Poynting vector in negative-index metamaterials

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Metamaterials are characterized by counterintuitive phenomena, which have been debated since their first practical realization. Due to their anomalous backward power flow, it has been recently discussed whether even the classic definition of the Poynting vector, defining the power flux per unit surface, should be modified when dealing with metamaterials with exotic properties. Here, we settle this issue and fully clarify the physics of negative refraction, showing that the Poynting vector and other power relations in artificial materials may be unambiguously defined consistently with classic formulas for homogeneous media.

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I. POWER FLUX IN MACROSCOPIC MEDIA

The Poynting vector represents one of the cornerstones of the monument of Maxwell's theory. It is usually introduced starting from the calculation of the instantaneous time variation of the electromagnetic energy stored within a certain volume, which in vacuum is equal to the flux of $\mathbf{s} = \mathbf{e} \times \mathbf{b}/\mu_0$ through the boundary of the considered region[1] (here $\mathbf{e}$ and $\mathbf{b}$ represent the microscopic local fields and μ_0 is the free-space permeability). Although it is arguable whether $\mathbf{s}$ may be considered a local electromagnetic power density vector,[1,2] its flux through an arbitrary closed surface is well known to represent the instantaneous rate of energy change in the interior volume. When dealing with an idealized continuous isotropic material other than vacuum, with local permittivity and permeability ε, μ, it is well established that its definition should be modified as $\mathbf{s} = \mathbf{e} \times \mathbf{b}/\mu$.[1] In practice, any natural material is actually formed by a collection of finite-sized atoms and molecules, and therefore proper care should be taken in the definition of such a "continuum" and in how fields and constitutive parameters are averaged over several of these basic elements. This classic definition of the Poynting vector, nevertheless, has been validated by over a century-long series of experiments on electromagnetic wave propagation and it is well consistent with the continuity of power flow across a boundary between two materials with different constitutive parameters.

A rigorous definition of the Poynting vector gets more challenging when dealing with metamaterials, i.e., artificial materials formed by arrays of subwavelength inclusions with exotic electromagnetic properties. One of the goals of metamaterial research consists of properly describing these structures as bulk materials, with homogeneous constitutive parameters and averaged fields, filtering out the irrelevant unit-cell field fluctuations.[3,4] Using rigorous homogenization techniques, it has been shown theoretically and experimentally how negative index metamaterials (NIMs), for which the average power flow bends toward a negative angle at a planar interface with a regular dielectric or vacuum, may be realized in given frequency bands. As originally predicted in the 1960s,[5] the average (effective) permittivity and permeability of such materials have a negative real part, consistent with the backward flow of $\mathbf{s}$ with respect to the phase velocity. Despite the success of various experiments proving these anomalous properties, the counterintuitive wave interaction of NIMs[5] has raised a series of concerns about the applicability of classic electromagnetic theorems to such values of homogenized parameters, and this debate has recently extended to the very definition of the Poynting vector.[6,7]

In the case of metamaterials, it is evident that we should first define a proper averaging procedure for the microscopic fields that filters out their irrelevant higher-order fluctuations around the inclusions.[1] If we define with capital letters the averaged source fields in the form $\mathbf{E} = \langle \mathbf{e} \rangle$ and $\mathbf{B} = \langle \mathbf{b} \rangle$, it has been argued that the proper definition of the averaged Poynting vector in metamaterials should be modified to neglect the artificial magnetic effects on which negative refraction is based, and the creative definition $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ has been put forward,[7] independent of the actual value of the effective permeability. Based on this definition of the Poynting vector, the very notion of negative refraction has been put into discussion.[7,8]

Since the Poynting vector is inherently based on a (nonlinear) quadratic expression, the relation $\langle \mathbf{e} \times \mathbf{b} \rangle = \langle \mathbf{e} \rangle \times \langle \mathbf{b} \rangle$ is generally incorrect and thus, in general, $\mathbf{E} \times \mathbf{B}/\mu_0$ may not have the same meaning as its microscopic counterpart (this is evident if one considers the interface between such material and a magnetic homogeneous material with $\mu \neq \mu_0$, for which power conservation would not be satisfied if one associates to this expression the meaning of power flux density, as discussed ahead with more detail). In the following, we clarify these issues and prove from first-principles considerations that it is indeed possible to put forward a meaningful and self-consistent definition of the averaged Poynting vector and other power relations in metamaterials, consistent with those in natural materials.

II. POYNTING IN A SELF-CONSISTENT DIRECTION

In this section, we consider fields with a time-harmonic variation $e^{-i\omega t}$, so that, in media with no magnetism ($\mu = \mu_0$), the (time-averaged) Poynting vector is $\mathbf{s}_c = \frac{1}{2}\mathrm{Re}\{\mathbf{e} \times \mathbf{b}^*/\mu_0\}$. We base our discussion on the homogenization results of Ref. 9, where it was proven that in low-loss periodic arrays of dielectric inclusions the spatially averaged microscopic Poynting vector,

$$\mathbf{S} = \frac{1}{V_{\mathrm{cell}}} \int_{\Omega} \mathbf{s}_c \, d^3\mathbf{r}, \tag{1}$$

associated with Bloch-periodic time-harmonic fields $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, and real-valued wave vector $\mathbf{k}$, may be *exactly* related to the macroscopic effective permittivity of the homogenized metamaterial by the following relation:

$$\mathbf{S}\cdot\hat{l}=\frac{1}{2}\mathrm{Re}\left\{\left(\mathbf{E}\times\frac{\mathbf{B}^*}{\mu_0}\right)\cdot\hat{l}\right\}-\frac{1}{4}\omega\mathbf{E}^*\cdot\frac{\partial\bar{\bar{\varepsilon}}_{\mathrm{eff}}}{\partial k_l}(\omega,\mathbf{k})\cdot\mathbf{E},\quad l=x,y,z, \tag{2}$$

where $\mathbf{E}=\langle\mathbf{e}\rangle$, $\mathbf{B}=\langle\mathbf{b}\rangle$, the effect of the spatial averaging operator $\langle\rangle$ is an ideal low-pass filter such that $\mathbf{E}=\mathbf{E}_{\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$ with $\mathbf{E}_{\mathrm{av}}=\frac{1}{V_{\mathrm{cell}}}\int_\Omega\mathbf{e}(\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}\,d^3\mathbf{r}$ (Refs. 4 and 9) and k_l is the lth component of $\mathbf{k}$. The nonlocal permittivity tensor $\bar{\bar{\varepsilon}}_{\mathrm{eff}}(\omega,\mathbf{k})$ includes the complete description of the metamaterial macroscopic response, with arbitrary spatial dispersion effects in its dependence on $\mathbf{k}$, including artificial magnetism. It is defined consistently with optical crystal theory,[10] and can be numerically calculated as explained in Refs. 4 and 9. For macroscopic fields with the time-space variation $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, the nonlocal permittivity tensor relates the macroscopic polarization vector $\mathbf{P}=\langle(\varepsilon-\varepsilon_0)\mathbf{e}\rangle$ [$\varepsilon=\varepsilon(\mathbf{r})$ is the permittivity of the inclusions] and the macroscopic electric field as $\mathbf{P}=(\bar{\bar{\varepsilon}}_{\mathrm{eff}}(\omega,\mathbf{k})-\varepsilon_0\bar{\bar{\mathbf{I}}})\cdot\mathbf{E}$.

