Floquet theory of Cooper pair pumping

Angelo Russomanno,[1] Stefano Pugnetti,[1] Valentina Brosco,[2] and Rosario Fazio[1]
[1]*NEST, Scuola Normale Superiore and Istituto Nanoscienze – CNR, 56126 Pisa, Italy*
[2]*ISC – CNR and Dipartimento di Fisica, Universitá La Sapienza, P.le A. Moro 2, 00185 Roma, Italy*
(Received 24 December 2010; revised manuscript received 18 March 2011; published 6 June 2011)

We derive a general formula for the charge pumped in a superconducting nanocircuit. Our expression generalizes previous results in several ways; it is applicable in both the adiabatic and in the nonadiabatic regimes and it takes into account also the effect of an external environment. More specifically, by applying Floquet theory to Cooper pair pumping, we show that under a cyclic evolution the total charge transferred through the circuit is proportional to the derivative of the associated Floquet quasi-energy with respect to the superconducting phase difference. In the presence of an external environment the expression for the transferred charge acquires a transparent form in the Floquet representation. It is given by the weighted sum of the charge transferred in each Floquet state, the weights being the diagonal components of the stationary density matrix of the system expressed in the Floquet basis. To test the power of this formulation we apply it to the study of pumping in a Cooper pair sluice. We reproduce the known results in the adiabatic regime and we show new data in the nonadiabatic case.

DOI: 10.1103/PhysRevB.83.214508 PACS number(s): 85.25.Cp, 03.65.Yz, 03.65.Vf

I. INTRODUCTION

In a mesoscopic conductor a dc charge current can be obtained, in the absence of applied voltages, by cycling in time two (or more) external parameters, e.g., gate voltages or magnetic fluxes, which govern the transport properties of the system.[1] Adiabatic charge pumping refers to the regime when the variation of the external parameters is slow compared with the characteristic time scale of the system.

In the scattering approach to quantum transport the pumped charge in an adiabatic cycle can be expressed in terms of derivatives of the scattering amplitudes with respect to the pumping parameters.[2] In the opposite regime of a Coulomb blockade, with several metallic islands connected to each other by small tunnel junctions, a periodic modulation of the externally applied gate voltages leads to a periodic lifting of the Coulomb blockade, thus enabling the transfer of exactly one electron per period through the device. Experimental evidence for parametric charge pumping in normal metallic systems in the regime of a Coulomb blockade has been obtained in Refs. 3 and 4. Over the last decades charge pumping has attracted the interest of many research groups working on very different aspects of this phenomenon, ranging from its metrological applications to its intimate relation with the fundaments of quantum theory (see Ref. 5 and references therein).

Originally motivated by the aim of achieving quantized charge pumping in the gigahertz range, a great deal of attention has been devoted in the last two decades to superconducting systems. The first experiment in this context, performed by Geerligs *et al.*,[6] showed that the degree of quantization of the pumped charge was not as good as in the normal case. As later discussed by Pekola *et al.*,[7] the main source of inaccuracies is related to the overall coherence of the superconducting system. This apparent disadvantage (appropriate designs of the superconducting circuit may overcome this difficulty), however, turned out to be a precious source for the investigation of fundamental properties of quantum theory in macroscopic systems.

If only superconducting leads are present, at a low-enough temperature, pumping is due to the adiabatic transport of Cooper pairs. Besides the dependence of the pumped charge on the details of the cycle, in superconducting pumps there is an additional dependence on the superconducting phase difference since the overall process is coherent. Cooper pair pumping has been thoroughly investigated[7–16] in the last decades. In a series of experiments the Helsinki group[17,18] has shown the coherent properties of Cooper pair pumping, and, very importantly, it has provided the first experimental demonstration of the relation between Cooper pair pumping and the Berry phase acquired by the system during its cyclic evolution. A connection between Berry phase and pumped charge in superconducting nanocircuits has been already established theoretically in Refs. 9, 12, and 13 (see also Refs. 19 and 20, where this relation was found for mesoscopic normal conductors).

Berry phases in macroscopic systems such as superconducting circuits have been studied in Refs. 13 and 21–23 and very recently experimentally demonstrated by the Zurich group.[24] The large body of theoretical understanding and the spectacular experimental control which leads to unveiling the coherent properties of pumping and its relation to geometric phases are important steps toward the implementation of geometric quantum computation[25,26] with superconducting devices.

While a lot has been found concerning the relation between geometric phases and pumping in closed quantum systems, the role of an external environment constitutes, with the notable exception of a few very recent papers,[27–29] an almost unexplored territory. The study of geometric phases in the presence of decoherence and dissipation has started only recently, though with a few exceptions, certainly prompted by the interest in quantum computation (see for example Ref. 30). Together with many features common in the theory of open quantum systems, the analysis of decoherence in geometric interferometry raises several distinct issues that are of interest as fundamental questions in both quantum mechanics and in quantum computation. The adiabatic evolution, for example,

cannot occur arbitrarily slow, as decoherence would destroy any interference. This implies that the decoherence processes should be analyzed in close connection with nonadiabatic corrections.[31]

Because Cooper pair pumping is a geometric quantum effect, it is natural to ask oneself to which extent an external environment modifies its characteristics. Not only is this question relevant for a detailed comparison with experimental data where an external bath is unavoidably present, but also it may shed additional light on the role of dissipation on geometric quantum phenomena. It is not *a priori* obvious, for example, that a relation between pumping and Berry phases (of any sort) will survive in an open system. As already mentioned, till now this problem was tackled in Refs. 27–29, where a generalized master equation to consistently account for the combined action of the driving and dissipation was derived in the adiabatic limit. Application to the Cooper pair sluice[18] showed that in the zero temperature limit the ground-state dynamics and consequently pumping are not affected by the environment.

Stimulated by the results obtained in Refs. 27–29, in this paper we further investigate the relation between pumping and geometric phases. We derive an expression for the pumped charge which generalizes previous results in several aspects. It is valid also under nonadiabatic conditions and in the presence of an external environment. The key to our approach is to apply Floquet theory to Cooper pair pumping. Floquet theory can be employed for the description of driven dissipative systems, and has recently found several applications[32] (a clear introduction of this method can be found for instance in Ref. 33; another interesting class of open driven systems is reviewed in Ref. 34). We will show that for a closed system under cyclic evolution the total charge transferred through the circuit is proportional to the derivative of the associated Floquet quasi-energy with respect to the superconducting phase difference (a result which is valid also in the case of a nonadiabatic evolution). In the presence of an external environment the expression for the transferred charge is easily generalized in the Floquet representation. It is given by the weighted sum of the charge transferred in each Floquet state, the weights being the diagonal components of the stationary density matrix of the system expressed in the Floquet basis. The central result of our work is Eq. (20); it embraces all the limits considered so far in the literature and allows us to investigate new regimes. Furthermore it suggests the use of a series of well-known numerical schemes to compute the pumped charge.

