional to the gradient of the confining potential. The terms with $\Delta(x)$ are nonzero only where the edge state lies under the superconducting leads. They account for the Andreev coupling arising from the two-dimensional electron gas (2DEG) wave functions reaching up to touch the superconductor as they drift under the electrodes (see Fig. 2). In contrast to the usual proximity effect, the topological protection of the QH edge modes means that this interaction *cannot* open a gap-but it may, for example, convert a charge-(e) right-going spin-up electron into a charge-($-e$) right-going spin-up hole, and in the process transfer a spin-singlet pair of charge-(e) electrons from the Hall bar to the superconductor where they merge with the S-wave condensate.

We have not included a Zeeman-energy term to split the energy between the spin-up and spin-down edge modes. Such

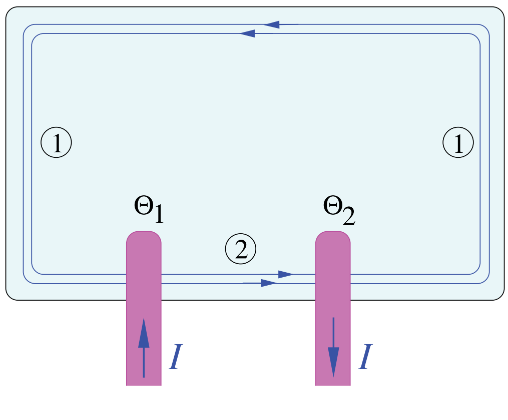

FIG. 1. (Color online) A Hall bar with superconducting probes passing a current I through the edge modes. The circled numbers label the regions (1) "outside the leads," and (2) "between the leads."

a term adds only a multiple of the identity matrix to the Bogoliubov–de Gennes (BdG) operator, and so has no effect on the subsequent analysis. Further, we assume that the energy scales of relevance are smaller than the energy gap of the superconducting leads. We therefore regard the parameters $|\Delta|$ as being externally imposed, and do not depend on the energy of the Hall-bar electrons or on the temperature.

We can rewrite H in the BdG form

$$H = \int dx \left\{ (\psi_\uparrow^\dagger, \psi_\downarrow) \times \begin{bmatrix} -iv_d(\partial_x - ieA) & |\Delta(x)|e^{i\theta(x)} \\ |\Delta(x)|e^{-i\theta(x)} & -iv_d(\partial_x + ieA) \end{bmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^\dagger \end{pmatrix} \right\} + \text{const.} \tag{2}$$

Here we have used an integration by parts together with the anticommutation property of the Fermi fields to write

$$\int \{\psi_\downarrow^\dagger[-iv_d(\partial_x - ieA)]\psi_\downarrow\}dx = \int \{\psi_\downarrow[-iv_d(\partial_x + ieA)]\psi_\downarrow^\dagger\}dx + \text{const.} \tag{3}$$

This rewriting is essentially a charge-conjugation transformation that makes manifest the particle-hole symmetry of the linearized edge spectrum. In particular, it reveals that the charge-$(-e)$ spin-up holes created by $\psi_\downarrow$ move in the *same* direction as the charge-(e) spin-up electrons created by $\psi_\uparrow^\dagger$. The "constant" contains the truly constant ground-state energy of the spin-down electrons, but also the term $-v_d e \int \delta(0)A(x)dx$ that subtracts a background electric charge. This charge gets discarded as we switch to the charge-conjugate picture in which charge-$(-e)$ holes occupy the states that are not occupied by electrons. Keeping track of the constant restores the physical charge when needed.

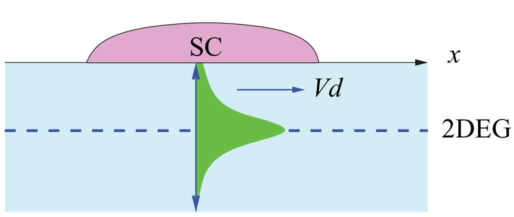

FIG. 2. (Color online) The wave function for an electron in a 2DEG is confined in the vertical direction, but there is some amplitude for the vertically oscillating electron to touch the superconductor. As a slowly drifting Landau-level wave packet passes under the superconducting lead, there will be many opportunities for Andreev reflection to turn the electron into a hole.

The vector potential A acts as a chemical potential and controls the location of the Fermi energy. In much of our discussion we will assume that when $\Delta = 0$ the Fermi energy lies midway between two edge-mode energy levels. This assumption is for illustrative purposes only. Indeed the detailed current/phase relation will depend sensitively on the exact location of the Fermi energy relative to the edge modes because varying θ can make a level cross the Fermi energy, change its occupation, and cause a jump in the Josephson current. The sensitivity will manifest itself as Bohm-Aharonov oscillations in the Josephson current as a function of the magnetic flux through the Hall bar.[11]

For our midspaced E_F we can make a gauge transformation to set $A \to 0$ at the expense of changing periodic boundary conditions to antiperiodic ones, and simultaneously redefining $\theta(x)$. We assume that we have done this. The BdG equation for the eigenmodes is therefore

$$\left[-iv_d \frac{\partial}{\partial x} + |\Delta(x)|e^{i\sigma_3\theta(x)}\sigma_1 \right] \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}. \tag{4}$$

Equation (4) has a path-ordered exponential solution

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = e^{iEx/v_d} P \exp\left\{ -i \int_0^x K(\xi)\, d\xi \right\} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, \tag{5}$$

where $K(x) = |\Delta(x)|e^{i\sigma_3\theta(x)}\sigma_1/v_d$ is a Hermitian matrix. Note that, in distinction to the usual BdG case, we did *not* double the number of degrees of freedom when we constructed the BdG operator, so *all* the BdG eigenmodes are needed.