Equation (2) shows that the proper expression for $\mathbf{S}$ needs to be corrected to include the spatial dispersion in $\bar{\bar{\varepsilon}}_{\mathrm{eff}}$, and it is written in perfect analogy to the definition of power flux in natural optical crystals with spatially dispersive properties.[10] This is not necessarily surprising as this homogenization procedure may apply also to natural materials. Indeed, even though Ref. 9 considered only the case of metamaterials, we show in the Appendices that the validity of (2) extends even to the case of crystalline natural materials, such that the interaction of atoms and molecules with the local field can be described by an electric polarizability tensor $\bar{\bar{\alpha}}_e=\bar{\bar{\alpha}}_e(\omega)$ (electric dipole approximation).

Specifically, within such a classical framework the spatially averaged microscopic Poynting vector is linked to the macroscopic response of the material as in Eq. (2). The relation is mathematically exact as long as the Sipe-Kranendonk lossless condition $\mathrm{Im}\{\bar{\bar{\alpha}}_e^{-1}\}=-\frac{1}{6\pi}(\frac{\omega}{c})^3\bar{\bar{\mathbf{I}}}$ holds,[11,12] i.e., provided the power extracted by the dipoles from the local field is equal to the radiated power. This ensures that there is neither absorption nor generation of energy by the system. The proof of these results is presented in Appendices A and B. Obviously, for the case of natural dielectrics the nonlocal effects are typically quite insignificant, and thus Eq. (2) establishes that in such circumstances $\langle\frac{1}{2}\mathrm{Re}\{\mathbf{e}\times\mathbf{b}^*/\mu_0\}\rangle\approx\frac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{B}^*/\mu_0\}$.

These considerations clearly demonstrate that for low-loss and time-harmonic fields it is possible to define the macroscopic Poynting vector in either natural media or metamaterials self-consistently with the standard definition of the Poynting vector in vacuum, $\mathbf{s}=\mathbf{e}\times\mathbf{b}/\mu_0$, i.e., in such a way that the macroscopic Poynting vector $\mathbf{S}$, calculated within the framework of an effective-medium theory, is exactly coincident with the spatially and time-averaged microscopic Poynting vector in vacuum.

Even though Ref. 9 considered only the case of Bloch natural modes, these theoretical results remain valid even when a macroscopic source (described by an arbitrary external current $\mathbf{j}_e$) is embedded in the composite material. Particularly, as demonstrated in Appendix A, Eq. (2) still holds in the limit of vanishing loss, even when the material is excited by a distributed current $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$, with $\mathbf{J}_{e,\mathrm{av}}$ an arbitrary complex vector (the microscopic fields excited by such an external current also have Bloch properties, but are not the natural modes of the array). It is interesting to highlight that a more general, arbitrarily localized source can always be written as a superposition of currents of the form $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$ (associated with different wave vectors and amplitudes), and consequently the fields created by such a localized source may be Fourier expanded in terms of the fields excited by the elementary excitations $\mathbf{j}_e=\mathbf{J}_{e,\mathrm{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$, due to the linearity of the problem.

After having established that the result (2) may be applied to arbitrarily excited metamaterials with a general form of spatial dispersion, let us apply it specifically to the case wherein the macroscopic description of the metamaterial may be characterized by effective local relative permittivity $\varepsilon_{\mathrm{eff}}(\omega)$ and permeability $\mu_{\mathrm{eff}}(\omega)$ (for simplicity, we restrict our discussion to the case of isotropic media and assume no magnetoelectric coupling), as in NIMs. As discussed in Refs. 4, 9, 13, and 14, the generalized permittivity tensor under these assumptions has the form

$$\frac{\bar{\bar{\varepsilon}}_{\mathrm{eff}}}{\varepsilon_0}(\omega,\mathbf{k})=\varepsilon_{\mathrm{eff}}\bar{\bar{\mathbf{I}}}+\left(\mu_{\mathrm{eff}}^{-1}-1\right)\frac{c^2}{\omega^2}\mathbf{k}\times\bar{\bar{\mathbf{I}}}\times\mathbf{k}, \tag{3}$$

where $\bar{\bar{\mathbf{I}}}$ is the identity dyadic. Substituting the above formula into (2), we readily recuperate the classic textbook formula for a time-averaged Poynting vector in magnetodielectric media:[9]

$$\mathbf{S}=\tfrac{1}{2}\mathrm{Re}\{\mathbf{E}\times\mathbf{H}^*\}, \tag{4}$$

where $\mathbf{H}\equiv\mu_0^{-1}\mu_{\mathrm{eff}}^{-1}\mathbf{B}$, ensuring that the only meaningful definition of power flux vector density in metamaterials coincides with the well-established one for natural materials. Formula (4) remains valid even if the metamaterial has a bianisotropic reciprocal response, as shown in Ref. 9 (the definition of $\mathbf{H}$ must however be modified to take into account the magnetoelectric coupling). This shows that proper spatial averaging of the microscopic power flux vector $\mathbf{s}_c$, in general affected by the spatial dispersion in $\bar{\bar{\varepsilon}}_{\mathrm{eff}}(\omega,\mathbf{k})$, requires considering the second-order spatial dispersion effects associated with the effective metamaterial permeability. This expression applies equally well to negative values of μ_{eff}, i.e., it holds for NIM.

In order to highlight the importance of our findings, let us consider an arbitrary surface Σ that may intersect one or more

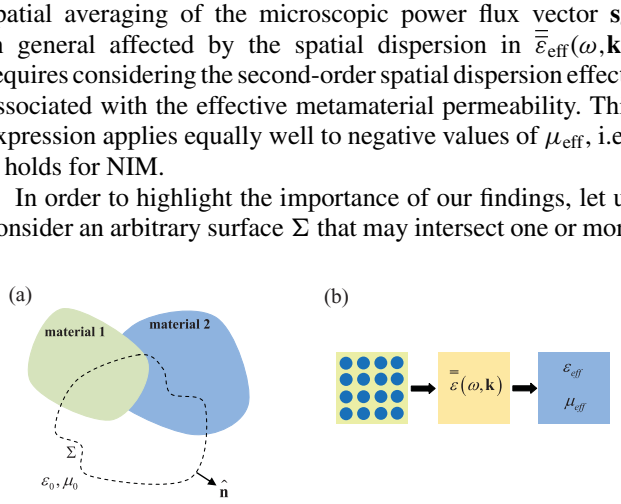

FIG. 1. (Color online) (a) A closed surface Σ encloses a nonuniform region. (b) Extraction of the effective-medium parameters of a composite material based on the theory of Ref. 4.