This paper is organized as follows. In Secs. II and III we will introduce the basic ingredients needed in the derivation of the pumping formula. In Sec. II we formulate the problem of Cooper pair pumping in superconducting circuits, while in Sec. III we provide the necessary tools of the Floquet theory of driven quantum systems both in the closed and open cases. The formula for the pumped charge will be derived in Sec. IV. Here we will discuss in which aspects our results generalize previous works. As an example we will apply our derivation to the Cooper pair sluice which was experimentally realized by the Helsinki group. In Sec. V various different limits will be discussed. Section VI will contain the conclusions of the present work.

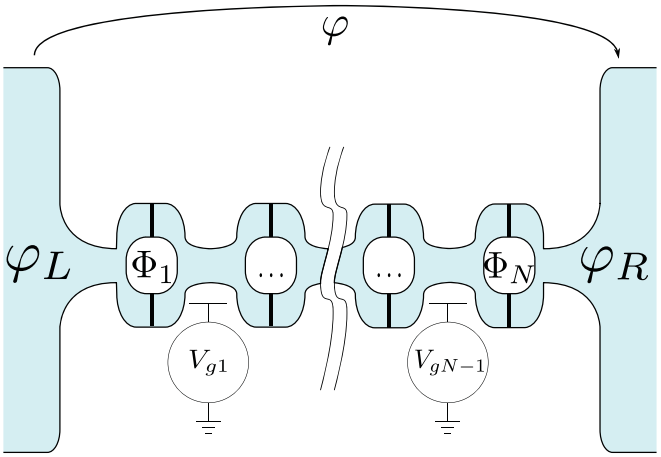

FIG. 1. (Color online) A generic setup of a Cooper pair pump. Two superconducting leads, kept at a phase bias $\varphi = \varphi_R - \varphi_L$, are connected through a Josephson network. The system is operated in a regime where quantum effects are important. To have a pumped charge, some external parameters (e.g., gate voltages or magnetic fluxes) are varied periodically.

II. COOPER PAIR PUMPING

Charge pumping in Josephson networks consists of a coherent periodic manipulation of the collective state of the Cooper pairs in the array. In this section we define the setting and review, for later convenience, the relation between Cooper pair pumping and Berry phases.

A Cooper pair pump consists of a Josephson network connected through Josephson junctions to two superconducting leads (see a sketch of the setup in Fig. 1). The system is phase biased, i.e., the two superconducting electrodes are kept at a finite phase difference $\varphi = \varphi_R - \varphi_L$, where $\varphi_{R/L}$ is the phase of the superconducting order parameter of the right/left lead. The Cooper pair pump operates by changing adiabatically in time some external parameters, such as gate voltages, to tune the charging energies, or magnetic fluxes, to vary the effective Josephson couplings. We will label this set of external parameters by the vector $\vec{\lambda}(t) = \{V_{gi}(t), \Phi_i(t)\}$. In the absence of an external environment, charge transport is a purely coherent phenomenon. The Hamiltonian of the pump depends on the superconducting phases of each island of the network φ_i $(i = 1, \ldots, N)$, its conjugate momenta (i.e., the charge on each island n_i), the phase difference across the pump, and all the external parameters, $H(t) = H[\varphi_1, \ldots, \varphi_N; n_1, \ldots, n_N; \vec{\lambda}(t), \varphi]$. The state of the system is denoted by $|\Psi(t)\rangle = |\Psi(t, \vec{\lambda}(t), \varphi)\rangle$.

By changing the control parameters in time a charge $Q^{(tr)}$ will be transferred from the left to the right electrode. The total charge transferred through the pump during the period T is given by

$$Q^{(tr)} = -2e\frac{1}{\hbar}\int_0^T \langle\Psi(t)|\frac{\partial H}{\partial \varphi}|\Psi(t)\rangle \mathrm{d}t. \qquad (1)$$

Assuming that there are no degeneracies in the spectrum and in the adiabatic limit, it was shown[9,13] that $Q^{(tr)}$ can be expressed

in terms of the total phase accumulated by the system after the cycle:

$$\frac{Q^{(tr)}}{2e} = \frac{\partial \gamma_D}{\partial \varphi} + \frac{\partial \gamma_B}{\partial \varphi}. \tag{2}$$

In the previous expression $\gamma_{D/B}$ are the dynamical/geometric contribution to the accumulated phase. The dynamical contribution corresponds to the charge transferred through the circuit due to the supercurrent flow. The geometric contribution is the pumped charge. In the latter case the charge is an even function of the superconducting phase difference while the contribution due to the supercurrent flow is odd in φ. The different symmetry under reflection of the phase bias is fundamental for the experimental detection of the pumped charge. A generalization to the non-Abelian case, i.e., in the presence of degeneracies in the spectrum, has been given in Ref. 15.

To investigate nonadiabatic corrections and to generalize this result to the dissipative case, it is useful to reexpress the pumped charge using the Floquet formalism. In the next sections, after introducing the basic definitions of Floquet theory, we will find an expression for the pumped charge which in the adiabatic limit and in the absence of an external environment reduces to Eq. (2).

III. FLOQUET THEORY

As will become clear in the continuation of this paper, Floquet formalism is naturally suited to study Cooper pair pumping. It is important to stress already now that this is not a mere reformulation of what has been done so far. We will show that, on the contrary, the Floquet approach, treating on the same footing adiabatic and nonadiabatic regimes, provides a transparent and general expression for the pumped charge in the case in which the superconducting network is coupled to an external environment. A Floquet scattering theory has been developed by Moskalets and Büttiker to study pumping in mesoscopic conductors.[35] Here we employ the Floquet approach to study Cooper pair pumping.

In the next sections we introduce the necessary ingredients of Floquet theory and its use in quantum dissipative systems. The presentation closely follows Refs. 33 and 36.

A. Basics of Floquet theory

Given a system whose dynamics is governed by a periodic Hamiltonian $\hat{H}(t) = \hat{H}(t+T)$, the Floquet theorem states that solutions to the Schrödinger equation exist which have the (Floquet) form

$$|\Psi_\alpha(t)\rangle = e^{-i\epsilon_\alpha t/\hbar}|\Phi_\alpha(t)\rangle, \tag{3}$$

where the state $|\Phi_\alpha(t)\rangle$ is called the Floquet mode; it is periodic in time ($|\Phi_\alpha(t+T)\rangle = |\Phi_\alpha(t)\rangle$) and the corresponding quasi-energy ϵ_α is real and unique up to multiples of $\hbar\Omega$, with $\Omega = 2\pi/T$. There are as many distinct such solutions as the dimension of the Hilbert space $\mathcal{H}$. These solutions are linearly independent and form a basis of the Hilbert space. An eigenvalue equation for the quasi-energy ϵ_α can be obtained by defining the operator $\overline{H}(t) \equiv \hat{H}(t) - i\hbar\partial_t$:

$$\overline{H}(t)|\Phi_\alpha(t)\rangle = \epsilon_\alpha|\Phi_\alpha(t)\rangle. \tag{4}$$

The Floquet modes $|\Phi_{\alpha,n}(t)\rangle = |\Phi_\alpha(t)\rangle \exp(-in\Omega t)$, where the integer n leads to a solution identical to the one given in Eq. (3), but with shifted quasi-energy $\epsilon_\alpha \to \epsilon_{\alpha,n} = \epsilon_\alpha - n\hbar\Omega$; hence the eigenvalues $\{\epsilon_\alpha\}$ can be mapped in the first Brillouin zone obeying $-\hbar\Omega/2 \leqslant \epsilon \leqslant \hbar\Omega/2$.

