Ising M-p-spin mean-field model for the structural glass: Continuous versus discontinuous transition

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The critical behavior of a family of fully connected mean-field models with quenched disorder, the $M - p$ Ising spin glass, is analyzed, displaying a crossover between a continuous and a random first order phase transition as a control parameter is tuned. Due to its microscopic properties the model is straightforwardly extendable to finite dimensions in any geometry.

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I. INTRODUCTION

Since the work of Kirkpatrick, Thirumalai, and Wolynes[1–6] a certain set of mean-field spin-glass models have been shown to own the salient properties of the behavior of structural glasses. In particular, these models display dynamic equations that are equivalent to those predicted by the mode coupling theory (MCT)[7–9] above the so-called mode coupling temperature $T_{\rm mc}$ at which ergodicity breaking occurs. Two kinds of transitions are predicted: a dynamic one at $T_d = T_{\rm mc}$ and a thermodynamic phase transition at a lower T, often referred to as the Kauzmann transition. Many mean-field models exhibiting structural glass features are characterized by multibody microscopic interactions and their thermodynamics is self-consistently described by implementing a discontinuous replica symmetry breaking (RSB) ansatz (usually one step: 1RSB).

The dynamic transition is due to the presence of a large number of metastable excited glassy states, represented as local minima of the free energy landscape, growing exponentially with the size N of the system. In the mean-field approximation, barriers between minima grow with size and in the thermodynamic limit the relaxing dynamics to equilibrium of the system at $T \leqslant T_d$ remains stuck inside the first "meta"stable state in where it ends up. In real glassy systems, instead, a slow dynamics occurs through activated processes. The dynamic arrest at (and below) T_d is an artifact due to the mean-field approximation.

In finite dimensions the glass transition occurs because, at a given glass temperature T_g, the time scales of observation are shorter than the characteristic time scales of the slowest structural processes (α relaxation) taking place in the glass-former sample. Metastable states really have a finite time life, even though (much) longer than the experimental time of observation. The effect of activated processes in spin-glass 1RSB mean-field models has been analyzed by working at finite N in the fully connected random orthogonal model (ROM)[10–13] and finding a glass behavior, similar to the one observed in computer glasses (i.e., those models implemented to be studied by means of numerical simulations, like Lennard-Jones and soft spheres mixtures, cf., Refs. 14 and 15).

As already mentioned, another property occurring in the glass-like mean-field models is a thermodynamic transition between the supercooled liquid (below T_d) and a thermodynamically stable glass.[16–18] This can occur with a jump in the order parameter, but without discontinuity in the internal energy (no latent heat is exchanged). This mixture of first order and continuous phase transition in the presence of disorder has been termed random first order transition (RFOT).[6]

One of the most accredited theories, the Adam-Gibbs–Di-Marzio entropic theory[19,20] predicts the existence of a thermodynamic transition to an *ideal glass* phase, the so-called Kauzmann transition. The Kauzmann temperature is generally associated with the asymptote of the Vogel-Fulcher law[21,22] of the relaxation time. It is thus related to the transition that one might observe in an infinitely slow cooling of a glass former. Because of the difficulty of experimental measurements of glass relaxation in those conditions, the very existence of the Kauzmann point and the nature of that transition is still a matter of debate.[23–27]

So far, attempts to track the properties envisaged in mean-field models in realistic systems have faced the problem of finding a meaningful way to generalize such models and embed the microscopic features into a given finite dimensional geometry (e.g., three-dimensional cubic lattice) without altering the discontinuous nature of the transition. This strongly hinders the chance of falsifying/verifying the hyphothesis of RFOT in finite-dimensional systems.

Let us consider, for example, the Ising mean-field p-spin model ($p > 2$), displaying a RFOT to a 1RSB stable phase.[28] In Ref. 29 a generalization of the $p = 3$-spin model with $M = 2$ Ising spins on each site was numerically studied on a $D = 4$ hypercubic lattice finding evidence for a continuous phase transition. The same continuous behavior was recently found, *in the mean-field regime*, in the same $p = 3$, $M = 2$ model in a $D = 1$ chain on a "Levy lattice".[30] Just starting from the observation that in some models with multibody interactions the transition can be continuous, both in mean-field and in finite D, the work of Moore, Drossel, and Yeo[31–33] showed that this is equivalent to the critical behavior of the Edwards-Anderson model in a field where the transition line is called the de-Almeida–Thouless (dAT) line. Applying droplet theory (which rules out the existence of a dAT line outside the limit of validity of mean-field theory) it is thus inferred that no thermodynamic random first order transition can occur in real structural glasses.[34] We notice that the above-mentioned approach is initially based on a small overlap expansion around the critical point, thus relying on a self-consistency check of the hypothesis of continuity of the overlap. With that expansion, however, potential discontinuous transitions cannot be detected and one cannot rule them out, in principle.

The motivation of the present work is to investigate the connection between the above-mentioned models (presenting spin-glass like transition in mean field) and the original mean-field p-spin model (displaying structural-glass-like transition) and to devise and study a *mean-field* lattice model able to interpolate between the two behaviors at the mean-field level and that, in principle, can be put on a lattice of any D without changing any constitutive features.

We will now focus on deriving a mean-field class of models, to which the Ising p-spin model[16,28] belongs, whose critical behavior shifts from continuous to discontinuous in a controlled way. The model consists of N sites, each one containing M spins interacting with spins on other sites in p-uples. We will see how, changing p and the number M of spins staying on a single site, it is possible to move from systems displaying a second order phase transition to systems displaying a random first order transition.