Only a part Ω (the union of the two regions under the SC electrodes) of the perimeter of the Hall bar is in contact with the superconductor, and we set

$$U = P \exp\left\{ -i \int_\Omega K(\xi)\, d\xi \right\} \in \text{SU}(2). \tag{6}$$

As the perimeter of the Hall bar forms a closed loop, it was reasonable to impose periodic boundary conditions, but recall that these were changed to antiperiodic boundary conditions by the gauge transformation that removed $A(x)$. The eigenmodes of the BdG operator Hamiltonian are therefore determined from the eigenvalues of U by requiring that

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = -e^{iE_nL/v_d} U \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{7}$$

Here L is the length of the Hall-bar perimeter. Now the eigenvalues of U will be of the form $e^{\pm i\phi}$ and so the energy eigenvalues are given by the requirement that $(E_nL/v_d) \pm \phi = \pi(2n+1)$ or

$$E_n = \frac{v_d}{L}[\pi(2n+1) \mp \phi]. \tag{8}$$

Note that if $(u,v)^T$ is an eigenvector of U with eigenvalue $e^{i\phi}$, then $-i\sigma_2(u^*,v^*) = (-v^*,u^*)$ is an eigenvector of U with eigenvalue $e^{-i\phi}$. Consequently, if $(u_n(x),v_n(x))^T$ is an eigenfunction of the BdG operator corresponding to eigenvalue E_n, then $(-v_n^*(x),u_n^*(x))^T$ is an eigenfunction corresponding to energy $-E_n$. These facts follow from

$$(i\sigma_2)\sigma_i(-i\sigma_2) = -\sigma_i^* \Longrightarrow (i\sigma_2)U^*(-i\sigma_2) = U \qquad (9)$$

and give rise to the usual antilinear S-wave BdG particle-hole symmetry "C" with $C^2 = -\mathrm{Id}$. This symmetry must be distinguished from the approximate particle-hole symmetry arising from our linearization of the quantum Hall edge-mode spectrum.

If the phase of the order parameter is constant in segments $\Omega_{1,2}$ (the superconducting leads) then $U = U_2U_1$, where

$$U_a = \begin{bmatrix} \cos\, D_a & -ie^{i\theta_a}\sin\, D_a \\ -ie^{-i\theta_a}\sin\, D_a & \cos\, D_a \end{bmatrix}, \quad a = 1,2. \qquad (10)$$

Here $D_a = |\Delta|w_a/v_d$, where w_a is the width of lead a. The eigenvalues of U are $e^{\pm i\phi}$, and by taking the trace of U, we see that ϕ is given by the spherical cosine rule:

$$\cos\,\phi = \cos\, D_1\,\cos\, D_2 - \cos\theta\,\sin\, D_1\,\sin\, D_2. \qquad (11)$$

The spherical triangle (see Fig. 3) arises because the matrices U_1 and U_2 are the spinor representations of successive SO(3) rotations through angles $2D_1$ and $2D_2$ about axes separated by the angle θ. It is shown in Ref. 13 that such rotations can be combined through the use of mirrors that form the geodesic sides of the triangle.

From now on we understand by "ϕ,' the solution of Eq. (11) that lies in the range $0 \leqslant \phi \leqslant \pi$, and by the vector $(u,v)^T$ the corresponding eigenvector of U. We similarly take "E_n" to mean the combination

$$E_n = \frac{v_d}{L}[2\pi(n+1/2) - \phi]. \qquad (12)$$

Now we make the Bogoliubov transformation

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \left\{ b_{n\uparrow} \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} + b_{n\downarrow}^\dagger \begin{pmatrix} -v_n^*(x) \\ u_n^*(x) \end{pmatrix} \right\}. \qquad (13)$$

In order not to overcount, we ensure that the modes are those that, after passing the superconductor, take the form $(u_n(x),v_n(x)) = e^{i(E_nx/v_d+\phi)}(u,v)$ and $(-v_n^*(x),u_n^*(x)) = e^{-i(E_nx/v_d+\phi)}(-v^*,u^*)$. The Fermionic anticommutation

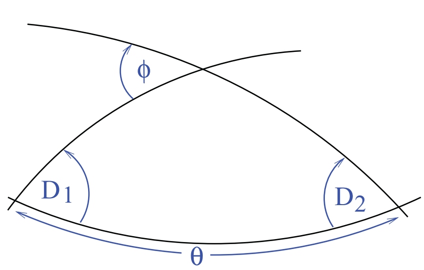

FIG. 3. (Color online) The spherical triangle that relates the eigenphase ϕ to the order-parameter phase difference $\theta = \theta_2 - \theta_1$.

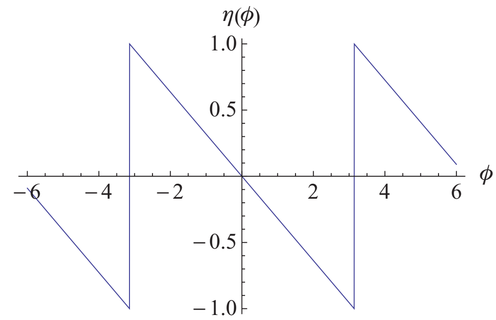

FIG. 4. (Color online) A plot of $\eta(\phi)$ showing the 2π periodicity.

relations coupled with the BdG eigenfunction completeness relations then require that

$$\{b_{n\downarrow},b_{m\downarrow}\} = \{b_{n\uparrow},b_{m\uparrow}\} = \{b_{n\downarrow},b_{m\uparrow}\} = \{b_{n\downarrow}^\dagger,b_{m\uparrow}^\dagger\} = 0 \qquad (14)$$

and

$$\{b_{n\downarrow}^\dagger,b_{m\downarrow}\} = \{b_{n\uparrow}^\dagger,b_{m\uparrow}\} = \delta_{nm}. \qquad (15)$$

The Bogoliubov transformation simplifies H to

$$H = \sum_{n=-\infty}^{\infty} E_n(b_{n\uparrow}^\dagger b_{n\uparrow} - b_{n\downarrow}b_{n\downarrow}^\dagger) + \text{const}, \qquad (16)$$

the constant being the same one that was introduced earlier. It is not really a constant as it depends on the gauge field A, but it is independent of $\theta(x)$. Recall that the A dependence accounts for the total charge of the spin-down Fermi sea that was discarded when we made the particle-hole interchange for this spin component. The minimum-energy state is defined by the properties

$$\begin{aligned} b_{n\uparrow}|0\rangle &= 0, & E_n &> 0, \\ b_{n\uparrow}^\dagger|0\rangle &= 0, & E_n &< 0, \\ b_{n\downarrow}|0\rangle &= 0, & E_n &> 0, \\ b_{n\downarrow}^\dagger|0\rangle &= 0, & E_n &< 0. \end{aligned} \qquad (17)$$

Using these, we compute the ground-state energy to be

$$E_{\text{ground}} = \sum_{E_n<0} E_n - \sum_{E_n>0} E_n. \qquad (18)$$

The quantity E_{ground} is formally divergent, but the physics resides entirely in the variation of E_{ground} with the phase difference $\theta = \theta_2 - \theta_1$. Now as we vary θ, all E_n move in the same direction. The energy dependence on θ largely cancels between the two sums. In order to extract the small, but nonzero, residuum we will have to regulate the sums in a controlled manner. This we do in the next section.

