Superconductivity from repulsive interactions in the two-dimensional electron gas

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We present a well-controlled perturbative renormalization-group treatment of superconductivity from short-ranged repulsive interactions in a variety of model two-dimensional electronic systems. Our analysis applies in the limit where the repulsive interactions between the electrons are small compared to their kinetic energy.

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I. INTRODUCTION

In a variety of recently discovered materials, superconductivity apparently arises directly from the electron correlations themselves. However, these materials are complex, and many material-specific details can play a role in the mechanism of superconductivity. The problem is greatly simplified in the weak-coupling limit, where we recently showed[1] that an asymptotically exact treatment of the problem is possible, valid in cases in which the superconducting state emerges at low temperatures from a well-formed Fermi liquid. Nonetheless, even under these circumstances, the character of the superconducting state and the transition temperature depend in a complicated way on details of the band structure, both near and far from the Fermi surface.

To the extent that there are basic principles at work underlying the mechanism of unconventional superconductivity, it would be a great advance to find simple model systems which exhibit such behavior. Here, we consider the possibility of unconventional superconductivity in some model systems with particularly simple electronic structures, where controlled theory is possible, and where, conceivably, experimental tests of the theory are feasible. Specifically, we consider circumstances in which superconductivity may occur in a two-dimensional electron gas (2DEG) in a high mobility heterostructure. Here, due to the the stiffness of the lattice and the limited phase space for electron-phonon scattering, electron-phonon coupling is probably negligible, and the single-particle dynamics can be treated accurately within a rotationally invariant effective mass approximation. Moreover, the strength of the correlations can, to a large extent, be tuned by varying the electron density.

The possibility of an electronic pairing mechanism in systems with rotational invariance was first put forth in a seminal paper by Kohn and Luttinger.[2] Although U, the bare interactions among electrons, are repulsive, there are *effective* attractive interactions that arise at $\mathcal{O}(U^2)$. Kohn and Luttinger focused on the portion of the effective attractions associated with the nonanalyticities in $\chi(q)$, the particle-hole susceptibility, at momentum $q = 2k_F$ which reflect the sharpness of the Fermi surface at zero temperature. More generally, what is required for this mechanism to work is strong q dependence of $\chi(q)$ for $q \leqslant 2k_F$. Indeed, the Kohn-Luttinger instability of a three-dimensional rotationally invariant system results in the formation of a p-wave superconducting ground state due to the peak in $\chi(q)$ near $q = 0$.[3,4] While this result is valid only in the weak-coupling regime where $U \ll E_F$, it is widely believed that the p-wave ground state obtained this way is adiabatically connected to the more realistic (and more strongly correlated) example of helium-3.[5]

However, the Kohn-Luttinger effect is exponentially weaker in a rotationally invariant 2DEG,[6–8] due to the fact $\chi(q)$ is independent of momentum for momenta $q \leqslant 2k_F$. It was later shown that at $\mathcal{O}(U^3)$, the 2DEG does exhibit a pairing instability.[9] Still, at least in weak coupling, electronically mediated superconductivity in the 2DEG is negligible.

In this paper, we show that by perturbing the 2DEG, it is possible to significantly enhance the superconducting transition temperature by engineering circumstances in which instabilities arise at $\mathcal{O}(U^2)$ in perturbation theory. We present asymptotically exact[1] weak-coupling solutions of the superconducting instability in several systems that are variants of the simplest, rotationally invariant 2DEG. As a first example, we show that partially spin polarizing the 2DEG produces a nonunitary $p + ip$ superconductor. Kagan and Chubukov have previously addressed this problem using an expansion in powers of the electron concentration,[10] and their result reduces to ours in the weak-coupling limit. As a second example, we consider the 2DEG in a semiconductor heterostructure quantum well with two populated subbands. We show that this system can possess both p-wave and d-wave ground states and present the phase diagram of this system.

This paper is organized as follows. In the next section, we review the method developed in Ref. 1 and discuss its straightforward generalization needed for the present context. In Sec. III, the effect of partially polarizing the 2DEG is studied. In Sec. IV, we consider two subbands in a 2DEG quantum well. Technical details of the various calculations are presented in the Appendix. In a forthcoming paper[11] we will consider a variety of slightly more complex situations pertinent to particular semiconductor heterostructures.

