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$\mathbb{Z}_2$ spin liquids in the $S = \frac{1}{2}$ Heisenberg model on the kagome lattice: A projective symmetry-group study of Schwinger fermion mean-field states

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Due to strong geometric frustration and quantum fluctuation, the $S = 1/2$ quantum Heisenberg antiferromagnet on the kagome lattice has long been considered as an ideal platform to realize a spin liquid (SL), a phase exhibiting fractionalized excitations without any symmetry breaking. A recent numerical study (Yan *et al.*, e-print arXiv:1011.6114) of the Heisenberg $S = 1/2$, kagome lattice model (HKLM) shows, in contrast to earlier results, that the ground state is a singlet-gapped SL with signatures of $\mathbb{Z}_2$ topological order. Motivated by this numerical discovery, we use the projective symmetry group to classify all 20 possible Schwinger fermion mean-field states of $\mathbb{Z}_2$ SLs on the kagome lattice. Among them we found only one gapped $\mathbb{Z}_2$ SL (which we call the $\mathbb{Z}_2[0,\pi]\beta$ state) in the neighborhood of the U(1) Dirac SL state. Since its parent state, i.e., the U(1) Dirac SL, was found [Ran *et al.*, Phys. Rev. Lett. **98**, 117205 (2007)] to be the lowest among many other candidate U(1) SLs, including the uniform resonating-valence-bond states, we propose this $\mathbb{Z}_2[0,\pi]\beta$ state to be the numerically discovered SL ground state of the HKLM.

I. INTRODUCTION

At zero temperature, all degrees of freedom tend to freeze, and usually a variety of different orders, such as superconductivity and magnetism, will develop in different materials. However, in a quantum system with a large zero-point energy, one may expect a liquidlike ground state to exist even at $T = 0$. In a system consisting of localized quantum magnets, we call such a quantum-fluctuation-driven disordered ground state a quantum spin liquid (SL).[1] It is an exotic phase with "fractionalized" excitations carrying only a fraction of the electron quantum number, e.g., spinons, which carry spin but no charge. The internal structures of these SLs are so rich that they are beyond the description of Landau's symmetry-breaking theory[2] of conventional ordered phases. Instead they are characterized by long-range quantum entanglement[3,4] encoded in the ground state, which is called "topological order"[5,6] in contrast to the conventional symmetry-breaking order.

Geometric frustration in a system of quantum magnets would lead to a huge degeneracy of classical ground-state configurations. The quantum tunneling among these classical ground states provides a mechanism to realize quantum SLs. The quest for quantum SLs in frustrated magnets (for a recent review, see Ref. 7) has been pursued for decades. Among them the Heisenberg $S = 1/2$ kagome lattice model (HKLM)

$$H_{\text{HKLM}} = J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j \qquad (1)$$

has long been thought a promising candidate. Here $\langle i,j \rangle$ denotes i,j being a nearest-neighbor pair. Experimental evidence[8–11] of SLs has been observed in $ZnCu_3(OH)_6Cl_2$ (called herbertsmithite), a spin-1/2 antiferromagnet on the kagome lattice. Theoretically, lacking an exact solution of the two-dimensional (2D) quantum Hamiltonian (1) in the

thermodynamic limit, in previous studies either a honeycomb valence-bond crystal[12–16] (HVBC) with an enlarged (6×6)-site unit cell, or a gapless SL (Ref. 17) were proposed as the ground state of the HKLM. However, recently an extensive density-matrix-renormalization-group (DMRG) study[18] on the HKLM revealed the ground state of the HKLM as a gapped SL, which substantially lowers the energy compared to the HVBC. In addition, the authors also observed numerical signatures of $\mathbb{Z}_2$ topological order in the SL state.

Motivated by this important numerical discovery, we try to discover the nature of this gapped $\mathbb{Z}_2$ SL. Different $\mathbb{Z}_2$ SLs on the kagome lattice have been previously studied using the Schwinger boson representation.[19,20] Here we propose candidate states for symmetric $\mathbb{Z}_2$ SLs on the kagome lattice by the Schwinger fermion mean-field approach.[21–27] Following is the summary of our results. First we use the projective symmetry group[6] (PSG) to classify all 20 possible Schwinger fermion mean-field *Ansätze* of $\mathbb{Z}_2$ SLs which preserve all the symmetry of the HKLM, as shown in Table I. We analyze these 20 states and rule out some obviously unfavorable ones: e.g., gapless states, and those states whose first-nearest-neighbor (NN) mean-field amplitudes must vanish due to symmetry. Then we focus on those $\mathbb{Z}_2$ SLs in the neighborhood of the U(1) Dirac SL.[28] In Ref. 28 it is shown that the U(1) Dirac SL has a significantly lower energy compared with other candidate U(1) SL states, such as the uniform resonating-valence-bond (RVB) state (or the U(1) SL-[0,0] state in the notation of Ref. 28). We find that there is only one gapped $\mathbb{Z}_2$ SL, which we label as $\mathbb{Z}_2[0,\pi]\beta$, in the neighborhood of (or continuously connected to) the U(1) Dirac SL. Therefore we propose this $\mathbb{Z}_2[0,\pi]\beta$ state as a promising candidate for the ground state of the HKLM. The mean-field ansatz of $\mathbb{Z}_2[0,\pi]\beta$ state is shown in Fig. 1(b). Our work also provides guidelines for choosing variational states in future numeric studies of the SL ground state on the kagome lattice.

TABLE I. Mean-field *Ansätze* of 20 possible $\mathbb{Z}_2$ SLs on a kagome lattice. In our notation of mean-field amplitudes $\langle x,y,s|0,0,u\rangle \equiv [x,y,s]$, this table summarizes all symmetry-allowed mean-field bonds up to third nearest neighbor, i.e., first-nearest-neighbor bond $u_\alpha = [0,0,v]$, second-nearest-neighbor bond $u_\beta = [0,1,w]$, and third-nearest-neighbor bonds $u_\gamma = [1,0,u]$ and $\tilde{u}_\gamma = [1,-1,u]$ as shown in Fig. 1(a). Λ_s denote the on-site chemical potential terms which enforce the constraint (11). τ^0 is a 2×2 identity matrix while $\tau^{1,2,3}$ are three Pauli matrices. $\tau^{0,3}$ denote hopping while $\tau^{1,2}$ denote pairing terms. 0 means that the corresponding mean-field amplitudes must vanish due to symmetry. The shortest mean-field bonds necessary to realize a $\mathbb{Z}_2$ SL are shown in italic type. In other words, the mean-field amplitudes in italics break the U(1) gauge redundancy down to $\mathbb{Z}_2$ through the Higgs mechanism. So in states nos. 3, 19, and 7–12 a $\mathbb{Z}_2$ SL cannot be realized with up to thrid-NN mean-field amplitudes. Note that state no. 15 needs only a third-nearest-neighbor bond u_γ to realize a $\mathbb{Z}_2$ SL ($\tilde{u}_\gamma$ is not necessary), while state no. 20 needs only $\tilde{u}_\gamma$ to realize a $\mathbb{Z}_2$ SL (u_γ is not necessary). Notice that when $\eta_{12} = -1$ the mean-field *Ansatz* (instead of the SL itself) will break translational symmetry and double the unit cell. There are six $\mathbb{Z}_2$ SLs, i.e., nos. 7–12 that do not allow any first-NN mean-field bonds. Among the other 14 $\mathbb{Z}_2$ SLs with nonvanishing first-NN mean-field bonds, only five $\mathbb{Z}_2$ SL states, i.e., nos. 1, 2, 5, 13, and 15, have gapped spinon spectra. No. 2 (in bold type) or the $\mathbb{Z}_2[0,\pi]\beta$ state in the neighborhood of the U(1) Dirac SL is the most promising candidate of the $\mathbb{Z}_2$ SLs for the HKLM ground state.

No.	η_{12}	Λ_s	u_α	u_β	u_γ	$\tilde{u}_\gamma$	Label	Gapped?
1	$+1$	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	$Z_2[0,0]A$	Yes
2	-1	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	τ^2,τ^3	0	$\mathbb{Z}_2[0,\pi]\beta$	**Yes**
3	$+1$	0	τ^2,τ^3	0	0	0	$Z_2[\pi,\pi]A$	No
4	-1	0	τ^2,τ^3	0	0	τ^2,τ^3	$Z_2[\pi,0]A$	No
5	$+1$	τ^3	τ^2,τ^3	τ^3	τ^3	τ^3	$Z_2[0,0]B$	Yes
6	-1	τ^3	τ^2,τ^3	τ^3	τ^3	τ^2	$Z_2[0,\pi]\alpha$	No
7	$+1$	0	0	τ^2,τ^3	0	0	–	–
8	-1	0	0	τ^2,τ^3	0	0	–	–
9	$+1$	0	0	0	τ^2,τ^3	0	–	–
10	-1	0	0	0	τ^2,τ^3	0	–	–
11	$+1$	0	0	τ^2	τ^2	0	–	–
12	-1	0	0	τ^2	τ^2	0	–	–
13	$+1$	τ^3	τ^3	τ^2,τ^3	τ^3	τ^3	$Z_2[0,0]D$	Yes
14	-1	τ^3	τ^3	τ^2,τ^3	τ^3	0	$Z_2[0,\pi]\gamma$	No
15	$+1$	τ^3	τ^3	τ^3	τ^2,τ^3	τ^3	$Z_2[0,0]C$	Yes
16	-1	τ^3	τ^3	τ^3	τ^2,τ^3	0	$Z_2[0,\pi]\delta$	No
17	$+1$	0	τ^2	τ^3	0	0	$Z_2[\pi,\pi]B$	No
18	-1	0	τ^2	τ^3	0	τ^3	$Z_2[\pi,0]B$	No
19	$+1$	0	τ^2	0	τ^2	0	$Z_2[\pi,\pi]C$	No
20	-1	0	τ^2	0	τ^2	τ^3	$Z_2[\pi,0]C$	No

II. SCHWINGER FERMION CONSTRUCTION OF SPIN LIQUIDS AND PROJECTIVE SYMMETRY GROUP

A. Schwinger fermion construction of symmetric spin liquids

In the Schwinger fermion construction,[21–26] we represent a spin-1/2 operator at site i by fermionic spinons $\{f_{i\alpha}, \alpha = \uparrow, \downarrow\}$:

$$\vec{S}_i = \frac{1}{2} f_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} f_{i\beta}. \qquad (2)$$
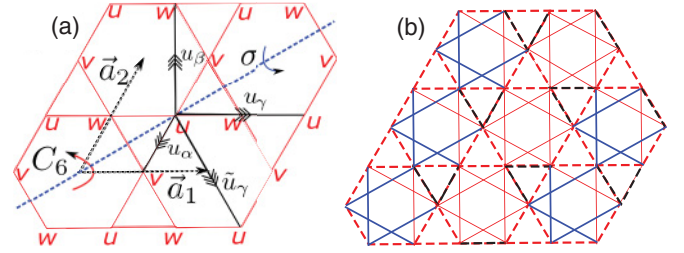

FIG. 1. (Color online) (a) Kagome lattice and the elements of its symmetry group. $\vec{a}_{1,2}$ are the translation unit vectors, C_6 denotes $\pi/3$ rotation around the honeycomb center, and σ represents mirror reflection along the dashed blue line. Here u_α and u_β denote first- and second-nearest-neighbor (NN) mean-field bonds while u_γ and $\tilde{u}_\gamma$ represent two kinds of independent third-NN mean-field bonds. (b) Mean-field ansatz of the $\mathbb{Z}_2[0,\pi]\beta$ state up to second nearest neighbor. Colors in general denote the sign structure of the mean-field bonds. Dashed lines denote first-NN real hopping terms $\chi_1 \sum_{\langle i,j \rangle \alpha} (v_{ij} f_{i\alpha}^\dagger f_{j\alpha} + \text{H.c.})$: red ones have $v_{ij} = 1$ and bold black ones have $v_{ij} = -1$. Solid lines stand for second-NN hopping $\chi_2 \sum_{\langle\langle ij \rangle\rangle \alpha} v_{ij}(f_{i\alpha}^\dagger f_{j\alpha} + \text{H.c.})$ and singlet pairing $\sum_{\langle\langle ij \rangle\rangle \alpha\beta} \epsilon_{\alpha\beta} v_{ij}(\Delta_2 f_{i\alpha}^\dagger f_{j\beta}^\dagger + \text{H.c.})$: again red ones have $v_{ij} = 1$ and bold blue ones have $v_{ij} = -1$. Here $\chi_{1,2}$ and Δ_2 are real parameters after choice of a proper gauge.