We mention that moving from mean-field to finite dimensions, also standard Ising p-spin and Potts models, might conserve the random first order nature of the transition and keep reproducing basic features of structural glasses. The latter, in particular, can be straightforwardly defined on a hypercubic lattice. Nevertheless, no numerical evidence has been collected so far for a RFOT in finite-dimensional disordered Potts models with the number of states $p_{\rm Potts} = 5,6,10$ (Refs. 35–38). Actually, we found no argument to infer a significant limit for the candidate control parameter $p_{\rm Potts}$ in finite-dimensional lattice cases that could be kept under control to ensure that a finite-dimensional Potts model recovers the mean-field properties in that limit.

On the contrary, as we will show in the following, the model considered in the present work has the advantage of reducing to an exact mean-field p-spin model for the RFOT as $M \to \infty$ even in finite dimension (and finite size), for any values of p. Moreover, in the mean-field theory, we can work out a sufficient criterion to determine the smallest value of M above which continuous spin-glass-like transitions cannot occur.

The manuscript is organized as follows. In Sec. II we will study the statistical mechanics of the model. In Sec. III we show that the large M limit corresponds to standard p-spin and in Sec. IV, expanding near criticality, we build the corresponding field-theory, compute the coupling constants, and study the relevance of terms competing for continuous/discontinuous transition. In Sec. V we present our conclusions.

II. THE MODEL

The model consists of N sites, each one hosting a set of M spins. Two sites interact through a p-body interaction involving spins belonging to the two sets of M spins. The Hamiltonian reads

$$\mathcal{H} = -\sum_{\langle x,y\rangle}\sum_{g(x,y)} J_g \prod_{\mu\in g} s_\mu, \qquad (1)$$

where $\langle x,y\rangle$ indicates the sum over all couples of sites and $g(x,y)$ are all the possible p-uplets among the $2M$ spins, with an exception if $p \leqslant M$: Those p-uplets completely pertaining to a single site are excluded. This choice actually *defines* our model when $p \leqslant M$, as we will discuss in the following.

The disordered interactions are Gaussian independent and identically distributed (i.i.d.) variables, with distribution

$$P(J_g) = \frac{1}{\sqrt{2\pi\sigma_J^2}} e^{-\frac{J_g^2}{2\sigma_J^2}}, \qquad (2)$$

where, to provide the right thermodynamic convergence of the free energy, the variance scales like

$$\sigma_J^2 = \frac{1}{N M^{p-1}}, \qquad (3)$$

Replicating n times the system we compute the average over quenched disorder of the replicated partition function

$$\overline{Z^n} = \int \prod_{\langle x,y\rangle}^{1,N} \prod_{g(x,y)} P(J_g)\, dJ_g \mathrm{Tr}_{[s]} \times \exp\left[\beta \sum_{a=1}^{n}\sum_{\langle x,y\rangle}^{1,N}\sum_{g(x,y)} J_g \prod_{\mu\in g} s_\mu^a\right], \qquad (4)$$

yielding

$$\overline{Z^n} = \mathrm{Tr}_{[s]} \exp\left[\frac{\beta^2}{4NM^{p-1}} \sum_{x\neq y}^{1,N}\sum_{g(x,y)}\sum_{a,b}^{1,n}\prod_{\mu\in g} s_\mu^a s_\mu^b\right]. \qquad (5)$$

Explicitly separating those spins belonging to site x from those on site y one can obtain a general expression for the partition function valid both for $p > M$ and $p \leqslant M$

$$\overline{Z^n} = \mathrm{Tr}_{\{s(x),s(y)\}} \exp\Bigg[\frac{\beta^2}{4NM^{p-1}} \sum_{a,b}\sum_{x\neq y}\sum_{k} \sum_{i_1<\cdots<i_k} s_{i_1}^a(x)s_{i_1}^b(x),\ldots,s_{i_k}^a(x)s_{i_k}^b(x) \sum_{i_{k+1}<,\cdots,<i_p} s_{i_{k+1}}^a(y)s_{i_{k+1}}^b(y),\ldots,s_{i_p}^a(y)s_{i_p}^b(y)\Bigg], \qquad (6)$$

with

$$k = p-M,\ldots,M \quad \text{if} \quad p > M,$$
$$k = 1,\ldots,p-1 \quad \text{if} \quad p \leqslant M.$$

In principle, it might be possible to include an extra term due to self-interaction: p out of M spins interact on a single site ("a single site standard p spin"). As already mentioned, in the present work we will consider a model *without* site self-interaction. We now introduce a set of multi-overlaps between k spins on the same site x in two replicas

$$Q_{ab}^{(k)} \equiv \frac{1}{NM^k}\sum_{x=1}^{N}\sum_{i_1<,\ldots,<i_k} s_{i_1}^a(x)s_{i_1}^b(x),\ \ldots,\ s_{i_k}^a(x)s_{i_k}^b(x). \qquad (7)$$

By means of multi-overlaps we can write the replicated partition function Eq. (7) as

$$\overline{Z^n} = e^{NC} \int D\underline{Q}\, \mathrm{Tr}_{\{s(x),s(y)\}},$$
$$\exp\left[\frac{\beta^2 NM}{4}\sum_k\sum_{a\neq b} Q_{ab}^{(k)}Q_{ab}^{(p-k)}\right]$$
$$\times\prod_k\prod_{a<b}\delta\Bigg(NMQ_{ab}^{(k)} - \frac{1}{M^{k-1}}\sum_x\sum_{i_1<\dots<i_k}$$
$$\times\, s_{i_1}^a(x)s_{i_1}^b(x),\dots,s_{i_k}^a(x)s_{i_k}^b(x)\Bigg), \qquad (8)$$

where the parameter C, proportional to minus the paramagnetic free energy, reads

$$\frac{C}{n} = \frac{\beta^2}{4M^{p-1}}\left[\sum_k\binom{M}{k}\binom{M}{p-k}\right]$$
$$= \frac{\beta^2}{4M^{p-1}}\begin{cases}\frac{1}{\Gamma(1+p)}\left[\frac{\Gamma(1+2M)}{\Gamma(1+2M-p)} - \frac{2\Gamma(1+M)}{\Gamma(1+M-p)}\right] & p\leqslant M\\ \frac{1}{\Gamma(1+2M-p)}\left[\frac{2\Gamma(1+M)}{\Gamma(1-M+p)} - \frac{\Gamma(1+2M)}{\Gamma(1+p)}\right] & p>M\end{cases}. \qquad (9)$$

