Vortex-induced dissipation in narrow current-biased thin-film superconducting strips

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(Received 10 February 2011; published 27 April 2011)

A vortex crossing a thin-film superconducting strip from one edge to the other, perpendicular to the bias current, is the dominant mechanism of dissipation for films of thickness d on the order of the coherence length ξ and of width w much narrower than the Pearl length $\Lambda \gg w \gg \xi$. At high bias currents $I^* < I < I_c$ the heat released by the crossing of a single vortex suffices to create a belt-like normal-state region across the strip, resulting in a detectable voltage pulse. Here I_c is the critical current at which the energy barrier vanishes for a single vortex crossing. The belt forms along the vortex path and causes a transition of the entire strip into the normal state. We estimate I^* to be roughly $I_c/3$. Furthermore, we argue that such "hot" vortex crossings are the origin of dark counts in photon detectors, which operate in the regime of metastable superconductivity at currents between I^* and I_c. We estimate the rate of vortex crossings and compare it with recent experimental data for dark counts. For currents below I^*, that is, in the stable superconducting but resistive regime, we estimate the amplitude and duration of voltage pulses induced by a single vortex crossing.

DOI: 10.1103/PhysRevB.83.144526 PACS number(s): 74.78.−w, 85.25.Pb

I. INTRODUCTION

Dissipation in superconducting wires thinner than the coherence length ξ have been thoroughly studied both theoretically[1,2] and experimentally.[3] In these one-dimensional (1D) superconductors the dissipation arises due to 2π-phase slips occurring in segments of length ξ of a wire that becomes temporarily normal. Langer and Ambegaokar[1] treated the problem of dissipation in 1D wires with ring geometry within the theory of nucleation rates of current-reducing fluctuations in a superconductor. The transition between states with different currents in a ring occurs via the nonstationary state described by the saddle point solution of the Ginzburg-Landau (GL) functional. Langer and Ambegaokar[1] found such a solution and the corresponding free-energy difference or barrier $\mathcal{U}$ between the original metastable state with current and the saddle point state (see also Ref. 4). Later McCumber and Halperin derived the attempt frequency Ω in the phase-slip rate $R = \Omega \exp(-\mathcal{U}/T)$ using time-dependent GL theory.[2]

The problem of dissipation in superconducting thin-film strips with the thickness d much smaller than the London penetration depth λ, and of width w much smaller than the Pearl length $\Lambda = 2\lambda^2/d \gg w$, has been extensively discussed in the context of a possible Berezinsky-Kosterlitz-Thouless (BKT) transition in superconducting films.[5–7] The interest in current-carrying thin-film strips has been revived recently in search for quantum tunneling of vortices,[8–11] their dynamic behavior,[12] and the observation of so-called "dark counts" in superconductor-based photon detectors.[13,14] The detector consists of a long and thin superconducting strip carrying currents slightly below the critical value. Typically, in NbN photon detectors w is of the order of 100 nm or more and $d \approx 4$–6 nm, while the zero-temperature coherence length $\xi(0) \approx 4$ nm. The low-temperature London penetration depth $\lambda \approx 350$ nm so that the Pearl length[15] $\Lambda \approx 40\ \mu\text{m} \gg w$.

When a photon interacts with the strip it induces a hot spot in the film that drives a belt-like region across the strip into the normal state. Consequently, a voltage pulse caused by the current redistribution between the superconducting strip and a parallel shunt resistor is detected. After the normal belt of the strip cools down, the strip returns to superconducting state. Thus, single photons can be detected and counted by measuring voltage pulses. However, similar pulses are recorded even without photons (dark counts). These voltage pulses have peak amplitudes similar to photon-induced pulses.[16] Therefore, one can conclude that dark counts are also caused by nucleation of normal belts across the strip. In both cases and in the absence of a shunt, the entire strip undergoes transition into the normal state due to heat released by the bias current in the normal belt region.

In fact, the observation of dark counts means that the superconducting strip, at bias currents slightly below the critical current, is in a metastable state. Photons or fluctuations trigger the transition from this state to the normal state. Thus, the central question is what kind of fluctuations trigger the transition in the case of dark counts. The origin of dark counts is still debated (see Refs. 13 and 14). The problem of dark counts is related to the basic question of dissipation in thin films and wires and is of technological relevance because fluctuations resulting in the formation of normal belt across the strip limit the ability of superconducting circuits to carry supercurrents, in general, and the accuracy of photon detectors, in particular. In the literature, dark counts are treated either within the formal framework of 1D phase slips in thin wires or within the picture of vortex-antivortex unbinding near the BKT transition (see Refs. 13 and 14). Vortices crossing the strip were employed to explain dc current-voltage characteristics of thin-film strips.[11,17]

In this paper we discuss three types of possible fluctuations in superconducting strips which result in dissipation. Each one causes transition to the normal state from the metastable superconducting state when currents are close to the critical value I_c.

(a) Spontaneous nucleation of a normal-state belt across the strip with 2π-phase slip as in thin wires.

(b) Spontaneous nucleation of a single vortex near the edge of the strip and its motion across to the opposite edge accompanied by a voltage pulse.

(c) Spontaneous nucleation of vortex-antivortex pairs and their unbinding as they move across the strip to opposite edges due to the Lorentz force, as well as the opposite process of nucleation of vortices and antivortices at the opposite edges and their annihilation in the strip middle.

The energy barrier for the nucleation of a temporary normal phase-slip belt is too high to be of importance because the belt volume $\gtrsim d\xi w$ is large. We will show that such a barrier remains large at any current in the superconducting state. Consequently, belt-like 2π-phase slips appear with extremely low probability. On the other hand, as proposed in Refs. 11 and 17, thermally induced vortex crossings in current-carrying strips result in 2π-phase changes along the strip just as in the 1D scenario and hence cause dissipation. For the case of quantum tunneling this mechanism of dissipation was discussed in Refs. 8–11. The free energy barrier for vortex crossing is much lower than for belt-like 2π-phase slips since the vortex core volume is $d\xi^2 \ll d\xi w$. The energy cost of creating a vortex and moving it over the barrier is w/ξ times smaller than for creating a belt-like phase slip. An important point is that such a barrier for vortex crossing vanishes as the current approaches I_c, whereas the barrier for the belt-like phase slip remains nonzero at any current. As to the vortex-antivortex process of the point (c), we show in the following that the corresponding barrier is twice as high as for the single vortex process.