The Heisenberg Hamiltonian $H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$ is represented as $H = \sum_{\langle ij \rangle} -\frac{1}{2} J_{ij} (f_{i\alpha}^\dagger f_{j\alpha} f_{j\beta}^\dagger f_{i\beta} + \frac{1}{2} f_{i\alpha}^\dagger f_{i\alpha} f_{j\beta}^\dagger f_{j\beta})$. This construction enlarges the Hilbert space of the original spin system. To obtain the physical spin state from a mean-field state of f spinons, we need to enforce the following one-f-spinon-per-site constraint:

$$f_{i\alpha}^\dagger f_{i\alpha} = 1, \quad f_{i\alpha} f_{i\beta} \epsilon_{\alpha\beta} = 0. \qquad (3)$$

The mean-field parameters of symmetric SLs are $\Delta_{ij} \epsilon_{\alpha\beta} = -2\langle f_{i\alpha} f_{j\beta} \rangle$, $\chi_{ij} \delta_{\alpha\beta} = 2\langle f_{i\alpha}^\dagger f_{j\beta} \rangle$, where $\epsilon_{\alpha\beta}$ is the completely antisymmetric tensor. Both terms are invariant under global SU(2) spin rotations. After a Hubbard-Stratonovich transformation, the Lagrangian of the spin system can be written as

$$L = \sum_i \psi_i^\dagger \partial_\tau \psi_i + \sum_{\langle ij \rangle} \frac{3}{8} J_{ij} \left[\frac{1}{2} \text{Tr}(U_{ij}^\dagger U_{ij}) \right.$$
$$\left. - (\psi_i^\dagger U_{ij} \psi_j + \text{H.c.}) \right] + \sum_i a_0^l(i) \psi_i^\dagger \tau^l \psi_i, \qquad (4)$$

where the two-component fermion notation $\psi_i \equiv (f_{i\uparrow}, f_{i\downarrow}^\dagger)$ is introduced for reasons that will be explained shortly. We use τ^0 to denote the 2×2 identity matrix and $\tau^{1,2,3}$ are the three Pauli matrices. U_{ij} is a matrix of mean-field amplitudes:

$$U_{ij} = \begin{pmatrix} \chi_{ij}^\dagger & \Delta_{ij} \\ \Delta_{ij}^\dagger & -\chi_{ij} \end{pmatrix}. \qquad (5)$$

$a_0^l(i)$ are the local Lagrangian multipliers that enforce the constraints Eq. (3).

In terms of ψ, the Schwinger fermion representation has an explicit SU(2) gauge redundancy: a transformation $\psi_i \to W_i \psi_i$, $U_{ij} \to W_i U_{ij} W_j^\dagger$, $W_i \in$ SU(2) leaves the action invariant. This redundancy originates from the representation Eq. (2): this local SU(2) transformation leaves the spin

operators invariant and does not change the physical Hilbert space. One can try to solve Eq. (4) by a mean-field (or saddle-point) approximation. At mean-field level, U_{ij} and a_0^l are treated as complex numbers, and a_0^l must be chosen such that the constraints (3) are satisfied at the mean-field level: $\langle \psi_i^\dagger \tau^l \psi_i \rangle = 0$. The mean-field *Ansatz* can be written as

$$H_{MF} = -\sum_{\langle ij \rangle} \psi_i^\dagger \langle i|j \rangle \psi_j + \sum_i \psi_i^\dagger a_0^l \tau^l \psi_i, \qquad (6)$$

where we defined $\langle i|j \rangle \equiv \frac{3}{8} J_{ij} U_{ij}$. Under a local SU(2) gauge transformation $\langle i|j \rangle \rightarrow W_i \langle i|j \rangle W_j^\dagger$, but the physical spin state described by the mean-field *Ansatz* $\{\langle i|j \rangle\}$ remains the same. By construction the mean-field *Ansatz* does not break spin rotation symmetry, and the mean-field solutions describe SL states if lattice symmetry is preserved. Different $\{\langle i|j \rangle\}$ *Ansätze* may be in different SL phases. The mathematical language to classify different SL phases is the projective symmetry group (PSG).[6]

B. Projective symmetry group classification of topological orders in spin liquids

The PSG characterizes the topological order in Schwinger fermion representation: SLs described by different PSGs are different phases. It is defined as the collection of all combinations of symmetry group and SU(2) gauge transformations that leave mean-field *Ansätze* $\{\langle i|j \rangle\}$ invariant (as a_0^l are determined self-consistently by $\{\langle i|j \rangle\}$, these transformations also leave a_0^l invariant). The invariance of a mean-field *Ansatz* $\{\langle i|j \rangle\}$ under an element of PSG $G_U U$ can be written as

$$G_U U(\{\langle i|j \rangle\}) = \{\langle i|j \rangle\},$$
$$U(\{\langle i|j \rangle\}) \equiv \{\langle \tilde{i}|j \rangle = \langle U^{-1}(i)|U^{-1}(j) \rangle\}, \qquad (7)$$
$$G_U(\{\langle i|j \rangle\}) \equiv \{\langle \tilde{i}|j \rangle = G_U(i) \langle i|j \rangle G_U(j)^\dagger\},$$
$$G_U(i) \in \mathrm{SU}(2).$$

Here $U \in SG$ is an element of the symmetry group (SG) of the corresponding SL. In our case of symmetric SLs on the kagome lattice, we use (x,y,s) to label a site with sublattice index $s = u,v,w$ and $x,y \in \mathbb{Z}$. The Bravais unit vectors are chosen as $\vec{a}_1 = a\hat{x}$ and $\vec{a}_2 = \frac{a}{2}(\hat{x} + \sqrt{3}\hat{y})$ as shown in Fig. 1(a). The symmetry group is generated by the time-reversal operation T, lattice translations $T_{1,2}$ along the $\vec{a}_{1,2}$ directions, a $\pi/3$ rotation C_6 around the honeycomb plaquette center, and the mirror reflection σ (for details, see Appendix A). For example, if $U = T_1$ is the translation along the $\vec{a}_1$ direction in Fig. 1(a), $T_1(\{x,y,s\}) = \{x+1,y,s\}$. G_U is the gauge transformation associated with U such that $G_U U$ leaves $\{\langle i|j \rangle\}$ invariant. Notice that this condition (7) allows us to generate all symmetry-related mean-field bonds from one by the following relation:

$$\langle i|j \rangle = G_U(i) \langle U^{-1}(i)|U^{-1}(j) \rangle G_U^\dagger(j). \qquad (8)$$

There is an important subgroup of the PSG, the invariant gauge group (IGG), which is composed of all the pure gauge transformations in the PSG: IGG $\equiv \{\{W_i\}|W_i \langle i|j \rangle W_j^\dagger = \langle i|j \rangle, W_i \in SU(2)\}$. In other words, $W_i = G_e(i)$ is the pure gauge transformation associated with the identity element $e \in SG$ of the symmetry group. One can always choose a

gauge in which the elements in IGG are site independent. In this gauge, the IGG can be the global $\mathbb{Z}_2$ transformations: $\{G_e(i) \equiv G_e = \pm\tau^0\}$, the global U(1) transformations: $\{G_e(i) \equiv e^{i\theta\tau^3}, \theta \in [0,2\pi]\}$, or the global SU(2) transformations: $\{G_e(i) \equiv e^{i\theta\hat{n}\cdot\vec{\tau}}, \theta \in (0,2\pi], \hat{n} \in S^2\}$, and we term them the $\mathbb{Z}_2$, U(1), and SU(2) states, respectively.

The importance of the IGG is that it controls the low-energy gauge fluctuations of the corresponding SL states. Beyond the mean-field level, fluctuations of $\langle i|j \rangle$ and a_0^l need to be considered and the mean-field state may or may not be stable. The low-energy effective theory is described by a fermionic spinon band structure coupled with a dynamical gauge field of the IGG. For example, a $\mathbb{Z}_2$ state with gapped spinon dispersion can be a stable phase because the low-energy $\mathbb{Z}_2$ dynamical gauge field can be in the deconfined phase.[29,30] Notice that the condition $\{G_e(i) \equiv G_e = \pm\tau^0\}$ for a $\mathbb{Z}_2$ SL leads to a series of consistent conditions for the gauge transformations $\{G_U(i)|U \in SG\}$, as shown in Appendix A. Gauge-inequivalent solutions of these conditions (A4)–(A11) lead to different $\mathbb{Z}_2$ SLs. Soon we will show that there are 20 $\mathbb{Z}_2$ SLs on the kagome lattice that can be realized by a Schwinger-fermion mean-field *Ansatz* $\{\langle i|j \rangle\}$.

III. $\mathbb{Z}_2$ SPIN LIQUIDS ON THE KAGOME LATTICE AND $\mathbb{Z}_2[0,\pi]\beta$ STATE

Following previous discussions, we use the PSG to classify all possible 20 $\mathbb{Z}_2$ SL states on the kagome lattice in this section. As will be shown later, among them there is one gapped $\mathbb{Z}_2$ SL labeled the $\mathbb{Z}_2[0,\pi]\beta$ state in the neighborhood of the U(1) Dirac SL. This $\mathbb{Z}_2[0,\pi]\beta$ SL state is the most promising candidate for the SL ground state of HKLM.

A. PSG classification of $\mathbb{Z}_2$ spin liquids on a kagome lattice

Applying the condition $G_e(i) \equiv G_e = \pm\tau^0$ to the kagome lattice with the symmetry group described in Appendix A, we obtain a series of consistent conditions for the gauge transformation $G_U(i)$, i.e., the conditions (A4)–(A11). Solving these conditions, we classify all the 20 different Schwinger fermion mean-field states of $\mathbb{Z}_2$ SLs on the kagome lattice, as summarized in Table I. These 20 mean-field states correspond to different $\mathbb{Z}_2$ SL phases, which cannot be continuously tuned into each other without a phase transition.