For the Hermitian operator $\overline{H}(t)$ it is convenient to introduce the composite Hilbert space[37] $\mathcal{H}\otimes\mathcal{T}$ made by the tensor product of the Hilbert space $\mathcal{H}$ of the vectors representing the state of the system and the space $\mathcal{T}$ of the periodic functions in t with period $T = 2\pi/\Omega$. In the space of the periodic functions of t we have a basis of Fourier vectors $\{\exp(-in\Omega t)\}$ which are orthonormal with respect to the scalar product given by

$$(m,n) \equiv \frac{1}{T}\int_0^T (e^{-im\Omega t})^* e^{-in\Omega t}\mathrm{d}t = \delta_{m,n}. \tag{5}$$

We define $|l\rangle \equiv e^{-il\Omega t}$ as vectors in $\mathcal{T}$. We can extend the scalar product on $\mathcal{H}$ to a scalar product on $\mathcal{H}\otimes\mathcal{T}$ defining

$$\langle\langle\Psi_1|\Psi_2\rangle\rangle \equiv \frac{1}{T}\int_0^T \langle\Psi_1(t)|\Psi_2(t)\rangle \mathrm{d}t. \tag{6}$$

B. Floquet states and geometric phases

The Floquet quasi-energies are intimately connected to geometric phases (in this paper we are interested only in the case in which their spectrum is nondegenerate). Indeed they are, up to a multiplying factor, the phases of the eigenvalues of the evolution operator $\hat{U}_S(t+T,t)$. The phase is defined up to $2n\pi$; hence an eigenvalue of $\hat{U}_S(t+T,t)$ corresponds to infinite Floquet exponents obtained through translations of $2n\pi\hbar/T$.

Noticing that Floquet states follow a cyclic evolution in the projective Hilbert space (we call $\hat{C}$ the closed path followed in the projective Hilbert space), it is possible to express the Aharonov–Anandan geometric phase acquired during a cyclic evolution, starting in the Floquet state α, by

$$\gamma_{AA}(\hat{C}) = -\frac{\epsilon_\alpha T}{\hbar} + \frac{1}{\hbar}\int_0^T \langle\Phi_\alpha(t)|\hat{H}(t)|\Phi_\alpha(t)\rangle \mathrm{d}t. \tag{7}$$

An equivalent expression, perhaps more useful in the computation, is

$$\gamma_{AA}(\hat{C}) = 2\pi\sum_k k\langle c_{\alpha,k}|c_{\alpha,k}\rangle, \tag{8}$$

where we used the Fourier expansion of the Floquet modes:

$$|\Phi_\alpha(t)\rangle = \sum_{l=-\infty}^{\infty} |c_{\alpha,l}\rangle e^{-il\Omega t}. \tag{9}$$

In the adiabatic limit the quasi-energy corresponding to the nth eigenstates can be expressed in terms of the dynamic γ_D and geometric γ_G phases:

$$\epsilon_n = -\frac{\hbar}{T}[\gamma_{D,n}(T) + \gamma_{G,n}(C)]. \tag{10}$$

C. Floquet–Born–Markov master equation

The Floquet basis is particularly useful for writing the master equation governing the dynamics of the reduced density matrix of a driven system when in contact with an external

environment. We consider below the case in which the Born–Markov approximation is applicable. Details and subtleties of the derivation of a master equation in this case are described in Ref. 33; here we merely state the end result for the master equation which will be used later to derive a formula for the pumped charge.

Given a quantum system interacting with an external reservoir, the Hamiltonian describing the system+reservoir is given by $\hat{H}_{S+R} = \hat{H}(t) + \hat{H}_R + \hat{V}$, where $\hat{H}(t)$ acts only on the system's degrees of freedom, and is periodic with period T, $\hat{H}_R$ acts only on the environment's degrees of freedom, and the interaction has the form $\hat{V} = \hat{X} \otimes \hat{Y}$, where the operator $\hat{X}$ acts on the environment and $\hat{Y}$ on the system.

The master equation for the reduced density matrix of the system can be presented in the form[33]

$$\dot{\rho}_{\alpha\beta}(t) = \sum_{\gamma\delta;k,k'} \Big[\Gamma^{-}_{\delta\beta\alpha\gamma,k'k} + \Gamma^{+}_{\delta\beta\alpha\gamma,k'k} - \sum_{\nu} (\delta_{\beta\delta}\Gamma^{+}_{\alpha\nu\nu\gamma,kk'} + \delta_{\alpha\gamma}\Gamma^{-}_{\delta\nu\nu\beta,kk'}) \Big] \rho_{\gamma\delta}(t) e^{i(\Delta_{\alpha\beta,k} - \Delta_{\gamma\delta,-k'})t}, \tag{11}$$

where

$$\Gamma^{+}_{\alpha\beta\gamma\delta,kk'} = \frac{1}{\hbar^2} Y_{\alpha\beta,k} Y_{\gamma\delta,k'} \gamma^{+}_{\gamma\delta,k'},$$

$$\Gamma^{-}_{\alpha\beta\gamma\delta,kk'} = \frac{1}{\hbar^2} Y_{\alpha\beta,k} Y_{\gamma\delta,k'} \gamma^{-}_{\alpha\beta,k},$$

$$Y_{\alpha\beta,k} = \frac{1}{T} \int_0^T \langle \Phi_\alpha(t) | \hat{Y} | \Phi_\beta(t) \rangle e^{i\Omega k t} \mathrm{d}t,$$

$$\gamma^{\pm}_{\alpha\beta,k} = \int_0^\infty \langle \hat{X}(\pm t'')\, \hat{X} \rangle \exp(-i\,\Delta_{\alpha\beta,k} t'') \mathrm{d}t'',$$

$$\Delta_{\alpha\beta,k} = \frac{1}{\hbar}(\epsilon_\alpha - \epsilon_\beta) - k\Omega.$$

As is evident from Eq. (11), Floquet theory allows one to treat the time-periodic case with a formalism which is formally identical to the one used in the time-independent case. The relevant effects due to the periodic driving are captured by use of the Floquet basis.