III. COMPUTING THE CURRENT

Given a Dirac-like spectrum of energy levels $-\infty < E_n < \infty$, the associated ground-state charge and current can often be expressed in terms of the spectral asymmetry.[14] This quantity is defined[15,16] to be the regulated sum

$$\eta = \lim_{s\to 0}\left\{ -\sum_{n=-\infty}^{\infty} \mathrm{sgn}(E_n)e^{-s|E_n|}\right\}. \qquad (19)$$

For energies of our form, $E_n = \alpha\,[2\pi(n+1/2) - \phi]$ (where $\alpha = v_d/L$), a direct calculation shows that for $-\pi < \phi < \pi$, we have

$$\left\{ -\sum_{n=-\infty}^{\infty} \mathrm{sgn}(E_n) e^{-s|E_n|} \right\} = -\frac{\phi}{\pi} - \frac{1}{6\pi}(\phi^3 - \phi\pi^2)(\alpha s)^2 + O(s^4). \quad (20)$$

Thus

$$\eta(\phi) = -\frac{\phi}{\pi}, \qquad -\pi < \phi < \pi \quad (21)$$

and extends with 2π periodicity in ϕ (see Fig. 4).

We may similarly define and compute an analytically regulated version of the ground-state energy (18):

$$(E_{\text{ground}})_{\text{reg}} = \lim_{s\to 0} \left\{ -\sum_{n=-\infty}^{\infty} \mathrm{sgn}(E_n) E_n e^{-s|E_n|} + \frac{1}{\pi\alpha s^2} \right\} = \alpha\left(\frac{\phi^2}{2\pi} - \frac{\pi}{6}\right), \qquad -\pi < \phi < \pi.$$

This quantity also extends periodically outside the range $-\pi < \phi < \pi$ (see Fig. 5). The subtraction needed for the existence of the limit is independent of ϕ, and the constant $-\alpha\pi/6$ is the same as would be obtained by ζ-function regularization.[17] Let us also compute

$$\left(\frac{dE_{\text{ground}}}{d\phi}\right)_{\text{reg}} \stackrel{\text{def}}{=} \lim_{s\to 0} \left\{ -\sum_{n=-\infty}^{\infty} \mathrm{sgn}(E_n) \left(\frac{dE_n}{d\phi}\right) e^{-s|E_n|} \right\} = \lim_{s\to 0} \left\{ \alpha \sum_{n=-\infty}^{\infty} \mathrm{sgn}(E_n) e^{-s|E_n|} \right\} = \alpha\frac{\phi}{\pi}$$

and observe that the regulated energy possesses the comforting property that

$$\frac{d}{d\phi}(E_{\text{ground}})_{\text{reg}} = \left(\frac{dE_{\text{ground}}}{d\phi}\right)_{\text{reg}}. \quad (22)$$

We will relate these energy derivatives to the ground-state expectation value of the divergence of the current operator.

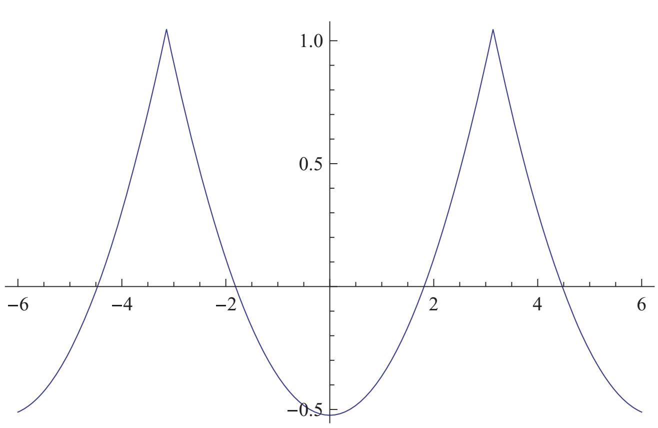

FIG. 5. (Color online) A plot of $\alpha^{-1}(E_{\text{ground}}(\phi))_{\text{reg}}$ showing the 2π periodicity.

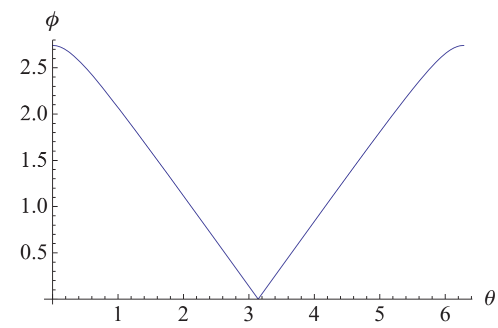

FIG. 6. (Color online) A plot of the eigenphase ϕ against θ for the case $D_1 = D_2 = \pi/2 - 0.2$. We are enforcing the condition $0 \leqslant \phi \leqslant \pi$ that is required by our Bogoliubov transformation.