As discussed in Appendix B 2, from PSG elements $G_U(i)$ one can obtain all other symmetry-related mean-field bonds from one using the symmetry condition (8). Therefore we use $u_\alpha \equiv \langle 0,0,v|0,0,u \rangle$ to represent first-nearest-neighbor mean-field bonds. $u_\beta \equiv \langle 0,1,w|0,0,u \rangle$ is the representative of second-NN mean-field bonds. There are two kinds of symmetry-unrelated third-NN mean-field bonds, represented by $u_\gamma = \langle 1,0,u|0,0,u \rangle$ and $\tilde{u}_\gamma = \langle 1,-1,u|0,0,u \rangle$. The symmetry conditions for these mean-field bonds are summarized in (B13)–(B16). In addition, the on-site chemical potential terms $\Lambda(i)$ [which guarantee the physical constraint (3) on the mean-field level] also satisfy symmetry conditions (B12). We can show that $\Lambda(x,y,s) \equiv \Lambda_s$ for these 20 $\mathbb{Z}_2$ SL states. The symmetry-allowed mean-field amplitudes and bonds are also summarized in Table I.

TABLE II. A summary of all 20 gauge-inequivalent PSGs with $G_T(x,y,s) = i\tau^1$ on the kagome lattice. Notice that there is a free $\mathbb{Z}_2$ integer $\eta_{12} = \pm 1$ in other PSG elements (B1), (B5), and (B7). They correspond to 20 different $\mathbb{Z}_2$ spin liquids on the kagome lattice.

No.	η_{12}	$g_\sigma(u)$	$g_\sigma(v)$	$g_\sigma(w)$	$g_{C_6}(u)$	$g_{C_6}(v)$	$g_{C_6}(w)$	Label
1	$+1$	τ^0	τ^0	τ^0	τ^0	τ^0	τ^0	$Z_2[0,0]A$
2	-1	τ^0	τ^0	τ^0	τ^0	τ^0	τ^0	$\mathbb{Z}_2[0,\pi]\beta$
3	$+1$	τ^0	τ^0	τ^0	τ^0	$-\tau^0$	$i\tau^1$	$Z_2[\pi,\pi]A$
4	-1	τ^0	τ^0	τ^0	τ^0	$-\tau^0$	$i\tau^1$	$Z_2[\pi,0]A$
5	$+1$	τ^0	τ^0	τ^0	$i\tau^3$	$i\tau^3$	$i\tau^3$	$Z_2[0,0]B$
6	-1	τ^0	τ^0	τ^0	$i\tau^3$	$i\tau^3$	$i\tau^3$	$Z_2[0,\pi]\alpha$
7	$+1$	$i\tau^1$	τ^0	$-\tau^0$	τ^0	$i\tau^1$	τ^0	$-$
8	-1	$i\tau^1$	τ^0	$-\tau^0$	τ^0	$i\tau^1$	τ^0	$-$
9	$+1$	$i\tau^1$	τ^0	$-\tau^0$	τ^0	$-i\tau^1$	$i\tau^1$	$-$
10	-1	$i\tau^1$	τ^0	$-\tau^0$	τ^0	$-i\tau^1$	$i\tau^1$	$-$
11	$+1$	$i\tau^1$	τ^0	$-\tau^0$	$i\tau^3$	$-i\tau^2$	$i\tau^3$	$-$
12	-1	$i\tau^1$	τ^0	$-\tau^0$	$i\tau^3$	$-i\tau^2$	$i\tau^3$	$-$
13	$+1$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$Z_2[0,0]D$
14	-1	$i\tau^3$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$Z_2[0,\pi]\gamma$
15	$+1$	$i\tau^3$	$i\tau^3$	$i\tau^3$	τ^0	τ^0	τ^0	$Z_2[0,0]C$
16	-1	$i\tau^3$	$i\tau^3$	$i\tau^3$	τ^0	τ^0	τ^0	$Z_2[0,\pi]\delta$
17	$+1$	$i\tau^3$	$i\tau^3$	$i\tau^3$	τ^0	τ^0	$i\tau^1$	$Z_2[\pi,\pi]B$
18	-1	$i\tau^3$	$i\tau^3$	$i\tau^3$	τ^0	τ^0	$i\tau^1$	$Z_2[\pi,0]B$
19	$+1$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$-i\tau^3$	$i\tau^2$	$Z_2[\pi,\pi]C$
20	-1	$i\tau^3$	$i\tau^3$	$i\tau^3$	$i\tau^3$	$-i\tau^3$	$i\tau^2$	$Z_2[\pi,0]C$

From Table I we can see there are six states, i.e., nos. 7–12, that do not allow nonzero first-NN mean-field amplitudes due to symmetry. Moreover, they cannot realize $\mathbb{Z}_2$ SLs with up to third-NN mean-field amplitudes. Therefore they are unlikely to be the HKLM ground state. Ruling out these six $\mathbb{Z}_2$ SLs, we can see that the other 14 $\mathbb{Z}_2$ SL states fall into four classes. To be specific, they are continuously connected to different parent U(1) gapless SL states on the kagome lattice. These parent U(1) SL states in general have the following mean-field *Ansatz*:

$$H_{\text{U(1)SL}} = \chi_1 \sum_{\langle ij \rangle} \nu_{ij}(f^\dagger_{i\alpha} f_{j\alpha} + \text{H.c.}), \qquad (9)$$

where $\nu_{ij} = \pm 1$ characterizes the sign structure of hopping terms with $\chi_1 \in \mathbb{R}$. Different parent U(1) SL states feature the flux of f spinon hopping phases around the basic plaquette: honeycombs and triangles on the kagome lattice.

The simplest example is the so-called uniform RVB state with $\nu_{ij} \equiv +1$ for all first-NN mean-field bonds. The hopping phase around any plaquette is $1 = \exp[i0]$, and the corresponding flux is $[0,0]$ for [triangle, honeycomb] motifs. The four possible $\mathbb{Z}_2$ spin liquids in the neighborhood[31] of uniform RVB states [i.e., the U(1) SL $[0,0]$ state in Ref. 28) are classified in Appendix D. They are nos. 1, 5, 15, and 13 in Tables I and II. We label them as $Z_2[0,0]A$, $Z_2[0,0]B$, $Z_2[0,0]C$, and $Z_2[0,0]D$ states. They all have gapped spectra of spinons.

The *Ansätze* of two other parent U(1) SLs are shown in Fig. 2. They both have π flux piercing through a triangle basic plaquette. Following the above notations of the hopping phase in [triangle,honeycomb] motifs, with either π or 0 flux through the honeycomb plaquette, they are called the U(1) SL $[\pi,\pi]$
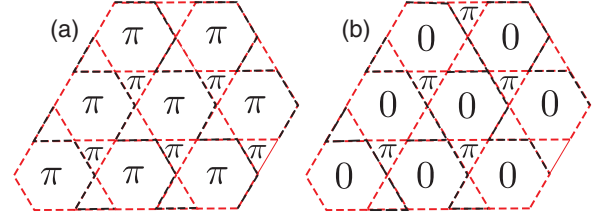

FIG. 2. (Color online) Mean-field *Ansätze* of (a) U(1) SL $[\pi,\pi]$ state and (b) U(1) SL $[\pi,0]$ state, with first-NN real hopping terms $H_{MF} = \chi_1 \sum_{\langle ij \rangle \alpha} (\nu_{ij} f^\dagger_{i\alpha} f_{j\alpha} + \text{H.c.})$. Colors again denote the sign structure of mean-field bonds: red dashed lines have $\nu_{ij} = +1$ and black dashed lines have $\nu_{ij} = -1$.

state and the U(1) SL $[\pi,0]$ state. There are three $\mathbb{Z}_2$ SLs in the neighborhood of both U(1) SL states, i.e., nos. 3, 17, and 19 around the U(1) SL $[\pi,\pi]$ state and nos. 4, 18, and 20 around the U(1) SL $[\pi,0]$ state. All these six $\mathbb{Z}_2$ SLs have gapless spinon spectra, inherited from the two parent gapless U(1) SLs. To be precise, the spinon band structures of these six $\mathbb{Z}_2$ SL states feature a doubly degenerate flatband and a Dirac cone at the Brillouin-zone center. This is in contrast to the numerically observed gap in the two-spinon spectrum;[18] thus we can also rule out these six $\mathbb{Z}_2$ SLs for the HKLM ground state.

Another U(1) SL state is the so-called U(1) Dirac SL or U(1) SL $[0,\pi]$ state. Its mean-field *Ansatz* is shown by the first-NN bonds in Fig. 1(b). Clearly π flux pierces through certain triangle plaquettes with no flux through the honeycomb plaquette. According to variational Monte Carlo studies,[28,32] this U(1) Dirac SLs have substantially lower energy compared to many other competing phases, including the uniform RVB state. Therefore we shall focus on those $\mathbb{Z}_2$ SLs in the neighborhood of the U(1) Dirac SL in our search for the HKLM ground state. We need to mention that, although it is unlikely, the four $\mathbb{Z}_2$ SLs in the neighborhood of the uniform RVB state or the U(1) SL $[0,0]$ state might potentially be the HKLM ground state.

In a previous study using the PSG in Schwinger boson representation,[20] it was shown that there are eight different Schwinger boson mean-field *Ansätze* of $\mathbb{Z}_2$ SLs on the kagome lattice which preserve all lattice symmetry. However, these eight $\mathbb{Z}_2$ SLs may or may not preserve time-reversal symmetry. One can show that requiring all lattice symmetry *and time-reversal symmetry*, there are 16 different Schwinger-boson $\mathbb{Z}_2$ SLs on the kagome lattice. The relation between the 20 $\mathbb{Z}_2$ SLs in Schwinger fermion representation (see Table I) and the 16 $\mathbb{Z}_2$ SLs in Schwinger boson representation is not clear. To clarify the relation between SL states in these two different representations, one can compare the neighboring (ordered) phases of the SLs, e.g., by computing the vison quantum numbers[34] of SL states.

B. $\mathbb{Z}_2[0,\pi]\beta$ state as a promising candidate for the HKLM ground state

How can we find those $\mathbb{Z}_2$ SLs in the neighborhood of (or continuously connected to) the U(1) Dirac SL? Naively, we expect that the mean-field *Ansätze* of these $\mathbb{Z}_2$ SLs can be obtained from that of the U(1) Dirac SL by adding

an infinitesimal perturbation. To be specific, we require an infinitesimal spinon pairing term on top of the U(1) Dirac SL mean-field *Ansatz* (9) or (C1) to break the IGG from U(1) to $\mathbb{Z}_2$ through the Higgs mechanism. Mathematically, we need to find those $\mathbb{Z}_2$ SL states whose PSG is a subgroup of the U(1) Dirac SL's PSG.[31] Such $\mathbb{Z}_2$ SL states are defined to be in the neighborhood of the U(1) Dirac SL. A similar criterion applies to the neighboring $\mathbb{Z}_2$ SL states of any parent U(1) or SU(2) SL state.

