Temperature dependence of the lower critical field in underdoped cuprates

F. Horváth and R. Hlubina
Department of Experimental Physics, Comenius University, Mlynská Dolina F2, SK-842 48 Bratislava, Slovakia
(Received 6 December 2010; published 11 May 2011)

Recently, we have suggested that the underdoped cuprate superconductors should be described by a modified Ginzburg-Landau theory which takes into account their reduced phase stiffness. In the present paper, (i) we calculate physical properties, such as the surface energy, which have not been discussed previously within this scheme, (ii) we propose a unified description of the pseudogap and of the superconducting state of the cuprates, and (iii) by analyzing the temperature dependence of the penetration depth and of the lower critical field we provide additional evidence for the applicability of the modified Ginzburg-Landau theory to underdoped cuprates.

DOI: 10.1103/PhysRevB.83.174508 PACS number(s): 74.20.De, 74.72.−h, 74.25.Op, 74.72.Kf

I. INTRODUCTION

The Ginzburg-Landau (GL) theory is a phenomenological theory of superconductivity which has been successfully applied to a handful of problems in conventional superconductivity.[1–3] Recently we have argued that, if we want to describe the superconducting state of underdoped cuprates, the conventional GL theory should be modified.[4] The reason is simple: It seems well established that when we heat up an underdoped cuprate above its T_c, the "normal" state immediately above the superconducting state is a kind of phase-disordered superconductor[5–7] or a somewhat related phase-disordered liquid of resonating valence bonds, which roughly may be thought of as Cooper pairs.[8,9] In our language this means that the stiffness of the superconducting amplitude above T_c is finite, whereas the superconducting phase stiffness vanishes in the pseudogap region. By continuity, it seems reasonable to assume that below T_c the superconducting phase stiffness, while being finite, is still much lower than the amplitude stiffness. The conventional GL theory does not have this property, but it is straightforward to modify it so that it contains two different stiffnesses, one for the amplitude and another for the phase. Precisely such a theory has been constructed in Ref. 4. The modified GL theory is characterized by three length scales, instead of the two length scales of the conventional theory. These length scales are the penetration depth λ, the amplitude coherence length ξ, and the phase coherence length $\xi_\perp$. The theory is therefore characterized by two dimensionless parameters: the GL parameter $\kappa = \lambda/\xi$ and a novel parameter $s = \xi_\perp/\xi$. It is worth pointing out that due to the large difference of the energy gap and of the transition temperature in the underdoped region, the possibility of introducing two different coherence lengths has been repeatedly pointed out also by different groups.[10–12]

In order to clarify the region of applicability of the modified GL theory, let us first note that within the non-superconducting pseudogap phase we have $\xi_\perp = 0$ and therefore $s = 0$. On the other hand, deep inside the superconducting dome we expect that the conventional GL theory applies and therefore $s = 1$ in this region. Thus we expect that within the shaded region of the schematic phase diagram shown in Fig. 1, the parameter s continuously changes from the value $s = 1$ deep in the superconducting phase to $s = 0$ along the phase boundary with the "normal" state. In this region of phase space the modified GL theory should be used. In order to prove this, in Ref. 4 we have introduced a dimensionless combination $\mathcal{H}_{c1}$ of the lower critical field H_{c1} and of the penetration depth,

$$\mathcal{H}_{c1} = \frac{4\pi\mu_0\lambda^2 H_{c1}}{\Phi_0}, \tag{1}$$

where $\Phi_0 = h/(2e)$ is the superconducting flux quantum, and we have studied the experimental doping dependence of $\mathcal{H}_{c1}$ in the low-temperature limit [path (a) in Fig. 1]. In agreement with our expectations, we have found that the large increase of $\mathcal{H}_{c1}$ upon approaching the non-superconducting region can be nicely explained by a parameter s interpolating between 0 and 1.

The plan of the present paper is as follows. In Sec. II we review the results obtained within the modified GL theory in Ref. 4 and we calculate further physical properties, such as the surface energy and the critical current of a thin wire, within this framework. In Sec. III we propose a unified description of the pseudogap and of the superconducting state of the cuprate superconductors. Finally, by comparing to a set of experimental data on a set of $YBa_2Cu_3O_{7-x}$ (YBCO) samples, in Sec. IV we present further experimental evidence for the validity of the modified GL theory. We start by presenting an improved analysis of the doping dependence along path (a) in Fig. 1 and then we compare the modified GL theory to the temperature dependence of $\mathcal{H}_{c1}$ for optimally doped and extremely underdoped YBCO samples [paths (b) and (c) in Fig. 1, respectively].

II. MODIFIED GL THEORY

Let us start by summarizing the main ideas of Ref. 4. The central object of both the conventional and the modified GL theories is the free-energy density

$$\delta\mathcal{F} = \frac{1}{2\mu_0}(\nabla \times \mathbf{A})^2 + \delta\mathcal{F}_s,$$

which is a sum of the trivial electromagnetic term and of the superconducting term $\delta\mathcal{F}_s$, which is a functional of the macroscopic center-of-mass wave function $\psi(\mathbf{r})$ of the Cooper pairs and of the vector potential $\mathbf{A}(\mathbf{r})$. In what follows we will write $\psi(\mathbf{r}) = \psi_\infty f(\mathbf{r}) e^{i\theta(\mathbf{r})}$, where ψ_∞ is the wave function in a homogeneous piece of a superconductor in zero applied magnetic field, $f(\mathbf{r})$ is a dimensionless amplitude field, and