Further simplifications can be made if the secular approximation holds[33] (as we will assume in the rest of this paper). As in the time-independent case, the equations for the populations decouple from those for the (off-diagonal) coherences. In the steady state the coherences vanish. The populations are given by a "detailed balance" condition,

$$W_{\nu\to\alpha} \rho^{\mathrm{st}}_{\nu\nu} = W_{\alpha\to\nu} \rho^{\mathrm{st}}_{\alpha\alpha}, \tag{12}$$

with

$$W_{\delta\to\alpha} = \frac{1}{\hbar^2} \sum_k |Y_{\delta\alpha,k}|^2 g(\Delta_{\delta\alpha,k}), \tag{13}$$

$$g(\omega) \equiv \int_{-\infty}^{+\infty} \langle \hat{X}(t)\, \hat{X} \rangle e^{i\omega t} \mathrm{d}t. \tag{14}$$

This property of the stationary solution when expressed in the Floquet representation, is crucial to obtain the pumped charge also in the dissipative case. In Ref. 27 the secular approximation in the adiabatic basis was shown to lead to unphysical results, in particular to charge nonconservation; the secular approximation in the Floquet basis does not break charge conservation, as we show in the appendix.

IV. FLOQUET APPROACH TO COOPER PAIR PUMPING

Equipped with the Floquet formalism outlined above, we now derive an expression for the pumped charge both in the absence and in the presence of an external environment. We first consider the case of a unitary evolution where, in the adiabatic case, we should recover the known relation, Eq. (2), between pumping and geometric phases.

A. Pumped charge in a closed system

At first we ignore any coupling with the external environment. The dynamics is unitary and governed by a time-periodic Hamiltonian. It is meaningful to compute the pumped charge in a given cycle only for those states that, up to a phase, do come back to their initial value. These are the Floquet states (one should keep in mind that no assumption of an adiabatic dynamics is done at this point). By employing the Schrödinger equation in Eq. (1) it is possible to write it in the form

$$\begin{aligned} Q^{(tr)} &= -\frac{2e}{\hbar} \int_0^T \left[\langle \Psi | \partial_\varphi (H|\Psi\rangle) - \langle \Psi | H \partial_\varphi | \Psi \rangle \right] \mathrm{d}t \\ &= -2ie \int_0^T \partial_t \left[\langle \Psi(t) | \partial_\varphi | \Psi(t) \rangle \right] \mathrm{d}t. \end{aligned} \tag{15}$$

Since the average is performed over a Floquet state, defined in Eq. (3), it is straightforward to obtain

$$Q^{(tr)} = -2e \frac{T}{\hbar} \partial_\varphi \epsilon_\alpha(\varphi). \tag{16}$$

Equation (16) is the first result of this paper. It gives a general formula for the transferred charge in a superconducting circuit which is valid under *both* adiabatic and non-adiabatic conditions. Obviously it reduces to Eq. (2) in the adiabatic case.

The pumped contribution to the transferred charge can be obtained by subtracting from Eq. (16) the supercurrent term (associated with the dynamical phase):

$$Q_p = 4\pi e \sum_k k\, \partial_\varphi \langle c_{\alpha,k}(\varphi) | c_{\alpha,k}(\varphi) \rangle. \tag{17}$$

B. Pumped charge of a superconducting circuit coupled to an external environment

The Floquet approach allows for a very simple and appealing generalization of Eq. (17) to the dissipative case. For simplicity we consider the case in which the secular approximation holds. The transferred charge for a system defined by a density matrix is given by

$$Q^{(tr)} = \frac{2e}{\hbar} \int_0^T \mathrm{Tr}((\partial_\varphi \hat{H}(t,\varphi)) \hat{\rho}^{\mathrm{st}}(t,\varphi)) \mathrm{d}t \tag{18}$$

where $\hat{\rho}^{\mathrm{st}}$ is the reduced density matrix of the system in the steady state (we are interested in obtaining a suitable expression for the pumped charge in the stationary limit after all transient effects have disappeared). Noting that (see the previous section) in the Floquet basis all the coherences vanish

in the long time limit, the expression for the transferred charge takes the form

$$Q^{(tr)} = -2e\frac{T}{\hbar}\sum_{\nu}\rho^{\rm st}_{\nu\nu}\partial_{\varphi}\epsilon_{\nu}(\varphi). \tag{19}$$

Hence, the charge passing through the circuit is the weighted average of the charge which would have passed if the system had been in a pure Floquet state, see Eq. (16). The weights are the populations of these Floquet states in the quasi-stationary case. The previous expression can be split into a geometric and a dynamic part, and the pumped charge is given by

$$Q_p = 4\pi e\sum_{k,\nu} k\rho^{\rm st}_{\nu\nu}(\varphi)\partial_{\varphi}\langle c_{\nu,k}(\varphi)|c_{\nu,k}(\varphi)\rangle. \tag{20}$$

This is the central result of our work. Equation (20) reduces to all known cases in the corresponding limits. In addition it allows us to explore regimes that have not been considered so far. The form in the dissipative case is self-explaining; it is the average of the corresponding expression in the noiseless case weighted by the populations of the Floquet states.

In the rest of this paper we will apply the general result of Eq. (20) to the Cooper pair sluice[18] which, as we will see in the next section, can be described by a two-level Hamiltonian. It is therefore useful to give explicit formulas in the case of a two-dimensional Hilbert space. In this case we have only two independent Floquet states that we call $|\Psi_\alpha\rangle$ and $|\Psi_\beta\rangle$. The populations in the stationary state are given by

$$\begin{aligned}\rho^{\rm st}_{\alpha\alpha} &= \frac{W_{\beta\to\alpha}}{W_{\beta\to\alpha}+W_{\alpha\to\beta}},\\ \rho^{\rm st}_{\beta\beta} &= \frac{W_{\alpha\to\beta}}{W_{\beta\to\alpha}+W_{\alpha\to\beta}}.\end{aligned} \tag{21}$$

Replacing this in Eq. (20) and exploiting the relation $\epsilon_\alpha = -\epsilon_\beta$ we obtain

$$Q_p = 4\pi e\frac{W_{\beta\to\alpha}(\varphi)-W_{\alpha\to\beta}(\varphi)}{W_{\alpha\to\beta}(\varphi)+W_{\beta\to\alpha}(\varphi)}\sum_{k} k\partial_{\varphi}\langle c_{\alpha,k}(\varphi)|c_{\alpha,k}(\varphi)\rangle. \tag{22}$$

V. PUMPING IN THE COOPER PAIR SLUICE

The Cooper pair sluice[18] is a superconducting transistor where the pumping effect is achieved by a modulation of the Josephson couplings and the gate voltages. A sketch of the sluice is shown in Fig. 2. The central island is connected to the two superconducting leads by two tunable Josephson

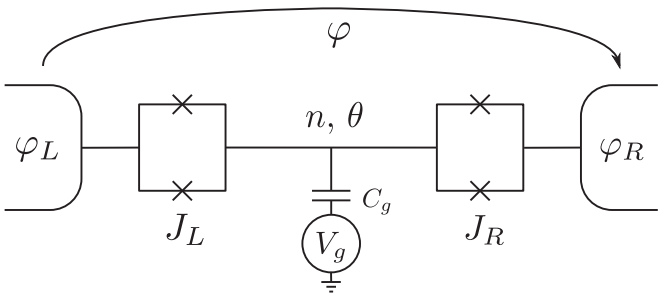

FIG. 2. Circuit scheme of the Cooper pair sluice.