We evaluate the amplitude of a voltage pulse and its duration assuming that the belt-like area around the vortex path remains superconducting. We call this process a "cold" pulse. This is not always the case, because vortex motion excites quasiparticles along the vortex path and their energies depending on the bias current may suffice for creation of a normal-state belt across the strip. This will result in redistribution of current from the superconducting strip to the shunt with the accompanied voltage pulse much bigger than for cold pulses. Such a "hot" pulse will be similar to the one induced by photons. In the following we will estimate at what minimum bias current I^* a single vortex crossing can trigger a hot voltage pulse and a corresponding dark count.

Thus we argue that dissipation and corresponding voltage pulses in strips are caused predominantly by vortex crossings. At high-bias currents such crossings release energy sufficient for the formation of a normal belt along the vortex trajectory [see Fig. 1(a)]. Such a belt triggers the transition of the whole strip into the normal state in the absence of a shunt resistor, as well as the redistribution of the bias current into the shunt in the case of photon detectors. Note that a similar process happens when a photon creates a normal hot spot on the strip. When this spot is sufficiently large it destroys the superconducting path for the transport current and the current redistribution leads to a voltage pulse, the photon count. If the hot spot does not disrupt completely the superconducting path, it will nevertheless lead to a decrease of the energy barrier for subsequent vortex crossings. At high bias currents, a hot vortex crossing can happen directly [see Fig. 1(a)], or through a hot spot area

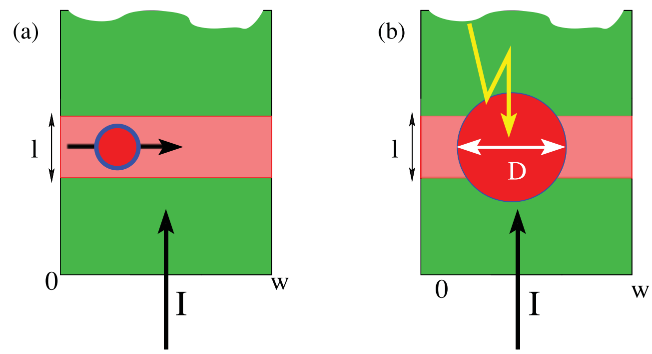

FIG. 1. (Color online) Sketch of a segment of the strip in the presence of a bias current I. (a) A single vortex (blue circle) causes a hot crossing (pink belt). The width of the belt ℓ is of the order of superconducting coherence length. (b) A single photon creates a hotspot (red disk) and induces a subsequent hot vortex crossing (pink belt). Both processes result in detectable voltage pulses in a superconducting nanowire single photon detector (SNSPD).

created by photon and forming a normal belt, which will result in signal detection [see Fig. 1(b)].

The layout of this paper is as follows. In Sec. II we discuss three energy barrier scenarios for vortex crossings. In Sec. III we derive dc current-voltage characteristics and evaluate the magnitude of induced voltage pulses. The concept of cold and hot vortex crossings is introduced in Sec. IV. In Sec. V we compare our results with data for dark count rates in NbN films.[18] We summarize our results in Sec. VI.

II. ENERGY BARRIERS AND VORTEX CROSSINGS

In this section we derive energy barriers for three dissipative processes mentioned within the GL theory. Consider a thin-film strip of width $w \ll \Lambda$ and of length $L \gg w$. We choose the coordinates so that $0 \leqslant x \leqslant w$ and $-L/2 \leqslant y \leqslant L/2$. Since we are interested in bias currents which may approach depairing values, the suppression of the superconducting order parameter must be taken into account. We use the standard GL functional with respect to the order parameter $\Psi(\boldsymbol{r})$ (normalized to its zero-field value in the absence of current) and the vector potential $\boldsymbol{A}$:

$$\mathcal{F}[\Psi(\boldsymbol{r}),\boldsymbol{A}] = d\int d\boldsymbol{r}\left\{\frac{H_c^2}{4\pi}\left[-|\Psi|^2+\frac{1}{2}|\Psi|^4 + \xi^2\left|\left(\nabla + i\frac{2\pi}{\Phi_0}\right)\Psi\right|^2\right]+\frac{B^2}{8\pi}\right\}. \quad (1)$$

Here Φ_0 is the flux quantum, $\boldsymbol{r}=(x,y)$ is a point on the film, $\boldsymbol{\nabla}$ is the 2D gradient, and $H_c = \Phi_0/2\sqrt{2}\pi\lambda\xi$ is the thermodynamic critical field. The order parameter in the presence of a uniform bias current I in zero applied magnetic fields and with no vortices present can be found by minimizing the GL functional and disregarding the current self-field, as is done, for example, in Ref. 2. As discussed in the next section, this is an accurate approximation for $w \ll \Lambda$. Thus we obtain the solution

$$\Psi_\kappa(\boldsymbol{r}) = (1-\kappa^2)^{1/2}e^{-i\kappa y/\xi + i\varphi_0}, \quad (2)$$

$$I = \frac{2w}{\pi\xi} I_0 \kappa (1-\kappa^2), \quad I_0 = \frac{c\Phi_0}{8\pi\Lambda}, \tag{3}$$

where φ_0 is an arbitrary constant phase. The parameter κ is proportional to the phase gradient and describes the order parameter suppression due to bias current. As a function of κ the bias current in the superconducting state is limited to the depairing current $I_{\max} = I_0(4/3\pi\sqrt{3})(w/\xi)$, corresponding to $\kappa_{\max} = 1/\sqrt{3}$, as for the case of the 1D wire.

A. Phase slip in the normal belt

When dealing with the situation of fixed uniform current I instead of vector potential $\boldsymbol{A}$ it is more suitable to work with the Gibbs free-energy functional rather than the free-energy functional [Eq. (1)]. We perform the usual Legendre transform (see Ref. 3) to obtain the corresponding free-energy density

$$f_I\{\Psi\} = \frac{H_c^2}{4\pi}\left[-|\Psi|^2 + \frac{1}{2}|\Psi|^4 + \left(\frac{I\pi\xi}{2wI_0}\right)^2 |\Psi|^{-2}\right]. \tag{4}$$

The equilibrium Gibbs free-energy density for a given current is obtained by minimization with respect to Ψ. It jumps at the maximum current $I_{\max}$ from $f_I(I_{\max}) = -(2/9)(H_c^2/8\pi)$ to zero at $I_{\max}$ as expected for a first-order transition. Hence, the free-energy barrier $\mathcal{U}$ for creation of a belt-like normal-state area with volume $V = \ell w d$ (ℓ is the width of the belt along the y axis) decreases from $(H_c^2/8\pi)V$ to $(2/9)(H_c^2/8\pi)V$ as the bias current increases from 0 to $I_{\max}$. The barrier never vanishes in this interval ("overheating" with respect to bias current is absent). Note that for $w \gg \xi$ and $\ell \gtrsim \xi$ the barrier remains very high in comparison with the temperature at all bias currents $I < I_c$ resulting in low probability for phase slips, except for temperatures close to T_c, where the barrier vanishes as $(1 - T/T_c)^2$.