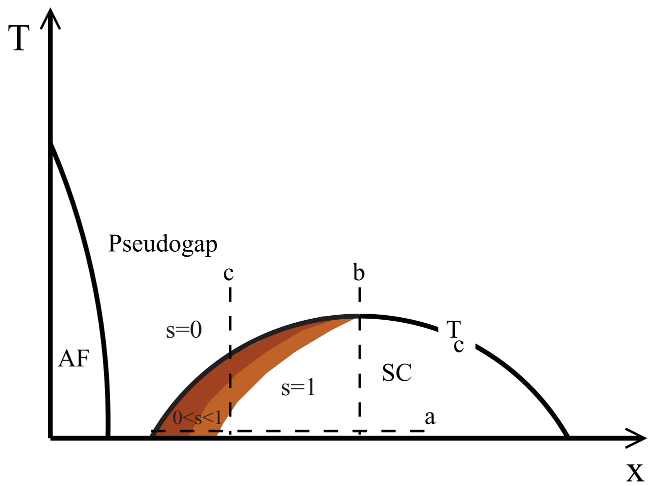

FIG. 1. (Color online) Schematic phase diagram of the cuprates in the doping vs temperature plane. We expect that within the shaded region $0 < s < 1$ holds and therefore the modified GL theory applies. Three different checks of the theory on a set of YBCO samples were performed. Path (a): Doping dependence of the dimensionless lower critical field $\mathcal{H}_{c1}$ in the low-temperature limit (see also Ref. 4). Paths (b), (c): Temperature dependence of $\mathcal{H}_{c1}$ for optimally doped and extremely underdoped samples.

$\theta(\mathbf{r})$ is a phase field. In Ref. 4 we have postulated the following form of the superconducting term $\delta\mathcal{F}_s$:

$$\frac{\delta\mathcal{F}_s}{\mu_0 H_c{}^2} = -f^2 + \frac{1}{2}f^4 + \xi^2(\nabla f)^2 + s^2\xi^2 f^2\left(\nabla\theta + \frac{2\pi}{\Phi_0}\mathbf{A}\right)^2.$$

Note that due to the presence of the factor $s < 1$, phase fluctuations are cheaper in energy than amplitude fluctuations, which is in fact the only difference of our theory with respect to the conventional GL theory. It is also worth pointing out that within our theory, the superconducting state is characterized by three parameters: the thermodynamic critical field H_c, the coherence length ξ, and the parameter s measuring the reduction of the phase stiffness.

Our choice of the free-energy functional represents a minimal extension of the conventional GL theory which allows for different phase and amplitude stiffness. Other terms are allowed by symmetry, but we do not consider them for the following reasons. The linear term in f is not taken into account, since a weak-coupling instability should gain energy at least quadratically in the order parameter.[14] The cubic term is not considered, since there is no experimental evidence for first-order transitions in the pseudogap region. As regards the mixed-gradient term $\nabla f \cdot (\nabla\theta + 2\pi\mathbf{A}/\Phi_0)$, it is not allowed in time-reversal-invariant superconductors.

Minimizing the functional $\delta\mathcal{F}$ with respect to f we obtain the modified GL equation

$$-\xi^2\triangle f + s^2\xi^2\left(\nabla\theta + \frac{2\pi}{\Phi_0}\mathbf{A}\right)^2 f = f - f^3. \quad (2)$$

Minimization with respect to θ results in the continuity equation $\nabla\cdot\mathbf{j} = 0$, where the supercurrent density is given by the usual expression

$$\mathbf{j} = -\frac{f^2}{\mu_0\lambda^2}\left(\mathbf{A} + \frac{\Phi_0}{2\pi}\nabla\theta\right). \quad (3)$$

One checks easily that in weak applied magnetic fields $f \approx 1$ and therefore λ is the weak-field penetration depth. When expressed in terms of the parameters H_c, ξ, and s, it reads

$$\lambda = \frac{\Phi_0}{2\pi\sqrt{2}\mu_0 H_c \xi s}. \quad (4)$$

Finally, minimization with respect to the vector potential $\mathbf{A}$ results in the Maxwell equation $\nabla\times\mathbf{B} = \mu_0\mathbf{j}$.

Within the above-mentioned formalism, in Ref. 4 we have calculated the lower and upper critical fields H_{c1} and H_{c2}, as well as the equilibrium magnetization curve. In particular, for the dimensionless lower critical field in the large-κ limit we have found

$$\mathcal{H}_{c1} = \frac{1}{s(1+s^2)} + \ln\kappa. \quad (5)$$

Note that as already mentioned in the Introduction, $\mathcal{H}_{c1}$ diverges when $s \to 0$.

In what follows we will calculate further physical properties, which have not been discussed in Ref. 4. To this end, we will rewrite the modified GL equations as a coupled set of equations for f and for the reduced magnetic field $\mathbf{b} = \mathbf{B}/(\mu_0 H_c)$:

$$-\xi^2\triangle f + \frac{\lambda^2}{2}\frac{(\nabla\times\mathbf{b})^2}{f^3} = f - f^3,$$

$$\nabla\times(\nabla\times\mathbf{b}) + f^2\left(\frac{1}{\lambda^2}\mathbf{b} + S\right) = \frac{2}{f}\nabla f\times(\nabla\times\mathbf{b}),$$

$$S = \frac{\sqrt{2}s}{\kappa}\nabla\times\nabla\theta.$$

Let us also note that $\delta\mathcal{F}_s$ can be written in terms of the fields $f(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$ which solve the above equations as

$$\frac{\delta\mathcal{F}_s}{\mu_0 H_c{}^2} = -f^2 + \frac{1}{2}f^4 + \xi^2(\nabla f)^2 + \frac{\lambda^2}{2f^2}(\nabla\times\mathbf{b})^2.$$

It is worth pointing out that for a nonsingular phase field $\theta(\mathbf{r})$, the source term $S = 0$ and the equations for f and $\mathbf{b}$ are equivalent to the conventional GL equations for a superconductor with coherence length ξ and penetration depth λ; i.e., the parameter s does not enter these equations. This means that for problems with $S = 0$ the results of the modified GL theory should be, when expressed in terms of H_c, ξ and λ, equal to the conventional GL results.