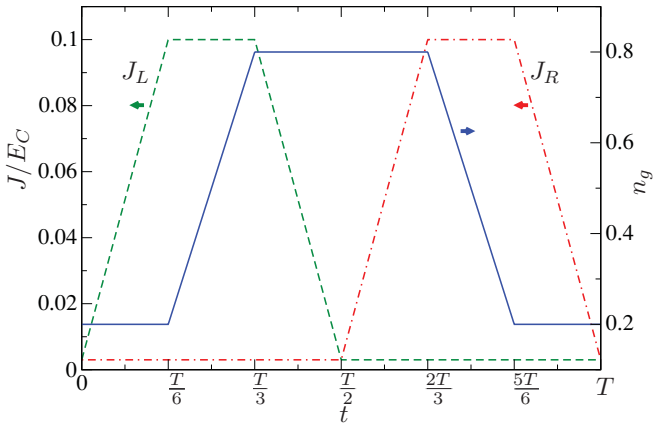

FIG. 3. (Color online) The variation in a cycle of n_g, J_L, and J_R. These latter are expressed in units of E_C. In this graph n_g varies between $n_{g\,\rm min} = 0.2$ and $n_{g\,\rm max} = 0.8$, J_L and J_R vary between $J_{\rm min} = 0.003E_C$ and $J_{\rm max} = 0.1E_C$.

junctions. Its charging energy can be tuned by means of a gate voltage. The Hamiltonian of the sluice is

$$\hat{H} = E_C\big(\hat{n}-n_g\big)^2 - \sum_{i=L/R} J_i\cos(\hat{\theta}-\varphi_i), \tag{23}$$

where $E_C = (2e)^2/(2C_\Sigma)$ is the charging energy of the central island, J_i are the Josephson couplings to the left ($i = L$) and right ($i = R$) electrodes, and $n_g = C_gV_g/(2e)$ is the gate charge which can be modulated by changing the gate voltage V_g, as shown in Fig. 2 (C_Σ is the total capacitance of the island, including the gate capacitance, the capacitance to the leads, and other possible contributions). To tune the Josephson couplings the junctions are replaced by superconducting quantum interference devices (SQUIDs) which behave as single Josephson junctions with an effective coupling which can be varied by changing the flux piercing the loop $J(\Phi) = J^{(0)}\cos(\pi\Phi/\Phi_0)$ (where Φ is the flux through the loop and Φ_0 is the flux quantum). The charge $\hat{n}$ on the central island and the phase $\hat{\theta}$ are canonically conjugated variables. The Hamiltonian is varied along a cyclic path; we change periodically Φ_L, Φ_R, and V_g, determining a periodic variation of n_g, J_L, and J_R. In all the cases considered here, these parameters are assumed to depend on time in the same way as in Pekola *et al.*[27] The time dependence of the parameters is shown in Fig. 3. Notice that when J_L is maximum J_R is minimum and vice versa, meaning that when a SQUID is open the other is closed and vice versa. In an ideal situation, where the minimum value of the Josephson couplings could be reduced to zero, the charge passing through the system in one cycle would be exactly quantized in units of $2e$ (the supercurrent contribution vanishes in this setup). In a realistic case where the SQUID loops do not close perfectly the pumped charge is given by[18]

$$Q_p \sim -2e\Big(1-2\frac{J_{\rm min}}{J_{\rm max}}\cos\varphi\Big). \tag{24}$$

If during the cycle n_g stays close enough to the degeneracy value $1/2$, we can describe the system by using the two-level Hamiltonian

$$H = \tfrac{1}{2}(B_x\hat{\sigma}_x + B_y\hat{\sigma}_y + B_z\hat{\sigma}_z), \tag{25}$$

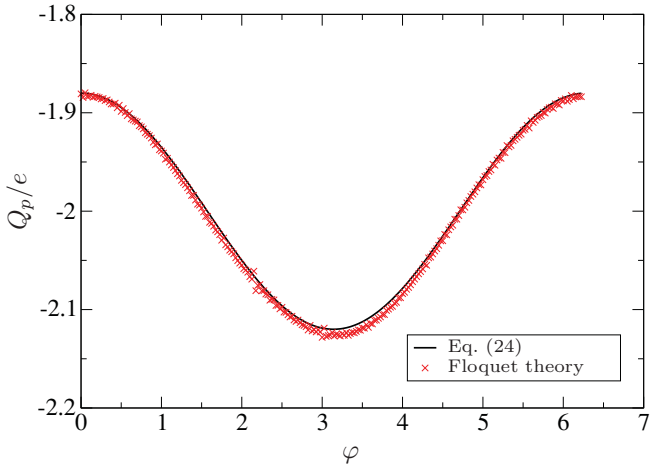

FIG. 4. (Color online) Pumped charge vs. φ in the adiabatic case computed with the Floquet theory (red crosses) confronted with the same quantity computed with the analytical expression (24) (black solid line), $0 \leqslant \varphi \leqslant 2\pi$. It is $J_{\min}/J_{\max} = 0.03$, $J_{\max} = 0.1E_C$, $\mathrm{TE}_C/\hbar = 8400$. Charge is in units of e. The slight scattering of the points is a numerical artifact due to intrinsic difficulties in looking at the system dynamics for very long times.

where $B_x = E_C(1 - 2n_g)$, $B_y = J_R \sin\varphi$, and $B_z = (J_L + J_R \cos\varphi)$.

In the following sections we will use the two-level approximation to present our results for the pumped charge in the Cooper pair sluice.

1. Unitary evolution

We first consider the case in which the environment is absent. We compute numerically the Floquet exponents for the Hamiltonian defined by Eq. (25), and by means of Eq. (22) we obtain the pumped charge. The first case we consider is the adiabatic limit to compare our approach with the known results. The behavior of the pumped charge as a function of the phase bias is shown in Fig. 4.

The results of the Floquet approach are tested against the analytic result,[18] Eq. (24). We chose $\mathrm{TE}_C/\hbar = 8400$, a value which guarantees amply to be in the adiabatic limit. The reason for this large value is due only to the simplicity to compute numerically the Floquet exponents. Obviously we do not expect any changes in the results as long as we are in the adiabatic regime. The numerical calculations agree well with the analytical expression of Eq. (24).

As we already discussed, the Floquet approach to pumping allows us to go beyond the adiabatic regime. An example is shown in Fig. 5 for $\mathrm{TE}_C/\hbar = 2.1$. In this particular case the pumped charge is much smaller than that obtained in the adiabatic regime. This example was indeed chosen just to demonstrate the power of the Floquet approach. There are cases, for a suitable choice of the parameters' loop, in which it is possible to obtain charge quantization also under nonadiabatic conditions.