B. Single vortex crossing

A vortex crossing from one strip edge to the opposite one induces a phase slip without creating a normal region across the strip width. We will treat the vortex as a particle moving in the energy potential formed by the superconducting currents around vortex center inside the strip and by the Lorentz force induced by the bias current. We will derive the energy potential and find the vortex crossing rate (phase slips and corresponding voltage pulses) in the framework of Langevin equation for viscous vortex motion and invoke the known solution of the corresponding Fokker-Planck equation.

In the presence of a vortex, the order parameter in the current-carrying strip, disregarding its suppression in the vortex core, reads

$$\Psi(\boldsymbol{r},\boldsymbol{r}_v) = \mu \exp\{i[\varphi(\boldsymbol{r},\boldsymbol{r}_v) - \kappa y/\xi + \varphi_0]\}, \tag{5}$$

$$\mu^2 = 1 - \kappa^2. \tag{6}$$

In this approximation the vortex affects mainly the phase $\varphi(\boldsymbol{r},\boldsymbol{r}_v)$ of the order parameter. To describe voltage pulses we need to know how the phase changes when the vortex moves across the strip. Consider a vortex at a point $\mathbf{r} = (x_v, y_v)$; y_v can be taken to zero. As we ignore the change of the order parameter amplitude in the vortex core, the current distribution is governed by the London equation (integrated over the film thickness)

$$h_z + 2\pi(\Lambda/c)\,\mathrm{curl}_z \boldsymbol{g} = \Phi_0\,\delta(\boldsymbol{r} - \boldsymbol{r}_v), \tag{7}$$

where $\boldsymbol{g}$ is the sheet current density.

For narrow strips $w \ll \Lambda$, the field h_z is of the order g/c, whereas the term with derivatives is of the order $\Lambda g/c$. Hence, supercurrents can be found by disregarding h_z and the corresponding vector potential of the order w/Λ.[19,20] Introducing the scalar stream function $G(\boldsymbol{r})$ such that

$$\boldsymbol{g} = \mathrm{curl}(G\,\hat{z}) \tag{8}$$

we reduce the problem to solving the Poisson equation

$$\nabla^2 G = -(c\Phi_0/2\pi\Lambda)\delta(\boldsymbol{r} - \boldsymbol{r}_v). \tag{9}$$

Since the boundary condition at the strip edges requires vanishing normal components of the current, we have $G = 0$ at $x = 0, w$. Therefore the problem is equivalent to one in 2D electrostatics: a linear charge at $\boldsymbol{r}_v$ between two parallel grounded plates at $x = 0, w$ with the known solution[21]

$$G(\boldsymbol{r}) = \frac{I_0\mu^2}{\pi} \ln\frac{\cosh Y - \cos(X + X_v)}{\cosh Y - \cos(X - X_v)},$$

where capitals stand for coordinates in units of w/π, that is, $x = X\,w/\pi$, $y = Y\,w/\pi$.

The energy of a vortex at $x = x_v$ and $y = 0$ is

$$\epsilon_v = \frac{\Phi_0}{2c} G(x_v, 0), \tag{10}$$

with the standard cutoff ξ at the vortex core.[19] In the presence of a uniform bias current the energy barrier reads

$$\mathcal{U}(X_v) = \mu^2\epsilon_0\left[\ln\left(\frac{2w}{\pi\xi}\sin X_v\right) - \frac{I}{\mu^2 I_0}X_v\right], \tag{11}$$

$$\epsilon_0 = \frac{\Phi_0^2}{8\pi^2\Lambda} = \frac{H_c^2}{8\pi}(2\pi\xi^2)d, \tag{12}$$

where ϵ_0 is the characteristic energy of a vortex in thin films. The vortex energy $\mathcal{U}(X_v)$ is maximum at $X_s = \tan^{-1}(\mu^2 I_0/I)$ and the energy barrier is given by

$$\frac{\mathcal{U}}{\mu^2\epsilon_0} = -\frac{1}{2}\ln\left[\frac{\pi^2\xi^2}{4w^2}\left(1 + \frac{I^2}{\mu^4 I_0^2}\right)\right] - \frac{I}{\mu^2 I_0}\tan^{-1}\frac{\mu^2 I_0}{I}. \tag{13}$$

This barrier decreases with increasing current and turns zero at a critical value on the order of the depairing GL current:

$$I_c = \frac{2\mu_c^2 w I_0}{\pi e \xi} = \frac{c\Phi_0\mu_c^2}{8\pi^2 e\lambda^2\xi}\,wd, \tag{14}$$

here $e = 2.718$ and $\mu_c = 1 - \kappa_c^2$ with $\kappa_c = 1/e$. One can see that the critical current I_c is slightly smaller than $I_{\max}$ discussed above.

Since the vortex mass is negligibly small, we use the equation of purely diffusive motion (only includes first-order time derivative) for describing the vortex propagation between $x = 0$ and $x = w$:

$$\gamma\frac{dX_v}{dt} = -\frac{d\mathcal{U}(X_v)}{dX_v} + F(t), \tag{15}$$

where $\gamma = w^2\eta/\pi^2$ and

$$\eta = \frac{\Phi_0^2}{2\pi\xi^2 c^2 R_\square} \qquad (16)$$

is the Bardeen-Stephen drag coefficient for film with $R_\square = \rho_n/d$ being the film's sheet resistance slightly above T_c. $F(t)$ is the Langevin random force obeying statistical averages $\langle F(t)\rangle = 0$ and $\langle F(t)F(t')\rangle = 2\gamma T\delta(t-t')$.