As a first example of a problem with $S = 0$, let us discuss the critical current density of a thin wire. In this case, we can assume that the amplitude field f is constant and the free-energy density of the wire with prescribed gauge-invariant phase gradient $\mathbf{q} = \nabla\theta + 2\pi\mathbf{A}/\Phi_0$ is[2]

$$\delta\mathcal{F}_s = \mu_0 H_c{}^2\left[-(1 - s^2\xi^2 q^2)f^2 + \frac{1}{2}f^4\right].$$

Minimizing $\delta\mathcal{F}_s$ at fixed q we find an optimal amplitude $f^2 = 1 - s^2\xi^2 q^2$. Plugging this value into Eq. (3) and maximizing with respect to q, we find the critical current density of a thin wire:

$$j_{\max} = \frac{1}{3\pi\sqrt{3}}\frac{\Phi_0}{\mu_0\lambda^2\xi s} = \left(\frac{2}{3}\right)^{3/2}\frac{H_c}{\lambda}. \qquad (6)$$

Note that the first form differs from the prediction of the conventional GL theory, since it depends on the parameter s. However, when $j_{\max}$ is expressed in terms of H_c and λ making use of Eq. (4), the conventional GL result is recovered. Since $S = 0$, this is consistent with the general argument.

As the next example, let us consider the critical field H_{c3} for surface superconductivity. As shown for example in Ref. 4, if we choose the Landau gauge $\mathbf{A} = (0,\mu_0 Hx,0)$ and if we take for the phase field $\theta = ky$, the linearized version of Eq. (2) reduces to an equation for a harmonic oscillator. Note that also in this case $S = 0$. Introducing instead of x a dimensionless coordinate $t = x/L$ where $L^2 = \Phi_0/(2\pi\mu_0 Hs)$, this equation can be rewritten as the following eigenvalue problem:

$$-\frac{d^2 f}{dt^2} + (t - t_0)^2 f = \Lambda f, \qquad \Lambda = \frac{\Phi_0}{2\pi\mu_0 H\xi^2 s},$$

where $t_0 = -k\Phi_0/(2\pi\mu_0 HL)$. The bulk upper critical field H_{c2} can be determined by inspecting the lowest eigenvalue $\Lambda_{\min}$ of this problem with the boundary conditions $f \to 0$ as $t \to \pm\infty$. It is well known that in this case $\Lambda_{\min} = 1$, leading to

$$H_{c2} = \frac{\Phi_0}{2\pi\mu_0\xi^2 s} = \kappa\sqrt{2}H_c. \qquad (7)$$

On the other hand, the critical field H_{c3} for surface superconductivity can be determined from the lowest eigenvalue $\Lambda_{\min}$ of the same problem with different boundary conditions $f \to 0$ as $t \to \infty$ and $f' = 0$ for $t = 0$, as appropriate for a superconductor-insulator interface. In this case $\Lambda_{\min} = 0.59$, and therefore

$$H_{c3} = 1.695 H_{c2}. \qquad (8)$$

Note that although both H_{c2} and H_{c3} depend on the parameter s, when expressed in terms of H_c and $\kappa = \lambda/\xi$, the formulas are formally the same as in the conventional GL theory. This is again a consequence of the fact that $S = 0$.

Finally, also the surface energy can be calculated in a gauge where the wave function $\psi(\mathbf{r})$ is real and thus $S = 0$. According to our general argument the fields f and $\mathbf{b}$ are equal to those calculated within the conventional GL theory; i.e., they do not depend on the value of s. Within the modified GL theory, it is therefore only the value of κ which distinguishes materials with positive and negative surface energies with the same critical value $\kappa_c = 1/\sqrt{2}$ as in the conventional GL theory.[13] Note that the same critical value of κ is predicted also by the second form of the result for the upper critical field Eq. (7).

On the other hand, in a vortex the phase field is nontrivial and thus $S \neq 0$. Therefore the dependence of H_{c1} on the parameter s in general cannot be eliminated by making use of the thermodynamic critical field H_c in conventional formulas. In particular, our result for the large-κ limit can be written as

$$\frac{H_{c1}}{H_c} = \frac{1}{\kappa\sqrt{2}}\left[\frac{1}{1+s^2} + s\ln\kappa\right].$$

Nevertheless, generalizing the argument presented in Ref. 13 it is possible to show that irrespective of the value of s, for $\kappa = \kappa_c$ the lower critical field equals the thermodynamic field, $H_{c1} = H_c$. In fact, the following modified GL equations for a single vortex were derived in Ref. 4:

$$-\xi^2\left(f'' + \frac{1}{r}f'\right) + \frac{s^2\xi^2}{r^2}(1-\phi)^2 f + f^3 = f, \qquad (9)$$

$$\phi'' - \frac{1}{r}\phi' + \frac{f^2}{\lambda^2}(1-\phi) = 0, \qquad (10)$$

where $\phi(r)$ is the dimensionless magnetic flux threading a ring with radius r around the vortex center. It is worth pointing out in passing that Eq. (9) predicts an unconventional vortex shape, as can be seen, e.g., by noting that $f(r) \propto r^s$ for $r \to 0$.[4]

One checks easily that if we assume that Eq. (10) holds and if we make the ansatz

$$\phi'(r) = \frac{r}{2s\lambda^2}(1 - f^2), \qquad (11)$$

then, assuming $\kappa = \kappa_c$, we can derive Eq. (9). Thus the ansatz Eq. (11) is consistent with Eqs. (9) and (10). Using the results of Ref. 4, H_{c1} can be calculated as

$$H_{c1} = \frac{2\pi\mu_0 H_c^2}{\Phi_0}\int_0^\infty dr r(1 - f^2) = \kappa\sqrt{2}H_c.$$

In the last step we have made use of Eq. (11), of the boundary conditions $\phi(\infty) = 1$, $\phi(0) = 0$, and of Eq. (4). If we finally take into account that $\kappa = \kappa_c$, we do obtain the equality $H_{c1} = H_c$.