The current operator is

$$j(x) = -\frac{\delta H}{\delta A(x)}. \quad (23)$$

If we include the contribution from the A-dependent constant when taking the functional derivative, then the ground-state current is

$$\langle j(x)\rangle = ev_d\langle 0|\psi_\uparrow^\dagger(x)\psi_\uparrow(x) + \psi_\downarrow^\dagger(x)\psi_\downarrow(x)|0\rangle = 2ev_d\left(\sum_{E_n<0}|u_n(x)|^2 + \sum_{E_n>0}|v_n(x)|^2\right). \quad (24)$$

If we ignore the constant, the current becomes

$$\langle j(x)\rangle = ev_d\langle 0|\psi_\uparrow^\dagger(x)\psi_\uparrow(x) - \psi_\downarrow(x)\psi_\downarrow^\dagger(x)|0\rangle = ev_d\sum_{E_n<0}[|u_n(x)|^2 - |v_n(x)|^2] - ev_d\sum_{E_n>0}[|u_n(x)|^2 - |v_n(x)|^2]. \quad (25)$$

These two currents differ only by the subtraction of $\sum_n[|u_n(x)|^2 + |v_n(x)|^2]$ in the second case. This divergent sum is "$\delta(0)$" and independent of x by eigenvector completeness. Therefore, when it comes to computing the current flowing in and out at the leads, we can use either expression. The second expression is the most convenient, and so we define

$$\langle j(x)\rangle_{\text{reg}} = \lim_{s\to 0}\left\{ -ev_d\sum_{n=-\infty}^{\infty}\mathrm{sgn}(E_n)[|u_n(x)|^2 - |v_n(x)|^2]e^{-s|E_n|}\right\}. \quad (26)$$

In our simple model $|u_n|^2(x)$ and $|v_n|^2(x)$ are independent of n, but *do* depend on whether x lies between the superconducting leads or not. This means that the edge-current differs in the two regions, and the difference is due to the Josephson current flowing in and out *via* the SC leads. We could compute $|u_n|^2$ and $|v_n|^2$ in the two regions by diagonalizing the matrix U, but it is simpler, and more revealing, to relate the difference in the currents to the variation of the ground-state energy with θ.

To do this we observe that

$$\begin{bmatrix} e^{i\chi/2} & 0 \\ 0 & e^{-i\chi/2} \end{bmatrix} \begin{bmatrix} -iv_d(\partial_x - ieA) & |\Delta|e^{i\theta} \\ |\Delta|e^{-i\theta} & -iv_d(\partial_x + ieA) \end{bmatrix} \times \begin{bmatrix} e^{-i\chi/2} & 0 \\ 0 & e^{+i\chi/2} \end{bmatrix} = \begin{bmatrix} -iv_d[\partial_x - i(eA + \chi'/2)] & |\Delta|e^{i(\theta+\chi)} \\ |\Delta|e^{-i(\theta+\chi)} & -iv_d[\partial_x + i(eA + \chi'/2)] \end{bmatrix}.$$

As the similarity transformation does not change the eigenvalues of the BdG operator, we see that

$$E_n[\theta, A] = E_n[\theta + \chi, eA + \chi'/2]. \quad (27)$$

The effect on the energy eigenvalue of changing $\theta(x) \to \theta(x) + \delta\theta(x)$ is therefore identical to changing $eA \to eA - (\delta\theta)'/2$. By first-order perturbation theory we compute the latter effect to give

$$\begin{aligned} \delta E_n &= \langle n|\delta H|n\rangle \\ &= -v_d \int dx[|u_n(x)|^2 - |v_n(x)|^2]\delta A \\ &= \frac{1}{2}v_d \int dx[|u_n(x)|^2 - |v_n(x)|^2]\frac{\partial}{\partial x}\delta\theta(x) \\ &= -\frac{1}{2}v_d \int dx \left\{\frac{\partial}{\partial x}[|u_n(x)|^2 - |v_n(x)|^2]\right\}\delta\theta(x). \end{aligned} \quad (28)$$

Now, on combining this last result with Eqs. (22) and (25), we find that

$$\begin{aligned} \delta(E_{\rm ground})_{\rm reg} &= -\frac{1}{2e}\int \langle\nabla\cdot j\rangle_{\rm reg}\delta\theta(x)\,dx \\ &= \frac{1}{2e}I_{\rm Josephson}(\delta\theta_2 - \delta\theta_1). \end{aligned} \quad (29)$$

Thus we see that the general result

$$I_{\rm Josephson} = \left(\frac{2e}{\hbar}\right)\frac{dE_{\rm ground}}{d\theta} \quad (30)$$

is consistent with our regularization scheme.

From

$$E_{\rm ground} = \frac{v_d}{L}\left(\frac{\phi^2}{2\pi} - \frac{\pi}{6}\right), \qquad 0 \leqslant \phi \leqslant \pi, \quad (31)$$

we have

$$I_{\rm Josephson} = 2e\frac{d}{d\theta}(E_{\rm ground})_{\rm reg} = 2e\frac{d}{d\phi}(E_{\rm ground})_{\rm reg}\frac{d\phi}{d\theta}. \quad (32)$$

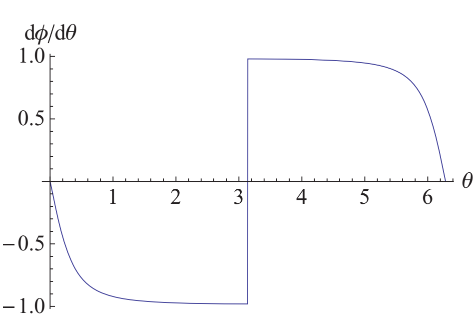

FIG. 7. (Color online) A plot of $d\phi/d\theta$ for $D_1 = D_2 = \pi/2 - 0.2$.

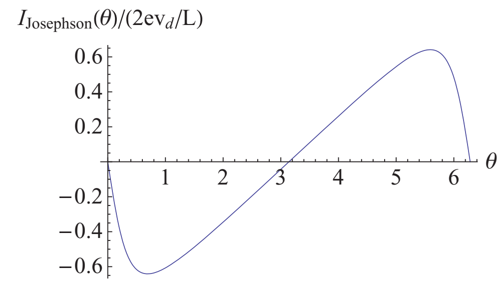

FIG. 8. (Color online) A plot of $I_{\rm Josephson}/(2ev_d/L)$ against θ for $D_1 = D_2 = \pi/2 - 0.2$. Observe how the discontinuities combine to give a smooth result. As $D_{1,2}$ approach perfect coupling at $D_1 = D_2 = \pi/2$, the drops at $\theta = 0,2\pi$ steepen, and become level-crossing discontinuities.

Figures 6–8 show how theses ingredients assemble to give the current/phase relation.