The vortex motion described by Eq. (15) takes place in the interval $a < x < w - a$, where a is of the order of ξ [the energy of the system cannot be described by the potential (11) in the intervals $w - a < x < w$ and $0 < x < a$]. The most crucial interval for vortex motion is near the point $x_s = X_s w/\pi$, where the vortex should overcome the potential barrier. Thus x_s should be inside the interval $(a, w-a)$, that is, the conditions $\xi \ll w$ and $I < (e/2)I_c$ should be fulfilled to consider the motion of vortex in the interval $0 < x < w$. To compute the average velocity in the interval $0 < x < w$, we consider the diffusion problem of a single particle that propagates in the interval $-\infty < x < \infty$ under the effect of the periodic potential $\epsilon_v(x) = \epsilon_v(x+w)$ and the Lorentz force. The average velocity is obtained from the known stationary solution for this periodic model (see Ref. 22). This approach was previously used by Gurevich and Vinokur.[17]

The corresponding Fokker-Planck equation (Smoluchowski equation) for the probability current in the case of the periodic potential has a stationary solution with the statistical average vortex velocity $\overline{v}$ given by[22]

$$\gamma\overline{v} = \frac{\pi T P}{Z_+(\pi)Z_-(\pi) - P\int_0^\pi dx e^{-\mathcal{U}(x)/T} Z_+(x)}, \qquad (17)$$

$$Z_\pm(x) = \int_0^x du\, e^{\pm\mathcal{U}(u)/T}, \quad P = 1 - e^{-\pi p}, \qquad (18)$$

where $\overline{v} \equiv \overline{\dot{X}}$ and $p = \nu I/\mu^2 I_0$. Except for temperatures close to T_c the parameter $\nu = \mu^2\epsilon_0/T \gg 1$. At large ν the function $\exp[\mathcal{U}(x)/T]$ has a sharp maximum between 0 and w, while the function $\exp[-\mathcal{U}(x)/T]$ has two sharp maxima at the edges of this interval. Since the integral $Z_+(\pi)$ has the analytic solution[23]

$$\int_0^\pi dx\, e^{-px}\sin^\nu x = \frac{\pi\exp(-\pi p/2)\Gamma(\nu+1)}{2^\nu|\Gamma(1+\nu/2+ip/2)|^2}, \qquad (19)$$

where $\Gamma(x)$ is the Gamma-function and $\nu > -1$, we obtain the asymptotic solution for $\nu \gg 1$:

$$Z_+(\pi) \approx \left(\frac{2w}{\pi\xi}\right)^\nu \sqrt{\frac{2\pi}{\nu}}\left(1 + \frac{p^2}{\nu^2}\right)^{-\frac{\nu+1}{2}} e^{-p\tan^{-1}(\nu/p)}. \qquad (20)$$

Evaluating $Z_-(\pi)$ we note that the main contribution comes from the regions near the edges, where we approximate $\sin(x) = \sin(\pi - x) \approx x$ and replace the low-integration limit by $\pi\xi/w$ and the upper one by $\pi - \pi\xi/w$. We obtain the asymptotic limit

$$Z_-(\pi) \approx \left(\frac{2w}{\pi\xi}\right)^{-\nu}\left(\frac{w}{\pi\xi}\right)^{\nu-1}\frac{e^{\pi p}+1}{\nu-1}. \qquad (21)$$

In the integral $\int_0^\pi dx e^{-\mathcal{U}(x)/T} Z_+(x)$ the function $Z_+(x)$ reaches maximum at $x = \pi$ and is small at low x. Hence the main contribution to this integral comes from the region near $x = \pi$:

$$\int_0^\pi dx\, e^{-\mathcal{U}(x)/T} Z_+(x) \approx \left(\frac{2w}{\pi\xi}\right)^{-\nu}\left(\frac{w}{\pi\xi}\right)^{\nu-1}\frac{e^{\pi p}}{\nu-1}Z_+(\pi). \qquad (22)$$

It then follows that the dependence of the average vortex velocity $\overline{v}$ on I at large p and ν is given by

$$\gamma\overline{v} \approx T\left(\frac{\pi\nu^3}{2}\right)^{1/2}\left(\frac{\pi\xi}{w}\right)^{\nu-1} Y\left(\frac{I}{\mu^2 I_0}\right), \qquad (23)$$

$$Y(z) = (1+z^2)^{(\nu+1)/2}\exp[\nu z\tan^{-1}(1/z)]. \qquad (24)$$

Note the strong power-law dependence of $\overline{v}$ on the strip width w.

For large currents $I \gg I_0$, this expression reduces to

$$\gamma\overline{v} \approx T\left(\frac{\pi\nu^3}{2}\right)^{1/2}\left(\frac{w}{\pi\xi}\right)^2\left(\frac{I}{I_c}\right)^{\nu+1}, \qquad (25)$$

with I_c given by Eq. (14). Note that the average velocity changes drastically near the critical current I_c, where the energy barrier vanishes. Such defined critical current is about 16% smaller than the standard depairing current $I_{\max}$ defined for 1D wires (vanishing energy barrier for phase slips in wires, see Ref. 2).[24]

In the case of multiple simultaneous vortex crossings happening in different parts of the strip, we must account for their interactions. The interaction of vortices situated at $(X_1, 0)$ and (X_2, Y) has been evaluated in Ref. 19:

$$\epsilon_{\rm int} = \epsilon_0 \ln\frac{\cosh Y - \cos(X_1 + X_2)}{\cosh Y - \cos(X_1 - X_2)}. \qquad (26)$$

If vortices are separated by $y > w$ along the strip, the interaction is exponentially weak and their crossings are uncorrelated. Accounting for both vortex and antivortex crossings (which are equivalent by symmetry), we estimate the rate for multiple vortex crossings at $I < I_c$ as $R \approx (2L/\pi w)\overline{v}$.

Finally, we obtain the asymptotic estimate for the rate

$$R \approx \frac{4Tc^2 R_\square L}{\Phi_0^2 w}\left(\frac{\pi\nu^3}{2}\right)^{1/2}\left(\frac{\pi\xi}{w}\right)^{\nu+1} Y\left(\frac{I}{\mu^2 I_0}\right). \qquad (27)$$

In obtaining this result we disregarded vortices crossing in the direction opposite to the Lorentz force, the corresponding probability for such processes is $\propto e^{-2p} \ll 1$. We note that Gurevich and Vinokur took L/ξ as the number of statistically independent vortex crossings.[17] It differs by a factor $\xi/w \ll 1$ from our estimated number L/w of independent crossings. Therefore, Ref. 17 overestimates the rate.