II. PERTURBATIVE RENORMALIZATION GROUP TREATMENT OF SUPERCONDUCTVITY

In this section, we review the prescription of Ref. 1 and discuss its generalization to the present context. We integrate out high-energy modes in two steps. In the first step, we integrate out all modes outside a narrow range of energies Ω about the Fermi energy. Ω is not a physical energy in the problem, but rather a calculational device. It is chosen large enough so that the interactions can be treated perturbatively but small enough that it can be set to zero in all nonsingular expressions without causing significant error, i.e., it is chosen to satisfy the inequalities $\rho U^2 \gg \Omega \gg \mu \exp\{-[1/\rho U]\}$, where ρ is the density of states at the Fermi energy and μ is the Fermi energy. The effective interactions generated in the process then serve as the "bare" interactions in a second step, in which the remaining problem is solved using the perturbative renormalization group procedure of Shankar and Polchinski.[12,13] T_c is, up to an unknown multiplicative constant, given by the energy scale, T^*, at which a relevant interaction grows to be of order 1. It was shown by careful analysis of perturbative expressions up to fourth order in the interaction strength that the resulting expression for T^* is independent of Ω.

The analysis of Ref. 1 leads to the following prescription for computing the leading-order asymptotic behavior of T_c for weak interactions: First, compute the effective interaction in the Cooper channel at energy scale Ω, $\Gamma^{(a)}(\hat{k},\hat{k}')$, to second order in the interactions. Here, $\hat{k}$ and $\hat{k}'$ denote points on the Fermi surface, and Γ is the vertex for scattering a pair of electrons with momenta $\hat{k}$ and $-\hat{k}$ to states with momenta $\hat{k}'$ and $-\hat{k}'$, where if there are multiple band indices, the subband index is implicitly determined depending on whether the momenta are on one Fermi surface or the other, and where there is a different matrix depending on whether the electron pair forms a spin singlet ($\Gamma^{(s)}$) or a spin triplet ($\Gamma^{(t)}$). We then construct the related dimensionless matrix

$$g^{(a)}_{\hat{k},\hat{k}'} \equiv \rho\sqrt{\bar{v}/v(\hat{k})}\Gamma^{(a)}(\hat{k},\hat{k}')\sqrt{\bar{v}/v(\hat{k}')}, \tag{1}$$

where $v(\hat{k})$ is the magnitude of the Fermi velocity on the Fermi surface of the corresponding subband, and ρ is the total density of states at the Fermi energy. Manifestly, g is a real, symmetric, hence Hermitian matrix, so it has a complete set of eigenstates and eigenvalues,

$$\sum_{\hat{k}'} g^{(a)}_{\hat{k},\hat{k}'}\phi^{(a,m)}_{\hat{k}'} = \lambda^{(a,m)}\phi^{(a,m)}_{\hat{k}}. \tag{2}$$

Among all the possible solutions, we identify the most negative eigenvalue,

$$\lambda \equiv \min[\lambda^{(a,m)}], \qquad \lambda < 0. \tag{3}$$

Then,

$$T_c \sim \mu \exp[-1/|\lambda|]. \tag{4}$$

III. PARTIALLY POLARIZED FERMI SURFACE

As a first example, we consider a partially spin polarized 2DEG with short-ranged repulsive interactions:

$$H = H_0 + H_1,$$
$$H_0 = \sum_\sigma \int \frac{d^2k}{(2\pi)^2} E_{\sigma,\sigma'}(\boldsymbol{k})\psi^\dagger_\sigma(\boldsymbol{k})\psi_{\sigma'}(\boldsymbol{k}), \tag{5}$$
$$H_1 = U\int \frac{d^2k_1 d^2k_2 d^2k_3}{(2\pi)^6}\psi^\dagger_\uparrow(\boldsymbol{k}_1)\psi^\dagger_\downarrow(\boldsymbol{k}_2)\psi_\downarrow(\boldsymbol{k}_3)\psi_\uparrow(\boldsymbol{k}_4),$$

where $\boldsymbol{k}_4 = \boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3$,

$$E_{\sigma,\sigma'} = \epsilon_{\boldsymbol{k}}\delta_{\sigma,\sigma'} + \boldsymbol{h}\cdot\boldsymbol{\tau}_{\sigma\sigma'}, \tag{6}$$

and $\boldsymbol{h}$ is a mean field that renders the ground state spin polarized. Such a partially polarized system can occur in a narrow-well semiconductor heterostructure in the presence of a parallel magnetic field (in which case $\boldsymbol{h} = g\mu_B \boldsymbol{H}_\parallel$), or in a ferromagnetic phase with spontaneously broken symmetry, such as probably occurs in the Hubbard model away from half-filling in the strong-coupling limit $U \gg t$.[14,15] (However, in the latter case, it requires something of an intuitive leap to treat the residual interactions beyond those that produce the mean field $\boldsymbol{h}$ as "weak.")