We find all four $\mathbb{Z}_2$ SLs in the neighborhood of U(1) Dirac SLs in Appendix C. They are states nos. 6, 2, 14, and 16 in Table I, labeled the $Z_2[0,\pi]\alpha$, $\mathbb{Z}_2[0,\pi]\beta$, $\mathbb{Z}_2[0,\pi]\gamma$, and $Z_2[0,\pi]\delta$ states, respectively. Since the effective theory of a U(1) Dirac SL is an eight-component Dirac fermion coupled with a dynamical U(1) gauge field,[28,33] we can find all symmetry-allowed mass terms that can open up a gap in the Dirac-like spinon spectrum. Following detailed calculations in Appendix C, we can see that among the four $\mathbb{Z}_2$ SLs around the U(1) Dirac SL, only one state, i.e., $\mathbb{Z}_2[0,\pi]\beta$ (state 2 in Tables I and II), can generate a mass gap in the spinon spectrum. In the other three states the Dirac cone in the spinon spectrum is protected by symmetry. The mean-field *Ansatz* of the $\mathbb{Z}_2[0,\pi]\beta$ SL state up to the second NN is shown in Fig. 1(b):

$$
\begin{aligned}
H_{MF} = \sum_i &\left(\lambda_3 \sum_\alpha f^\dagger_{i\alpha} f_{i\alpha} + \lambda_1 f^\dagger_{i\uparrow} f^\dagger_{i\downarrow} + \text{H.c.} \right) \\
&+ \chi_1 \sum_{\langle ij\rangle \alpha} v_{ij}(f^\dagger_{i\alpha} f_{j\alpha} + \text{H.c.}) + \sum_{\langle\langle ij\rangle\rangle} v_{ij}\left(\chi_2 \sum_\alpha f^\dagger_{i\alpha} f_{j\alpha} \right. \\
&\left. + \Delta_2 \sum_{\alpha\beta} \epsilon^{\alpha\beta} f^\dagger_{i\alpha} f^\dagger_{j\beta} + \text{h.c.} \right),
\end{aligned} \tag{10}
$$

where $\epsilon^{\alpha\beta}$ is the completely antisymmetric tensor. We only list up to second-NN mean-field amplitudes because, as shown in Table I (see also Appendix C), this $\mathbb{Z}_2[0,\pi]\beta$ state needs only second-NN pairing terms to realize a $\mathbb{Z}_2$ SL. We can always choose a proper gauge so that the mean-field parameters $\chi_{1,2}$ and Δ_2 are all real. The sign structure of $v_{ij} = \pm 1$ is shown in Fig. 1(b), with red denoting $v_{ij} = +1$ and other colors with bold lines representing $v_{ij} = -1$. As discussed in Appendix C, the second-NN singlet-pairing term $\Delta_2 \neq 0$ not only breaks the U(1) gauge symmetry down to $\mathbb{Z}_2$, but also opens up a mass gap in the spinon spectrum. The on-site chemical potentials $\lambda_{1,3}$ are self-consistently determined by the following constraint:

$$
\begin{aligned}
\sum_i \langle f^\dagger_{i\uparrow} f^\dagger_{i\downarrow}\rangle = \sum_i \langle f_{i\uparrow} f_{i\downarrow}\rangle &= 0, \\
\sum_i \left(\sum_{\alpha=\uparrow,\downarrow} f^\dagger_{i\alpha} f_{i\alpha} - 1 \right) &= 0.
\end{aligned} \tag{11}
$$

For further NN mean-field ansatz see discussions in Appendix C.

IV. CONCLUSION

To summarize, motivated by the strong evidence of a $\mathbb{Z}_2$ SL as the HKLM ground state in a recent DMRG study,[18] we

classify all possible $\mathbb{Z}_2$ SL states in the Schwinger fermion mean-field approach using the PSG. We found 20 different Schwinger fermion mean-field states of $\mathbb{Z}_2$ SLs on the kagome lattice, among which six states are unlikely due to vanishing first-NN mean-field amplitude. Among the other 14 $\mathbb{Z}_2$ SLs only five possess a gapped spinon spectrum, which is observed in the DMRG result.[18] These five symmetric $\mathbb{Z}_2$ SL states are all in the neighborhood of certain parent U(1) gapless SLs. To be precise, four are in the neighborhood of gapless uniform RVB [or U(1) SL [0,0]] state, while the other one, i.e., $\mathbb{Z}_2[0,\pi]\beta$, is in the neighborhood of a gapless U(1) Dirac SL]or U(1) SL [0,π]] state. A previous variational Monte Carlo study[28] showed that the gapless U(1) Dirac SL has a substantially lower energy in comparison to the uniform RVB state. This suggests that the $\mathbb{Z}_2$ SLs in the neighborhood of U(1) Dirac SLs should have lower energy compared to those in the neighborhood of uniform RVB states. Therefore we propose this $\mathbb{Z}_2[0,\pi]\beta$ state with the mean-field *Ansatz* (10) shown in Fig. 1(b) as the HKLM ground state numerically detected in Ref. 18. Our work provides important insight for future numeric study, e.g., variational Monte Carlo studies of Gutzwiller-projected wave functions.

APPENDIX A: SYMMETRY GROUP OF THE KAGOME LATTICE AND ALGEBRAIC CONDITIONS FOR $\mathbb{Z}_2$ SPIN LIQUIDS

As shown in Fig. 1(a), we label the three lattice sites in each unit cell with sublattice indices $\{s = u,v,w\}$. Choosing the Bravais unit vector as $\vec{a}_1 = a\hat{x}$ and $\vec{a}_2 = \frac{a}{2}(\hat{x} + \sqrt{3}\hat{y})$, the positions of the three atoms in a unit cell labeled by indices $i = (x,y,s)$ are

$$
\begin{aligned}
\vec{r}(x,y,u) &= \left(x + \tfrac{1}{2}\right)\vec{a}_1 + \left(y + \tfrac{1}{2}\right)\vec{a}_2, \\
\vec{r}(x,y,v) &= \left(x + \tfrac{1}{2}\right)\vec{a}_1 + y\vec{a}_2, \\
\vec{r}(x,y,w) &= x\vec{a}_1 + \left(y + \tfrac{1}{2}\right)\vec{a}_2.
\end{aligned} \tag{A1}
$$

The symmetry group of such a two-dimensional kagome lattice is generated by the following operations:

$$
\begin{aligned}
T_1 &: (x,y,s) \rightarrow (x+1,y,s), \\
T_2 &: (x,y,s) \rightarrow (x,y+1,s), \\
\sigma &: (x,y,u) \rightarrow (y,x,u), \\
&\quad (x,y,v) \rightarrow (y,x,w), \\
&\quad (x,y,w) \rightarrow (y,x,v); \\
C_6 &: (x,y,u) \rightarrow (-y-1,x+y+1,v), \\
&\quad (x,y,v) \rightarrow (-y,x+y,w), \\
&\quad (x,y,w) \rightarrow (-y-1,x+y,u),
\end{aligned} \tag{A2}
$$

together with time reversal T.

The symmetry group of a kagome lattice is defined by the following algebraic relations between its generators:

$$\boldsymbol{T}^2 = \boldsymbol{\sigma}^2 = (C_6)^6 = \boldsymbol{e},$$
$$g^{-1}\boldsymbol{T}^{-1}g\boldsymbol{T} = \boldsymbol{e}, \quad \forall \; g = T_{1,2}, \boldsymbol{\sigma}, C_6,$$
$$T_2^{-1}T_1^{-1}T_2T_1 = \boldsymbol{e},$$
$$\boldsymbol{\sigma}^{-1}T_1^{-1}\boldsymbol{\sigma}T_2 = \boldsymbol{e},$$
$$\boldsymbol{\sigma}^{-1}T_2^{-1}\boldsymbol{\sigma}T_1 = \boldsymbol{e}, \qquad \text{(A3)}$$
$$C_6^{-1}T_2^{-1}C_6T_1 = \boldsymbol{e},$$
$$C_6^{-1}T_2^{-1}T_1C_6T_2 = \boldsymbol{e},$$
$$\boldsymbol{\sigma}^{-1}C_6\boldsymbol{\sigma}C_6 = \boldsymbol{e},$$

where $\boldsymbol{e}$ stands for the identity element in the symmetry group. Therefore the consistent conditions for a generic $\mathbb{Z}_2$ PSG on a kagome lattice are written as

$$[G_T(i)]^2 = \eta_T \tau^0, \qquad \text{(A4)}$$

$$G_\sigma(\sigma(i))G_\sigma(i) = \eta_\sigma \tau^0, \qquad \text{(A5)}$$

$$G_{T_1}^\dagger(i)G_T^\dagger(i)G_{T_1}(i)G_T\big(T_1^{-1}(i)\big) = \eta_{T_1T}\tau^0, \qquad \text{(A6)}$$

$$G_{T_2}^\dagger(i)G_T^\dagger(i)G_{T_2}(i)G_T\big(T_2^{-1}(i)\big) = \eta_{T_2T}\tau^0, \qquad \text{(A7)}$$

$$G_\sigma^\dagger(i)G_T^\dagger(i)G_\sigma(i)G_T(\sigma^{-1}(i)) = \eta_{\sigma T}\tau^0, \qquad \text{(A8)}$$

$$G_{C_6}^\dagger(i)G_T^\dagger(i)G_{C_6}(i)G_T\big(C_6^{-1}(i)\big) = \eta_{C_6T}\tau^0, \qquad \text{(A9)}$$

$$G_{T_2}^\dagger\big(T_1^{-1}(i)\big)G_{T_1}^\dagger(i)G_{T_2}(i)G_{T_1}\big(T_2^{-1}(i)\big) = \eta_{12}\tau^0, \quad \text{(A10)}$$

$$G_{C_6}\big(C_6^{-1}(i)\big)G_{C_6}\big(C_6^{-2}(i)\big)G_{C_6}\big(C_6^3(i)\big)G_{C_6}\big(C_6^2(i)\big)$$
$$\times G_{C_6}\big(C_6^2(i)\big)G_{C_6}(C_6(i))G_{C_6}(i) = \eta_{C_6}\tau^0, \qquad \text{(A11)}$$

$$G_\sigma^\dagger\big(T_2^{-1}(i)\big)G_{T_2}^\dagger(i)G_\sigma(i)G_{T_1}(\sigma(i)) = \eta_{\sigma T_1}\tau^0, \quad \text{(A12)}$$

$$G_\sigma^\dagger\big(T_1^{-1}(i)\big)G_{T_1}^\dagger(i)G_\sigma(i)G_{T_2}(\sigma(i)) = \eta_{\sigma T_2}\tau^0, \quad \text{(A13)}$$

$$G_\sigma^\dagger(C_6(i))G_{C_6}(C_6(i))G_\sigma(i)G_{C_6}(\sigma(i)) = \eta_{\sigma C_6}\tau^0, \quad \text{(A14)}$$

$$G_{C_6}^\dagger\big(T_2^{-1}(i)\big)G_{T_2}^\dagger(i)G_{C_6}(i)G_{T_1}\big(C_6^{-1}(i)\big) = \eta_{C_6T_1}\tau^0, \quad \text{(A15)}$$

$$G_{C_6}^\dagger\big(T_2^{-1}T_1(i)\big)G_{T_2}^\dagger(T_1(i))G_{T_1}(T_1(i))$$
$$\times G_{C_6}(i)G_{T_2}\big(C_6^{-1}(i)\big) = \eta_{C_6T_2}\tau^0, \qquad \text{(A16)}$$

for any lattice site $i = (x,y,s)$. Here all η's are $\mathbb{Z}_2$ integers characterizing different SLs: different (gauge-inequivalent) choices of these $\mathbb{Z}_2$ integers (different $\mathbb{Z}_2$ PSGs) correspond to different $\mathbb{Z}_2$ SLs. Notice that under a local gauge transformation $W(i) \in SU(2)$ the PSG element $G_U(i)$ transforms as

$$G_U(i) \rightarrow W(i)G_U(i)W^\dagger(U^{-1}(i)). \qquad \text{(A17)}$$

APPENDIX B: CLASSIFICATION OF ALL $\mathbb{Z}_2$ SPIN LIQUIDS ON THE KAGOME LATTICE

1. Classification of $\mathbb{Z}_2$ algebraic PSGs on the kagome lattice

In this section we classify all possible $\mathbb{Z}_2$ spin liquids on a kagome lattice. Mathematically we need to find all gauge-inequivalent solutions of the algebraic conditions (A4)–(A15) for $\mathbb{Z}_2$ PSGs.