C. Vortex-antivortex pair scenario

The energy of a vortex-antivortex pair (vortex-antivortex interaction included) was derived in Ref. 19 and is

$$\frac{\epsilon_p}{\mu^2\epsilon_0} = \ln\left[\frac{4W^2}{\pi^2\xi^2}\sin X_1 \sin X_2 \frac{\cosh Y - \cos(X_1 - X_2)}{\cosh Y - \cos(X_1 + X_2)}\right]. \qquad (28)$$

This energy increases with increasing separation Y, so that one expects the lowest barriers for $Y = 0$:

$$\frac{\epsilon_p}{\mu^2\epsilon_0} = \ln\left[\frac{4w^2}{\pi^2\xi^2}\sin X_1 \sin X_2 \frac{\sin^2[(X_1 - X_2)/2]}{\sin^2[(X_1 + X_2)/2]}\right]. \tag{29}$$

One can show that if a pair is formed at X_0 and the pair members are pushed apart a distance $2b$, the lowest energy increase (for a given b) corresponds to the initial position $X_0 = \pi/2$ in the middle of the strip. The energy barrier for such a pair, in the presence of bias current I, is obtained by setting $X_{1,2} = \pi/2 \mp b$ and adding the Lorentz force contribution:

$$\mathcal{U}_p(b) = 2\mu^2\epsilon_0\left[\ln\frac{w\sin(2b)}{\pi\xi} - \frac{Ib}{\mu^2 I_0}\right]. \tag{30}$$

This energy is maximum if $2b = \tan^{-1}(2\mu^2 I_0/I)$ so that the energy barrier for vortex-antivortex pairs is given by

$$\frac{\mathcal{U}_p}{\mu^2\epsilon_0} = -\ln\left[\frac{\pi^2\xi^2}{w^2}\left(1 + \frac{I^2}{4\mu^2 I_0^2}\right)\right] - \frac{I}{\mu^2 I_0}\tan^{-1}\frac{2\mu^2 I_0}{I}. \tag{31}$$

For $I \gg I_0$ this barrier is twice as large than that for a single vortex crossing [Eq. (13)] and the ratio of these barriers increases for smaller currents. Note also that the core contribution to the pair energy (neglected here) is at least twice that for a single vortex.

Based on our estimates for the three different fluctuation scenarios presented here, we conclude that single vortex crossings are the main source for dark counts.

III. VOLTAGE INDUCED BY VORTEX CROSSING

Let us now find how the phase of the order parameter varies when a vortex crosses the strip. The current is expressed either in terms of the gauge invariant phase φ or via the stream function G: $\boldsymbol{g} = -(c\Phi_0/4\pi^2\Lambda)\nabla\varphi = \text{curl}[G\hat{z}]$. Written in components, this gives the Cauchy-Riemann relations for functions $[4\pi^2\Lambda_0/c\Phi_0\mu^2]G(\boldsymbol{r})$ and $\varphi(\boldsymbol{r})$. Hence they are real and imaginary parts of an analytic function of complex argument $z = x + iy$:[21]

$$\mathcal{G}(Z) = \ln\frac{\sin[(X_v + Z)/2]}{\sin[(X_v - Z)/2]} \tag{32}$$

(recall the capitals are coordinates in units of w/π, so that $0 < X < \pi$, etc.). We then obtain

$$\varphi(\boldsymbol{r},\boldsymbol{r}_v) = \text{Im}[\mathcal{G}(Z)] = \tan^{-1}\frac{\sin X_v \sinh(Y - Y_v)}{\cos X - \cosh(Y - Y_v)\cos X_v}. \tag{33}$$

Note that the characteristic length of variations for φ in both x and y directions is w. For long strips of interest, $L \gg w$, and for distances $|Y - Y_v| \gg 1$, we have at the strip ends $\varphi(\pm L/2) = \mp X_v$. Hence when the vortex moves from the strip edge at $X_v = 0$ to the opposite edge at $X_v = \pi$ and $|L/2 - Y_v| \gg 1$, the phase difference at the ends of the strip changes by $\varphi(L/2) - \varphi(-L/2) = 2X_v = 2\pi$, that is, a vortex crossing results in a global phase slip of 2π.

A. dc voltage

The motion of vortices causes the phase difference at the strip ends to vary in time. Using the Josephson relation for the phase, we obtain the induced voltage due to a single vortex crossing:

$$V(t) = \frac{\Phi_0}{2\pi c}\frac{d}{dt}[\varphi(L/2) - \varphi(-L/2)] = \frac{\Phi_0 v(t)}{cw}, \tag{34}$$

where the vortex velocity is $v(t) = dx_v/dt = (w/\pi)dX_v/dt$ and we used $\varphi(L/2) - \varphi(-L/2) = 2X_v$. A quasistatic approach employed here is justified as long as the characteristic crossing time $\Delta t = w/v$ is large compared to L/c. Note that for each crossing, that is, for each voltage pulse between time t and $t + \Delta t$, the relation

$$\int_t^{t+\Delta t} dt'\, V(t') = \frac{\Phi_0}{c} \tag{35}$$

is satisfied as in the case of voltage pulses due to phase slips in 1D wires.[2] Thus we obtain the average (dc) voltage

$$V_{\text{dc}} = \frac{\Phi_0}{c}R. \tag{36}$$

This relation also follows directly from comparing the dissipated power $V_{\text{dc}}I$ with the work per unit time done by the Lorentz force $(\Phi_0 I/cw)wR$. It is worth to remember that we have derived the crossing rate assuming an isothermal strip. In continuous measurements of current-voltage characteristics at currents of the order of the critical one, the strip temperature is certainly higher than that of the bath. In principle, this heating may be reduced using short bias current pulses.