Introducing the integral representation for the delta functions in Eq. (8) one obtains

$$\overline{Z^n} = e^{NC}\int D\underline{Q}\,D\underline{\Lambda}\,\exp[-NG(\underline{Q},\underline{\Lambda})], \qquad (10)$$

$$G(\underline{Q},\underline{\Lambda}) = -\frac{\beta^2 M}{4}\sum_k\sum_{a\neq b}Q_{ab}^{(k)}Q_{ab}^{(p-k)} + \frac{M}{2}\sum_k\sum_{a\neq b}\Lambda_{ab}^{(k)}Q_{ab}^{(k)} - \log Z(\underline{\Lambda}), \qquad (11)$$

$$Z(\underline{\Lambda}) = \mathrm{Tr}_{[s]}e^{S(\underline{\Lambda})}$$
$$S(\underline{\Lambda}) = \frac{1}{2}\sum_k\sum_{a\neq b}\frac{\Lambda_{ab}^{(k)}}{M^{k-1}}\sum_{i_1<,\dots,<i_k}s_{i_1}^a s_{i_1}^b,\dots,s_{i_k}^a s_{i_k}^b, \qquad (12)$$

$$D\underline{Q} = \prod_k\prod_{a<b}dQ_{ab}^{(k)},$$
$$D\underline{\Lambda} = \prod_k\prod_{a<b}d\Lambda_{ab}^{(k)}. \qquad (13)$$

The stationarity equations in Λ and Q are

$$Q_{ab}^{(k)} = \frac{1}{Z(\underline{\Lambda})}\mathrm{Tr}_{[s^a]}\frac{1}{M^k}\times\sum_{i_1<\dots<i_k}s_{i_1}^a s_{i_1}^b,\dots,s_{i_k}^a s_{i_k}^b e^{S(\underline{\Lambda})}, \qquad (14)$$
$$\Lambda_{ab}^{(k)} = \beta^2\,Q_{ab}^{(p-k)}. \qquad (15)$$

Substituting the saddle point value for Λ in the effective action we obtain

$$G(\underline{Q}) = \frac{\beta^2 M}{4}\sum_k\sum_{a\neq b}Q_{ab}^{(k)}Q_{ab}^{(p-k)} - \log\mathrm{Tr}_{[s^a]}e^{S(\underline{Q})}, \qquad (16)$$

$$S(\underline{Q}) = \frac{\beta^2}{2}\sum_k\sum_{a\neq b}\frac{Q_{ab}^{(p-k)}}{M^{k-1}}\sum_{i_1<,\dots,<i_k}s_{i_1}^a s_{i_1}^b,\dots,s_{i_k}^a s_{i_k}^b.$$

The physical meaning of the overlap matrix at saddle point value is the usual one and more precisely

$$Q_{ab}^{(k)} = \frac{1}{NM^k}\sum_{x=1}^N\sum_{i_1<i_2<,\dots,<i_k}\overline{\langle s_{i_1}(x),\dots,s_{i_k}(x)\rangle^2}$$
$$= \lim_{n\to 0}\frac{2}{n(n-1)}\sum_{a<b}Q_{ab}^{(k)}|_{\mathrm{SP}}. \qquad (17)$$

III. LARGE M LIMIT: STANDARD p-SPIN

For large M, neglecting diagonal terms in the sum over $i_1,\dots,i_k$, in Eq. (16), the log Tr term can be rewritten as

$$S(\underline{Q}) = M\frac{\beta^2}{2}\sum_{k=1}^{p-1}\sum_{a\neq b}Q_{ab}^{(p-k)}\frac{1}{k!}\left(\frac{1}{M}\sum_{i=1}^M s_i^a s_i^b\right)^k. \qquad (18)$$

Performing the saddle point for large M, rather than N, and introducing the auxiliary parameter

$$q_{ab} \equiv \frac{1}{M}\sum_{i=1}^M s_i^a s_i^b, \qquad (19)$$

we obtain, for the free energy Eq. (16)

$$G(\underline{Q}) = M\Bigg[\frac{\beta^2}{4}\sum_{k=1}^{p-1}\sum_{a\neq b}Q_{ab}^{(k)}Q_{ab}^{(p-k)} - \frac{\beta^2}{2}\sum_{k=1}^{p-1}\frac{1}{k!}\sum_{a\neq b}Q_{ab}^{(p-k)}q_{ab}^k + \lambda_{ab}q_{ab} - \log\mathrm{Tr}_{[s^a]}\exp\left\{\sum_{a\neq b}\lambda_{ab}s^a s^b\right\}\Bigg]. \qquad (20)$$

The saddle point self-consistency equation with respect to $Q^{(p-k)}$ yields

$$Q_{ab}^{(k)} = \frac{1}{k!}q_{ab}^k. \qquad (21)$$

Substituting Eq. (21) in Eq. (20), we obtain the expression

$$\frac{G(\underline{q},\underline{\lambda})}{M} = -\frac{\beta^2}{4}\sum_{a\neq b}\sum_{k=1}^{p-1}\frac{q_{ab}^p}{k!(p-k)!} + \lambda_{ab}q_{ab} - \log\mathrm{Tr}_{[s^a]}\exp\left\{\sum_{a\neq b}\lambda_{ab}s^a s^b\right\}, \qquad (22)$$

that is, the standard formal free energy of the fully connected Ising p-spin model

$$\frac{G(\underline{q},\underline{\lambda})}{M} = -\frac{\beta^2}{4}\frac{2^p-2}{p!}\sum_{a\neq b}q_{ab}^p + \lambda_{ab}q_{ab} - \log\mathrm{Tr}_{[s^a]}\exp\left\{\sum_{a\neq b}\lambda_{ab}s^a s^b\right\}, \qquad (23)$$

whose saddle point equations read

$$q_{ab} = \langle s^a s^b \rangle, \tag{24}$$

$$\lambda_{ab} = \frac{p\beta^2}{2} q_{ab}^{p-1}. \tag{25}$$

The method of replicating lattice site variables a large number of times—tending to infinity ($M \to \infty$)—is a standard way of obtaining the mean-field limit of finite-dimensional systems, as well as the corrections to it, as loop expansions in $1/M$. For a pedagogical instance the reader can look at Appendix A21.2 of Ref. 39. For references explicitly related to the Ising p-spin model with quenched disorder one can look at Refs. 40 and 41.