To gain further insight, consider the case of "perfect coupling," where $\sin D_a = 1$ and $\phi = \pm(\theta_2 - \theta_1 + \pi)$. In this case

$$\begin{aligned} U &= \begin{bmatrix} 0 & -ie^{i\theta_2} \\ -ie^{-i\theta_2} & 0 \end{bmatrix}\begin{bmatrix} 0 & -ie^{i\theta_1} \\ -ie^{-i\theta_1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} -e^{i(\theta_2-\theta_1)} & 0 \\ 0 & -e^{-i(\theta_2-\theta_1)} \end{bmatrix}, \end{aligned} \quad (33)$$

and so $\phi = (\theta_2 - \theta_1) + \pi$. In the absence of relaxation, each 2π turn of θ would put another particle into both the spin-up and spin-down sea. In equilibrium, however, the state ceases to be occupied as soon at its energy becomes positive. This change in occupation leads to a jump in the Josephson current as the state crosses the Fermi energy and its contribution is lost. The maximum possible current occurs just before or after the jump and has $I_{\rm max} = \pm 2ev_d/L$. For $v_d \sim 10^6$ m/s and a perimeter of about 1 mm we get an upper bound on the Josephson current of about 1 nA. This is consistent with the estimate of Ma and Zyuzin.[11]

A physical picture for this upper bound is as follows: At the phase difference corresponding to the "jump," we have a spin-up/spin-down pair of levels lying exactly at the Fermi energy. At perfect coupling, the extreme equilibrium currents correspond to two possible cases: (i) between the leads both zero-energy levels are empty, while outside they are occupied and (ii) between the leads both zero-energy levels are occupied and outside they are empty. Levels in the Dirac sea that are not at the Fermi energy cannot be left empty by a passage under a lead, as this would lead to the energy being different in different regions and this is not possible in an energy eigenstate. Only the topmost energy level can contribute to the equilibrium Josephson current therefore, and this is the reason why the Josephson current is so small. To estimate its magnitude we note that in case (i), in each passage around the perimeter of the Hall bar, a pair of electrons passes from the Hall bar to the first lead and is returned to the Hall bar from the second lead. In case (ii) in each orbit a pair of electrons passes from the first lead to the Hall bar and is collected from the Hall bar at the second lead. This physical picture shows that the two possible Josephson currents are equal and opposite and have magnitude $|I_{\rm max}| = 2ev_d/L$. (Because it is easy to get confused by Bogoliubov transformations, we provide, in Appendix A, a more detailed description of what happens to the particle content of the many-body eigenstates as they pass under the superconducting leads.)

IV. COMPARISON WITH PERTURBATION THEORY

The analytic regularization method used in the computations in the previous sections is standard in relativistic field theory,[14] but is perhaps less familiar in superconducting applications. As a check on its validity it is worthwhile (and nontrivial) to compare our all-orders in D_1 and D_2 calculations with conventional perturbation theory.

In the weak-coupling regime, where D_1 and D_2 are small, the spherical cosine rule reduces to

$$\phi^2 = D_1^2 + D_2^2 + 2D_1D_2 \cos\theta + O(D^3). \tag{34}$$

In this limit the ground-state energy and zero-temperature and Josephson current become

$$E_{\text{ground}}(\theta) = \frac{v_d}{L}\frac{1}{2\pi}\left(D_1^2 + D_2^2 + 2D_1D_2 \cos\theta\right), \tag{35}$$

and hence

$$I_{\text{Josephson}} = -\frac{2ev_d}{\pi L} D_1 D_2 \sin\theta. \tag{36}$$

We begin by verifying that Eq. (35) is correctly reproduced by the perturbation expansion.

The Euclidean chiral propagator for zero-temperature and antiperiodic spatial boundary conditions is

$$\begin{aligned}&\langle 0|T\psi_a^\dagger(z_1)\psi_b(z_2)|0\rangle\\ &\quad= \delta_{ab}G(z_1 - z_2) = \frac{1}{2iL}\frac{\delta_{ab}}{\sin[\pi(z_1 - z_2)/L]},\end{aligned} \tag{37}$$

where $a,b = \uparrow, \downarrow$ and $z = x + iv_d\tau$. The change in the $\Delta = 0$ ground-state energy due to the interaction

$$H_{\text{int}} = \int |\Delta(x)|[e^{i\theta(x)}\psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x) + e^{-i\theta(x)}\psi_\downarrow(x)\psi_\uparrow(x)]dx \tag{38}$$

occurs at second order in $|\Delta|$ and is

$$\begin{aligned}\delta E_{\text{ground}} = &-\int dx_1 \int dx_2 \int_{-\infty}^{\infty} d\tau |\Delta(x_1)||\Delta(x_2)|e^{\theta(x_1)}\\ &\times e^{-i\theta(x_2)}\langle 0|T\psi_\uparrow^\dagger(z_1)\psi_\downarrow^\dagger(z_1)\psi_\downarrow(z_2)\psi_\uparrow(z_2)|0\rangle.\end{aligned} \tag{39}$$

Here $\tau = \tau_2 - \tau_1$ is the Euclidean time interval between z_2 and z_1. Now

$$\langle 0|T\psi_\uparrow^\dagger(z_1)\psi_\downarrow^\dagger(z_1)\psi_\downarrow(z_2)\psi_\uparrow(z_2)|0\rangle = [G(z_1 - z_2)]^2 \tag{40}$$

by Wick's theorem, and

$$\begin{aligned}&\frac{1}{4L^2}\int_{-\infty}^{\infty}\frac{1}{(\sin[\pi(x_1 - x_2 + iv_d\tau)/L])^2}d\tau\\ &\quad= \left(\frac{1}{2\pi L v_d}\right)\end{aligned} \tag{41}$$

is independent of the separation $x_1 - x_2$ unless $x_1 - x_2 = 0$ (mod L). The perturbation integral has four contributing regions: (i) both x_1 and x_2 in lead 1, (ii) both x_1 and x_2 in lead 2, (iii) x_1 in lead 1, x_2 in lead 2, and (iv) x_1 in lead 2, x_2 in lead 1. Recalling that $D_a = |\Delta|w_a/v_d$, these combine to give

$$\begin{aligned}\delta E_{\text{ground}} &= v_d^2\left(D_1^2 + D_2^2 + 2D_1D_2 \cos\theta\right)\frac{1}{2\pi L v_d}\\ &= \frac{v_d}{2\pi L}\left(D_1^2 + D_2^2 + 2D_1D_2 \cos\theta\right).\end{aligned} \tag{42}$$

This expression coincides with the weak-coupling limit of the all-orders calculation.