B. Voltage pulses

In this section we consider the time evolution of the voltage pulse $V(t)$ induced by single vortex crossing. Here we use the equation of vortex motion [Eq. (15)] for $X > X_s$ and neglect random forces (thermal noise). Therefore the velocity is

$$v \equiv \dot{x}_v = \frac{\pi\epsilon_0}{\eta w}\left(\frac{I}{I_0} - \mu^2\cot X\right). \tag{37}$$

This can be written in the form

$$\dot{X} = \beta\,(\cot X_s - \cot X)\,, \quad \beta = \frac{\pi^2\epsilon_0\mu^2}{\eta w^2}, \tag{38}$$

which is valid for $X > X_s$. It is worth noting that for currents of the order of I_c the saddle point is very close to the strip edge,

$$X_s \approx \frac{I_0\mu^2}{I} = \frac{e\pi}{2}\frac{\mu^2 I_c}{\mu_c^2 I}\frac{\xi}{w} \ll 1\,. \tag{39}$$

Integration of Eq. (38) results in an implicit solution for $X(t)$:

$$X(t)\cos X_s + \sin X_s \ln\sin[X(t) - X_s] = \frac{\beta\,(t - t_0)}{\sin X_s}. \tag{40}$$

We choose the constant t_0 so that $t = 0$ corresponds to the vortex exit at $X = \pi$. Note that any instant for which $0 < X(t) < X_s$ is beyond this approximation, because in this early time interval the process is described by thermal activation rather than by the equation of motion (37) with random force omitted. The instant for which $X(t) = X_s$ is also inappropriate

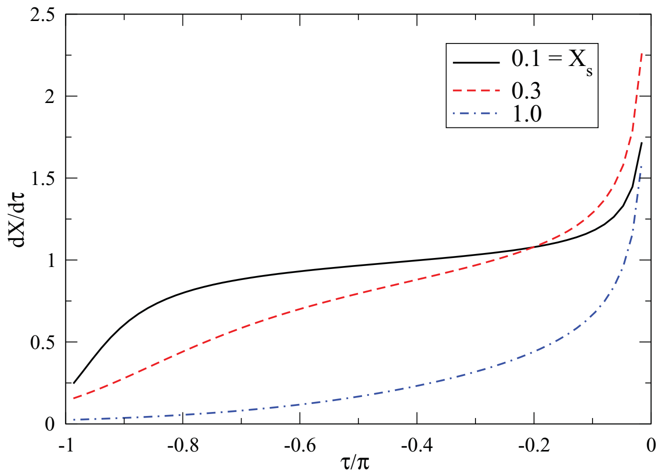

FIG. 2. (Color online) The dimensionless vortex velocity $dX/d\tau$ vs τ for parameters $X_s = 0.1,\ 0.3$, and 1.0. Note that $dX/d\tau = v/\bar{v}$, where v is the velocity in common units and $\bar{v}$ is the average velocity, which is identical to the one solely due to the Lorentz force.

as an initial moment, because at this point the velocity vanishes $\dot{X} = 0$. Thus Eq. (40) can be written as

$$[X(t) - \pi]\cos X_s + \sin X_s \ln \frac{\sin[X(t) - X_s]}{\sin X_s} = \frac{\beta t}{\sin X_s}. \tag{41}$$

Clearly $X(0) = \pi$ and $X(t \to -\infty) = X_s$. Hence, formally, the motion from the saddle point X_s to the edge takes infinite time because the velocity goes to zero as $X \to X_s$. In reality, the dynamic viscous vortex motion starts at some distance from the saddle point where the vortex is kicked by random force (an activation driven process) and the total "time-of-flight" is finite. To see this, consider the situation of large currents for which X_s is given by Eq. (39) and

$$X(t) - \pi + X_s \ln \frac{\sin[X(t) - X_s]}{\sin X_s} = \frac{\beta t}{X_s}. \tag{42}$$

Denote as δX a small distance from the saddle at X_s and evaluate the time τ_0 of motion from $X_s + \delta X$ to the edge $X = \pi$:

$$-\frac{\beta \tau_0}{X_s} = X_s + \delta X - \pi + X_s \ln \frac{\delta X}{X_s}.$$

Since both δX and X_s are small, all terms on the right-hand side, except for π, are negligible and we obtain

$$\tau_0 \approx \frac{\pi X_s}{\beta} = \frac{c\eta w^2}{\Phi_0 I} = \frac{w^2 \Phi_0}{2\pi \xi^2 c R_\Box I}, \tag{43}$$

so that the time-of-flight τ_0 does not depend on a particular choice of δX. In fact, this estimate coincides with the time it takes a vortex to cross the strip being pushed solely by the Lorentz force.

Solving numerically Eq. (41) for $X(t)$ and substituting the result in Eq. (38) we obtain $v(t)$. The result is shown in Fig. 2. For convenience, we use X_s/β as the unit of time. The dimensionless time $\tau = \beta t/X_s$ varies between $-\pi < \tau < 0$.

The divergence at the edge $x = w$ must be cut off at distances of the order of ξ from the edge. We obtain from Eq. (38) an estimate for the maximum velocity at the exit,

$$v_{max} \approx \frac{\phi_0}{cw\eta}\left(I + \frac{e I_c}{2}\right), \tag{44}$$

where the critical current is given by Eq. (14).

For large currents, $X_s \ll 1$, we solve Eq. (42) perturbatively: $X = X_1 + \delta X$ with $X_1 = \pi + \beta t/X_s$ and $\delta X \ll X_1$:

$$X = \pi + \tau - X_s \ln \frac{\sin(X_s - \tau)}{\sin X_s}. \tag{45}$$

Thus the velocity for $X_s \ll 1$ is

$$\frac{dX}{d\tau} = 1 + X_s \cot(X_s - \tau), \tag{46}$$

the unity corresponds to a constant velocity due to the Lorentz force, whereas the second term is caused by the vortex potential.