different materials and may enclose sources of radiation, as illustrated in Fig. 1(a). The power flow toward the region outside Σ can be unambiguously calculated through the microscopic Poynting vector as $P = \int_\Sigma \hat{\mathbf{n}} \cdot \mathbf{s}_c \, ds$. Obviously, because of the strong fluctuations of the microscopic fields near the portions of Σ that intersect regions with matter, the use of $\mathbf{e}$ and $\mathbf{b}$ to calculate the power flow is impractical. However, provided the wavelength of radiation is much larger than the characteristic granularity of the materials, it is possible to average out the fluctuations of the microscopic fields and use the effective-medium theory. Indeed, assuming that Σ encloses a macroscopic region, we can write

$$P = \int_\Sigma \hat{\mathbf{n}} \cdot \mathbf{s}_c \, ds \approx \int_\Sigma \hat{\mathbf{n}} \cdot \mathbf{S} \, ds, \tag{5}$$

where $\mathbf{S}$ is the spatially averaged Poynting vector, which, as discussed before, in the case of local materials may be written in terms of the average fields and effective parameters as in Eq. (4). It follows that $\mathbf{S}$ can be really regarded as an averaged flux of electromagnetic power, and the power flow through the generic surface Σ remains the same, independent of the one using the microscopic or the macroscopic fields. Actually, this property is true even if Σ is not a closed surface, because the averaging operator $\langle\rangle$ establishes an intrinsically local relation between macroscopic and microscopic fields.

From the previous discussion it should also be clear that for the Poynting vector to be really regarded as a flux of electromagnetic energy within an effective-medium description, then $\mathbf{S}$ needs to be defined self-consistently as in Eq. (4) in every material, because any acceptable definition of a macroscopic Poynting vector is constrained to satisfy Eq. (5). In particular, the definition of the macroscopic Poynting vector in a given material cannot be made independent of the definition of the (macroscopic) Poynting vector in vacuum, and it certainly cannot be simultaneously compatible with different forms, such as $\mathbf{D} \times \mathbf{B}$, $\mathbf{E} \times \mathbf{H}$, etc.[6]

III. POWER FLOW AND ENERGY RELATIONS IN NEGATIVE-INDEX METAMATERIALS: NUMERICAL VERIFICATION

In order to validate our theory in a practical example, consider a two-dimensional (2D) metamaterial array formed by high-index cylindrical dielectric inclusions with radius R, permittivity $\varepsilon_d = \varepsilon' + i\varepsilon''$, and permeability $\mu = \mu_0$, embedded in a plasmonic background material. The inclusions are arranged in a square lattice with period a, as in the inset of Fig. 2(d), excited with electric field in the xoy plane and magnetic field along z, the cylinder axis. The permittivity of the host medium follows the Drude dispersion model

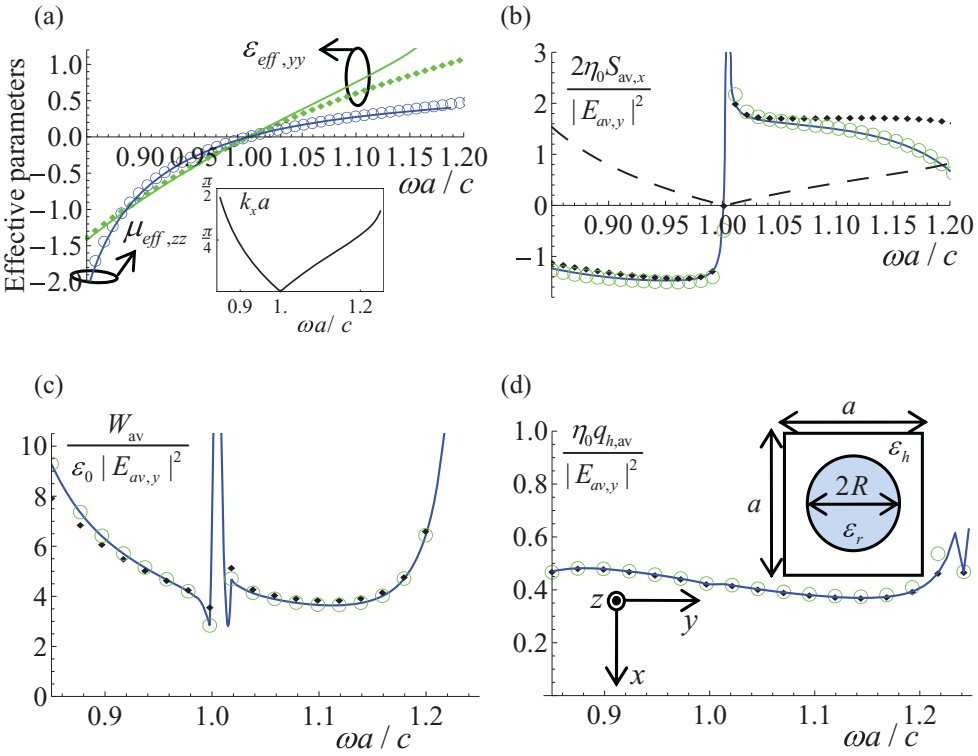

FIG. 2. (Color online) (a) Effective permittivity ε_{eff} (green curves) and permeability μ_{eff} (blue curves) versus normalized frequency $\omega a/c$. The discrete symbols correspond to the full wave homogenization values (Ref. 13) and the solid lines are obtained using Clausius-Mossotti formulas. The inset shows the wave vector k_x as a function of frequency. (b) x component of the Poynting vector calculated using (i) averaged microscopic Poynting vector (solid blue) [Eq. (1)], (ii) nonlocal homogenization model (circles) [Eq. (2)], (iii) local effective parameters (diamonds) [Eq. (4)], and (iv) results based on the definition of Poynting vector of Ref. 7. (c) Stored energy calculated using (i) averaged microscopic stored energy (solid blue) [Eq. (6)], (ii) nonlocal homogenization model (circles) [Eq. (7)], and (iii) local effective parameters (diamonds) [Eq. (8)]. (d) Heating rate calculated using (i) averaged microscopic heating rate (solid blue) [Eq. (9)], (ii) nonlocal homogenization model (circles) [Eq. (10)], and (iii) local effective parameters (diamonds) [Eq. (11)]. The unit-cell geometry is shown in the inset.

$\varepsilon_h = 1 - \frac{\omega_p^2}{\omega(\omega+i\Gamma)}$, where $\omega_p a/c = 1.0$ is the plasma frequency and Γ is the collision frequency. Consider first the case in which the background material is lossless, i.e., $\Gamma/\omega_p = 0$. We choose the cylinder permittivity $\varepsilon_d \approx 50.47\varepsilon_0$ and the normalized radius $R/a = 0.435$ to ensure that $\varepsilon_{\text{eff}}(\omega) \approx \mu_{\text{eff}}(\omega) \approx -1$ at $\omega a/c = 0.87$, providing a negative index of refraction and good matching with free space in this frequency range, reproducing with good approximation the properties of a Veselago-Pendry metamaterial lens.[5]

Figure 2(a) shows the local effective parameters $\varepsilon_{\text{eff}}(\omega)$, $\mu_{\text{eff}}(\omega)$, extracted as $\varepsilon_{\text{eff}}(\omega) = \hat{\mathbf{x}} \cdot \overline{\overline{\varepsilon}}_{\text{eff}}(\omega, \mathbf{k} = 0) \cdot \hat{\mathbf{x}}/\varepsilon_0$ and $\mu(\omega) = 1 + (1 - \frac{1}{2\varepsilon_0}\frac{\omega^2}{c^2}\hat{\mathbf{y}} \cdot \frac{\partial^2 \overline{\overline{\varepsilon}}_{\text{eff}}}{\partial k_x^2} \cdot \hat{\mathbf{y}}|_{\mathbf{k}=0})^{-1}$ (Ref. 4) from the nonlocal dielectric function $\overline{\overline{\varepsilon}}_{\text{eff}}(\omega, \mathbf{k})$, evaluated numerically[13] [the homogenization method is summarized in Fig. 1(b)]. As seen in Fig. 2(a), both permittivity and permeability are simultaneously negative for $0.85 < \omega a/c < 1$. The solid lines represent the effective parameters predicted by Clausius-Mossotti mixing formulas,[1] which agree fairly well with the exact data extracted numerically (discrete symbols), confirming that a quasistatic local description of such a NIM is accurate.