It is interesting to note that in the nonadiabatic case the pumped charge is phase dependent also in the case in which $J_{\min} = 0$. This case is very similar to the Cooper pair shuttle.[38,39]

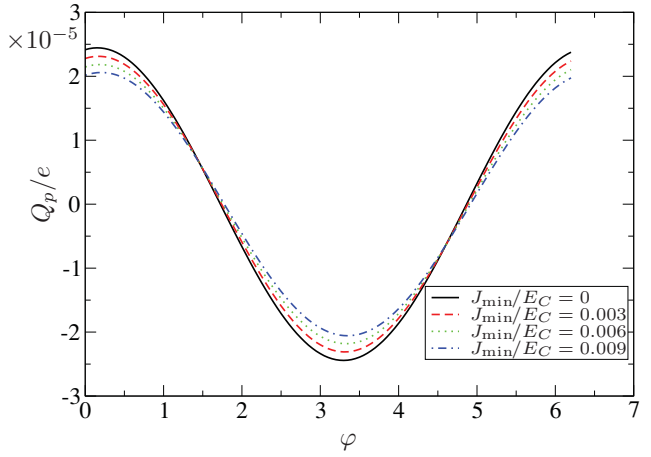

FIG. 5. (Color online) Pumped charge vs. φ with $\mathrm{TE}_C/\hbar = 2.1$ for different values of $J_{\min}$, $0 \leqslant \varphi \leqslant 2\pi$. Charge is in units of $10^{-5}e$.

2. Influence of the external environment

The second application of our pumping formula deals with the case in which the sluice is coupled to an external environment. The main source of decoherence is due to charge fluctuations. In terms of the coupling Hamiltonian introduced in Sec. III C, the operator of the system is $\hat{Y} \propto \sigma_z$ and the operator of the environment is $\hat{X} = E_C\, \delta\hat{n}_g$, where $\delta\hat{n}_g = C_g \delta\hat{V}_g/2e$ expresses the fluctuations of the gate voltage (C_g is the gate capacitance). As usual, we assume that the charge fluctuations are due to the thermal noise of a resistance R put in series with C_g. There are also some fluctuations in the fluxes Φ_L and Φ_R, but these are coupled to the Josephson energies J_L and J_R which are $J_L, J_R \ll E_C$; hence these fluctuations are much smaller than the charge fluctuations which are coupled to E_C and we can neglect them. If the fluctuations of the gate voltage induced by R can be described by the Caldeira–Leggett model, the function $g(\omega) = \int_{-\infty}^{\infty} \langle \hat{X}(t'')\, \hat{X} \rangle e^{i\omega t''} \mathrm{d}t''$ which appears in Eq. (13) reads[40]

$$g(\omega) = \hbar\omega R \left(e\frac{C_g}{C_\Sigma}\right)^2 \left(\coth\left(\frac{\beta\hbar\omega}{2}\right) + 1\right). \qquad (26)$$

In all the subsequent calculations $R = 300\,\mathrm{k}\Omega$.

There are several methods to compute the Floquet quasi-spectrum.[33] For the far nonadiabatic regime, we diagonalized numerically the Floquet operator $\bar{H}(t)$ in the composite Hilbert space[37] as briefly discussed in Sec. III A; in the adiabatic limit we found it more convenient to compute the quasi-energies as the phases of the eigenvalues of the time-evolution operator over one period, times $\hbar/T$.[41] From the knowledge of the Floquet eigenvalues and eigenvectors it is possible to compute the steady-state populations in each Floquet mode. These latter are shown in Fig. 6. The parameters are chosen to be such that the pumping is adiabatic.

By means of Eq. (22) the pumped charge in a dissipative case is readily obtained. An example is shown in Fig. 7.

As should be expected, on increasing the temperature the pumped charge decreases. The pumped charge associated with the two Floquet states is opposite for a given cycle. Therefore the progressive mixture of the two states suppresses the size of the pumping.

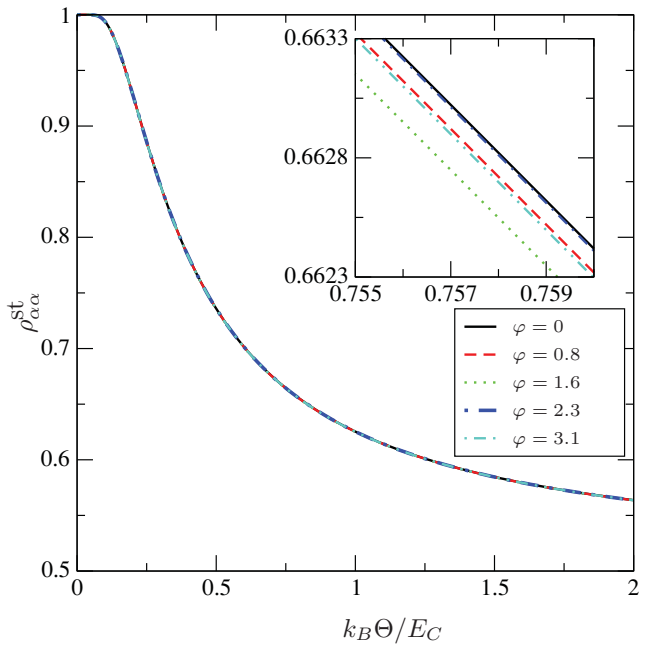

FIG. 6. (Color online) Population $\rho^{\rm st}_{\alpha\alpha}$ of the Floquet state with the lower quasi-energy vs. temperature Θ (in its dimensionless form) for different values of φ. We are in the adiabatic limit ($\mathrm{TE}_C/\hbar = 8400$ and $\Delta E_{\rm min} \simeq E_C/10$). Temperature is expressed in units of E_C/k_B. Inset: The same graph magnified to show the dependence on φ.

3. *Floquet states in the adiabatic limit*

In the limiting case of an adiabatic cycle our approach to pumping should reduce to the one discussed in Ref. 27. In the remainder of this section we will address this point by analyzing the Floquet states in the adiabatic approximation. Floquet states are the eigenstates of the operator $\hat{H}(t) - i\hbar\partial_t$. We assume a nondegenerate spectrum with instantaneous eigenvectors and eigenvalues given respectively by $\{|k(t)\rangle\}$ and $\{E_k(t)\}$. Adiabatic condition requires that $\alpha \equiv 1/(T \min \omega_{kl}) \ll 1$, where $\omega_{kl}(t) \equiv \big(E_k(t) - E_l(t)\big)/\hbar$. On the basis of the instantaneous eigenstates the operator to be diagonalized reads $E_k(t)\delta_{kl} - \hbar w_{kl}(t)$, where $w_{kl}(t) = i\langle k(t)|\dot{l}(t)\rangle$. To first order in α the corresponding eigenvectors are

$$|\Phi_k(t)\rangle = |k(t)\rangle - \sum_{l\neq k} \frac{w_{lk}(t)}{\omega_{kl}(t)}|l(t)\rangle + \mathcal{O}(\alpha^2), \tag{27}$$

with eigenvalues

$$\epsilon_k = \frac{1}{T}\int_0^T \big(E_k(t) - \hbar w_{kk}(t)\big)\mathrm{d}t + \mathcal{O}(\alpha^2). \tag{28}$$

It can be shown that these eigenvalues are invariant under gauge transformations. These states are eigenstates of the operator

$$\tilde{H}(t) = \hat{D}(t)^\dagger \hat{H}(t)\hat{D}(t) - i\hbar\hat{D}(t)^\dagger\dot{\hat{D}}(t), \tag{29}$$

where $\hat{D}$ is the transformation from a given (time-independent) basis of the system Hamiltonian. It was argued in Ref. 27 that the system relaxes in the eigenbasis of $\tilde{H}$, Eq. (29). This is what is contained in the result of Eq. (20) for the pumped charge.