IV. ANALYSIS OF THE CRITICAL POINT

Our aim is to find the transition point and to study its thermodynamic nature as M and p are changed. In particular, we will verify that, at given p (M) there are threshold values of M (p) beyond which the transition switches from continuous to discontinuous.

First, to identify the critical point we expand the stationarity equation (14) to first order in $Q_{ab}^{(k)}$, obtaining

$$Q_{ab}^{(k)} = \frac{\beta^2}{M^k} \frac{Q_{ab}^{(p-k)}}{M^{k-1}} \binom{M}{k}. \tag{26}$$

There are "multi'critical temperatures for the "multi"-overlaps, whose expressions read

$$\beta_c(k) = \frac{M^{\frac{p-1}{2}}}{\binom{M}{k}^{\frac{1}{4}} \binom{M}{p-k}^{\frac{1}{4}}}. \tag{27}$$

The largest critical temperature is obtained for $k = p/2$ if p is even, and for $k = (p+1)/2\,, (p-1)/2$ if p is odd. The overlap corresponding to the smallest β_c (slightly above β_c) is nonzero and of order $\tau \propto (T_c - T)/T_c$, while the others are at least of order τ^2.

Proceeding to the second order expansion of Eq. (14) we have

$$\begin{aligned} Q_{ab}^{(k)} = {} & \frac{\beta^2}{M^k} \frac{Q_{ab}^{(p-k)}}{M^{k-1}} \binom{M}{k} \\ & + \mathrm{Tr}_{[s^a]} \frac{1}{M^k} \sum_{i_1<,\ldots,<i_k} s_{i_1}^a s_{i_1}^b, \ldots, s_{i_k}^a s_{i_k}^b \frac{\beta^4}{4\times 2!} \\ & \times \sum_{l,m} \sum_{c\neq d, e\neq f} \frac{Q_{cd}^{(p-l)}}{M^{l-1}} \frac{Q_{ef}^{(p-m)}}{M^{m-1}} \sum_{j_1<,\ldots,<j_l} \\ & \times \sum_{t_1<,\ldots,<t_m} s_{j_1}^c s_{j_1}^d, \ldots, s_{j_l}^c s_{j_l}^d\, s_{t_1}^e s_{t_1}^f, \ldots, s_{t_m}^e s_{t_m}^f. \end{aligned} \tag{28}$$

We will focus only on the equations for the overlaps corresponding to the largest critical temperature [cf., Eq. (27)], that is, on the terms of the type

$$Q_{ab}^{(p/2)} Q_{ab}^{(p/2)}, \quad \text{for} \quad \text{even } p,$$

or

$$Q_{ab}^{(\frac{p\pm1}{2})} Q_{ab}^{(\frac{p\pm1}{2})}, \quad \text{for} \quad \text{odd } p.$$

More specifically, we are interested in the terms of the series at the right-hand side (r.h.s.) of Eq. (28) with $k = l = m = p/2$, if p is even, or with $k,l,m = \frac{p\pm1}{2}$, if p is odd.

It is interesting to notice that we would have the same physics considering a model in which $p/2$-uples on each site interacting with $p/2$ on another site (p even) or $(p+1)/2$-uples on a site interact with $(p-1)/2$ on another site (p odd).

In Eq. (28) each spin in each replica has to be matched with another one in another replica to get a nonzero result from the trace. At second order we are, thus, left with only two kinds of possible matching, yielding terms

$$\sum_c Q_{ac}^{(\times)} Q_{cb}^{(\times)} = (Q^{(\times)})^2_{ab}, \quad \text{and} \quad (Q_{ab}^{(\times)})^2.$$

We will see how, depending on the parity of p, the multiplicity of such terms will change, leading to different expressions of their coefficients as functions of p and M.

Using the above results, Eq. (16) can then be approximated to the third order in Q as

$$G(\underline{Q}) = \frac{\tau}{2} \sum_{a,b} Q_{ab}^2 + \frac{w_1}{6} \mathrm{Tr} Q^3 + \frac{w_2}{6} \sum_{a,b} Q_{ab}^3, \tag{29}$$

where Q_{ab} stays for $Q_{ab}^{(\times)}$.

As already noticed by Gross, Kanter, and Sompolinsky[17] in the Potts model (threshold was $p_{\mathrm{Potts}} = 4$ colors) and in Ref. 42, it can be shown (see Appendix A) that if the ratio w_2/w_1 between coupling constants on the nonlinear terms is larger than one the phase transition cannot be continuous.

We will now proceed to the computation of the coupling constants for the M-p Ising spin model. Since, as already mentioned, the computation of the third order coefficients will yield different functional expressions depending on the values of p, we have to distinguish between four cases

$$p = \begin{cases} 4a & \mathbf{A}, \\ 4a-2 & \mathbf{B}, \\ 4a-1 & \mathbf{C}, \\ 4a-3 & \mathbf{D}, \end{cases} \quad a \in \mathbb{N}^+, \tag{30}$$

and we will analyze them separately.