First, from condition (A10) we can always choose a proper gauge so that

$$G_{T_1}(x,y,s) = \eta_{12}^y\tau^0, \qquad G_{T_2}(x,y,s) \equiv \tau^0. \qquad \text{(B1)}$$

From (A12) and (A13) we can see that $G_\sigma(x,y,s) = \eta_{\sigma T_1}^y \eta_{\sigma T_2}^x \eta_{12}^{xy} g_\sigma(s)$. Condition (A5) further determines $\eta_{\sigma T_1} = \eta_{\sigma T_2}$ and therefore we have

$$G_\sigma(x,y,s) = \eta_{\sigma T_1}^{x+y}\eta_{12}^{xy}g_\sigma(s),$$

where the SU(2) matrices $g_\sigma(s)$ satisfy

$$g_\sigma(w)g_\sigma(v) = [g_\sigma(u)]^2 = \eta_\sigma\tau^0. \qquad \text{(B2)}$$

Notice that we can always choose a proper global $\mathbb{Z}_2$ gauge on $G_{T_1}(x,y,s)$ (which does not change the mean-field *Ansatz*) so that $\eta_{C_6T_2} = 1$ in (A16). From (A15) and (A16) it is straightforward to show that $G_{C_6}(x,y,u/v) = \eta_{C_6T_1}^{x+y}\eta_{12}^{xy+x(x+1)/2}g_{C_6}(u/v)$ and $G_{C_6}(x,y,w) = \eta_{C_6T_1}^{x+y}\eta_{12}^{x+y+xy+x(x+1)/2}g_{C_6}(w)$. It is condition (A14) that determines $\eta_{C_6T_1} = \eta_{\sigma T_1}\eta_{12}$, and finally we have

$$G_{C_6}(x,y,u/v) = \eta_{\sigma T_1}^{x+y}\eta_{12}^{xy+\frac{x(x+1)}{2}}g_{C_6}(u/v),$$

$$G_{C_6}(x,y,w) = (\eta_{12}\eta_{\sigma T_1})^{x+y}\eta_{12}^{xy+\frac{x(x+1)}{2}}g_{C_6}(w),$$

where the SU(2) matrices $g_{C_6}(s)$ satisfy

$$\big[g_{C_6}(w)g_{C_6}(v)g_{C_6}(u)\big]^2 = \eta_{12}\eta_{C_6}\tau^0, \qquad \text{(B3)}$$

$$\big[g_\sigma(v)g_{C_6}(w)\big]^2 = g_\sigma(w)g_{C_6}(v)g_\sigma(u)g_{C_6}(u) = \eta_\sigma\eta_{\sigma C_6}\tau^0 \qquad \text{(B4)}$$

according to (A11) and (A14).

Now through a gauge transformation $W(x,y,s) = \eta_{\sigma T_1}^y$ we can fix $\eta_{\sigma T_{1,2}} = 1$ and the PSG elements become

$$G_\sigma(x,y,s) = \eta_{12}^{xy}g_\sigma(s), \qquad \text{(B5)}$$

$$G_{C_6}(x,y,u/v) = \eta_{12}^{xy+\frac{x(x+1)}{2}}g_{C_6}(u/v),$$
$$G_{C_6}(x,y,w) = \eta_{12}^{xy+x+y+\frac{x(x+1)}{2}}g_{C_6}(w). \qquad \text{(B6)}$$

According to (A4), (A6), and (A7) we can see that $G_T(x,y,s) = \eta_{T_1T}^x\eta_{T_2T}^y g_T(s)$. (A9) and (A8) further determine $\eta_{T_1T} = \eta_{T_2T} = 1$ and by choosing a proper gauge we have

$$G_T(x,y,s) = g_T(s) \equiv \begin{cases} \tau^0, & \eta_T = 1, \\ i\tau^1, & \eta_T = -1, \end{cases} \qquad \text{(B7)}$$

which satisfy

$$g_\sigma(u)g_T(u) = \eta_{\sigma T}g_T(u)g_\sigma(u),$$
$$g_\sigma(v)g_T(w) = \eta_{\sigma T}g_T(v)g_\sigma(v), \qquad \text{(B8)}$$
$$g_\sigma(w)g_T(v) = \eta_{\sigma T}g_T(w)g_\sigma(w),$$
$$g_{C_6}(u)g_T(w) = \eta_{C_6T}g_T(u)g_{C_6}(u),$$
$$g_{C_6}(v)g_T(u) = \eta_{C_6T}g_T(v)g_{C_6}(v), \qquad \text{(B9)}$$
$$g_{C_6}(w)g_T(v) = \eta_{C_6T}g_T(w)g_{C_6}(w),$$

according to (A9) and (A8).

In the following we find all the gauge-inequivalent solutions of SU(2) matrices $g_{T,\sigma,C_6}(s)$ satisfying the above conditions. They are summarized in Table II.

(I) $g_T(s) = \tau^0$ and therefore $\eta_T = \eta_{\sigma T} = \eta_{C_6 T} = 1$. Conditions (B8) and (B9) are automatically satisfied.

(i) $\eta_\sigma = 1$. Notice that under a global gauge transformation $W(x,y,s) \equiv W_s \in SU(2)$ the PSG elements transform as

$$g_\sigma(u) \to W_u g_\sigma(u) W_u^\dagger,$$
$$g_\sigma(v) \to W_v g_\sigma(v) W_w^\dagger,$$
$$g_\sigma(w) \to W_w g_\sigma(w) W_v^\dagger,$$
$$g_{C_6}(u) \to W_u g_{C_6}(u) W_w^\dagger,$$
$$g_{C_6}(v) \to W_v g_{C_6}(v) W_u^\dagger,$$
$$g_{C_6}(w) \to W_w g_{C_6}(w) W_v^\dagger.$$

Thus from (B2) and (B4) we can always have $g_\sigma(s) = \tau^0$ and $g_{C_6}(u) = \tau^0$, $g_{C_6}(v) = \eta_{\sigma C_6}\tau^0$ by choosing a proper gauge.

(A) $\eta_{\sigma C_6} = \eta_{12}\eta_{C_6} = 1$. From (B3) we have $g_{C_6}(w) = \tau^0$.

(B) $\eta_{\sigma C_6} = \eta_{12}\eta_{C_6} = -1$. From (B3) we have $g_{C_6}(w) = i\tau^3$ by gauge fixing.

(ii) $\eta_\sigma = -1$. From (B2) we have $g_\sigma(v) = -g_\sigma(w) = \tau^0$ and $g_\sigma(u) = i\tau^3$ by gauge fixing. Also from (B4) we can choose a gauge so that $g_{C_6}(u) = \tau^0$ and $g_{C_6}(v) = -i\eta_{\sigma C_6}\tau^3$.

(A) $\eta_{\sigma C_6} = -1$. In this case (B4) requires $g_{C_6}(w) = \tau^0$ and thus $\eta_{12}\eta_{C_6} = -1$ according to (B3).

(B) $\eta_{\sigma C_6} = 1$.

(a) $\eta_{12}\eta_{C_6} = -1$. Now from (B4) and (B3) we have $g_{C_6}(w) = i\tau^1$ by gauge fixing.

(b) $\eta_{12}\eta_{C_6} = 1$. By (B4) and (B3) we must have $g_{C_6}(w) = i\tau^3$.

To summarize, there are $2 \times (2 + 3) = 10$ different algebraic PSGs with $\eta_T = 1$ and $g_T(s) = \tau^0$.

(II) $g_T(s) = i\tau^1$ and $\eta_T = -1$

(i) $\eta_\sigma = 1$. According to (B2) and (B8), by choosing a proper gauge we can have $g_\sigma(s) = \tau^0$ and $\eta_{\sigma T} = 1$. From (B3) and (B4) we also have $[g_{C_6}(w)]^2 = g_{C_6}(v)g_{C_6}(u) = \eta_{\sigma C_6}\tau^0 = \eta_{12}\eta_{C_6}\tau^0$.

(A) $\eta_{12}\eta_{C_6} = \eta_{\sigma C_6} = 1$. From (B9), (B3), and (B4), by choosing the gauge we have $g_{C_6}(s) = \tau^0$ and $\eta_{C_6 T} = 1$.

(B) $\eta_{12}\eta_{C_6} = \eta_{\sigma C_6} = -1$.

(a) $\eta_{C_6 T} = 1$. In this case we have $g_{C_6}(u) = -g_{C_6}(v) = \tau^0$ and $g_{C_6}(w) = i\tau^1$ by choosing a proper gauge.

(b) $\eta_{C_6 T} = -1$. In this case we can have $g_{C_6}(s) = i\tau^3$ by choosing a proper gauge.

(ii) $\eta_\sigma = -1$.

(A) $\eta_{\sigma T} = 1$. From (B8) and (B2) we have $g_\sigma(u) = i\tau^1$ and $g_\sigma(v) = -g_\sigma(w) = \tau^0$ by proper gauge fixing. Also from (B4) we know that $[g_{C_6}(w)]^2 = -\eta_{\sigma C_6}\tau^0$ and $g_{C_6}(u)g_{C_6}(v) = -i\eta_{\sigma C_6}\tau^1$.

(a) $\eta_{\sigma C_6} = -1$. From (B9), (B4), and (B3) it is clear that $\eta_{C_6 T} = 1$, $g_{C_6}(u) = g_{C_6}(w) = \tau^0$, and $g_{C_6}(v) = i\tau^1$ through gauge fixing. Also we have $\eta_{12}\eta_{C_6} = -1$.

(b) $\eta_{\sigma C_6} = 1$.

(b1) $\eta_{C_6 T} = 1$. In this case $\eta_{12}\eta_{C_6} = 1$, and we can always choose a proper gauge so that $g_{C_6}(u) = \tau^0$, $g_{C_6}(w) = -g_{C_6}(v) = i\tau^1$.

(b2) $\eta_{C_6 T} = -1$. In this case $\eta_{12}\eta_{C_6} = -1$, and we can always choose a proper gauge so that $g_{C_6}(v) = -i\tau^2$, $g_{C_6}(u) = g_{C_6}(w) = i\tau^3$.

(B) $\eta_{\sigma T} = -1$. Conditions (B8) and (B2) assert that $g_\sigma(s) = i\tau^3$ by proper choice of gauge.

(a) $\eta_{\sigma C_6} = -1$. In this case from (B4) we know that $g_{C_6}(w) = i\tau^3$; hence $\eta_{C_6 T} = -1$. Then we can always choose a gauge so that $g_{C_6}(u) = g_{C_6}(v) = i\tau^3$ and so $\eta_{12}\eta_{C_6} = -1$ from (B3).

(b) $\eta_{\sigma C_6} = 1$.