Therefore, within the modified GL theory, it is only the value of κ which distinguishes type-I from type-II behavior, irrespective of the value of s. Three different criteria, namely, the sign of the surface energy, the comparison of H_{c1} with H_c, and the comparison of H_c with H_{c2}, all lead to the same critical value $\kappa_c = 1/\sqrt{2}$, as in the conventional GL theory.

III. UNIFIED DESCRIPTION OF THE PSEUDOGAP AND OF THE SUPERCONDUCTING STATE

It is instructive to express the superconducting free-energy density $\delta\mathcal{F}_s$ in terms of the wave function $\psi(\mathbf{r})$:

$$\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{2m^*}\left[(\nabla|\psi|)^2 + s^2|\psi|^2\left(\nabla\theta + \frac{2\pi}{\Phi_0}\mathbf{A}\right)^2\right].$$

Note that in this formulation, $\delta\mathcal{F}_s$ is parametrized by s and by three conventional GL parameters: α, β, and the Cooper pair mass m^*. One checks easily that this formulation is equivalent with the amplitude-phase formulation, if we require

$$\alpha = -\frac{\mu_0 H_c^2}{\psi_\infty^2}, \quad \beta = \frac{\mu_0 H_c^2}{\psi_\infty^4}, \quad \psi_\infty^2 = \frac{2m^*}{\hbar^2}\mu_0 H_c^2\xi^2. \quad (12)$$

Observe that the triplet of parameters α, β, and m^* can be replaced by the triplet H_c, ξ, and ψ_∞, and vice versa. It should be noted, however, that none of the measurable quantities discussed in Sec. II depends on the value of ψ_∞. This is in fact a consequence of the well-known arbitrariness in the choice of m^* emphasized by de Gennes.[14]

In what follows we will further develop our interpretation of the modified GL theory. To this end, we will use the "unobservability" of ψ_∞ in constructing a unified description of the pseudogap and of the superconducting state. Our key assumption is that in zero applied field, ψ_∞^2 has a finite value in the whole temperature range between absolute zero and the pseudogap formation temperature T_0 which we assume to be higher than T_c. This assumption makes our theory substantially different from the standard GL theory in which the precise value of ψ_∞^2 is arbitrary, but ψ_∞^2 is definitely finite for $T < T_c$ and strictly zero for $T > T_c$. We will show later that this choice is made possible by the existence of the additional parameter s in the modified GL theory.

To be more specific, we will assume that ψ_∞^2 minimizes the following GL-type functional for the pseudogap:

$$\delta\mathcal{F}_0 = \alpha_0\psi_\infty^2 + \frac{\beta_0}{2}\psi_\infty^4,$$

where α_0 changes sign at $T = T_0$ and $\beta_0 > 0$ is roughly independent of temperature. The temperature dependence of ψ_∞^2 and of the pseudogap "condensation energy density" are therefore given by

$$\psi_\infty^2 = -\frac{\alpha_0}{\beta_0}, \qquad \frac{1}{2}\mu_0 H_0^2 = \frac{\alpha_0^2}{2\beta_0}.$$

Before proceeding, we would like to point out that the field ψ_∞ is truly "unobservable" only strictly within the mean-field GL-type description. It definitely is observable in all sorts of spectroscopies at the very least. Furthermore, we will argue that a nonzero value of ψ_∞ implies the existence of a vortex liquid.

Let us next discuss the temperature dependence of physical quantities slightly below the true superconducting transition temperature T_c. Since ψ_∞^2 is postulated to be finite at T_c, using Eq. (12) we have to require that $H_c\xi$ is finite as well. On the other hand, superconductivity with its associated Meissner effect disappears as $T \to T_c$. Therefore, in this limit, we have to require that $\lambda \to \infty$ and $H_{c1} \to 0$. Making use of Eqs. (1), (4), and (5), we conclude that $s \to 0$ as the temperature approaches T_c. Moreover, we require that H_{c2} stays finite at T_c. Since $H_{c2} \propto \xi^{-2}s^{-1}$, we are therefore forced to assume that $\xi \propto 1/\sqrt{s}$. It should be emphasized that a finite value of H_{c2} does not mean true superconductivity in our theory. In fact, in the field region $H_{c1} < H < H_{c2}$ we can only say that the sample is filled by vortex matter. Finally, since also $H_c\xi$ should be constant at T_c, we have to require $H_c \propto \sqrt{s}$. Our predictions for the scaling of various quantities with s in the vicinity of T_c are shown in Table. I.

Summarizing the above analysis, we postulate that the free-energy density

$$\delta\mathcal{F}_s = s\alpha_0|\psi|^2 + s\frac{\beta_0}{2}|\psi|^4 + \frac{\hbar^2}{2m^*}\left[(\nabla|\psi|)^2 + s^2|\psi|^2\left(\nabla\theta + \frac{2\pi}{\Phi_0}\mathbf{A}\right)^2\right] \qquad (13)$$

describes both the superconducting and the pseudogap phases. For $T_c < T < T_0$, the minimization with respect to ψ should be performed at a finite value of s and only afterward the limit $s \to 0^+$ should be taken. This way we can reproduce our previous description of the pseudogap phase.