A. Even p and $p/2$, $p = 4a$

The only surviving term in the sum over l and m in the r.h.s. of Eq. (28) is for $l = m = k = p/2$. The trace term turns out to be

$$w_1 \sum_{c=1}^{n} Q_{ac}^{(p/2)} Q_{cb}^{(p/2)} = \frac{\beta^4}{M^{3p/2-2}} \binom{M}{p/2} \sum_{c=1}^{n} Q_{ac}^{(p/2)} Q_{cb}^{(p/2)}. \tag{31}$$

The squared term is

$$w_2 \big(Q_{ab}^{(p/2)}\big)^2 = \frac{1}{2M^{\frac{3}{2}p-2}} \binom{M}{p/2} \binom{p/2}{p/4} \times \binom{M-p/2}{p/4} \left(Q_{ab}^{(p/2)}\right)^2, \tag{32}$$

and the ratio

$$\frac{w_2}{w_1} = \frac{1}{2} \binom{p/2}{p/4} \binom{M-p/2}{p/4}. \tag{33}$$

B. Even p, odd $p/2$, $p = 4a - 2$

In the r.h.s. of Eq. (29) only the coefficient in front of the Tr Q_{ab}^3 term survives, whereas $w_2 = 0$ always. The ratio is

$$\frac{w_2}{w_1} = 0. \tag{34}$$

According to the small Q expansion, Eqs. (28) and (29), when p is even and $p/2$ is odd the transition at the largest critical temperature $1/\beta_c(p/2)$ [cf., Eq. (27)] turns out to be consistent with a continuous one, no matter how many spins M stay on each site. This might appear at contrast with the large M limit leading to Eq. (23) that is equivalent to an Ising p-spin mean-field model for any $p > 2$. There, however, no perturbative expansion was carried out, while Eq. (34) is the outcome of an expansion for small overlap values that cannot help in identifying discontinuous transitions with finite jumps in Q. Indeed, the condition expressed by Eq. (A7) is sufficient but not necessary to rule out a continuous transition. As mentioned in the Introduction, finding a continuous transition by means of a small Q expanded action does not rule out the possibility of a discontinuous transition in Q.

C. Odd p, even $(p+1)/2$, $p = 4a - 1$

When p is odd we have to deal with two relevant overlaps $Q^{(\frac{p-1}{2})}$ and $Q^{(\frac{p+1}{2})}$ and one critical temperature. To determine the coupling constants of the cubic terms one thus has to diagonalize a 2×2 matrix. In Appendix A we report the details of the computation leading to

$$w_1(p,M) = \frac{\sqrt{2M-p+1}+\sqrt{p+1}}{4M^{p-3/2}\sqrt{p+1}}\sqrt{\binom{M}{\frac{p-1}{2}}}, \tag{35}$$

for the coefficient of the cubic trace term in the action, Eq. (29). The expression for the coupling constant depends further on $(p+1)/2$ being even or odd. For $p = 4a - 1$ we obtain

$$w_2(p,M) = \frac{1}{8M^{p-3/2}}\frac{5M-3p+2}{2M-p+1} \times\sqrt{\binom{M}{\frac{p-1}{2}}}\binom{\frac{p+1}{2}}{\frac{p+1}{4}}\binom{M-\frac{p-1}{2}}{\frac{p+1}{4}}, \tag{36}$$

and, eventually, the ratio is

$$\frac{w_2}{w_1} = \frac{2+5M-3p}{2+4M-2p}\frac{\sqrt{p+1}}{\sqrt{p+1}+\sqrt{2M-p+1}} \times\binom{M-\frac{p-1}{2}}{\frac{1+p}{4}}\binom{\frac{1+p}{2}}{\frac{1+p}{4}}. \tag{37}$$

D. Odd p, even $(p-1)/2$, $p = 4a - 3$

In this last case the coupling of the trace cubic term is still given by Eq. ((37)) and the second nonlinear coupling constant is expressed as

$$w_2(p,M) = \frac{1}{4M^{p-3/2}}\frac{6-p-5p^2+9M+7pM}{(p+3)(2M-p+1)} \times\sqrt{\binom{M}{\frac{1+p}{2}}}\binom{M-\frac{p-1}{2}}{\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}, \tag{38}$$

TABLE I. Ratio values for small p and M around the threshold 1.

p	M	w_2/w_1
3	2	$3(1-1/\sqrt{2}) = 0.878\,68$
3	3	2
4	2	0
4	3	1
4	4	2
5	3	$(\sqrt{3}-1)/2 = 0.366\,025$
5	4	$13(\sqrt{3/2}-1) = 2.921\,68$
6	any	0

yielding the ratio

$$\frac{w_2}{w_1} = \frac{6-p-5p^2+9M+7pM}{(p+3)(2M-p+1)} \times\frac{\sqrt{2M-p+1}}{\sqrt{p+1}+\sqrt{2M-p+1}}\binom{M-\frac{p-1}{2}}{\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}. \tag{39}$$

E. Summary

We now have a complete description of the critical behavior of the M-p system. Already at the mean-field level to have a discontinuous transition a $p > 2$ interaction between spins is not enough. We find that for each given p one needs a minimal number of spins $M_{\rm disc}$ on each site to have a random first order phase transition, corresponding to the lowest integer M for which $w_2/w_1 > 1$ [cf., Eqs. (33), (37), or (39) depending on the parity of p and $(p+1)/2$].

In Table I we report some values of the ratios for systems with small p and M. In Fig. 1 we plot the $M_{\rm disc}(p)$ behavior.