(b1) $\eta_{C_6 T} = 1$. In this case from (B8),(B4) we have $g_{C_6}(u) = g_{C_6}(v) = \tau^0$ by a proper gauge choice. Meanwhile, conditions (B3) and (B4) become $[g_{C_6}(w)]^2 = \eta_{12}\eta_{C_6}\tau^0$ and $[i\tau^3 g_{C_6}(w)]^2 = -\tau^0$.

(b.1.1) $\eta_{12}\eta_{C_6} = 1$. Here we have $g_{C_6}(w) = \tau^0$.

(b.1.2) $\eta_{12}\eta_{C_6} = -1$. Here we have $g_{C_6}(w) = i\tau^1$.

(b2) $\eta_{C_6 T} = -1$. In this case from (B8) and (B4) we can always choose a proper gauge so that $g_{C_6}(u) = -g_{C_6}(v) = i\tau^3$. We also have $g_{C_6}(w) = i\tau^2$ and $\eta_{12}\eta_{C_6} = -1$ from (B3).

To summarize there are $2 \times (3 + 7) = 20$ different algebraic PSGs with $\eta_T = -1$ and $g_T(s) = i\tau^1$.

So in summary we have $10 + 20 = 30$ different $\mathbb{Z}_2$ algebraic PSGs satisfying conditions (A4)–(A16). Among them there are at most 20 solutions that can be realized by a mean-field *Ansatz*, since those PSGs with $g_T(s) = \tau^0$ would require all mean-field bonds to vanish due to (B11). As a result there are 20 different $\mathbb{Z}_2$ spin liquids on a kagome lattice.

A. Symmetry conditions on the mean-field *Ansatz*

Let us denote the mean-field bonds connecting sites $(0,0,u)$ and (x,y,s) as $[x,y,s] \equiv \langle x,y,s|0,0,u\rangle$. Using (8) we can generate any other mean-field bonds through symmetry operations (such as translations $G_{T_{1,2}} T_{1,2}$ and mirror reflection $G_\sigma \sigma$) from $[x,y,s]$. However, these mean-field bonds cannot be chosen arbitrarily since they possess symmetry relation (8):

$$\langle i|j\rangle = G_U(i)\langle U^{-1}(i)|U^{-1}(j)\rangle G_U^\dagger(j), \qquad \text{(B10)}$$

where U is any element in the symmetry group. Notice that for time reversal $\boldsymbol{T}$ we have

$$G_T(i)\langle i|j\rangle G_T^\dagger(j) = -\langle i|j\rangle. \qquad \text{(B11)}$$

We summarize these symmetry conditions on the mean-field bonds here:

(i) For $s = u$,

$$\boldsymbol{T} : g_T[x,y,u]g_T^\dagger = -[x,y,u],$$
$$T_1^x T_2^{-x} \boldsymbol{\sigma} : [x, -x, u] \to [x, -x, u]^\dagger,$$
$$T_1^{x+1} T_2^{y+1} C_6^3 : [x,y,u] \to [x,y,u]^\dagger,$$
$$\boldsymbol{\sigma} : [x,x,u] \to [x,x,u].$$

(ii) For $s = v$,

$$\boldsymbol{T} : g_T[x,y,v]g_T^\dagger = [x,y,v],$$
$$T_2^{y+1} \boldsymbol{\sigma} C_6^2 : [0,y,v] \to [0,y,v]^\dagger,$$
$$T_1^{2-2y} T_2^{y-1} \boldsymbol{\sigma} C_6^{-1} : [1-2y,y,v] \to [1-2y,y,v]^\dagger.$$

(iii) For $s = w$,

$$\boldsymbol{T} : g_T[x,y,w]g_T^\dagger = [x,y,w],$$
$$T_1^{x-1} T_2^{2-2x} \boldsymbol{\sigma} C_6 : [x,1-2x,w] \to [x,1-2x,w]^\dagger,$$
$$T_1^{x+1} \boldsymbol{\sigma} C_6^{-2} : [x,0,w] \to [x,0,w]^\dagger.$$

Now let us consider several very simple examples. First, the on-site chemical potential terms $\Lambda(x,y,s) = \Lambda_s$ satisfy the following consistent conditions:

$$
\begin{aligned}
\tau^1 \Lambda_s \tau^1 &= -\Lambda_s, \\
g_\sigma(u) \Lambda_u g_\sigma^\dagger(u) &= \Lambda_u, \\
g_\sigma(v) \Lambda_w g_\sigma^\dagger(v) &= \Lambda_v, \\
g_\sigma(w) \Lambda_v g_\sigma^\dagger(w) &= \Lambda_w, \\
g_{C_6}(u) \Lambda_w g_{C_6}^\dagger(u) &= \Lambda_u, \\
g_{C_6}(v) \Lambda_u g_{C_6}^\dagger(v) &= \Lambda_v, \\
g_{C_6}(w) \Lambda_v g_{C_6}^\dagger(w) &= \Lambda_w.
\end{aligned} \tag{B12}
$$

In fact in all 20 $\mathbb{Z}_2$ spins on a kagome lattice we have $\Lambda_u = \Lambda_v = \Lambda_w \equiv \Lambda_s$ with a proper gauge choice.

All the first-NN mean-field bonds can be generated from $u_\alpha \equiv [0,0,v]$. For a generic $\mathbb{Z}_2$ spin liquid with PSG elements $G_T(x,y,s) = i\tau^1$ and (B1), (B5), and (B7), the bond $u_\alpha = [0,0,v]$ satisfies the following consistent conditions:

$$
\begin{aligned}
\tau^1 u_\alpha \tau^1 &= -u_\alpha, \\
g_\sigma(u) g_{C_6}(u) g_{C_6}(w) u_\alpha g_{C_6}^\dagger(v) g_{C_6}^\dagger(w) g_\sigma^\dagger(v) &= u_\alpha^\dagger.
\end{aligned} \tag{B13}
$$

It follows immediately that for six $\mathbb{Z}_2$ spin liquids, i.e., nos. 7–12 in Table II, all NN mean-field bonds must vanish since $u_\alpha = 0$ as required by (B13). Therefore it is unlikely that the $\mathbb{Z}_2$ spin liquid realized in the kagome Hubbard model will be one of these six states. In the following we study the remaining 14 $\mathbb{Z}_2$ spin liquids on the kagome lattice.

All second-NN mean-field bonds can be generated from $u_\beta \equiv [0,1,w]$, which satisfies the following symmetry conditions:

$$
\begin{aligned}
\tau^1 u_\beta \tau^1 &= -u_\beta, \\
g_\sigma(u) g_{C_6}(u) u_\beta g_{C_6}^\dagger(v) g_\sigma^\dagger(w) &= u_\beta^\dagger.
\end{aligned} \tag{B14}
$$

There are two kinds of third-NN mean-field bond. The first kind can all be generated by $u_\gamma \equiv [1,0,u]$, which satisfies

$$
\begin{aligned}
\tau^1 u_\gamma \tau^1 &= -u_\gamma, \\
g_{C_6}(u) g_{C_6}(v) g_{C_6}(w) u_\gamma \big[g_{C_6}(u) g_{C_6}(v) g_{C_6}(w) \big]^\dagger &= u_\gamma^\dagger.
\end{aligned} \tag{B15}
$$

The second kind can all be generated by $\tilde{u}_\gamma \equiv [1,-1,u]$, which satisfies

$$
\begin{aligned}
\tau^1 \tilde{u}_\gamma \tau^1 &= -\tilde{u}_\gamma, \\
g_\sigma(u) \tilde{u}_\gamma g_\sigma^\dagger(u) &= \tilde{u}_\gamma^\dagger,
\end{aligned} \tag{B16}
$$

$$
g_{C_6}(u) g_{C_6}(w) g_{C_6}(v) \tilde{u}_\gamma \big[g_{C_6}(u) g_{C_6}(w) g_{C_6}(v) \big]^\dagger = \eta_{12} \tilde{u}_\gamma^\dagger.
$$

APPENDIX C: $\mathbb{Z}_2$ SPIN LIQUIDS IN THE NEIGHBORHOOD OF THE U(1) SL $[0,\pi]$ STATE

1. Mean-field *Ansatz* of U(1) SL $[0,\pi]$ state

Following the SU(2) Schwinger fermion formulation with $\psi_i \equiv (f_{i\uparrow}, f_{i\downarrow}^\dagger)^T$, we focus on those $\mathbb{Z}_2$ spin liquids (SLs) in the neighborhood of the U(1) SL $[0,\pi]$ state with the following mean-field *Ansätze*:

$$
\langle x,y,u | x,y,v \rangle = -\langle x,y,u | x,y,w \rangle = (-1)^x \chi \tau^3,
$$

$$
\begin{aligned}
\langle x+1,y,w | x,y,u \rangle &= \langle x,y+1,v | x,y,u \rangle \\
&= -\langle x,y,v | x,y,w \rangle \\
&= \langle x+1,y-1,w | x,y,v \rangle = \chi \tau^3,
\end{aligned} \tag{C1}
$$

where χ is a real hopping parameter. We define mean-field bonds $\langle x,y,s | x',y',s' \rangle$ in the following way:

$$
H_{MF} = \sum_{i,j} \psi_i^\dagger \langle i | j \rangle \psi_j + \text{H.c.} \tag{C2}
$$

For convenience of later calculation we implement the following gauge transformation:

$$
\psi_{x,y,u} \to i\tau^3 \psi_{x,y,u} \tag{C3}
$$

and the original mean-field *Ansatz* (C1) transforms to

$$
\begin{aligned}
\langle x,y,u | x,y,v \rangle &= -\langle x,y,u | x,y,w \rangle = i(-1)^x \chi \tau^0, \\
\langle x+1,y,w | x,y,u \rangle &= \langle x,y+1,v | x,y,u \rangle = -i\chi \tau^0, \\
-\langle x,y,v | x,y,w \rangle &= \langle x+1,y-1,w | x,y,v \rangle = \chi \tau^3.
\end{aligned} \tag{C4}
$$

The projected symmetry group corresponding to the above mean-field *Ansatz* (C4) is

$$
\begin{aligned}
G_T(x,y,v) &= G_T(x,y,w) = -G_T(x,y,u) = g_T, \\
&\quad g_T \tau^3 g_T^\dagger = -\tau^3, \\
G_{T_2}(x,y,s) &= g_{T_2}, \qquad g_{T_2} \tau^3 g_{T_2}^\dagger = \tau_3, \\
G_{T_1}(x,y,v) &= G_{T_1}(x,y,w) = -G_{T_1}(x,y,u) \\
&= (-1)^{x+y} g_{T_1}, \qquad g_{T_1} \tau^3 g_{T_1}^\dagger = \tau_3, \\
G_\sigma(x,y,v) &= G_\sigma(x,y,w) = (-1)^{x+y+1} G_\sigma(x,y,u) \\
&= (-1)^{(x+y)(x+y+1)/2} g_\sigma, \quad g_\sigma \tau^3 g_\sigma^\dagger = \tau^3, \\
G_{C_6}(x,y,u) &= (-1)^{\frac{x(x+1)+y(y-1)}{2}} g_{C_6}, \\
G_{C_6}(x,y,v) &= -(-1)^{\frac{x(x-1)+y(y-1)}{2}} g_{C_6}, \\
G_{C_6}(x,y,w) &= i(-1)^{\frac{x(x-1)+y(y-1)}{2}} g_{C_6} \tau^3, \\
&\quad g_{C_6} \tau^3 g_{C_6}^\dagger = \tau^3,
\end{aligned} \tag{C5}
$$

so that the mean-field *Ansätze* satisfy (8).