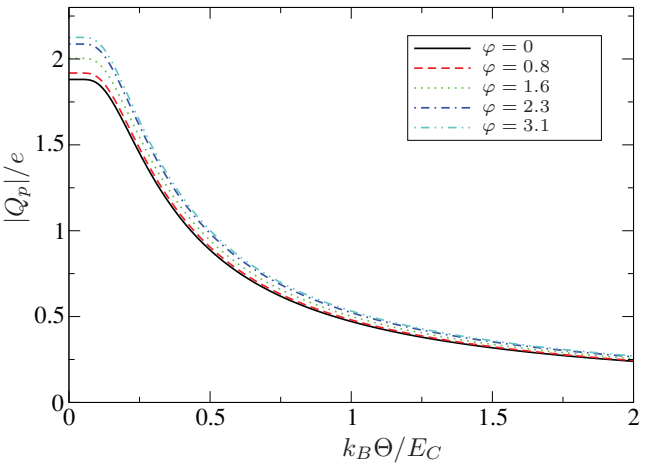

FIG. 7. (Color online) Modulus of the pumped charge vs. temperature for different values of φ. We are in the adiabatic limit ($\mathrm{TE}_C/\hbar = 8400$ and $\Delta E_{\rm min} \simeq E_C/10$). Temperature is expressed in units of E_C/k_B and charge in units of e.

VI. CONCLUSIONS

In this paper we apply, for the first time to our knowledge, the Floquet theory of periodically driven quantum systems to Cooper pair pumping. We found that the pumped charge can be expressed in a very natural way in terms of the Floquet states (there is a clear connection between Floquet exponents and states and the geometric phase acquired by the system). This approach does not require the adiabatic limit to hold, and can be used to work out the pumped charge out of the adiabatic regime (see Fig. 5). We further extended this to the dissipative case, where we provided a general formula to compute pumping. To demonstrate the power of our approach we applied it to the case of Cooper pair sluice. In the known limits we recovered previous results. We further discussed new regimes which are now addressable due to the expressions for the pumped charge given in Eq. (20).

ACKNOWLEDGMENTS

We acknowledge very useful discussions with A. Shnirman and J.P. Pekola. This work was supported by EU through the projects GEOMDISS, QNEMS, SOLID, and NANOCTM.

APPENDIX: CHARGE CONSERVATION

The charge imbalance operator can be defined as

$$\delta\hat{Q} = -\frac{2e}{\hbar}\int_0^T \left(\frac{\partial}{\partial\varphi_R} + \frac{\partial}{\partial\varphi_L}\right)\hat{H}(t)\mathrm{d}t. \tag{A1}$$

Since physical quantities depend on only the phase difference $\varphi = \varphi_R - \varphi_L$, there must exist a unitary (gauge) transformation U such that $\hat{H}' = U\hat{H}U^\dagger$ depends on only $\varphi_R - \varphi_L$, so that $(\partial_{\varphi_R} + \partial_{\varphi_L})\hat{H}' = 0$. In our case $\hat{H}$ and $\hat{H}'$ are periodic functions of time with the same period.

If $|\psi(t)\rangle$ is a cyclic solution of the Schrödinger equation in the Floquet form, then the charge imbalance associated with

this state in one period reads

$$\begin{aligned}\delta Q &= -\frac{2e}{\hbar}\int_0^T \langle\psi|\left(\frac{\partial}{\partial\varphi_R}+\frac{\partial}{\partial\varphi_L}\right)\hat{H}(t)|\psi\rangle \mathrm{d}t \\ &= -\frac{2e}{\hbar}\int_0^T \langle\psi|U^\dagger U[(\partial_{\varphi_R}+\partial_{\varphi_L})\hat{H}]U^\dagger U|\psi\rangle \mathrm{d}t \\ &= -\frac{2e}{\hbar}\int_0^T \langle\psi'|(\partial_{\varphi_R}+\partial_{\varphi_L})\hat{H}'|\psi'\rangle \mathrm{d}t \\ &\quad +\frac{2e}{\hbar}\int_0^T \langle\psi|i[\hat{N},\hat{H}]|\psi\rangle \mathrm{d}t, \end{aligned} \tag{A2}$$

where $|\psi'\rangle = U|\psi\rangle$ and $\hat{N} = -iU^\dagger(\partial_{\varphi_R}+\partial_{\varphi_L})U$; we used the fact that $U^\dagger\partial U = -(\partial U^\dagger)U$ (the derivative is taken with respect to any variable), since $U^\dagger U = \mathbb{1}$.

The contribution on the third line of Eq. (A2) vanishes by definition of $\hat{H}'$. The contribution on the fourth line is proportional to the variation of the average value of $\hat{N}$ over one period; since the state varies periodically, this term also vanishes. Therefore, exact charge conservation holds for nondissipative driven systems in a cyclic state because of gauge invariance and the periodicity of the state.

For those open driven systems for which the secular approximation holds, the stationary state is a mixture of Floquet states, which are cyclic solutions of the Schrödinger equation; therefore

$$\delta Q = -\frac{2e}{\hbar}\sum_\alpha \rho^{\mathrm{st}}_{\alpha\alpha}\int_0^T \langle\psi_\alpha|(\partial_{\varphi_R}+\partial_{\varphi_L})H|\psi_\alpha\rangle \mathrm{d}t = 0, \tag{A3}$$

since each term of the sum separately vanishes; we can conclude that application of the secular approximation to the Floquet description of Cooper pair pumping does not produce violation of charge conservation.