In order to characterize the Poynting vector in the metamaterial, we have fixed the wave vector at each frequency ω so that $k_x = \omega/c\sqrt{|\varepsilon_{\text{eff}}\mu_{\text{eff}}|}$, as reported in the inset of Fig. 2(a). The structure is excited by an external current density distribution $\mathbf{j}_e = j_e\hat{\mathbf{y}}e^{+ik_x x}$, which effectively excites the natural Bloch mode of this array, and the corresponding microscopic fields are determined numerically.[13] The macroscopic averaged electric and induction fields are then calculated as described above. Figure 2(b) shows the computed x component of the Poynting vector calculated using (i) the spatial average of its exact microscopic definition $\mathbf{s}$ [Eq. (1)] (solid blue line), (ii) the macroscopic (averaged) fields and generalized permittivity tensor [Eq. (2)] (discrete circles), (iii) the local effective parameters [Eq. (4)] (discrete diamonds), and (iv) the incorrect definition of Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$ proposed in Ref. 7. Consistent with Ref. 9, the results obtained with microscopic and macroscopic (averaged) fields are coincident. Even the results obtained with a local model, using the standard Poynting vector definition (4), follow very closely the exact curves. In particular, it is seen that all three curves indeed flip the sign of power flow crossing the frequency ω_p, ensuring that for negative-index propagation ($\omega < \omega_p$) power univocally flows backward. Finally, the curve obtained assuming the wrong definition in Ref. 7 yields a completely different response, since it neglects the artificial magnetism introduced by high-permittivity cylinders, and it cannot be interpreted as an average power flux density. Its value is always positive for $k_x > 0$, incorrectly implying that no negative refraction and backward propagation would be available in such metamaterial.[7]

Our theory establishes that Eq. (2) yields *exactly* the same result as the spatial average of $\mathbf{s}$ in the limit of vanishing loss, ensuring that indeed Eq. (2) represents the correct macroscopic definition of the Poynting vector in an arbitrary metamaterial. In addition, it proves that Eq. (4) is the correct definition for local (meta)materials, as in the present example. As is well known, effects of loss are unavoidable in realistic metamaterials, and thus it is interesting to study how the

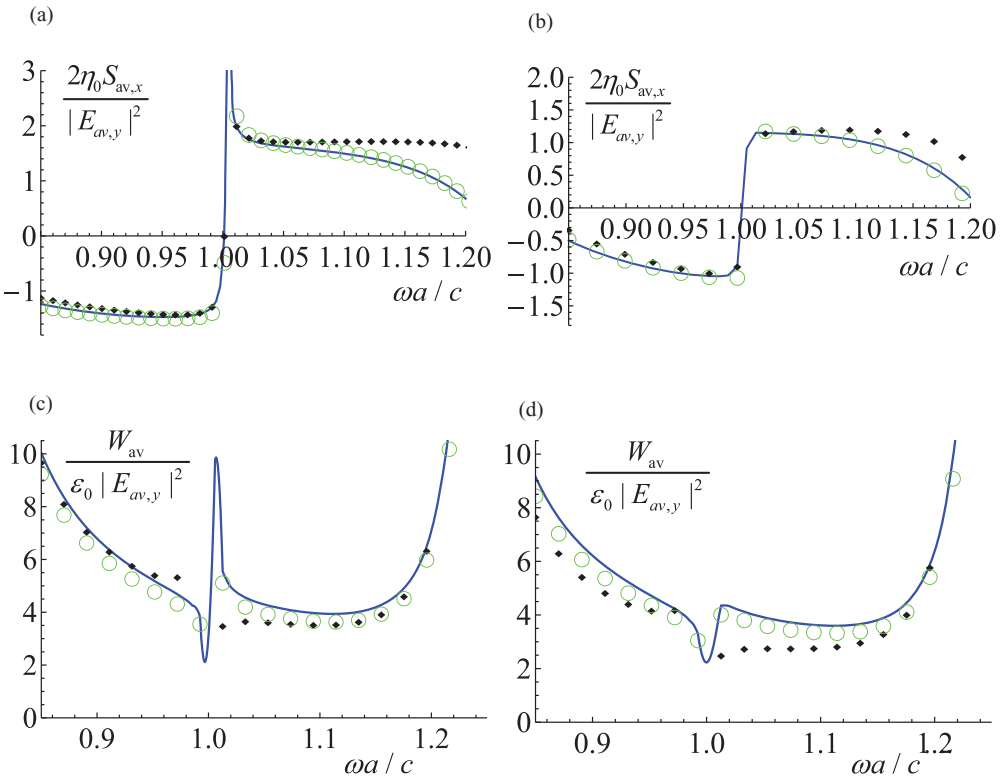

FIG. 3. (Color online) (a), (b) Similar to Fig. 2(b), except that losses are taken into account: (a) $\Gamma/\omega_p = 0.1$, (b) $\Gamma/\omega_p = 2.0$. (c), (d) Similar to Fig. 2(c), except that losses are taken into account: (c) $\Gamma/\omega_p = 0.1$, (d) $\Gamma/\omega_p = 0.5$.

energy flux vectors compare in a more realistic scenario, as in Figs. 3(a) and 3(b), where we consider $\Gamma/\omega_p = 0.1$ and $\Gamma/\omega_p = 2.0$, respectively. Despite the presence of strong loss, the general agreement between Eqs. (1), (2), and (4) remains very good, confirming that even in lossy systems the macroscopic Poynting vector can be self-consistently defined (to a very good approximation) with the averaged microscopic Poynting vector.