2. Classification of $\mathbb{Z}_2$ spin liquids around the U(1) SL $[0,\pi]$ state

Plugging (C5) into the algebraic consistency conditions (A4)–(A15), we obtain four algebraic solutions of $\mathbb{Z}_2$ PSGs around the U(1) SL $[0,\pi]$ state. With choice of a proper gauge they all satisfy

$$
\begin{aligned}
g_T &= i\tau_1, \ g_{T_1} = g_{T_2} = \tau^0, \\
\eta_T &= \eta_{12} = \eta_{C_6 T_1} = -1, \\
\eta_{T_{1,2}T} &= \eta_{\sigma T_{1,2}} = \eta_{C_6 T_2} = 1.
\end{aligned} \tag{C6}
$$

The four $\mathbb{Z}_2$ PSGs near the U(1) SL $[0,\pi]$ state have the following features:

$$
\begin{aligned}
&\text{(No. 6) } Z_2[0,\pi]\alpha: \qquad g_\sigma = g_{C_6} = \tau^0, \\
&\qquad \eta_\sigma = \eta_{\sigma T} = 1, \\
&\qquad \eta_{\sigma C_6} = \eta_{C_6 T} = -\eta_{C_6} = -1;
\end{aligned} \tag{C7}
$$

$$
\text{(No. 2) } Z_2[0,\pi]\beta: \ g_\sigma = \tau^0, \ g_{C_6} = i\tau^3,
$$

$$\eta_\sigma = \eta_{\sigma T} = 1, \tag{C8}$$

$$\eta_{\sigma C_6} = \eta_{C_6 T} = -\eta_{C_6} = 1;$$

(No. 14) $Z_2[0,\pi]\gamma$: $g_\sigma = i\tau^3$, $g_{C_6} = \tau^0$,

$$\eta_\sigma = \eta_{\sigma T} = -1, \tag{C9}$$

$$\eta_{\sigma C_6} = \eta_{C_6 T} = -\eta_{C_6} = -1;$$

(No. 16) $Z_2[0,\pi]\delta$: $g_\sigma = g_{C_6} = i\tau^3$

$$\eta_\sigma = \eta_{\sigma T} = -1, \tag{C10}$$

$$\eta_{\sigma C_6} = \eta_{C_6 T} = -\eta_{C_6} = 1.$$

Of course they belong to the 20 $\mathbb{Z}_2$ spin liquids summarized in Table II.

3. Four possible $\mathbb{Z}_2$ spin liquids around the U(1) SL $[0,\pi]$ state: Mean-field *Ansätze*

a. Consistent conditions on mean-field bonds

Implementing the generic conditions mentioned earlier on several near-neighbor mean-field bonds with PSG (C6)–(C10), we obtain the following consistent conditions:

(0) For on-site chemical potential terms $\Lambda_s(x,y,s) = \vec{\lambda}(x,y,s) \cdot \vec{\tau}$, the translation operations $G_{T_{1,2}} T_{1,2}$ in the PSG guarantee that $\Lambda_s(x,y,s) = \lambda_s(0,0,s) \equiv \Lambda_s, s = u,v,w$. They satisfy

$$g_T \Lambda_s g_T^\dagger = -\Lambda_s,$$

$$g_\sigma \Lambda_u g_\sigma^\dagger = \Lambda_u, \quad g_\sigma \Lambda_v g_\sigma^\dagger = \Lambda_w, \quad g_\sigma \Lambda_w g_\sigma^\dagger = \Lambda_v;$$
$$g_{C_6} \Lambda_u g_{C_6}^\dagger = \Lambda_v, \quad (g_{C_6}\tau^3)\Lambda_v(g_{C_6}\tau^3)^\dagger = \Lambda_w, \tag{C11}$$
$$g_{C_6} \Lambda_w g_{C_6}^\dagger = \Lambda_u.$$

(I) For the first-neighbor mean-field bond $u_a \equiv [0,0,v]^\dagger$ (there is only one independent mean-field bond, meaning all other first-neighbor bonds can be generated from $[0,0,v]$ through symmetry operations),

$$g_T u_a^\dagger g_T^\dagger = u_a^\dagger,$$
$$(g_\sigma g_{C_6}^2 \tau^3) u_a^\dagger (g_\sigma g_{C_6}^2 \tau^3)^\dagger = -u_a. \tag{C12}$$

(II) For the second-neighbor mean-field bond $u_b \equiv [0,1,w]$ we have

$$g_T u_b g_T^\dagger = u_b,$$
$$(g_\sigma g_{C_6}) u_b (g_\sigma g_{C_6})^\dagger = -u_b^\dagger. \tag{C13}$$

(II) For third-neighbor mean-field bonds $u_{c1} \equiv [1,0,u]$ and $u_{c2} \equiv [1,-1,u]$ we have

$$g_T u_{c1} g_T^\dagger = -u_{c1},$$
$$(g_{C_6}^3 \tau^3) u_{c1} (g_{C_6}^3 \tau^3)^\dagger = u_{c1}^\dagger \tag{C14}$$

and

$$g_T u_{c2} g_T^\dagger = -u_{c2},$$
$$g_\sigma u_{c2} g_\sigma^\dagger = u_{c2}^\dagger, \tag{C15}$$
$$(g_{C_6}^3 \tau^3) u_{c2} (g_{C_6}^3 \tau^3)^\dagger = -u_{c2}^\dagger.$$

b. Mean-field Ansätze of the four $\mathbb{Z}_2$ spin liquids near the U(1) SL $[0,\pi]$ state

For the $Z_2[0,\pi]\alpha$ state with $g_\sigma = g_{C_6} = \tau^0$ the mean-field *Ansätze* are (up to third-neighbor mean-field bonds)

$$u_a = ia_0\tau^0 + a_1\tau^1, \quad u_b = ib_0\tau^0,$$
$$u_{c1} = c_3\tau^3, \quad u_{c2} = c_2\tau^2, \tag{C16}$$
$$\Lambda_s = \lambda_3\tau^3, \quad s = u,v,w.$$

Since we are considering a phase perturbed from the U(1) SL $[0,\pi]$ state, we shall always assume $a_0 \neq 0$ (first-neighbor hopping terms) in the following discussion. A $Z_2[0,\pi]\alpha$ spin liquid can be realized by first-neighbor mean-field singlet pairing terms with $a_1 \neq 0$.

For the $\mathbb{Z}_2[0,\pi]\beta$ state with $g_\sigma = \tau^0, g_{C_6} = i\tau^3$ the mean-field *Ansätze* are (up to third-neighbor mean-field bonds)

$$u_a = ia_0\tau^0 + a_1\tau^1, \quad u_b = ib_0\tau^0 + b_1\tau^1,$$
$$u_{c1} = c_2\tau^2 + c_3\tau^3, \quad u_{c2} = 0, \tag{C17}$$
$$\Lambda_u = \lambda_2\tau^2 + \lambda_3\tau^3, \quad \Lambda_{v,w} = -\lambda_2\tau^2 + \lambda_3\tau^3.$$

A $\mathbb{Z}_2[0,\pi]\beta$ spin liquid can be realized by second-neighbor pairing terms with $a_0b_1 - a_1b_0 \neq 0$.

For the $Z_2[0,\pi]\gamma$ state with $g_\sigma = i\tau^3, g_{C_6} = \tau^0$ the mean-field *Ansätze* are (up to third-neighbor mean-field bonds)

$$u_a = ia_0\tau^0, \quad u_b = ib_0\tau^0 + b_1\tau^1,$$
$$u_{c1} = c_3\tau^3, \quad u_{c2} = 0, \tag{C18}$$
$$\Lambda_s = \lambda_3\tau^3, \quad s = u,v,w.$$

A $Z_2[0,\pi]\gamma$ spin liquid can be realized by second-neighbor pairing terms with $b_1 \neq 0$.

For the $Z_2[0,\pi]\delta$ state with $g_\sigma = g_{C_6} = i\tau^3$, the mean-field *Ansätze* are (up to third-neighbor mean-field bonds)

$$u_a = ia_0\tau^0, \quad u_b = ib_0\tau^0,$$
$$u_{c1} = c_2\tau^2 + c_3\tau^3, \quad u_{c2} = 0, \tag{C19}$$
$$\Lambda_s = \lambda_3\tau^3, \quad s = u,v,w.$$

A $Z_2[0,\pi]\delta$ spin liquid can be realized by third-neighbor pairing terms with $c_2 \neq 0$.

4. Low-energy effective theory

The reciprocal unit vectors (corresponding to unit vectors $\vec{a}_{1,2}$) on a kagome lattice are $\vec{b}_1 = \frac{1}{a}(\hat{x} - \frac{1}{\sqrt{3}}\hat{y})$ and $\vec{b}_2 = \frac{1}{a}\frac{2}{\sqrt{3}}\hat{y}$, satisfying $\vec{a}_i \cdot \vec{b}_j = \delta_{i,j}$. In the mean-field *Ansatz* (C4) of U(1) SL $[0,\pi]$ the unit cell is doubled; the translation unit vectors are $\vec{A}_1 = 2\vec{a}_1$ and $\vec{A}_2 = \vec{a}_2$. Accordingly, the first BZ for such a mean-field *Ansatz* is only half of the original first BZ with the new reciprocal unit vectors being $\vec{B}_1 = \vec{b}_1/2$ and $\vec{B}_2 = \vec{b}_2$. Denoting the momentum as $\mathbf{k} \equiv (k_x,k_y)/a = k_1\vec{B}_1 + k_2\vec{B}_2$ with $|k_{1,2}| \leqslant \pi$, we have

$$k_1 = 2k_x, \quad k_2 = (k_x + \sqrt{3}k_y)/2. \tag{C20}$$

The two Dirac cones in the spectra of the U(1) SL $[0,\pi]$ state (C4) are located at $\pm\mathbf{Q}$ with

$$\mathbf{Q} = \left(0, \frac{\pi}{\sqrt{3}}\right) = \frac{\pi}{2}\vec{B}_2 \tag{C21}$$

with the proper chemical potential $\Lambda(i) = \langle i|i \rangle = \chi(\sqrt{3} - 1)\tau^3$ added to the mean-field *Ansatz* (C4).

For convenience we choose the following basis for the Dirac-like Hamiltonian obtained from expansion around $\pm \mathbf{Q}$:

$$
\phi_{+,\uparrow,A} = \frac{1}{\sqrt{6}} e^{-i\frac{1}{24}\pi}
$$
$$
\times \left(e^{-i\frac{11}{12}\pi}, 0, e^{i\frac{11}{12}\pi}, 0, 0, 0, e^{-i\frac{11}{12}\pi}, 0, e^{i\frac{5}{12}\pi}, 0, \sqrt{2}, 0 \right)^T,
$$
$$
\phi_{+,\uparrow,B} = \frac{1}{\sqrt{6}} e^{-i\frac{1}{24}\pi}
$$
$$
\times \left(1, 0, e^{-i\frac{4}{3}\pi}, 0, \sqrt{2}e^{-i\frac{11}{12}\pi}, 0, -1, 0, e^{-i\frac{5}{6}\pi}, 0, 0, 0 \right)^T,
$$
$$
\phi_{-,\uparrow,b} = R_{T_1}\left(k_1 = 0, k_2 = -\frac{\pi}{2} \right)\phi_{+,\uparrow,b},
$$
$$
\phi_{\pm,\downarrow,b} = R_T \phi_{\pm,\uparrow,b}, \tag{C22}
$$

where $\pm$ are the valley indices for two Dirac cones at $\pm \mathbf{Q}$ with Pauli matrices $\boldsymbol{\mu}$ and $b = A, B$ are band indices with Pauli matrices $\boldsymbol{\nu}$. The spin indices $\Sigma = \uparrow, \downarrow$ are as usual, with Pauli matrices $\boldsymbol{\sigma}$. The corresponding creation operators for these modes are $\Psi_{\pm,\Sigma,b}^{\dagger} = \psi_{\pm\mathbf{Q}}^{\dagger}\phi_{\pm,\Sigma,b}$ in the order of $(0,0,u),(0,0,v),(0,0,w),(1,0,u),(1,0,v),(1,0,w)$ for the six sites per doubled new unit cell.