We can extend the comparison to nonzero temperature. At temperature $T = \beta^{-1}$, the Josephson current can be written as

$$I_{\text{Josephson}} = \left(\frac{2e}{\hbar}\right)\frac{dF}{d\theta}, \tag{43}$$

where F is the free energy. For a general spectral shift ϕ, we use standard methods to write down the partition function

$$\begin{aligned}Z &= \exp\left\{-\beta F[\phi,\beta]\right\}\\ &= \exp\left\{-\frac{\beta v_d}{L}\left(\frac{\phi^2}{2\pi} - \frac{\pi}{6}\right)\right\}\prod_{N=1}^{\infty}(1 + wq^{2n-1})^2\\ &\quad\times(1 + w^{-1}q^{2n-1})^2 = (\eta(q))^{-2}\\ &\quad\times\left[\sum_{n=-\infty}^{\infty}\exp\left\{-\frac{v_d\beta}{2\pi L}\frac{1}{2}(2\pi n + \phi)^2\right\}\right]^2,\end{aligned} \tag{44}$$

where $q = \exp\{-\pi\beta v_d/L\}$, $w = \exp\{-\beta v_d\phi/L\}$, and

$$\eta(q) = q^{1/12}\prod_{n=1}^{\infty}(1 - q^{2n})$$

is the Dedekind η function. We used the Jacobi triple-product formula to pass from the second line to the third. The sum in the expression for Z is squared because there are two independent Fermi seas (spin up and spin down) and their contributions to the partition function are symmetric under the interchange of ϕ with $-\phi$. By using the Poisson summation formula, we can rewrite the partition function as

$$\begin{aligned}\exp\{-\beta F[\phi,\beta]\} &= (\eta(q))^{-2}\frac{L}{v_d\beta}\\ &\quad\times\left[\sum_{n=-\infty}^{\infty}\exp\left\{-\frac{1}{2}\frac{2\pi L}{v_d\beta}n^2 + in\phi\right\}\right]^2\\ &= (\eta(q))^{-2}\frac{L}{v_d\beta}[\theta_3(\phi/2\pi|iL/v_d\beta)]^2.\end{aligned} \tag{45}$$

Thus the free energy is given by

$$F[\phi,\beta] = c - \frac{2}{\beta}\ln\theta_3(\phi/2\pi|iL/v_d\beta), \tag{46}$$

where c does not depend on ϕ. For small spectral shifts ϕ, we can Taylor expand

$$F[\phi,\beta] = c - \frac{1}{\beta}\phi^2\frac{d^2}{d\phi^2}\ln\theta_3(\phi/2\pi|L/v_d\beta) + O(\phi^4). \tag{47}$$

We would now like to compare expression (47) with that obtained by perturbation theory. At finite temperature the chiral propagator becomes

$$\langle 0|T\psi_a^\dagger(z)\psi_b(0)|0\rangle \to G(z)$$
$$= \frac{1}{2\pi i L}\frac{\theta'(0|iv_d\beta/L)}{\theta(z/L|iv_d\beta/L)}\frac{\theta_3(z/L|iv_d\beta/L)}{\theta_3(0|iv_d\beta/L)}. \tag{48}$$

Here we are using the θ function definitions from Ref. 18, in which

$$\theta(z|\tau) = \sum_{m=-\infty}^{\infty} \exp\{i\pi\tau(m+1/2)^2 + 2\pi i(m+1/2)(z+1/2)\},$$
$$\theta_3(z|\tau) = \sum_{m=-\infty}^{\infty} \exp\{i\pi\tau m^2 + 2\pi i m z\}. \tag{49}$$

Thus $\theta(z|\tau)$ is odd under $z \leftrightarrow -z$, while $\theta_3(z|\tau)$ is even. These properties were the ingredients used to assemble (48), which is specified uniquely by requiring the propagator to be analytic, doubly anti-periodic

$$G(z+L) = -G(z), \qquad G(z+iv_d\beta) = -G(z), \tag{50}$$

and for small z to obey

$$G(z) \sim \frac{1}{2\pi i}\frac{1}{z}. \tag{51}$$

It is this last property that makes it a Green function.

In terms of $G(z)$ we now have

$$\delta E_{\text{ground}} = -\int dx_1 \int dx_2 \int_0^\beta d\tau |\Delta(x_1)||\Delta(x_2)|e^{\theta(x_1)} \times e^{-i\theta(x_2)}[G(x_1 - x_2 + iv_d\tau)]^2. \tag{52}$$

The x_a integrals are the same as before, and, although it is little more complicated, the integral over τ can still be evaluated in closed form. We begin by observing that $[2\pi i G(z)]^2$ is analytic, has a double pole $1/z^2$ at the origin, is doubly periodic with periods $\omega_1 = L$ and $\omega_2 = iv_d\beta$, and (from the $\theta_3(z|\tau)$ in the numerator) has a double zero at $z = \frac{1}{2}(\omega_1 + \omega_2)$. These properties are sufficient to show that

$$[2\pi i G(z)]^2 = \wp(z\,|\,\omega_1,\omega_2) - e_3, \tag{53}$$

where $\wp(z\,|\,\omega_1,\omega_2)$ is the Weierstrass elliptic function, and

$$e_3 \equiv \wp(\{\omega_1+\omega_2\}/2\,|\,\omega_1,\omega_1). \tag{54}$$

The Weierstrass ζ function is defined so that

$$\frac{d}{dz}\zeta(z\,|\,\omega_1,\omega_2) = -\wp(z\,|\,\omega_1,\omega_2), \tag{55}$$

together with initial condition

$$\lim_{z\to 0}\left\{\zeta(z) - \frac{1}{z}\right\} = 0. \tag{56}$$

We may therefore evaluate the τ integral in terms of tabulated functions:

$$\int_a^{a+\omega_2} [2\pi i G(z)]^2\, dz = -\zeta(a+\omega_2) + \zeta(a) - \omega_2 e_3$$
$$= -2\eta_2 - \omega_2 e_3$$
$$= \frac{1}{\omega_2}\frac{d^2}{dz^2}\ln\theta_3(z|-\omega_1/\omega_2)\bigg|_{z=0}. \tag{57}$$

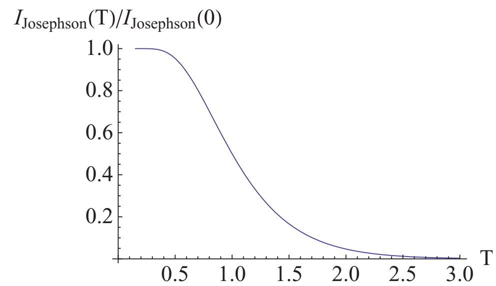

FIG. 9. (Color online) A plot of the effect of temperature on the perturbative Josephson current. The horizontal axis is temperature in units of $\hbar v_d/L$. We see an effect as soon as the temperature becomes comparable with the $2\pi\hbar v_d/L$ level spacing of the edge energy states.