To further validate this effective-medium theory, we have used CST Microwave Studio to study the refraction of a Gaussian beam by a finite metamaterial slab with the same microstructure. The considered metamaterial slab is finite along the x and y directions, with thicknesses $L_x = 17.8a$ and $L_y = 60.4a$, respectively. In the simulation, the effect of loss is taken into account by considering $\Gamma/\omega_p = 0.05$. A Gaussian beam illuminates the structure at an angle $\theta_i = 33^\circ$. Figure 4(a) shows a time snapshot of H_z at $\omega a/c = 0.87$, for which $\varepsilon_{\text{eff}} \approx \mu_{\text{eff}} \approx -1$. Negative refraction is evident in the plot and, in the corresponding time animation,[15] it is possible to verify that the phase velocity is antiparallel with the direction of energy flow inside the slab, which confirms that S_x and k_x have opposite signs, as correctly predicted by our theory [Fig. 2(b)]. In Fig. 4(b) we have analyzed the emergence of backward wave propagation and negative refraction in the presence of much higher loss, $\Gamma/\omega_p = 0.9$ (this value may model the response of some semiconductors). Notwithstanding the presence of strong absorption, the emergence of negative refraction is still evident, even though now the transmission level is quite weak here.

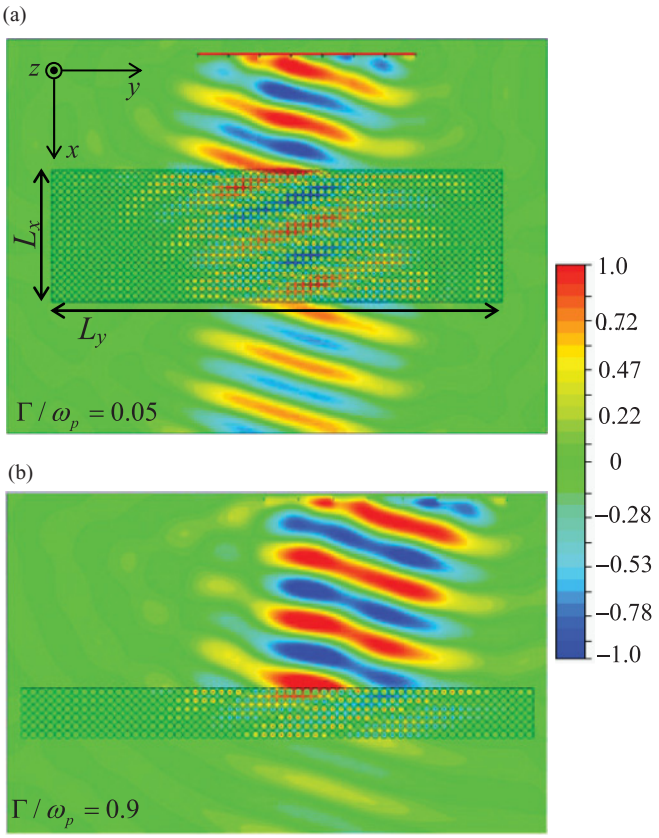

FIG. 4. (Color online) Time snapshot of H_z for a Gaussian beam illuminating the metamaterial slab ($\theta_i = 33^\circ$). (a) slab with dimensions $L_x = 17.8a$ and $L_y = 60.4a$ and plasmonic host with collision frequency $\Gamma/\omega_p = 0.05$. (b) Similar to (a) but for the parameters $L_x = 6a$, $L_y = 60a$, and $\Gamma/\omega_p = 0.9$.

Within this framework, we can also consider other spatially averaged energy quantities of interest. The expression for the averaged stored energy,

$$W_{\text{av}} = \frac{1}{4V_{\text{cell}}}\int_\Omega \frac{|\mathbf{b}|^2}{\mu_0} d^3\mathbf{r} + \frac{1}{4V_{\text{cell}}}\int_\Omega \frac{\partial}{\partial\omega}(\omega\varepsilon)|\mathbf{e}|^2\, d^3\mathbf{r}, \tag{6}$$

may be related to the macroscopic effective parameters in analogy with Eq. (2) as[9]

$$W_{\text{av}} = \frac{1}{4}\frac{|\mathbf{B}|^2}{\mu_0} + \frac{1}{4}\mathbf{E}^* \cdot \frac{\partial}{\partial\omega}(\omega\bar{\bar{\varepsilon}}_{\text{eff}}) \cdot \mathbf{E}. \tag{7}$$

Equations (6) and (7) are strictly valid for vanishingly small loss and time-harmonic Bloch microscopic fields. In the special case of local metamaterials, for which Eq. (3) applies, the stored energy expression becomes

$$W_{\text{av}} = \frac{1}{4}\frac{\partial}{\partial\omega}(\omega\mu_0\mu_{\text{eff}})|\mathbf{H}|^2 + \frac{1}{4}\frac{\partial}{\partial\omega}(\omega\varepsilon_0\varepsilon_{\text{eff}})|\mathbf{E}|^2, \tag{8}$$

which is consistent with classic textbook formulas for magnetoelectric dispersive media,[1] but extended here to local metamaterials and effective constitutive parameters.

Figure 2(c) shows the stored energy W_{av} in the metamaterial sample of Fig. 2(b), comparing the results based on Eqs. (6)–(8), in analogy with Fig. 1(b). It is seen that the exact averaged stored energy [Eq. (6)] (solid blue line) coincides with the stored energy computed from nonlocal effective parameters [Eq. (7)] (discrete circles), in agreement with Ref. 9. Moreover, the results obtained from local effective parameters [Eq. (8)] (discrete diamonds) follow reasonably well the two curves, confirming that the metamaterial response is local. When losses are taken into account [Figs. 3(c) and 3(d), consistent with the lossy scenario of Figs. 3(a) and 3(b)], the results obtained using Eqs. (6) and (7) partially lose their close agreement, but this is not surprising, since the same definition of macroscopic stored energy density loses much of its physical meaning in presence of losses (Ref. 10, p. 63). Still, the different curves show good agreement in the limit of low losses.

Finally, we can apply analogous considerations to the heating rate in metamaterials. The spatially averaged heating rate is analogously defined as

$$q_{\text{av}} = \frac{1}{V_{\text{cell}}}\int_\Omega \frac{\omega}{2}\varepsilon''(\mathbf{r})|\mathbf{e}(\mathbf{r})|^2\, d^3\mathbf{r}, \tag{9}$$

which may be written in terms of the generalized permittivity tensor as[9]

$$q_{\text{av}} = \tfrac{1}{2}\text{Re}\{-i\omega\mathbf{E}^* \cdot \bar{\bar{\varepsilon}}_{\text{eff}}(\omega,\mathbf{k}) \cdot \mathbf{E}\}. \tag{10}$$

The above formula exactly holds only in the case of real-valued wave vector $\mathbf{k}$, which, due to the presence of loss, can only be obtained if the metamaterial is excited by an external current distribution $\mathbf{j}_e = \mathbf{J}_{e,\text{av}}e^{i\mathbf{k}\cdot\mathbf{r}}$ (they are evidently not eigenmodes of a lossy array). For a local material, Eq. (10) simplifies into

$$q_{\text{av}} = \tfrac{1}{2}\omega\varepsilon_0\varepsilon''_{\text{eff}}(\omega)|\mathbf{E}|^2 + \tfrac{1}{2}\omega\mu_0\mu''_{\text{eff}}(\omega)|\mathbf{H}|^2, \tag{11}$$

again extending the textbook formulas for natural materials to local metamaterials. Figure 2(d) shows the comparison among these three definitions (9)–(11) for $\Gamma/\omega_p = 0.1$ and cylinder permittivity $\varepsilon_r = 50.47 + 0.1i$ (for simplicity, frequency dispersion in the dielectric is ignored). The results obtained using Eqs. (9) (solid blue line) and (10) (discrete circles) are virtually coincident, consistent with Ref. 9. Similarly, the results computed using the local model [Eq. (11)] (discrete diamonds) agree extremely well.