[1] D. J. Thouless, Phys. Rev. B **27**, 6083 (1983).
[2] P. W. Brouwer, Phys. Rev. B **58**, R10135 (1998).
[3] L. P. Kouwenhoven, A. T. Johnson, N. C. van der Vaart, C. J. P. M. Harmans, and C. T. Foxon, Phys. Rev. Lett. **67**, 1626 (1991).
[4] H. Pothier, P. Lafarge, C. Urbina, D. Esteve, and M. H. Devoret, Europhys. Lett. **17**, 249 (1992).
[5] F. Zhou, B. Spivak, and B. Altshuler, Phys. Rev. Lett. **82**, 608 (1999); Yu. Makhlin and A. D. Mirlin, *ibid.* **87**, 276803 (2001); O. Entin-Wohlman, A. Aharony, and Y. Levinson, Phys. Rev. B **65**, 195411 (2002); M. Moskalets and M. Büttiker, *ibid.* **66**, 205320 (2002); I. L. Aleiner and A. V. Andreev, Phys. Rev. Lett. **81**, 1286 (1998); M. Blaauboer and E. J. Heller, Phys. Rev. B **64**, 241301(R) (2001); B. L. Hazelzet, M. R. Wegewijs, T. H. Stoof, and Yu. V. Nazarov, *ibid.* **63**, 165313 (2001); R. Citro, N. Andrei, and Q. Niu, *ibid.* **68**, 165312 (2003); P. W. Brouwer, A. Lamacraft, and K. Flensberg, *ibid.* **72**, 075316 (2005); L. Arrachea, A. Levy Yeyati, and A. Martin-Rodero, *ibid.* **77**, 165326 (2008); A. R. Hernández, F. A. Pinheiro, C. H. Lewenkopf, and E. R. Mucciolo, *ibid.* **80**, 115311 (2009); J. Splettstoesser, M. Governale, J. König, and R. Fazio, Phys. Rev. Lett. **95**, 246803 (2005); E. Sela and Y. Oreg, *ibid.* **96**, 166802 (2006); D. Fioretto and A. Silva, *ibid.* **100**, 236803 (2008).
[6] L. J. Geerligs, S. M. Verbrugh, P. Hadley, J. E. Mooij, H. Pothier, P. Lafarge, C. Urbina, D. Esteve, and M. H. Devoret, Z. Phys. B **85**, 349 (1991).
[7] J. P. Pekola, J. J. Toppari, M. Aunola, M. T. Savolainen, and D. V. Averin, Phys. Rev. B **60**, 9931 (1999).
[8] J. P. Pekola and J. J. Toppari, Phys. Rev. B **64**, 172509 (2001).
[9] M. Aunola and J. J. Toppari, Phys. Rev. B **68**, 020502(R) (2003).
[10] R. Fazio, F. W. J. Hekking, and J. P. Pekola, Phys. Rev. B **68**, 054510 (2003).
[11] A. O. Niskanen, J. P. Pekola, and H. Seppä, Phys. Rev. Lett. **91**, 177003 (2003).
[12] M. Governale, F. Taddei, R. Fazio, and F. W. J. Hekking, Phys. Rev. Lett. **95**, 256801 (2005).
[13] M. Möttönen, J. P. Pekola, J. J. Vartiainen, V. Brosco, and F. W. J. Hekking, Phys. Rev. B **73**, 214523 (2006).
[14] R. Leone, L. P. Levy, and P. Lafarge, Phys. Rev. Lett. **100**, 117001 (2008).
[15] V. Brosco, R. Fazio, F. W. J. Hekking, and A. Joye, Phys. Rev. Lett. **100**, 027002 (2008).
[16] J.-M. Pirkkalainen, P. Solinas, J. P. Pekola, and M. Möttönen, Phys. Rev. B **81**, 174506 (2010).
[17] J. J. Vartiainen, M. Möttönen, and J. P. Pekola, Appl. Phys. Lett. **90**, 082102 (2007).
[18] M. Möttönen, J. J. Vartiainen, and J. P. Pekola, Phys. Rev. Lett. **100**, 177201 (2008).
[19] J. E. Avron, A. Elgart, G. M. Graf, and L. Sadun, Phys. Rev. B **62**, R10618 (2000).
[20] H.-Q. Zhou, S. Y. Cho, and R. H. McKenzie, Phys. Rev. Lett. **91**, 186803 (2003).
[21] G. Falci, R. Fazio, G. M. Palma, J. Siewert, and V. Vedral, Nature (London) **407**, 355 (2000).
[22] W. Xiang-bin and M. Keiji, Phys. Rev. B **65**, 172508 (2002).
[23] A. Blais and A. M. S. Tremblay, Phys. Rev. A **67**, 012308 (2003).
[24] P. J. Leek, J. M. Fink, A. Blais, R. Bianchetti, M. Gppl, J. M. Gambetta, D. I. Schuster, L. Frunzio, R. J. Schoelkopf, and A. Wallraff, Science **318**, 1889 (2007).
[25] J. Jones, V. Vedral, A. Ekert, and G. Castagnoli, Nature (London) **403**, 869 (2000).
[26] P. Zanardi and M. Rasetti, Phys. Lett. A **264**, 94 (1999).
[27] J. P. Pekola, V. Brosco, M. Möttönen, P. Solinas, and A. Shnirman, Phys. Rev. Lett. **105**, 030401 (2010).
[28] P. Solinas, M. Möttönen, J. Salmilehto, and J. P. Pekola, Phys. Rev. B **82**, 134517 (2010).
[29] J. Salmilehto, P. Solinas, J. Ankerhold, and M. Möttönen, Phys. Rev. A **82**, 062112 (2010).
[30] E. Sjöqvist, A. K. Pati, A. Ekert, J. S. Anandan, M. Ericsson, D. K. L. Oi, and V. Vedral, Phys. Rev. Lett. **85**, 2845 (2000); I. Fuentes-Guridi, J. Pachos, S. Bose, V. Vedral, and S. Choi, Phys. Rev. A **66**, 022102 (2002); G. De Chiara and G. M. Palma, Phys. Rev. Lett. **91**, 090404 (2003); A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, *ibid.* **90**, 160402 (2003); M. S. Sarandy and D. A. Lidar, *ibid.* **95**, 250503 (2005); R. S. Whitney, Y. Makhlin, A. Shnirman,

and Y. Gefen, *ibid.* **94**, 070407 (2005); P. Thunström, J. Åberg, and E. Sjöqvist, Phys. Rev. A **72**, 022328 (2005); G. Florio, P. Facchi, R. Fazio, V. Giovannetti, and S. Pascazio, *ibid.* **73**, 022327 (2006); D. Parodi, M. Sassetti, P. Solinas, P. Zanardi, and N. Zanghì, *ibid.* **73**, 052304 (2006).

[31]M. Wubs, K. Saito, S. Kohler, P. Hänggi, and Y. Kayanuma, Phys. Rev. Lett. **97**, 200404 (2006).

[32]V. Peano and M. Thorwart, Chem. Phys. **322**, 135 (2006); New J. Phys. **8**, 21 (2006); Europhys. Lett. **89**, 17008 (2010); Phys. Rev. B **82**, 155129 (2010); S.-K. Son, S. Han, and S. I. Chu, Phys. Rev. A **79**, 032301 (2009); J. Hausinger and M. Grifoni, *ibid.* **81**, 022117 (2010).

[33]M. Grifoni and P. Hanggi, Phys. Rep. **304**, 229 (1998).

[34]S. N. Shevchenko, S. Ashhab, and F. Nori, Phys. Rep. **492**, 1 (2010).

[35]M. Moskalets and M. Büttiker, Phys. Rev. B **66**, 205320 (2002); **70**, 245305 (2004).

[36]S. Guerin and H. R. Jauslin, Adv. Chem. Phys. **125**, 147 (2003).

[37]H. Sambe, Phys. Rev. A **7**, 2203 (1973).

[38]L. Y. Gorelik, A. Isacsson, Y. M. Galperin, R. I. Shekhter, and M. Jonson, Nature (London) **411**, 454 (2001).

[39]A. Romito, F. Plastina, and R. Fazio, Phys. Rev. B **68**, 140502(R) (2003).

[40]Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. **73**, 357 (2001).

[41]S. I. Chu, Adv. Chem. Phys. **73**, 739 (1989).