Here $2\eta_2 \equiv \zeta(a+\omega_2) - \zeta(a) = 2\zeta(\omega_2/2)$ is independent of a. The quantities in the second line of Eq. (57) are available in MATHEMATICA, and we use them to plot $I_{\text{Josephson}}(T)/I_{\text{Josephson}}(0)$ in Fig. 9.

It takes a little more work to obtain the logarithmic derivative appearing in the last line of Eq. (57), and so we relegate its derivation to Appendix B. Accepting that the claim is correct, and putting in the dimensionful constants, we confirm that our all-orders evaluation of the free energy coincides with the perturbation theory calculation in the weak-coupling regime.

V. DISCUSSION

We have shown that the maximum possible Josephson current for a pair of spin-up/spin-down QH edge states is rather small for typical Hall bar geometries. The bound is small because the relevant length and energy scales are set by the perimeter of the Hall device rather than the separation of the superconducting probes. Also, unlike a typical Josephson device, there is only one conduction channel per pair of edge modes. This last observation means that nothing is to be gained by making the superconducting leads overlay deeper into the Hall bar.

It may seem strange that we have so far discussed quantum Hall physics with no mention of the magnetic field that is necessary for its existence. The field, however, has only a few consequences for our discussion. Obviously the superconducting leads must be constructed of materials that remain superconducting in a field of few Tesla at temperatures of about 1 K, but this is not hard to achieve. The leads must also be narrow enough that the order-parameter phase does not vary widely within the part of the lead that is actively coupled to the 2DEG. A subtle point in this regard affects the claim that the Josephson current is independent of the separation of the leads. The phase difference θ that we have equated to $\theta_2 - \theta_1$ should be understood as the gauge-invariant quantity $\theta = \theta_2 - \theta_1 - 2e\int_{x_1}^{x_2} A\,dx$. Now a quantum of magnetic flux lies between each of the edge-state energy levels and if the effective "θ" is not to vary with the energy level index n, only a small fraction of this flux should pass between the leads. The leads should not be spaced apart by more than a small fraction of the perimeter. A more subtle effect is that of pair breaking in the leads due to the magnetic field.[20] Pair breaking will alter the Andreev-scattering phase matching between the normal and superconducting electrons, and being field dependent may complicate the pattern of Bohm-Aharonov oscillations.

Something that we have not considered here, and that may well allow for larger currents, is "edge reconstruction".[21–23] A reconstructed edge, with its alternating strips of compressible and incompressible 2DEG can allow many more levels to lie exactly at the Fermi energy and so have their occupation number changed without a change in energy. These levels have zero drift velocity, however, so it unlikely that they contribute significantly to the Josephson current.

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APPENDIX A

The maximum possible Josephson current occurs when we have both perfect coupling ($\sin D_1 = \sin D_2 = 1$) and $\cos\theta = 1$. In this special case we have

$$U_1 = U_2 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad U = U_2 U_1 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{A1}$$

The Bogoliubov mode expansion (13) then becomes

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \left\{ b_{n\uparrow} \frac{1}{\sqrt{L}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2\pi i n x/L} + b_{n\downarrow}^\dagger \frac{1}{\sqrt{L}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2\pi i n x/L} \right\} \tag{A2}$$

for x in region (1), and

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \left\{ b_{n\uparrow} \frac{1}{\sqrt{L}} \begin{pmatrix} 0 \\ -i \end{pmatrix} e^{2\pi i n x/L} + b_{n\downarrow}^\dagger \frac{1}{\sqrt{L}} \begin{pmatrix} -i \\ 0 \end{pmatrix} e^{-2\pi i n x/L} \right\} \tag{A3}$$

for x in region (2). (The numbering of the regions refers to Fig. 1.)

In these mode expansions, the operators $b_{n\uparrow}$ and $b_{n\downarrow}^\dagger$ annihilate or create *quasiparticles* with energy $|E_n| = 2\pi v_d |n|/L$. We compare these expansions with the free-particle plane-wave expansion

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix} = \sum_{n=-\infty}^{\infty} \left\{ a_{n\uparrow} \frac{1}{\sqrt{L}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2\pi i n x/L} + a_{n\downarrow}^\dagger \frac{1}{\sqrt{L}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2\pi i n x/L} \right\}, \tag{A4}$$

where the operators $a_{n\uparrow}$ and $a_{n\downarrow}^\dagger$ annihilate and create *electrons*. We see that we can identify

$$b_{n\uparrow} = a_{n\uparrow}, \qquad b_{n\uparrow}^\dagger = a_{n\uparrow}^\dagger,$$
$$b_{n\downarrow} = a_{n\downarrow}, \qquad b_{n\downarrow}^\dagger = a_{n\downarrow}^\dagger \tag{A5}$$

in region (1), and

$$b_{n\uparrow} = i a_{-n\downarrow}^\dagger, \qquad b_{n\uparrow}^\dagger = -i a_{-n\downarrow},$$
$$b_{n\downarrow} = -i a_{n\downarrow}^\dagger, \qquad b_{n\downarrow}^\dagger = +i a_{-n\uparrow} \tag{A6}$$

in region (2). We now use these identifications to examine what happens to the particle content of the many-body eigenstates as they drift under the superconducting leads.