Since the Fermi surfaces are spin polarized, singlet pairing is suppressed, so the leading superconducting instability will therefore be in the spin triplet channel. We first consider the limit in which there is no spin-orbit coupling, in which case, the two-particle scattering amplitude is a separate function for each spin polarization. As derived in the Appendix,

$$\Gamma_\uparrow(\hat{k},\hat{q}) = -U^2\chi_\downarrow(\vec{k}-\vec{q}),$$
$$\Gamma_\downarrow(\hat{k},\hat{q}) = -U^2\chi_\uparrow(\vec{k}-\vec{q}),$$

where χ_σ is the contribution of spin σ electrons to the susceptibility.

In the case of a rotationally invariant system with $\epsilon_{\vec{k},\sigma} = k^2/2m + \sigma h$, $v_{f,\sigma}(\hat{k}) = k_{f,\sigma}/m$, and $\rho_\sigma = \rho = m/2\pi$ is independent of the spin polarization. Therefore the matrix $g_{\hat{k},\hat{q}}$ defined in the previous section is

$$g^\sigma_{\hat{k},\hat{q}} = \rho\Gamma_\sigma(\hat{k},\hat{k}'). \tag{7}$$

The particle-hole susceptibility for this system has the following well-known form (see the Appendix):

$$\chi_\sigma(\vec{q}) = \frac{\rho}{2}\left[1 - \frac{\mathrm{Re}\sqrt{q^2-(2k_{F\sigma})^2}}{q}\right]. \tag{8}$$

Thus, $\chi_\sigma(q)$ is a constant for $q < 2k_{F\sigma}$, has a derivative discontinuity at $q = 2k_{F\sigma}$, and vanishes as $1/q^2$ when $q \gg 2k_{F\sigma}$.

The rotational invariance of the problem implies that the triplet eigenfunctions are of the form

$$\psi^{t,m}_\sigma(\hat{k}) = \psi(k_{F\sigma})\cos(m\theta_{\hat{k}}), \tag{9}$$

where m is an odd integer. The eigenvalue problem for this system therefore reduces to the integral expressions

$$\lambda^{(m,\uparrow)} = -\rho U^2\int\frac{d\theta}{2\pi}\chi_\downarrow(2k_{F\uparrow}|\sin(\theta/2)|)\cos(m\theta),$$
$$\lambda^{(m,\downarrow)} = -\rho U^2\int\frac{d\theta}{2\pi}\chi_\uparrow(2k_{F\downarrow}|\sin(\theta/2)|)\cos(m\theta), \tag{10}$$

where θ is the angle relative between $\hat{k}$ and $\hat{q}$.

Without loss of generality, we suppose that $k_{F\downarrow} < k_{F\uparrow}$. For any $\hat{k},\hat{q}$ on the smaller (spin-down) Fermi surface, $\hat{k}-\hat{q} < 2k_{F\downarrow} < 2k_{F\uparrow}$ so the effective interaction, $\sim\chi_\uparrow(\hat{k}-\hat{q})$, is a constant. Therefore, it follows that $\lambda_{m\downarrow} = 0$ for all m or in other words, the smaller Fermi surface has no superconducting instability to $\mathcal{O}(U^2)$. [Presumably, $\lambda_{m,\downarrow} \sim \mathcal{O}(U^3)$.]