V. CONCLUSION

In the present work we have performed an analytic computation of the critical behavior of a mean-field p-spin model that can display both a random first order and a continuous

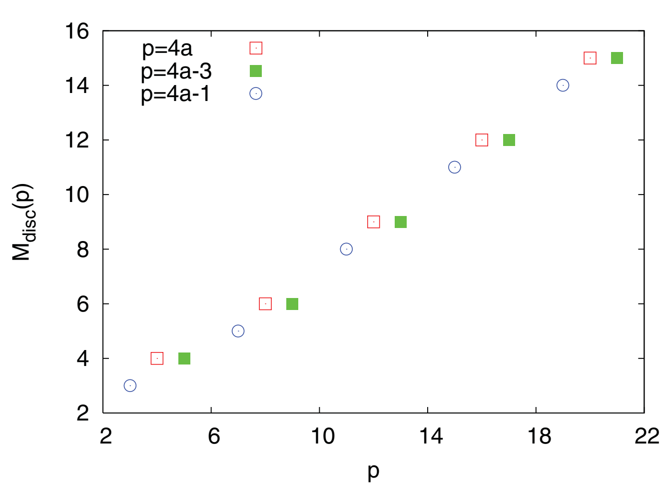

FIG. 1. (Color online) Lowest integer values of M at given p, for which a discontinuous transition is certainly expected (sufficient condition to have RFOT is $M \geqslant M_{\rm disc}$).

phase transition tuning the number of spins present on each site. The effective action is represented by Eq. (29) where the first two relevant terms are third order in the overlap parameter Q. The ratio of their coupling constants has been computed, thus providing the threshold values of $M_{\rm disc}(p)$ separating models compatible with a continuous critical behavior and models displaying discontinuous transitions. If $M \geqslant M_{\rm disc}$ this is sufficient to guarantee the discontinuity of the transition, though it is not necessary. A RFOT can also occur at lower values of M that cannot be identified with the probe based on the perturbative expansion for small Q [cf., Eq. (29)].

The particular case studied in Ref. 29, ($M = 2,\ p = 3$), yielded numerical evidence for a continuous phase transition in dimension four. This is consistent with the value of the $w_2/w_1 = 3[1 - 1/\sqrt{2}] = 0.878\,68$ as computed in the mean-field theory (cf., Table I and Ref. 43). The same applies to the model recently studied in Ref. 30, a one dimensional $(M,p) = (2,3)$ model on a Levy lattice.[44] *Already at the mean-field level* the $M = 2$, $p = 3$ model seems to display a Sherrington-Kirkpatrick-like transition rather than a RFOT.

The continuous-discontinuous crossover appears to be very similar to the one found in Potts[17] varying the number of colors, and in the spherical p-spin varying an external magnetic field.[18] The advantage of the present model is that it can be easily represented in finite dimensions on lattices of given geometry (e.g., on a cubic lattice with short-range interactions) and that it always displays a RFOT in the $M \to \infty$ limit for any kind of lattice. The finite dimension counterpart of the model under study might then be easily achieved since the p-spin interaction is always exchanged between two sites [e.g., nearest neighbors on a d-dimensional (hyper)cubic lattice]. It remains to be investigated if this finite-dimensional counterpart displays the RFOT or not.

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APPENDIX A: THRESHOLD VALUE FOR w_2/w_1

Starting from the self-consistency equation for small Q's [implying a continuous transition in Q, cf., Eq. (29) where also the quartic term is considered]

$$\tau Q_{ab} + w_1(Q^2)_{ab} + w_2 Q_{ab}^2 + y Q_{ab}^3 = 0, \tag{A1}$$

we have, in the RSB ansatz,

$$\tau q(x) - w_1\left[2\,q(x)\int_0^1 q(s)\,ds + \int_0^x [q(x)-q(s)]^2\,ds\right] + w_2 q(x)^2 + y q(x)^3 = 0. \tag{A2}$$

Deriving once Eq. (A2) with respect to x one has

$$q'(x)\left\{\tau - 2w_1\left[\int_x^1 q(s)\,ds + x q(x)\right] + 2w_2 q(x) + 3yq(x)^2\right\} = 0. \tag{A3}$$

If $q'(x) \neq 0$, deriving a second time with respect to x, one finds

$$q'(x)[-w_1 x + w_2 + 3yq(x)] = 0. \tag{A4}$$

If $y > 0$, then the overlap function around criticality can be written as

$$q(x) = \begin{cases} 0, & x < x_1, \\ \frac{1}{3y}(w_1 x - w_2), & x_1 < x < x_2, \\ q(1), & x > x_2, \end{cases} \tag{A5}$$

where, for continuity, $x_1 = w_2/w_1$ and

$$q(1) = \frac{\tau}{p}\,\frac{2}{1+\sqrt{1-6y\tau/p^2}}, \quad p \equiv w_1(1-x_1). \tag{A6}$$

As a consequence, to have a continuous transition it must be

$$\frac{w_2}{w_1} \leqslant 1. \tag{A7}$$

The argument for the threshold value of x still works also if $y \leqslant 0$ in Eq. (A1). In that case, rather than a continuous function, we simply have a 1RSB step function for $q(x) = \theta(x - x_1)q$, with

$$q = \frac{\tau}{p}\,\frac{2}{1+\sqrt{1-10y\tau/p^2}}, \tag{A8}$$

$$p \equiv w_1\left(1 - \frac{w_2}{w_1}\right), \qquad x_1 = \frac{w_2}{w_1} + \frac{3yq}{w_1}. \tag{A9}$$

APPENDIX B: COUPLING CONSTANTS WITH ODD p

When p is odd we have to deal with two relevant overlaps and one critical temperature. The second order equation, Eq. (28), has the structure

$$\mathcal{A}\mathbf{Q}_{ab} = \mathbf{F}(\{\mathbf{Q}\}), \tag{B1}$$

where $\mathbf{Q}_{ab} = \{Q_{ab}^{(p-M)}, \ldots, Q_{ab}^{(M)}\}$. Diagonalizing $\mathcal{A} \to \mathcal{D}_{\mathcal{A}} = \mathcal{P}^{-1}\mathcal{A}\mathcal{P}$ one obtains

$$\mathcal{P}^{-1}\mathcal{A}\mathcal{P}\mathcal{P}^{-1}\mathbf{Q}_{ab} = \mathcal{P}^{-1}\mathbf{F}(\{\mathbf{Q}_{ab}\}). \tag{B2}$$