The velocity $v(t)$ is peaked near the edge $x = w$ and it is of interest to estimate the width $\Delta\tau$ of this peak in the velocity and in the voltage $V(t) \propto v(t)$. The width $\Delta\tau$ is definition dependent. For example, one can define it as the time interval between instants when $v = v_{\max}$ and time τ_1 when $v = (v_{\max} + \overline{v})/2$, where $\overline{v}$ is the background velocity due to the current I. In dimensionless units, $\overline{v}$ corresponds to $dX/d\tau = 1$. Thus we obtain

$$\tau_1 \approx -X_s \frac{6wX_s + \pi\xi}{3wX_s - \pi\xi} \tag{47}$$

with

$$\frac{\pi\xi}{wX_s} = \frac{2\mu_c^2 I}{e\mu^2 I_c} < 1, \tag{48}$$

so that $\tau_1 < 0$. Since $|\tau_1| \sim X_s \gg \tau_m$, we estimate the width of the velocity peak near the edge as $\Delta t \sim \Delta\tau X_s/\beta \sim X_s^2/\beta$, where the fraction of order unity in Eq. (47) has been neglected. Therefore, the ratio of this width relative to the total crossing time τ_0 of Eq. (43) is

$$\frac{\Delta t}{\tau_0} \approx \frac{X_s}{\pi} \ll 1. \tag{49}$$

IV. "COLD" AND "HOT" VORTEX CROSSINGS

A vortex moving from the saddle point $x = x_s$ to the strip edge $x = w$, during the time $\tau_0 = w/\overline{v}$, excites quasiparticles along its path by the mechanism described by Larkin and Ovchinnikov.[25,26] This mechanism is appropriate for dirty superconductors (for clean and intermediate clean regimes see Refs. 27 and 28). Since NbN films are inherently dirty, we can safely disregard the latter mechanism. We estimate the total energy transferred to quasiparticles during the time τ_0 along the vortex path as

$$Q \approx (\Phi_0 I/c) \approx \frac{8\pi}{e}\frac{H_c^2}{8\pi}\frac{I}{I_c} w\xi d. \tag{50}$$

This is, in fact, the work done by the Lorentz force on the vortex path of the length $w - x_s$. This energy is distributed near uniformly along the path at currents close to the critical current, because the vortex velocity varies weakly for most of the crossing (see Fig. 1). In a belt of width ℓ along the y axis with the volume $V_b = \ell w d$, the energy increase per unit volume is $(8\pi\xi/e\ell)(H_c^2/8\pi)(I/I_c)$.

We now estimate the time τ_0. For a strip with resistivity $\rho(T_c) = 240\ \mu\Omega\,\mathrm{cm}$, $w = 120$ nm, $d = 4$ nm, $\Lambda = 45\ \mu$m, and a bias current of the order I_c, the crossing time is roughly $\tau_0 \sim 10$ ps and corresponding vortex speed is 12 km/s. This time is too short for any significant transfer of the electronic excitation energy into the substrate and surrounding strip area. Indeed, the phonon escape time was estimated as 160 ps in a strip of thickness $d = 20$ nm, whereas the electron-phonon relaxation time is about 17 ps.[29] During the time τ_0 quasiparticles diffuse away from the vortex path by a short distance $(D\tau_0)^{1/2} \approx 8$ nm as estimated from the electronic specific heat $C_e = 2.2\ \mathrm{kJ/m^3\,K}$ and the normal-state resistivity at 10 K.[30]

Hence quasiparticles remain practically within the belt of volume $V_b = \ell w d$ along the vortex path. The quasiparticle energy density within the belt is $(8\pi\xi/e\ell)(H_c^2/8\pi)(I/I_c)$. Taking $\ell \approx 3\xi$, we see that for $I > I^* \approx I_c/3$ such an energy is sufficient to turn the belt normal causing a dark count in the photon detector. We call this process at high currents $I > I^*$ a "hot" vortex crossing.

Therefore, we conclude that the superconducting strip with a bias current in the interval $I^* < I < I_c$ is unstable with respect to the transition into the normal state, that can be triggered by a vortex overcoming the barrier. Clearly photons can trigger such a transition as well. The photon efficiency increases as I approaches I_c and so does the rate of dark counts.

In fact, the true critical current of a strip, below which the strip remains superconducting, is I^*. At currents below I^*, the superconducting state is stable, but remains resistive due to the presence of quasiparticles in normal cores of vortices crossing the strip. In this scenario a single vortex crossing leaves the strip in the superconducting state and thus we call this process a "cold" vortex crossing.

V. COMPARISON WITH EXPERIMENTAL DATA

In Fig. 3 experimental dark count rates are shown for three different NbN samples of SNSPDs.[18] We fit the data using Eqs. (27) and (24) by writing

$$\ln(R/L) = \ln(a) + \ln[Y(\Phi_0 I/\pi\nu c T)], \quad (51)$$

$$a = \frac{4Tc^2 R_\square}{\Phi_0^2 w}\left(\frac{\pi\nu^3}{2}\right)^{1/2}\left(\frac{\pi\xi}{w}\right)^{\nu+1}. \quad (52)$$

The dimensions of samples 1, 2, and 3 are $d = 6$ nm, $w = 53.4,\ 82.9, 170.6$ nm, $L = 73.9, 145.1,\ 141.4\ \mu$m, respectively. The sheet resistance $R_\square = 445, 393, 431\ \Omega$, and data were taken at $T = 5.5$ K. According to Bartolf *et al.*, at low currents the data was dominated by electronic noise in the measurement circuit.[18] The data for samples 1 and 2 agree well with the theoretical results for high currents, while the data for sample 3 yield an unreasonably large exponent ν.

For sample 1 with fit parameter $\nu = \Phi_0^2/8\pi^2\Lambda$, we extract the Pearl length $\Lambda(5.5\ \ \mathrm{K}) = 57.1\ \mu$m, and from $\ln(a/L)$ we estimate the coherence length $\xi(5.5\ \ \mathrm{K}) = 3.9$ nm. The authors of Ref. 18 estimated $\xi(0) = 4$ nm from independent

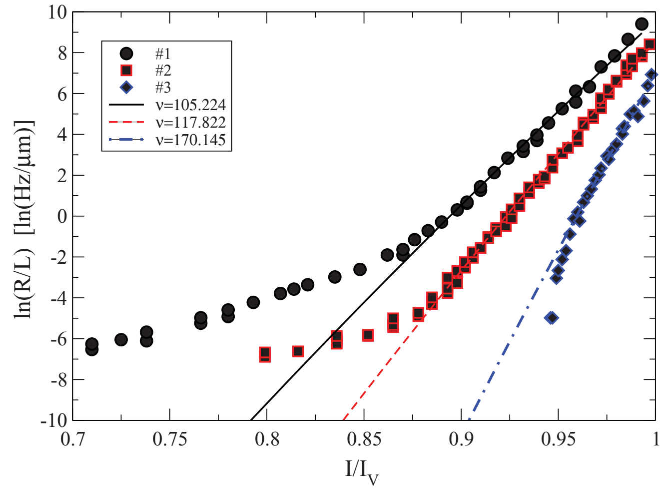

FIG. 3. (Color online) The dark count rates of three SNSPDs at 5.5 K by Bartolf *et al.*[18] and fits based on Eqs. (27) and (24). The current is in units $I_V = \alpha I_c$, where $\alpha = 0.72, 0.77, 0.60$ for samples 1, 2, and 3. I_c is the critical current defined as the current at which the barrier for vortex crossings vanishes. At low currents electronic noise in the measurement setup dominates over vortex crossings.