We first note that a minimum-energy eigenstate must be annihilated by $b_{n\uparrow}$ and $b_{n\downarrow}$ for $n > 0$, and by $b_{n\uparrow}^\dagger$ and $b_{n\downarrow}^\dagger$ for $n < 0$. Let us define the eigenstate $|0\rangle$ by requiring that it is killed by all these operators, and also by $b_{0\downarrow}$ and $b_{0\uparrow}$. Then the states

$$|0\rangle, \quad b_{0\uparrow}^\dagger|0\rangle, \quad b_{0\downarrow}^\dagger|0\rangle, \quad b_{0\downarrow}^\dagger b_{0\uparrow}^\dagger|0\rangle, \tag{A7}$$

all have the same energy, making the ground state fourfold degenerate.

With the operator identifications established above, we find that

$$|0\rangle = \prod_{n=-\infty}^{-1} (a_{n\downarrow}^\dagger a_{n\uparrow}^\dagger)|\text{empty}\rangle \tag{A8}$$

when x lies in region (1), but in region (2), where $b_{0\uparrow}$ and $b_{0\downarrow}$ are identified with $a_{0\downarrow}^\dagger$ and $a_{0\uparrow}^\dagger$, respectively, we must have

$$|0\rangle \propto a_{0\downarrow}^\dagger a_{0\uparrow}^\dagger \prod_{n=-\infty}^{-1} (a_{n\downarrow}^\dagger a_{n\uparrow}^\dagger)|\text{empty}\rangle, \tag{A9}$$

for it still to be annihilated by $b_{0\uparrow}$ and $b_{0\downarrow}$. We see that the occupation number of the energy levels for $n < 0$ are unchanged, but $|0\rangle$ picks up a pair of $n = 0$ electrons from the superconducting lead as it passes under it. Similarly the state $b_{0\downarrow}^\dagger b_{0\uparrow}^\dagger|0\rangle$ *loses* a pair from the $n = 0$ level.

The state $b_{0\uparrow}^\dagger|0\rangle$ is annihilated by $a_{0\uparrow}^\dagger$ and $a_{0\downarrow}$ in region (1) , and these become, respectively, $a_{0\downarrow}$ and $a_{0\uparrow}^\dagger$ in region (2). Therefore, the particle content of this state is unaffected by its passage under the lead. Similarly, $b_{0\downarrow}^\dagger|0\rangle$ retains its particle content.

Now consider an excited state, for example, $b_{m\uparrow}^\dagger b_{0\uparrow}^\dagger|0\rangle$ with $m > 0$. This state has energy $E = 2\pi v_d m/L$. In region (1) it has particle content

$$a_{m\uparrow}^\dagger a_{0\uparrow}^\dagger \prod_{n=-\infty}^{-1} (a_{n\downarrow}^\dagger a_{n\uparrow}^\dagger)|\text{empty}\rangle, \tag{A10}$$

and so consists of a Dirac sea together with an electron in a positive energy level. In region (2) it becomes

$$a_{-m\downarrow} a_{0\uparrow}^\dagger \prod_{n=-\infty}^{-1} (a_{n\downarrow}^\dagger a_{n\uparrow}^\dagger)|\text{empty}\rangle, \tag{A11}$$

which consists of a Dirac sea which has lost an electron from a negative energy level. Therefore, after passing the superconductor , the state has the same energy and spin, but the electron has become a hole.

APPENDIX B

We wish to establish the third line of Eq. (57), which reads

$$\int_a^{a+\omega_2}\{\wp(z|\omega_1,\omega_2)-e_3\}dz = \frac{1}{\omega_2}\frac{d^2}{dz^2}\ln\theta_3(z|-\omega_1/\omega_2)\Big|_{z=0}. \tag{B1}$$

This result follows indirectly from the related integral

$$\int_a^{a+\omega_1}\{\wp(z|\omega_1,\omega_2)-e_3\}dz = -2\eta_1-\omega_1 e_3 = \frac{1}{\omega_1}\frac{\theta_3''(0|\omega_2/\omega 1)}{\theta_3(0|\omega_2/\omega_1)} = \frac{1}{\omega_1}\frac{d^2}{dz^2}\ln\,\theta_3(z|\omega_2/\omega_1)\Big|_{z=0}. \tag{B2}$$

Here we require $\mathrm{Im}\,(\omega_2/\omega_1)>0$ for the θ functions to converge. To establish Eq. (B2) we observe that the second line follows from the first by combining two standard formulas:

$$e_3=\frac{1}{\omega_1^2}\left\{\frac{1}{3}\frac{\theta'''(0|\tau)}{\theta'(0|\tau)}-\frac{\theta_3''(0|\tau)}{\theta_3(0|\tau)}\right\} \tag{B3}$$

(Ref. 18, Eq. 5.2) and

$$2\eta_1=-\frac{1}{\omega_1}\frac{1}{3}\frac{\theta'''(0|\tau)}{\theta'(0|\tau)} \tag{B4}$$

(Ref. 19, Sec. 21.43). Here $\tau=\omega_2/\omega_1$ with $\mathrm{Im}\,\tau>0$. The third line of Eq. (B2) follows from the second because $\theta_3'(0|\tau)=0$.

To derive Eq. (B1), however, we need the integral over the $\omega_2=iv_d\beta$ imaginary period, and not over the $\omega_1=L$ real period. Because of the positivity condition on the imaginary part of τ, we cannot change the integration path by merely interchanging $\omega_1\leftrightarrow\omega_2$ in Eq. (B2). We need to be more subtle. By changing $(\omega_1,\omega_2)\to(-\omega_2,\omega_1)$ in Eq. (B2), we obtain

$$-\frac{1}{\omega_2}\frac{d^2}{dz^2}\ln\,\theta_3(z|-\omega_1/\omega_2)\Big|_{z=0} = \int_a^{a-\omega_2}\{\wp(z|-\omega_2,\omega_1)-e_3\}dz. \tag{B5}$$

This last equation is legitimate because $\mathrm{Im}(\omega_2/\omega_1)>0$ implies that $\mathrm{Im}(-\omega_1/\omega_2)>0$. We now manipulate

$$\mathrm{RHS}=-\int_a^{a+\omega_2}\{\wp(z|-\omega_2,\omega_1)-e_3\}dz = -\int_a^{a+\omega_2}\{\wp(z|\,\omega_1,\omega_2)-e_3\}dz, \tag{B6}$$

where the last line follows from the invariance of $\wp(z|\,\omega_1,\omega_2)$ under modular transformations

$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},\quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z}). \tag{B7}$$

From this we immediately deduce Eq. (B1).

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