TABLE I. Predictions of the modified GL theory for the scaling of different quantities with $s \to 0$ under approaching the superconductor-pseudogap (i.e., the vortex solid–vortex liquid) boundary.

ξ	λ	H_c	H_{c1}	$H_{c2} \approx H_0$	ψ_∞^2
$s^{-1/2}$	s^{-1}	$s^{1/2}$	s	const	const

On the other hand, for $T < T_c$, the parameter s is finite and Eq. (13) reproduces the modified GL theory introduced in Ref. 4 and further studied in Sec. II of the present paper, with $\alpha = s\alpha_0$ and $\beta = s\beta_0$. This can be checked easily, if we express the free-energy density in terms of H_c and ξ, making use of Eq. (12).

It is worth pointing out that Eq. (13) can be thought of as an effective model taking into account phase fluctuations in a conventional GL theory which obtains by setting $s = 1$ in Eq. (13). If we denote the thermodynamic critical field, coherence length, and penetration depth of that conventional GL theory as H_0, ξ_0, and λ_0, respectively, then we can write the relation between the parameters of the effective and conventional theories as

$$H_c = \sqrt{s}H_0, \qquad \xi = \xi_0/\sqrt{s}, \qquad \lambda = \lambda_0/s.$$

From here it follows that $H_{c2} = H_{c2}^0$ and, for small s, $H_{c1} \sim sH_{c1}^0$, where H_{c1}^0, H_{c2}^0 denote the lower and upper critical fields of the conventional GL theory.

The phase diagram of underdoped cuprates in the temperature vs magnetic field plane, as predicted by the modified GL theory, is schematically shown in Fig. 2. We would like to point out that our phase diagram differs from the various reported phase diagrams for vortex matter[3] in that our vortex liquid region extends to temperatures higher than T_c also at zero applied field. Therefore, according to our theory, the pseudogap state at $H = 0$ in the temperature range $T_c < T < T_0$ can be continuously deformed into the low-temperature high-field vortex liquid state recently observed in Ref. 15.

IV. COMPARISON WITH EXPERIMENTAL DATA

In Ref. 4 the result of the modified GL theory for the dimensionless lower magnetic field $\mathcal{H}_{c1}$ has been compared to experimental data on extremely underdoped YBCO samples acquired along path (a) in Fig. 1. For the doping dependence of the low-temperature limit of H_{c1}, we have used the empirical formula $H_{c1} = H_{c1}(T_c)$ proposed in Ref. 16. On the other hand, the doping dependence of the low-temperature limit of λ was taken from Ref. 17. However, when preparing this manuscript, we realized that in Fig. 4 of Ref. 17, the authors do not present the actually measured low-temperature values of the penetration depth, but instead they plot the values extrapolated to $T = 0$ from the linear part in the $\lambda^{-2} = \lambda^{-2}(T)$ curve. For our purpose, the actually measured values are more relevant. We have extracted them from Fig. 2 of Ref. 17 and, using these corrected data, we have repeated our analysis of $\mathcal{H}_{c1}$. The results are shown in Fig. 3. Note that our conclusions are the same as in Ref. 4; i.e., we find that the parameter s decreases to zero with decreasing sample T_c, in complete agreement with our expectations.

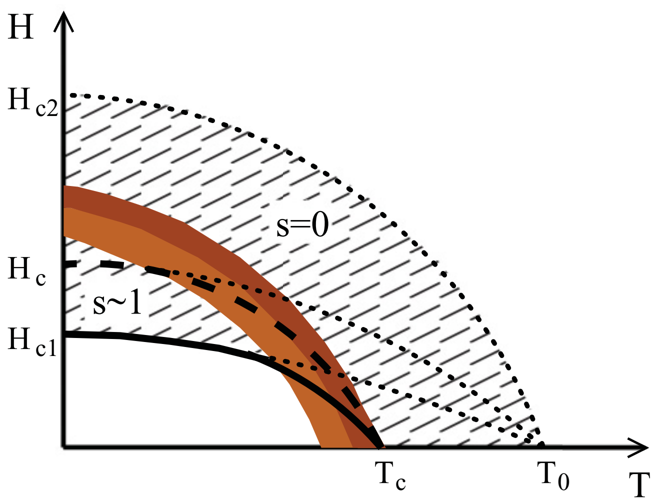

FIG. 2. (Color online) Schematic phase diagram of underdoped cuprates in the temperature vs magnetic field plane. The dotted lines denote the critical fields H_{c1}^0, H_0, and $H_{c2}^0 = H_{c2}$ of the conventional theory (bottom to top). The hatched region between H_{c1} and H_{c2} corresponds to a superconductor with vortex matter. Finer methods are needed to locate the position of the melting line between the vortex liquid ($s = 0$) and the vortex solid ($s > 0$) phases. The expected position of the melting line is at the upper margin of the shaded region. Inside the shaded region we expect $0 < s < 1$.

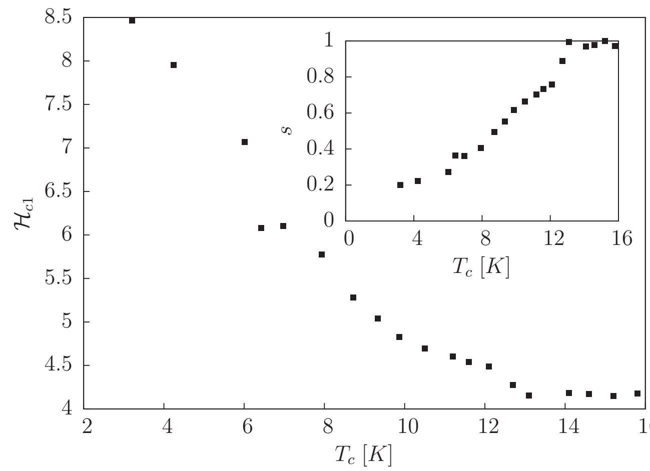

FIG. 3. Doping dependence of the dimensionless lower critical field $\mathcal{H}_{c1}$ in extremely underdoped YBCO samples in the limit $T = 0$. The data points were determined from the experimental data in Refs. 16,17, see the main text, using Eq. (1). The inset shows the doping dependence of the parameter s determined from the experimental values of $\mathcal{H}_{c1}$ using Eq. (5) and assuming $\ln\kappa = 3.65$.