IV. CONCLUSIONS

We have shown here how it is possible to self-consistently define the Poynting vector and energy relations in metamaterials with local constitutive parameters, even when the negative index of refraction or anomalous values of effective constitutive parameters are considered. We have proven from first-principles considerations based on the general theory derived in Ref. 9 that the correct definitions of the Poynting vector, stored energy, and heating rate in NIMs coincide with those in natural materials after properly defining macroscopic averaged fields and effective constitutive parameters, and that other proposed definitions are not physically meaningful. We have also validated our results with full-wave numerical simulations considering a 2D NIM, which indeed supports backward propagation and power flow antiparallel to phase velocity, showing excellent quantitative agreement with Eq. (4) and our theory. Moreover, within a purely classical framework, we have extended the theory of Ref. 9 to the case of natural dielectrics, showing that also in this case the spatially averaged microscopic Poynting vector can be written in terms of macroscopic fields. This further supports that, independent of whether we consider one or two levels of homogenization (over an atomic scale in natural media, and over an additional mesoscopic scale in metamaterials), the macroscopic Poynting vector can always be defined self-consistently with its form in vacuum, maintaining its physical meaning of a power flux density vector.

To conclude, we would like to point out that our theory implies (in time-harmonic regime) that if the electromagnetic momentum density is $\mathbf{g} = \varepsilon_0 \mathbf{e} \times \mathbf{b}$ at the microscopic (molecular) level, then the average electromagnetic momentum density in a macroscopic local medium is given by $\langle \mathbf{g} \rangle = \varepsilon_0 \langle \mathbf{e} \times \mathbf{b} \rangle = \mathbf{E} \times \mathbf{H}/c^2$, which is the so-called Abraham form.[16] Notice that at the microscopic level, if the fields are not averaged in any form, both the Minkowski and the Abraham forms are coincident. However, as discussed in detail in Ref. 16, the electromagnetic momentum of the fields does not give a complete description of a physical system: One also needs to consider the terms related to the motion of matter (e.g., the mechanical momentum of the electrons in the molecular dipoles).[1,16] Therefore, $\langle \mathbf{g} \rangle = \varepsilon_0 \langle \mathbf{e} \times \mathbf{b} \rangle = \mathbf{E} \times \mathbf{H}/c^2$ is insufficient to characterize the *total* momentum a physical system incorporating both electromagnetic waves and material media. Quite differently, the results of this work indicate that the (time-averaged) energy flux vector (understood as a flux of *electromagnetic energy*) can be fully described by $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ in low loss macroscopic media with a local effective response.

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APPENDIX A: SPATIALLY AVERAGED POYNTING VECTOR

In this appendix, we extend the theory of Ref. 9 to the cases where (i) the medium is excited with an external (macroscopic current) and (ii) the medium is formed by point dipole-type electrical scatterers. The latter case may model with good approximation (within the framework of a purely classical theory) the response of natural dielectrics. Let us consider a periodic material such that the microscopic fields (in time-harmonic regime) satisfy the Maxwell's equations

$$\nabla \times \mathbf{e} = i\omega \mathbf{b}; \quad \nabla \times \frac{\mathbf{b}}{\mu_0} = -i\omega\varepsilon_0 \mathbf{e} + \mathbf{j}_d + \mathbf{j}_e. \tag{A1}$$

In the above, $\mathbf{j}_e$ is the external (macroscopic) excitation (if any), and $\mathbf{j}_d$ is the microscopic density of current induced in the material. Since $\mathbf{j}_e$ is a macroscopic excitation, it must satisfy $\mathbf{j}_e = \langle \mathbf{j}_e \rangle$ and, assuming it has the Bloch property, it is necessarily of the form $\mathbf{j}_e = \mathbf{J}_{e,\mathrm{av}} e^{i\mathbf{k}\cdot\mathbf{r}}$, with $\mathbf{J}_{e,\mathrm{av}}$ a constant vector. The induced currents are of the form $\mathbf{j}_d = -i\omega(\varepsilon - \varepsilon_0)\mathbf{e}$, if the material is described at the microscopic level by a dielectric function $\varepsilon = \varepsilon(\mathbf{r})$ (e.g., metamaterials), or alternatively by $\mathbf{j}_d = \sum_{\mathbf{I}} -i\omega \mathbf{p}_{e,\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}_{\mathbf{I}})$, in the limit in which the material may be described as a collection of electric point dipoles at a microscopic level (e.g., classical description of natural materials); $\mathbf{r}_{\mathbf{I}}$ represents the sites of the dipoles and $\mathbf{p}_{e,\mathbf{I}}$ the corresponding electric dipole moments.

Let us now consider two Bloch solutions $\mathbf{e}_1$ and $\mathbf{e}_2$ of Eq. (A1), associated with the external currents $\mathbf{j}_{e,1} = \mathbf{J}_{e,1} e^{i\mathbf{k}_1\cdot\mathbf{r}}$ and $\mathbf{j}_{e,2} = \mathbf{J}_{e,2} e^{i\mathbf{k}_2\cdot\mathbf{r}}$, where the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are such that $\mathbf{k}_1 = \mathbf{k}_2^*$. In case $\mathbf{e}_1$ is a natural mode of the periodic structure the external current $\mathbf{j}_{e,1}$ is trivial ($\mathbf{j}_{e,1} = 0$). A similar statement can be made about the field $\mathbf{e}_2$.

Similar to the analysis of Ref. 9, Eq. (26), $\mathbf{e}_1$ can always be regarded as an element of a family $\mathbf{e}_{1F}(\mathbf{r};\mathbf{k})$ of solutions of Maxwell's equations associated to the external excitation $\mathbf{j}_{e,1} = \mathbf{J}_{e,1}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}$, such that

$$\mathbf{J}_{e,1}(\mathbf{k}) = -i\omega\varepsilon_0 \left[-\frac{1}{\varepsilon_0} \bar{\bar{\varepsilon}}_{\mathrm{eff}}(\omega,\mathbf{k}) + \frac{c^2}{\omega^2} k^2 \bar{\bar{\mathbf{I}}} - \frac{c^2}{\omega^2} \mathbf{k}\mathbf{k} \right] \cdot \mathbf{E}_{\mathrm{av},1}. \tag{A2}$$

Specifically, we have that $\mathbf{e}_1(\mathbf{r}) = \mathbf{e}_{1F}(\mathbf{r};\mathbf{k}_1)$. In the above, $\mathbf{E}_{\mathrm{av},1}$ is such that $\langle \mathbf{e}_1 \rangle = \mathbf{E}_{\mathrm{av},1} e^{i\mathbf{k}_1\cdot\mathbf{r}}$. It is interesting to note that the amplitude of the external excitation, $\mathbf{J}_{e,1}(\mathbf{k})$, is written in terms of the dielectric function of the effective medium. All the elements of the family $\mathbf{e}_{1F}(\mathbf{r};\mathbf{k})$ have the same amplitude after the microscopic fluctuations are filtered out: $\langle \mathbf{e}_{1F}(\cdot;\mathbf{k}) \rangle = \mathbf{E}_{\mathrm{av},1} e^{i\mathbf{k}\cdot\mathbf{r}}$ (with $\mathbf{E}_{\mathrm{av},1}$ independent of $\mathbf{k}$).[9] For simplicity, in what follows we will drop the subscript F from the family of

fields $\mathbf{e}_{1F}(\mathbf{r};\mathbf{k})$ and regard simply $\mathbf{e}_1 = \mathbf{e}_1(\mathbf{r};\mathbf{k})$ as a continuous function of the wave vector.