Introducing new variables Θ_{ab}, linear combinations of $\mathbf{Q}_{ab}$, the above expression can be rewritten as

$$\mathcal{D}_{\mathcal{A}}\Theta_{ab} = \mathcal{P}^{-1}\mathbf{F}(\{\mathcal{P}\Theta_{ab}\}). \tag{B3}$$

Rearranging the entries in a proper way, $\mathcal{A}$ can be written as a block matrix of 2×2 elements per block, and each block can be diagonalized separately, with eigenvalues

$$\lambda^{(k\pm)} = 1 \pm \beta^2\sqrt{f(k)f(p-k)}, \tag{B4}$$

and eigenvectors

$$v^{k\pm} = \left[\frac{1}{2\sqrt{f(p-k)}} \mp \frac{1}{2\sqrt{f(k)}}\right]. \tag{B5}$$

For each block of the matrix, labeled by k, the eigenvector matrix and its inverse, thus, are

$$\mathcal{P} = \begin{bmatrix} \frac{1}{2\sqrt{f(p-k)}} & \frac{1}{2\sqrt{f(p-k)}} \\ -\frac{1}{2\sqrt{f(k)}} & \frac{1}{2\sqrt{f(k)}} \end{bmatrix}, \tag{B6}$$

$$\mathcal{P}^{-1} = \begin{bmatrix} \sqrt{f(p-k)} & -\sqrt{f(k)} \\ \sqrt{f(p-k)} & \sqrt{f(k)} \end{bmatrix}, \tag{B7}$$

with

$$f(k) = \frac{1}{M^{2k-1}}\binom{M}{k}, \tag{B8}$$

and

$$\Theta^{(k+)} = \sqrt{f(p-k)}\,Q^{(k)} - \sqrt{f(k)}\,Q^{(p-k)}, \tag{B9}$$

$$\Theta^{(k-)} = \sqrt{f(p-k)}\,Q^{(k)} + \sqrt{f(k)}\,Q^{(p-k)}. \tag{B10}$$

In the present case, since p is odd, the only overlaps we need to consider are $Q^{(\frac{p-1}{2})}$ and $Q^{(\frac{p+1}{2})}$. Their self-consistency equation can be written in the form

$$\mathcal{A}\begin{bmatrix} Q_{ab}^{(\frac{p-1}{2})} \\ Q_{ab}^{(\frac{p+1}{2})} \end{bmatrix} = \begin{bmatrix} F_{\frac{p-1}{2}}(\mathbf{Q}) \\ F_{\frac{p+1}{2}}(\mathbf{Q}) \end{bmatrix}, \tag{B11}$$

with

$$\mathcal{A} = \begin{bmatrix} 1 & -\beta^2 f\left(\frac{p-1}{2}\right) \\ -\beta^2 f\left(\frac{p+1}{2}\right) & 1 \end{bmatrix}. \tag{B12}$$

The functions $F_{(p\pm1)/2}$ are two polynomials of degree two in all the $Q^{(k)}$'s. However, as mentioned above, to study the nature of the critical behavior (continuous or discontinuous) we only need the terms relevant at the highest critical temperature [cf., Eq. (27)] and we thus set to zero all the overlap matrices except for $Q^{(\frac{p-1}{2})}$ and $Q^{(\frac{p+1}{2})}$.

Depending on the parity of $(p+1)/2$ the relevant terms contributing to the nonlinear couplings in the action Eq. (29) differ. We now consider the two cases separately.

a. $(p+1)/2$ even

If $p = 4a - 1$ with $a \in \mathbb{N}$ the functions on the r.h.s. of Eq. (B11) read

$$F_{\frac{p-1}{2}}(\mathbf{Q}) = \frac{\beta^4}{M^{\frac{3}{2}p-\frac{7}{2}}}\binom{M}{\frac{p-1}{2}}\sum_{c=1}^{n} Q_{ac}^{(\frac{p+1}{2})}Q_{cb}^{(\frac{p+1}{2})} + \frac{\beta^4}{M^{\frac{3}{2}p-\frac{5}{2}}} \times \binom{M}{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{\frac{p+1}{4}}\binom{M-\frac{p-1}{2}}{\frac{p+1}{4}} Q_{ab}^{(\frac{p-1}{2})}Q_{ab}^{(\frac{p+1}{2})}, \tag{B13}$$

$$F_{\frac{p+1}{2}}(\mathbf{Q}) = \frac{\beta^4}{M^{\frac{3}{2}p-\frac{1}{2}}}\binom{M}{\frac{p+1}{2}}\sum_{c=1}^{n} Q_{ac}^{(\frac{p-1}{2})}Q_{cb}^{(\frac{p-1}{2})} + \frac{\beta^4}{2M^{\frac{3}{2}p-\frac{1}{2}}}\binom{M}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{\frac{p+1}{4}}\binom{M-\frac{p+1}{2}}{\frac{p+1}{4}}\left(Q_{ab}^{(\frac{p-1}{2})}\right)^2 + \frac{\beta^4}{2M^{\frac{3}{2}p-\frac{5}{2}}}\binom{M}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{\frac{p+1}{4}}\binom{M-\frac{p+1}{2}}{\frac{p-3}{4}}\left(Q_{ab}^{(\frac{p+1}{2})}\right)^2. \tag{B14}$$