In what follows we present further experimental tests of the modified GL theory against experimental data. In particular, we will study the temperature dependence of $\mathcal{H}_{c1}$ for optimally doped and extremely underdoped YBCO samples, i.e., along paths (b) and (c) schematically depicted in Fig. 1.

A. Classical superconductors

Before proceeding, let us start with the discussion of the function $\mathcal{H}_{c1} = \mathcal{H}_{c1}(T)$ in two well-studied low-temperature superconductors, Nb_3Sn and V_3Si. Experimental data for these classical type-II superconductors are presented in Fig. 4. First of all it should be pointed out that in neither case could we find data for H_{c1} and λ measured on the same sample; therefore some caution in interpretation of the data is necessary. Nevertheless the data do show common behavior. Namely, we find that in both cases $\mathcal{H}_{c1}$ is a weakly decreasing function of temperature.

Of course, even in conventional superconductors there is no reason why $\mathcal{H}_{c1}$ should not depend on temperature outside the GL region. This topic has been studied already in the 1960s and the results have been reviewed in Ref. 1. It has been pointed out there that for large-κ materials in magnetic fields close to H_{c1}, one can use the so-called temperature-dependent London approximation. Within this approach, it is useful to introduce a new dimensionless parameter κ_3 which is defined in terms of the measured values of the thermodynamic critical field H_c and of the penetration depth λ as follows:

$$\kappa_3 = \sqrt{2}\frac{2\pi\mu_0 H_c \lambda^2}{\Phi_0}. \tag{14}$$

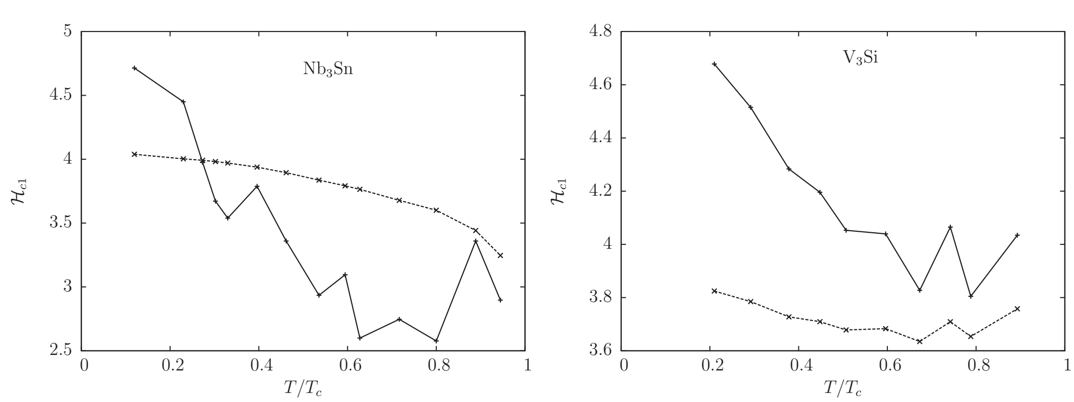

FIG. 4. Temperature dependence of the dimensionless lower critical field for classical superconductors. Solid lines: $\mathcal{H}_{c1}$ determined from H_{c1} and λ using Eq. (1). Dashed lines: $\mathcal{H}_{c1}$ determined from λ and H_c using Eq. (14) and $\mathcal{H}_{c1} = \ln\kappa_3 + 0.5$. Left panel: Nb_3Sn. The data for H_{c1}, λ, and H_c were taken from Refs. 18, 19, and 20, respectively. Right panel: V_3Si. The data for H_{c1}, λ, and H_c were taken from Refs. 21, 22, and 23, respectively.

Note that the parameter κ_3 reduces in the GL region to the usual GL parameter κ. According to Ref. 1, the leading term in the large-κ_3 expansion of the dimensionless lower critical field is $\mathcal{H}_{c1} \approx \ln \kappa_3$. In order to facilitate a smooth transition to the GL region, without proof we have extended this formula to $\mathcal{H}_{c1} = \ln \kappa_3 + 0.5$, which contains an additional constant term and which coincides with the conventional GL result Eq. (5) with $s = 1$. As shown in Fig. 4, the temperature-dependent London approximation does explain the decrease of $\mathcal{H}_{c1}$ with temperature, but the actually observed temperature dependence is slightly stronger than predicted by theory.

B. Cuprate superconductors

In the literature on cuprate superconductors, we could not find a simultaneous measurement of $H_{c1}(T)$ and $\lambda(T)$ on the same sample and we always had to compare data sets measured on samples with slightly varying T_c's. That is why we have used the reduced temperature $t = T/T_c$ instead of the absolute temperature T, which enabled us to construct the curves $\mathcal{H}_{c1} = \mathcal{H}_{c1}(t)$ and $\lambda^{-2} = \lambda^{-2}(t)$.