As in Ref. 9, we define an auxiliary vector field $\mathbf{s}_{1,2}$ as follows:

$$\mathbf{s}_{1,2} = \frac{1}{-4i\omega\mu_0}(\mathbf{e}_1 \times \nabla \times \mathbf{e}_2^* - \mathbf{e}_2^* \times \nabla \times \mathbf{e}_1). \quad \text{(A3)}$$

It is a simple exercise to show that if

$$\mathbf{e}_1 \cdot \mathbf{j}_{d,2}^* + \mathbf{e}_2^* \cdot \mathbf{j}_{d,1} = 0, \quad \text{(A4)}$$

then

$$\nabla \cdot \mathbf{s}_{1,2} = \frac{-1}{4}(\mathbf{e}_1 \cdot \mathbf{j}_{e,2}^* + \mathbf{e}_2^* \cdot \mathbf{j}_{e,1}). \quad \text{(A5)}$$

The condition (A4) holds if the material is lossless. Indeed, when the material is modeled at the microscopic level by a dielectric function $\varepsilon = \varepsilon(\mathbf{r})$ and $\mathbf{j}_d = -i\omega(\varepsilon - \varepsilon_0)\mathbf{e}$, Eq. (A4) is always satisfied provided $\varepsilon = \varepsilon(\mathbf{r})$ is real valued. The case where the material is formed by an array of electric dipoles at the microscopic level is trickier and is studied in Appendix B.

Our objective is to calculate the spatial average of $\mathbf{s}_{1,2}$:

$$(\mathbf{s}_{1,2})_{\text{av}} = \frac{1}{V_{\text{cell}}}\int_\Omega \mathbf{s}_{1,2}\, d^3\mathbf{r}. \quad \text{(A6)}$$

To this end, we calculate the derivative of both members of Eq. (A5) with respect to k_l ($l = x,y,z$), regarding $\mathbf{e}_1 = \mathbf{e}_1(\mathbf{r};\mathbf{k})$ and $\mathbf{j}_{e,1} = \mathbf{J}_{e,1}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$ as functions of $\mathbf{k}$, as explained before. Integrating the resulting equation over the unit cell it is found that

$$\frac{1}{V_{\text{cell}}}\int_\Omega \nabla \cdot \frac{\partial \mathbf{s}_{1,2}}{\partial k_l} d^3\mathbf{r} = -\frac{1}{4V_{\text{cell}}}\int_\Omega \left(\frac{\partial \mathbf{e}_1}{\partial k_l} \cdot \mathbf{j}_{e,2}^* + \mathbf{e}_2^* \cdot \frac{\partial \mathbf{j}_{e,1}}{\partial k_l}\right) d^3\mathbf{r}. \quad \text{(A7)}$$

Now it is simple to check that for $\mathbf{k} = \mathbf{k}_1$ the function $\frac{\partial \mathbf{s}_{1,2}}{\partial k_l} - ix_l\mathbf{s}_{1,2}$ is periodic. Thus, the integral of $\nabla \cdot (\frac{\partial \mathbf{s}_{1,2}}{\partial k_l} - ix_l\mathbf{s}_{1,2})$ over the unit cell vanishes, and we find that

$$\frac{i}{V_{\text{cell}}}\int_\Omega \nabla \cdot (x_l\mathbf{s}_{1,2}) d^3\mathbf{r} = -\frac{1}{4V_{\text{cell}}}\int_\Omega \left(\frac{\partial \mathbf{e}_1}{\partial k_l} \cdot \mathbf{j}_{e,2}^* + \mathbf{e}_2^* \cdot \frac{\partial \mathbf{j}_{e,1}}{\partial k_l}\right) d^3\mathbf{r}. \quad \text{(A8)}$$

The above identity holds for $\mathbf{k} = \mathbf{k}_1$ [where $\mathbf{k}$ is the parameter associated with the family of functions $\mathbf{e}_1 = \mathbf{e}_1(\mathbf{r};\mathbf{k})$]. After trivial manipulations, using again Eq. (A5), it is found that

$$(\mathbf{s}_{1,2})_{\text{av},l} = \frac{1}{4V_{\text{cell}}}\int_\Omega \left(i\frac{\partial \mathbf{e}_1}{\partial k_l} + x_l\mathbf{e}_1\right) \cdot \mathbf{j}_{e,2}^* + \mathbf{e}_2^* \cdot \left(i\frac{\partial \mathbf{j}_{e,1}}{\partial k_l} + x_l\mathbf{j}_{e,1}\right) d^3\mathbf{r}, \quad \text{(A9)}$$

where $(\mathbf{s}_{1,2})_{\text{av},l} = (\mathbf{s}_{1,2})_{\text{av}} \cdot \hat{\mathbf{u}}_l$. Since $i\frac{\partial \mathbf{j}_{e,1}}{\partial k_l} + x_l\mathbf{j}_{e,1} = i\frac{\partial \mathbf{J}_{e,1}}{\partial k_l}e^{i\mathbf{k}_1\cdot\mathbf{r}}$ and $i\frac{\partial \mathbf{e}_1}{\partial k_l} + x_l\mathbf{e}_1 = i[\frac{\partial}{\partial k_l}(\mathbf{e}_1 e^{-i\mathbf{k}\cdot\mathbf{r}})]e^{i\mathbf{k}\cdot\mathbf{r}}$ we have that

$$(\mathbf{s}_{1,2})_{\text{av},l} = \frac{i}{4}\frac{\partial \mathbf{J}_{e,1}}{\partial k_l} \cdot \mathbf{E}_{\text{av},2}^* + \frac{i}{4V_{\text{cell}}}\mathbf{J}_{e,2}^* \cdot \left(\frac{\partial}{\partial k_l}\int_\Omega \mathbf{e}_1 e^{-i\mathbf{k}\cdot\mathbf{r}} d^3\mathbf{r}\right)_{\mathbf{k}=\mathbf{k}_1} \quad \text{(A10)}$$

However, the integral $\frac{1}{V_{\text{cell}}}\int_\Omega \mathbf{e}_1 e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{r} = \mathbf{E}_{\text{av},1}$ is independent of $\mathbf{k}$, and thus the second term in the right-hand side of the above equation vanishes. Therefore we conclude that

$$(\mathbf{s}_{1,2})_{\text{av},l} = \frac{i}{4}\mathbf{E}_{\text{av},2}^* \cdot \frac{\partial \mathbf{J}_{e,1}}{\partial k_l}. \quad \text{(A11)}$$

Using now Eq. (A2) it is found after straightforward calculations that

$$(\mathbf{s}_{1,2})_{\text{av},l} = \frac{1}{4}\left(\mathbf{E}_{\text{av},1} \times \frac{\mathbf{B}_{\text{av},2}^*}{\mu_0} + \mathbf{E}_{\mathbf{av},2}^* \times \frac{\mathbf{B}_{\text{av},1}}{\mu_0}\right)_l - \frac{\omega}{4}\mathbf{E}_{\text{av},2}^* \cdot \frac{\partial \overline{\overline{\varepsilon}}_{\text{eff}}}{\partial k_l}(\omega,\mathbf{k}_1) \cdot \mathbf{E}_{\text{av},1}, \quad \text{(A12)}$$

which is in agreement with Ref. 9. In particular, when $\mathbf{k}_1$ is real valued we can choose $\mathbf{e}_1 = \mathbf{e}_2$ and this yields Eq. (2).