Here

$$
R_T \equiv I_{2\times 2} \otimes \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes g_T,
$$
$$
R_{T_2}(\mathbf{k}) = e^{-ik_2} I_{6\times 6} \otimes g_{T_2},
$$

and

$$
R_{T_1}(\mathbf{k}) = \begin{bmatrix} 0 & -e^{-ik_1} \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \otimes g_{T_1}
$$

are transformation matrices on 12-component eigenvectors for time reversal $\boldsymbol{T}$ and translation $T_{1,2}$ operations. By definition of the PSG the eigenvectors $\phi_{\mathbf{k}}$ with momentum $\mathbf{k} = k_1 \vec{B}_1 + k_2 \vec{B}_2 \equiv (k_1, k_2)$ and energy E have the following symmetric properties:

$$
\boldsymbol{T}: \tilde{\phi}_{(k_1,k_2)}(E) = R_T \phi_{(k_1,k_2)}(-E),
$$
$$
T_1: \tilde{\phi}_{(k_1,k_2)}(E) = R_{T_1}(k_1,k_2)\phi_{(k_1,k_2+\pi)}(E),
$$
$$
T_2: \tilde{\phi}_{(k_1,k_2)}(E) = R_{T_2}(k_1,k_2)\phi_{(k_1,k_2)}(E).
$$

$\tilde{\phi}$ and ϕ are the bases after and before the symmetry operations.

In such a set of bases the Dirac Hamiltonian obtained by expanding the U(1) SL $[0,\pi]$ mean-field *Ansatz* (C4) around the two cones at $\pm \mathbf{Q}$ is

$$
H_{\text{Dirac}} = \sum_{\mathbf{k}} \frac{\chi}{\sqrt{2}} \Psi_{\mathbf{k}}^{\dagger} \mu^0 \sigma^3 (-k_x \nu^1 + k_y \nu^2) \Psi_{\mathbf{k}}. \tag{C23}
$$

$\mathbf{k}$ should be understood as small momenta measured from $\pm \mathbf{Q}$. Possible mass terms are $\mu^{0,1,2,3}\sigma^{1,2}\nu^0$ and $\mu^{0,1,2,3}\sigma^{0,3}\nu^3$. However, not all of them are allowed by symmetry. Here we enumerate all symmetry operations and associated operator transformations:

spin rotation along the $\hat{z}$ axis by angle θ:

$$
\Psi_{\mathbf{k}}^{\dagger} \to \Psi_{\mathbf{k}}^{\dagger} e^{i\frac{\theta}{2}};
$$

spin rotation along the $\hat{y}$ axis by π:

$$
\Psi_{\mathbf{k}}^{\dagger} \to \Psi_{-\mathbf{k}}^{T} \mu^2 \sigma^2 \nu^2;
$$

time reversal $\boldsymbol{T}$:

$$
\Psi_{\mathbf{k}}^{\dagger} \to \Psi_{\mathbf{k}}^{\dagger}(-i\sigma^2);
$$

translation T_1:

$$
\Psi_{\mathbf{k}}^{\dagger} \to \Psi_{\mathbf{k}}^{\dagger}(-\mu^3);
$$

translation T_2:

$$
\Psi_{\mathbf{k}}^{\dagger} \to \Psi_{\mathbf{k}}^{\dagger}(-i\mu^3).
$$

Considering the above conditions, the only symmetry-allowed mass terms are $\sum_{\mathbf{k}} \Psi_{\mathbf{k}}^{\dagger} m_{1,2} \Psi_{\mathbf{k}}$ with $m_1 = \mu^0 \sigma^1 \nu^0$ and $m_2 = \mu^3 \sigma^3 \nu^3$.

The transformation rules for mirror reflection $\boldsymbol{\sigma}$ and $\pi/3$ rotation C_6 depend on the choice of g_{σ}, g_{C_6} in the PSG. In general we have

$$
\boldsymbol{\sigma}: \Psi_{\mathbf{k}}^{\dagger} \to \Psi_{\sigma\mathbf{k}}^{\dagger} M_{\sigma}(g_{\sigma}),
$$
$$
C_6: \Psi_{\mathbf{k}}^{\dagger} \to \Psi_{C_6\mathbf{k}}^{\dagger} M_{C_6}(g_{C_6}).
$$

Using the basis (C22) the 8×8 matrices M_{σ,C_6} can be expressed in terms of Pauli matrices $\boldsymbol{\mu} \otimes \boldsymbol{\sigma} \otimes \boldsymbol{\nu}$. For the four $\mathbb{Z}_2$ spin liquids we have

$$
M_{\sigma}(g_{\sigma} = \tau^0) = \mu^3 \otimes \sigma^0 \otimes \begin{pmatrix} 0 & e^{-i\frac{1}{12}\pi} \\ e^{-i\frac{5}{12}\pi} & 0 \end{pmatrix},
$$
$$
M_{\sigma}(g_{\sigma} = i\tau^3) = \mu^3 \otimes \sigma^3 \otimes \begin{pmatrix} 0 & e^{i\frac{5}{12}\pi} \\ e^{i\frac{1}{12}\pi} & 0 \end{pmatrix},
$$
$$
M_{C_6}(g_{C_6} = \tau^0) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \otimes \sigma^0 \otimes e^{i\frac{7}{6}\pi \nu^3},
$$
$$
M_{C_6}(g_{C_6} = i\tau^3) = \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} \otimes \sigma^0 \otimes e^{i\frac{1}{6}\pi \nu^3}.
$$

It turns out that in the $\mathbb{Z}_2[0,\pi]\beta$ state, only the first mass term $m_1 = \mu^0 \sigma^1 \nu^0$ is invariant under $\boldsymbol{\sigma}$ and C_6 operations. In the other three states neither of the mass terms $m_{1,2}$ is symmetry allowed. As a result we only have one gapped $\mathbb{Z}_2$ spin liquid, i.e., the $\mathbb{Z}_2[0,\pi]\beta$ state in the neighborhood of the U(1) Dirac SL $[0,\pi]$ state.

Let us consider mean-field bonds up to second neighbor for the *Ansatz* $\mathbb{Z}_2 [0,\pi]\beta$. Perturbations to the two Dirac cones of U(1) SL $[0,\pi]$ with $\lambda_3 = (\sqrt{3} - 1)a_0$ in general have the following form:

$$
\delta H_0 = [\lambda_3 - (\sqrt{3} - 1)a_0 - (\sqrt{3} + 1)b_0]\mu^0 \sigma^3 \nu^0
$$
$$
+ [(\sqrt{3} + 1)b_1 - \lambda_2 - (\sqrt{3} - 1)a_1]\mu^0 \sigma^1 \nu^0. \tag{C24}
$$

This means that we need either a first-neighbor (a_1) or a second-neighbor (b_1) pairing term to open up a gap in the spectrum. Meanwhile these pairing terms break the original U(1) symmetry down to $\mathbb{Z}_2$ symmetry.

APPENDIX D: $\mathbb{Z}_2$ SPIN LIQUIDS IN THE NEIGHBORHOOD OF THE UNIFORM RVB STATE

The mean-field *Ansatz* of the uniform RVB state is simple:

$$H_{MF} = \chi \sum_{\langle ij \rangle, \sigma} f^\dagger_{i,\sigma} f_{j,\sigma}, \tag{D1}$$

where χ is a real parameter and $\langle ij \rangle$ represents sites i, j that are nearest neighbors of each other. It is straightforward to show that the PSG elements of such a mean-field *Ansatz* are

$$G_U(x, y, s) = g_U, \quad U = T_{1,2}, \boldsymbol{T}, \boldsymbol{\sigma}, C_6 \tag{D2}$$

and the SU(2) matrices g_U satisfy

$$\begin{aligned} g_{\boldsymbol{T}} \tau^3 g^\dagger_{\boldsymbol{T}} &= -\tau^3, \\ g_U \tau^3 g^\dagger_U &= \tau^3, \quad U = T_{1,2}, \boldsymbol{\sigma}, C_6. \end{aligned} \tag{D3}$$

It turns out that there are only four gauge-inequivalent $\mathbb{Z}_2$ PSGs as solutions to (A4)–(A16) with the form (D2). In other words, there are only four different $\mathbb{Z}_2$ in the neighborhood of a uniform RVB state. With choice of a proper gauge they all satisfy $g_{\boldsymbol{T}} = i\tau^1$, $g_{T_{1,2}} = \tau^0$, and $\eta_{T_{1,2}\boldsymbol{T}} = \eta_{12} = \eta_{C_6 T_{1,2}} = \eta_{\boldsymbol{\sigma} T_{1,2}} = 1, \eta_{\boldsymbol{T}} = -1$. These four states are characterized by

$$\text{(No. 1)} \quad Z_2[0,0]A: \ g_{\boldsymbol{\sigma}} = g_{C_6} = \tau^0,$$
$$\eta_{\boldsymbol{\sigma} \boldsymbol{T}} = \eta_{C_6 \boldsymbol{T}} = \eta_{\boldsymbol{\sigma}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma} C_6} = 1. \tag{D4}$$

$$\text{(No. 5)} \quad Z_2[0,0]B: \ g_{\boldsymbol{\sigma}} = \tau^0, \ g_{C_6} = i\tau^3,$$
$$\eta_{\boldsymbol{\sigma} \boldsymbol{T}} = \eta_{\boldsymbol{\sigma}} = 1, \ \eta_{C_6 \boldsymbol{T}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma} C_6} = -1. \tag{D5}$$

$$\text{(No. 15)} \quad Z_2[0,0]C: \ g_{\boldsymbol{\sigma}} = i\tau^3, \ g_{C_6} = \tau^0,$$
$$\eta_{\boldsymbol{\sigma} \boldsymbol{T}} = \eta_{\boldsymbol{\sigma}} = -1, \ \eta_{C_6 \boldsymbol{T}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma} C_6} = 1. \tag{D6}$$

$$\text{(No. 13)} \quad Z_2[0,0]D: \ g_{\boldsymbol{\sigma}} = g_{C_6} = i\tau^3,$$
$$\eta_{\boldsymbol{\sigma} \boldsymbol{T}} = \eta_{C_6 \boldsymbol{T}} = \eta_{\boldsymbol{\sigma}} = \eta_{C_6} = \eta_{\boldsymbol{\sigma} C_6} = -1. \tag{D7}$$

It turns out that these four $\mathbb{Z}_2$ SLs around the uniform RVB state are all gapped as shown in Table II.

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