It is worth pointing out that the determination of $\mathcal{H}_{c1}$ at temperatures close to T_c is a delicate issue, since it is a product of H_{c1} which vanishes and of λ^2 which diverges in this limit. We have found that the resulting $\mathcal{H}_{c1}(t)$ curves depend sensitively on the precise T_c values of the samples used for measurements of H_{c1} and λ. As is evident, e.g., from the temperature dependence of λ^{-2} in Refs. 25, even the best samples exhibit some degree of inhomogeneity which shows up as a small tail of λ^{-2} above the average T_c, which is due to regions with a T_c higher than the average. For the same reason, determination of the critical temperature from the onset of the Meissner signal overestimates T_c, since it is sensitive to the regions with the highest T_c. Therefore we have determined the average T_c values of the studied samples by a careful fitting procedure described in Fig. 5.

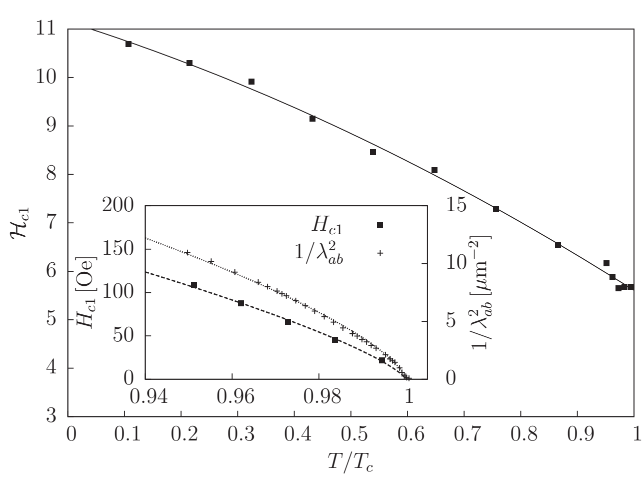

FIG. 5. Dependence of $\mathcal{H}_{c1}$ on the reduced temperature $t = T/T_c$ for optimally doped samples [path (b) in Fig. 1]. The data for H_{c1} and λ were taken from Refs. 24 and 25, respectively. The inset shows the fitting procedure which we used to determine the critical temperatures of the studied samples. Both the $H_{c1}(T)$ and the $1/\lambda^2(T)$ data were fitted to the function $a(T_c - T)^b$. We found $T_c = 92.5$ K, $b = 0.74$ for $H_{c1}(T)$ and $T_c = 92.13$ K, $b = 0.68$ for $1/\lambda^2(T)$.

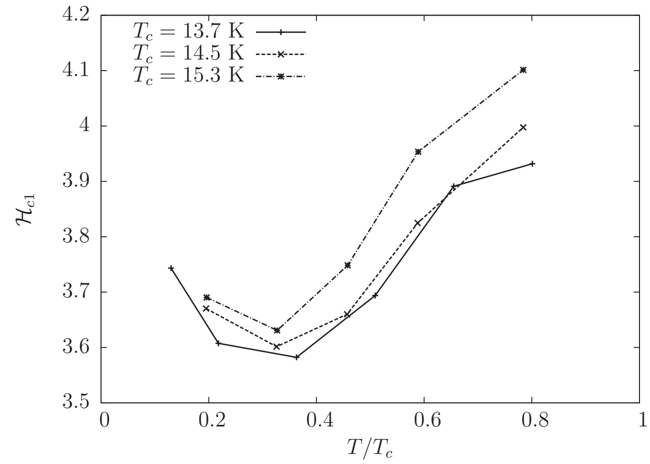

FIG. 6. Temperature dependence of $\mathcal{H}_{c1}$ for extremely underdoped samples [path (c) in Fig. 1]. The data for H_{c1} and λ were taken from Refs. 16 and 17, respectively.

Moreover, since YBCO is an orthorhombic material, both H_{c1} and λ are tensor quantities. We are interested in H_{c1} for magnetic fields perpendicular to the CuO_2 planes and in penetration depths for currents flowing within the planes. Note that penetration depths for currents flowing in the a and b directions may differ. Whenever the anisotropy has been measured, we have determined the penetration depth from the experimental data using $\lambda = \sqrt{\lambda_a \lambda_b}$.

Let us turn now to the discussion of the results. The temperature dependence of the dimensionless lower critical field $\mathcal{H}_{c1}$ for optimally doped samples is shown in Fig. 5. We find that $\mathcal{H}_{c1}$ decreases with increasing temperature, similarly as in the classical superconductors Nb_3Sn and V_3Si. This effect, while interesting, is not of direct interest to us in the present study. We conclude that optimally doped samples do not exhibit anomalous behavior in the vicinity of T_c and they should be described by the model with $s = 1$.

Temperature dependence of the dimensionless lower critical field $\mathcal{H}_{c1}$ of three extremely underdoped samples is shown in Fig. 6. Note that in this doping region, $\mathcal{H}_{c1}$ exhibits quite different behavior. The decrease of $\mathcal{H}_{c1}$ at low temperatures is very weak when compared with the optimally doped samples and, more importantly, in most of the measured temperature region $\mathcal{H}_{c1}$ increases with increasing temperature, in complete agreement with the predictions of the modified GL theory, which predicts that $\mathcal{H}_{c1}$ diverges as the temperature approaches T_c.

V. CONCLUSIONS

In this paper we have extended the GL theory for superconductors with reduced phase stiffness, which has been recently introduced in Ref. 4, in three independent directions.

First, we have presented a general argument which shows that if we are interested in physical phenomena described by a trivial phase field $S = 0$, then the predictions of the modified GL theory can be simply obtained from those of the conventional GL theory. This applies, e.g., to the surface energy, critical current of a thin wire, and critical fields H_{c2} and H_{c3}. On the other hand, nontrivial modifications of the conventional GL theory are necessary for $S \neq 0$. This is always the case if vortices are present.