APPENDIX B: PROOF OF EQ. (A4) FOR LOSSLESS POINT DIPOLES

Here, we demonstrate that, in case of a medium formed (at the microscopic level) by point-dipole electrical scatterers, condition (A4) is satisfied provided that

$$\text{Im}\{\overline{\overline{\alpha}}_e^{-1}\} = -\frac{1}{6\pi}\left(\frac{\omega}{c}\right)^3 \overline{\overline{\mathbf{I}}}, \quad \text{(B1)}$$

where $\overline{\overline{\alpha}}_e$ is the electric polarizability tensor of the dipoles, and the time variation $e^{-i\omega t}$ is implicit. The current density induced along a given dipole (which, without loss of generality, is assumed to be centered at the origin) is of the form $\mathbf{j}_d = -i\omega\mathbf{p}_e\delta(\mathbf{r})$ with $\mathbf{p}_e = \varepsilon_0\overline{\overline{\alpha}}_e \cdot \mathbf{e}_{\text{loc}}$; $\mathbf{e}_{\text{loc}}$ is the local field in the vicinity of the particle (i.e., the total field minus the self-induced field created by the particle itself).

To prove the enunciated result, we note that Eq. (A4) is equivalent to

$$\mathbf{e}_1(\mathbf{r}) \cdot \mathbf{p}_{e2}^*\delta(\mathbf{r}) - \mathbf{e}_2^*(\mathbf{r}) \cdot \mathbf{p}_{e1}\delta(\mathbf{r}) = 0. \quad \text{(B2)}$$

For true point dipoles the fields calculated at the dipole position are singular. To avoid such a pathologic situation first we will regard $\delta(\mathbf{r})$ not as the exact Dirac distribution, but instead as a smooth function (e.g., a Gaussian distribution) extremely localized at the origin. In these conditions the fields remain finite in the position of the dipole. Considering the decomposition $\mathbf{e}_1 = \mathbf{e}_{1,\text{loc}} + \mathbf{e}_{1,\text{self}}$, and using $\mathbf{p}_e = \varepsilon_0\overline{\overline{\alpha}}_e \cdot \mathbf{e}_{\text{loc}}$, we find that (using the fact that $\overline{\overline{\alpha}}_e$ is a symmetric tensor)

$$\mathbf{e}_{1,\text{self}}(\mathbf{r}) \cdot \mathbf{p}_{e2}^*\delta(\mathbf{r}) - \mathbf{e}_{2,\text{self}}^*(\mathbf{r}) \cdot \mathbf{p}_{e1}\delta(\mathbf{r}) + \frac{2i}{\varepsilon_0}\mathbf{p}_{e2}^* \cdot \text{Im}\{\overline{\overline{\alpha}}_e^{-1}\} \cdot \mathbf{p}_{e1}\delta(\mathbf{r}) = 0. \quad \text{(B3)}$$

The self-field is given by $\mathbf{e}_{\text{self}}(\mathbf{r}) = \frac{1}{\varepsilon_0}(\frac{\omega}{c})^2 \int \overline{\overline{G}}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{p}_e\delta(\mathbf{r}')d^3\mathbf{r}'$, where $\overline{\overline{G}} = (\overline{\overline{\mathbf{I}}} + c^2/\omega^2\nabla\nabla)\Phi_0(\mathbf{r} - \mathbf{r}')$ with $\Phi_0 = e^{i(\omega/c)r}/4\pi r$. Hence, we can write

$$\begin{aligned}
&\mathbf{e}_{1,\text{self}}(\mathbf{r}) \cdot \mathbf{p}_{e2}^* - \mathbf{e}_{2,self}^*(\mathbf{r}) \cdot \mathbf{p}_{e1} \\
&= \frac{1}{\varepsilon_0}\left(\frac{\omega}{c}\right)^2 \int (\mathbf{p}_{e2}^* \cdot \overline{\overline{G}}(\mathbf{r}|\mathbf{r}') \cdot \mathbf{p}_{e1} \\
&\quad -\mathbf{p}_{e1} \cdot \overline{\overline{G}}^*(\mathbf{r}|\mathbf{r}') \cdot \mathbf{p}_{e2}^*)\delta(\mathbf{r}')d^3\mathbf{r}' \\
&= \frac{1}{\varepsilon_0}\left(\frac{\omega}{c}\right)^2 2i \int \mathbf{p}_{e2}^* \cdot \text{Im}\{\overline{\overline{G}}(\mathbf{r}|\mathbf{r}')\} \cdot \mathbf{p}_{e1}\delta(\mathbf{r}')d^3\mathbf{r}', \quad \text{(B4)}
\end{aligned}$$

where we used the transpose symmetry of $\overline{\overline{G}}$. It can be easily checked that $\mathrm{Im}\{\overline{\overline{G}}(\mathbf{r}|\mathbf{r}')\}$ is a smooth function in the vicinity of $\mathbf{r}=\mathbf{r}'$. Hence, at this point it is possible to let $\delta(\mathbf{r})$ become the Dirac distribution. Therefore, Eq. (B3) yields

$$\left(\frac{\omega}{c}\right)^2 \mathbf{p}_{e2}^* \cdot \mathrm{Im}\{\overline{\overline{G}}(\mathbf{r}=\mathbf{r}')\} \cdot \mathbf{p}_{e1} + \mathbf{p}_{e2}^* \cdot \mathrm{Im}\left\{\overline{\overline{\alpha}}_e^{-1}\right\} \cdot \mathbf{p}_{e1} = 0. \tag{B5}$$

However, simple calculations show that $\mathrm{Im}\{\overline{\overline{G}}(\mathbf{r}=\mathbf{r}')\} = \frac{1}{6\pi}\frac{\omega}{c}\overline{\overline{\mathbf{I}}}$, and therefore we finally conclude that the condition (A4) is satisfied if and only if Eq. (B1) holds, as we wanted to prove.

It is interesting to note that Eq. (B1) is nothing more than the well-known Sipe-Kranendonk power-balance relation,[11] which guarantees that the power extracted by the dipole from the local field is equal to the radiated power.

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