b. $(p-1)/2$ even

If otherwise, $p = 4a + 1$ with $a \in \mathbb{N}$, one obtains

$$F_{\frac{p-1}{2}} = \frac{\beta^4}{M^{\frac{3}{2}p-\frac{7}{2}}}\binom{M}{\frac{p-1}{2}}\sum_{c} Q_{ac}^{(\frac{p+1}{2})}Q_{cb}^{(\frac{p+1}{2})} + \frac{\beta^4}{2M^{\frac{3}{2}p-\frac{7}{2}}}\binom{M}{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\binom{M-\frac{p-1}{2}}{\frac{p-1}{4}}\left(Q_{ab}^{(\frac{p+1}{2})}\right)^2 + \frac{\beta^4}{2M^{\frac{3}{2}p-\frac{3}{2}}}\binom{M}{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\binom{M-\frac{p-1}{2}}{\frac{p+3}{4}}\left(Q_{ab}^{(\frac{p-1}{2})}\right)^2, \tag{B15}$$

$$F_{\frac{p+1}{2}} = \frac{\beta^4}{M^{\frac{3}{2}p-\frac{1}{2}}}\binom{M}{\frac{p+1}{2}}\sum_{c} Q_{ac}^{(\frac{p-1}{2})}Q_{cb}^{(\frac{p-1}{2})} + \frac{\beta^4}{M^{\frac{3}{2}p-\frac{3}{2}}}\binom{M}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{\frac{p-1}{4}}\binom{M-\frac{p+1}{2}}{\frac{p-1}{4}} Q_{ab}^{(\frac{p-1}{2})}Q_{ab}^{(\frac{p+1}{2})}. \tag{B16}$$

To decouple Eqs. (B11) and (B12) we specify the two new variables, Eqs. (B9) and (B10), for $k = (p-1)/2$

$$\Theta_{ab}^{(+)} = \sqrt{f\left(\frac{p+1}{2}\right)}\,Q_{ab}^{(\frac{p-1}{2})} - \sqrt{f\left(\frac{p-1}{2}\right)}\,Q_{ab}^{(\frac{p+1}{2})}, \tag{B17}$$

$$\Theta_{ab}^{(-)} = \sqrt{f\left(\frac{p+1}{2}\right)}\,Q_{ab}^{(\frac{p-1}{2})} + \sqrt{f\left(\frac{p-1}{2}\right)}\,Q_{ab}^{(\frac{p+1}{2})}. \tag{B18}$$

Applying the diagonalization transformation described above [cf., Eqs. (B1)–(B3)], one finds

$$\lambda^{(+)}\Theta_{ab}^{(+)} = \sqrt{f\left(\frac{p+1}{2}\right)}\,F_{\frac{p-1}{2}} - \sqrt{f\left(\frac{p-1}{2}\right)}\,F_{\frac{p+1}{2}}, \tag{B19}$$

$$\lambda^{(-)}\Theta_{ab}^{(-)} = \sqrt{f\left(\frac{p+1}{2}\right)}\,F_{\frac{p-1}{2}} + \sqrt{f\left(\frac{p-1}{2}\right)}\,F_{\frac{p+1}{2}}, \tag{B20}$$

where the eigenvalues [cf., Eq. (B4)] are

$$\lambda^{(\pm)} = 1 \pm \beta^2\sqrt{f((p-1)/2)f((p+1)/2)}.$$

Since the F's depend on the Q's, we have to apply the inverse transformation to get equations in terms of the Θ's. The eigenvalue $\lambda^{(+)}$ is always positive, so that $\Theta^{(+)}$ plays the same role of the "noncritical" overlaps and can be put to zero. The inverse transformation, thus, reduces to

$$Q_{ab}^{(\frac{p-1}{2})} = \frac{\Theta_{ab}^{(-)}}{2\sqrt{f\left(\frac{p+1}{2}\right)}}, \quad Q_{ab}^{(\frac{p+1}{2})} = \frac{\Theta_{ab}^{(-)}}{2\sqrt{f\left(\frac{p-1}{2}\right)}}, \tag{B21}$$

so that Eq. (B20) decouples in

$$\lambda^{(-)}\Theta_{ab}^{(-)} = w_1(p,M)\sum_{c}\Theta_{ac}^{(-)}\Theta_{cb}^{(-)} + w_2(p,M)\left(\Theta_{ab}^{(-)}\right)^2. \tag{B22}$$

The constants w_1 and w_2 depend on p and M. The expression for w_1 is

$$w_1(p,M) = \frac{1}{4M^{p-3/2}}\left(\sqrt{\binom{M}{\frac{p-1}{2}}} + \sqrt{\binom{M}{\frac{p+1}{2}}}\right). \tag{B23}$$

The formula for w_2 changes depending on the parity of $(p+1)/2$.

For even $(p+1)/2$

$$\begin{aligned} w_2(p,M) = \frac{1}{8M^{p-3/2}}\sqrt{\binom{M}{\frac{p-1}{2}}}\Bigg[&\binom{\frac{p-1}{2}}{\frac{p+1}{4}}\binom{M-\frac{p-1}{2}}{\frac{p+1}{4}} \\ &+\binom{\frac{p+1}{2}}{\frac{p+1}{4}}\binom{M-\frac{p+1}{2}}{\frac{p+1}{4}} + \frac{2M-p+1}{1+p} \\ &\times\binom{\frac{p+1}{2}}{\frac{p+1}{4}}\binom{M-\frac{p+1}{2}}{\frac{p-3}{4}}\Bigg]. \end{aligned} \tag{B24}$$

If $(p+1)/2$ is odd it reads

$$\begin{aligned} w_2(p,M) = \frac{1}{8M^{p-\frac{3}{2}}}\sqrt{\binom{M}{\frac{p+1}{2}}}\Bigg[&\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\binom{M-\frac{p-1}{2}}{\frac{p-1}{4}} \\ &+\frac{1+p}{2M-p+1}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\binom{M-\frac{p-1}{2}}{\frac{p+3}{4}} \\ &+2\binom{\frac{p+1}{2}}{\frac{p-1}{4}}\binom{M-\frac{p+1}{2}}{\frac{p-1}{4}}\Bigg]. \end{aligned} \tag{B25}$$

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