Second, we have suggested a natural description of the pseudogap "phase" of the cuprates as a region where the GL wave function ψ_∞^2 is finite and where, at the same time, the phase stiffness vanishes since $s = 0$. This enabled us to construct the phase diagram Fig. 2 of the underdoped cuprates in the temperature vs magnetic field plane which shows that both the pseudogap region at high temperatures and low fields and the vortex liquid region at low temperatures and high fields belong to the same "phase."

Third, using Eq. (1) and the published experimental results for the penetration depth λ and for the lower critical field H_{c1}, we have calculated the dimensionless critical field $\mathcal{H}_{c1}$ for several type-II superconductors. We have shown that in conventional type-II superconductors, $\mathcal{H}_{c1}$ is a weak function of temperature which approaches a constant in the limit $T \to T_c$. Close to the transition temperature, we have found the same type of behavior also in optimally doped cuprate superconductors denoted as (b) in Fig. 1. On the other hand, in extremely underdoped cuprate superconductors denoted as (c) in Fig. 1, the dimensionless lower critical field $\mathcal{H}_{c1}$ increases sharply as the temperature approaches T_c. This behavior can be explained by assuming that the parameter s measuring the weakness of the phase stiffness approaches zero as $T \to T_c$, in agreement with our theory.

Further experimental work is needed to confirm the applicability of the modified GL theory to underdoped cuprates. Simultaneous measurements of the temperature dependence of H_{c1} and λ on the same sample would provide a more direct test of our picture than the analysis of different samples which we presented here. Moreover, measurements on samples with different doping levels could help to determine the precise shape of the shaded region in Fig. 1 where $0 < s < 1$. Alternatively, as discussed in detail in Ref. 4, the modified GL theory can be tested also by measuring the profile of the magnetic field in the superconducting vortex.

ACKNOWLEDGMENTS

We thank S. Takács for helpful discussions. This work was supported by the Slovak Research and Development Agency under Grants No. APVV-0432-07 and No. VVCE-0058-07, and by the Slovak Scientific Grant Agency under Grant No. 1/0096/08.

[1] A. Fetter and P. C. Hohenberg, in *Superconductivity*, edited by R. D. Parks (Marcel Dekker, New York, 1969), Vol. 2, p. 817.
[2] M. Tinkham, *Introduction to Superconductivity*, 2nd ed. (Dover, New York, 2004).
[3] G. Blatter and V. B. Geshkenbein, in *Superconductivity*, edited by K. H. Bennemann and J. B. Ketterson (Springer, Berlin, 2008), Vol. 1, p. 495.
[4] F. Horváth and R. Hlubina, Phys. Rev. B **81**, 134517 (2010).
[5] V. J. Emery and S. A. Kivelson, Nature (London) **374**, 434 (1995).
[6] Y. Wang, L. Li, and N. P. Ong, Phys. Rev. B **73**, 024510 (2006).
[7] J. Orenstein, J. Corson, S. Oh, and J. N. Eckstein, Ann. Phys. (Leipzig) **15**, 596 (2006).
[8] P. W. Anderson, P. A. Lee, M. Randeria, T. M. Rice, N. Trivedi, and F. C. Zhang, J. Phys. Condens. Matter **16**, R755 (2004).
[9] P. A. Lee, N. Nagaosa, and Xiao-Gang Wen, Rev. Mod. Phys. **78**, 17 (2006).
[10] P. A. Lee and Xiao-Gang Wen, Phys. Rev. Lett. **78**, 4111 (1997).
[11] L. B. Ioffe and A. J. Millis, Phys. Rev. B **66**, 094513 (2002).
[12] T. Senthil and P. A. Lee, Phys. Rev. B **79**, 245116 (2009).
[13] D. Saint-James, G. Sarma, and E. J. Thomas, *Type II Superconductivity* (Pergamon, New York, 1969).
[14] P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, New York, 1966).
[15] L. Li, J. G. Checkelsky, S. Komiya, Y. Ando, and N. P. Ong, Nature Phys. **3**, 311 (2007).
[16] R. Liang, D. A. Bonn, W. N. Hardy, and D. Broun, Phys. Rev. Lett. **94**, 117001 (2005).
[17] D. M. Broun, W. A. Huttema, P. J. Turner, S. Özcan, B. Morgan, R. Liang, W. N. Hardy, and D. A. Bonn, Phys. Rev. Lett. **99**, 237003 (2007).
[18] S. Ghamati, H. Suhl, W. Vernon, and G. Webb, IEEE Trans. Magn. **21**, 831 (1985).
[19] M. Perpeet and M. A. Hein, Phys. Rev. B **72**, 094502 (2005).
[20] V. Guritanu, W. Goldacker, F. Bouquet, Y. Wang, R. Lortz, G. Goll, and A. Junod, Phys. Rev. B **70**, 184526 (2004).
[21] D. Eckert and K. H. Berthel, Cryogenics **15**, 479 (1975).
[22] D. K. Christen, H. R. Kerchner, S. T. Sekula, and Y. K. Chang, Oak Ridge National Laboratory, Technical Report No. CONF-840842-26, doi: 10.2172/6887986.
[23] Y. Muto, N. Toyota, K. Noto, K. Akutsu, M. Isino, and T. Fukase, J. Low Temp. Phys. **34**, 617 (1979).
[24] Ruixing Liang, P. Dosanjh, D. A. Bonn, W. N. Hardy, and A. J. Berlinsky, Phys. Rev. B **50**, 4212 (1994).
[25] A. Hosseini, Saeid Kamal, D. A. Bonn, Ruixing Liang, and W. N. Hardy, Phys. Rev. Lett. **81**, 1298 (1998); See also [http://www.physics.ubc.ca/ supercon/datasets.html].