measurements of the upper critical field. They also estimated the Pearl length for NbN films of thickness $d = 6$ nm, $\Lambda(0) = 65.1\ \mu$m, from known resistivity ρ_n and the superconducting gap $\Delta(0) \sim 2$–3 meV.[31] By using Eq. (14), we find the critical current $I_c = 20.1\ \mu$A defined as the current at which the energy barrier vanishes for vortex crossings. The authors of Ref. 18 defined the "critical" current $I_V = 14.5\ \mu$A using the 1% voltage criterion (current at which resistance is 1% of the normal one). We see that the critical current defined through such a voltage criterion is less than the critical current defined by the current at which the energy barrier vanishes, $I_V \approx 0.72\,I_c$.

For sample 2 we find the coherence length $\xi(5.5\ \ \mathrm{K}) = 4.33$ nm and $\Lambda(5.5\ \ \mathrm{K}) = 51\ \mu$m. Independent estimates given in Ref. 18 are $\xi(0) = 4.2$ nm and $\Lambda(0) = 59.2\ \mu$m; the critical current $I_c = 31.5\ \mu$A, while $I_V = 0.77\,I_c$.[18] We conclude that our model for vortex crossing rates describes satisfactorily the dark count rates in samples 1 and 2.

Next we estimate the peak of the voltage pulse for I slightly below I^*:

$$V_{\mathrm{peak}} \approx \frac{c\Phi_0\xi R_\square}{\pi e\Lambda w}, \quad (53)$$

and the duration of the pulse is $\tau_{\mathrm{peak}} < \Phi_0/cV_{\mathrm{peak}}$. For sample 2 studied by Bartolf *et al.*[18] we estimate $V_{\mathrm{peak}} \approx 0.8$ mV, while $\tau_{\mathrm{peak}} \approx 3$ ps slightly below I^*. For comparison, dark counts are characterized by peak voltages of ≈1 mV and by durations of several nanoseconds (FWHM ∼2.5 ns[16]). For dark counts the duration of pulses is caused by the current redistribution and thus depends on the experimental setup used to detect the pulses. Note that pulse duration differs significantly from that caused by single-vortex crossing without formation of normal belt.

The following experiment could, in principle, distinguish between regimes at $I < I^*$ and at $I > I^*$: One induces a bias current in a thin-film ring and measures the magnetic flux in the ring as a function of time. For $I > I^*$, a single vortex crossing destroys superconductivity and the flux vanishes. The lifetime of this *persistent* current is $1/R$ and R is determined by Eq. (27). If $I < I^*$, the flux should decrease stepwise through multiple transitions between quantized current states I_n, each transition corresponds to a single vortex crossing. In this case, the lifetime for the current I_n is $1/R_n$ where R_n is given by Eq. (27) with $I = I_n$. The total decay time of the initial current I_N will be $\tau = \sum_{n=1}^{N} R_n^{-1}$. For 1D wires similar behavior due to phase slips was described by McCumber and Halperin.[2]

In comparing theory and experiment, the issue of possible inhomogeneities of the thickness d and the width w is often raised. We note that the model developed here is only valid for $w \ll \Lambda$. Each vortex in a narrow strip has mostly the kinetic energy of its supercurrents which are confined within an area of size $\sim w \times w$. In other words, the model is not sensitive to inhomogeneities of d and of the edge roughness on scales small relative to w.

Finally, it is worth mentioning that we assumed in this work that the strip temperature is equal to the bath temperature of the substrate. This may not always be the case in measurements of dark counts in photon detectors. After redistribution of the bias current, the normal belt induced by a crossing vortex cools down. The strip can carry the superconducting current equal to the bias current I only if the temperature drops below the value T^* defined by the condition $I^*(T^*) = I$. Slightly below T^* vortices can cross the strip inside the warmer belt whose temperature is close to T^* or inside the cooler areas whose temperature is that of the bath. The rate of vortex crossings is determined by both processes and the latter dominates only in the limit of very large L. Again, we emphasize that the measured rate is higher than the calculated rate, and the difference is larger for small currents because for them T^* is higher.

VI. CONCLUSIONS

In summary, we have found that the most plausible mechanism for dark counts in photon detectors is due to thermal fluctuations related to vortex crossings in the metastable current-carrying superconducting state, which is realized at bias currents above some value $I^* \sim I_c/3$. We conclude by listing our main results:

(a) Vortices crossing the current-biased strip due to thermal fluctuations induce voltage pulses which can be detected experimentally. The barrier for vortex crossings vanishes at the critical current defined by Eq. (14).

(b) In narrow and thin strips, the superconducting state is unstable in the current interval $I^* < I < I_c$ and a transition into the normal state is triggered by vortices crossing the strip accompanied by energy (heat) release.

(c) We estimated the threshold for hot vortex crossings to be roughly $I^* \approx I_c/3$.

(d) Dark counts in current-biased superconducting strips reported in the literature were observed in the regime of metastable superconducting state.

(e) At currents below I^*, vortex crossings do not induce transitions into the normal state, but still induce voltage pulses and the superconducting state is resistive due to the quasiparticles inside vortex cores of crossing vortices. We proposed a ring experiment, which allows to distinguish different decay processes of circular currents above and below I^*.

(f) We estimated the amplitude and duration of cold voltage pulses which can be detected below I^*.

Clearly it is desirable to test our theory by measuring *I*-*V* characteristics with a pulsed current technique to avoid heating. Furthermore, it will be interesting to study the rate and the shape of cold pulses at currents below I^* at different temperatures and see their evolution from thermally induced crossings to quantum tunneling.

ACKNOWLEDGMENTS

We are grateful to I. Martin, M. Rabin, D. Rosenberg, and J. Clem for many useful discussions. Work at the Los Alamos National Laboratory was performed under the auspices of the US DOE Contract No. DE-AC52-06NA25396 through the LDRD program. Work at the Ames Lab (V.K.) was supported by the DOE-Office of Basic Energy Sciences, Division of Materials Sciences and Engineering under Contract No. DE-AC02-07CH11358.

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