Conversely, the effective interaction between electrons on the larger (spin-up) Fermi surface is $\sim\chi_\downarrow(\hat{k}-\hat{q})$, which *does* depend on the relative position of the incoming and outgoing electrons on the Fermi surface. Using Eq. (8), the above expression for the eigenvalue on the larger Fermi surface becomes

$$\lambda_{m\uparrow}(\eta) = \frac{\rho^2 U^2}{\pi} \int_{\theta_c}^{\pi} d\theta \frac{\sqrt{\sin^2\frac{\theta}{2} - \eta^2}}{\sin\frac{\theta}{2}} \cos(m\theta), \tag{11}$$

where $\eta = (k_{F\downarrow}/k_{F\uparrow})$, $0 \leqslant \eta \leqslant 1$, and $\theta_c = 2\sin^{-1}\eta$. As can be seen from the equation above, $\lambda_{m\uparrow}(0) = \lambda_{m\uparrow}(1) = 0$. That is, in the limit where the Fermi surface is either completely polarized, or completely unpolarized, there is no superconducting instability to $\mathcal{O}(U^2)$. However, for intermediate values of the polarization, the integral above yields

$$\lambda_{1\uparrow}(\eta) = -\rho^2 U^2 \eta(1-\eta), \tag{12}$$

which is clearly negative for all intermediate values of η. This is the main result of this section: by polarizing the Fermi surfaces in two dimensions, there is a significant enhancement of p-wave superconductivity. The optimal pairing strength occurs when $\eta = 1/2$, so that

$$\max[\lambda_{m\uparrow}(\eta)] = \lambda_{1\uparrow}(\eta = 0.5) = -\frac{(\rho U)^2}{4}. \tag{13}$$

[Note that Eq. (7) of Ref. 10 reduces to this result in the limit of weak interaction.] For completeness, we quote the next leading eigenvalue, which corresponds to the f wave (i.e., $m = 3$) solution:

$$\lambda_{3\uparrow}(\eta) = -\rho^2 U^2 \eta[1 - \eta(3 - 4\eta^2 + 2\eta^4)], \tag{14}$$

which is not symmetric about the point $\eta = 0.5$.

Weak but nonvanishing spin-orbit coupling will generically change this situation, since superconductivity will be induced in the minority fluid by the proximity effect as soon as the majority fluid becomes superconducting. In 2D, this induced superconductivity will generally track the fundamental order parameter.

IV. TWO SUBBANDS IN A 2DEG

In this section, we consider the case of a 2DEG in a semiconductor heterostructure having two subbands, with Hamiltonian:

$$\begin{aligned} H &= H_0 + H_1, \\ H_0 &= \sum_{a=1,2} \sum_{\sigma} \int \frac{d^2k}{(2\pi)^2} \epsilon_{\boldsymbol{k},a} \psi^\dagger_{a,\sigma}(\boldsymbol{k}) \psi_{a,\sigma}(\boldsymbol{k}), \\ H_1 &= \sum_{a} \sum_{\sigma,\sigma'} V^{\sigma,\sigma'}_{ab,cd} \int \frac{d^2k_1 d^2k_2 d^2k_3}{(2\pi)^6} \\ &\quad \times [\psi^\dagger_{a,\sigma}(\boldsymbol{k}_1) \psi^\dagger_{b,\sigma'} \psi_{c,\sigma'}(\boldsymbol{k}_3) \psi_{d,\sigma}(\boldsymbol{k}_4)], \end{aligned} \tag{15}$$

where a is the subband index and is used to distinguish the smaller ($a = 1$) and larger ($a = 2$) Fermi surface, and $\epsilon_{\boldsymbol{k},a} = k^2/2m + \delta_a$ with $\delta_1 = 0$ and $\delta_2 > 0$. The interactions are assumed to be short ranged, consisting of an intraband repulsion U, an interband repulsion V, and an interband pair-hopping amplitude J. The interaction matrix in the basis $(1\sigma 1\sigma', 1\sigma 2\sigma', 2\sigma 1\sigma', 2\sigma 2\sigma')$ is thus

$$V^{\sigma,\sigma'}_{ab,cd} = \begin{pmatrix} U_{\sigma\sigma'} & 0 & 0 & J_{\sigma\sigma'} \\ 0 & 0 & V_{\sigma\sigma'} & 0 \\ 0 & V_{\sigma\sigma'} & 0 & 0 \\ J_{\sigma\sigma'} & 0 & 0 & U_{\sigma\sigma'} \end{pmatrix}, \tag{16}$$

where

$$\begin{aligned} U_{\sigma\sigma'} &= U(1 - \delta_{\sigma\sigma'}), \\ J_{\sigma\sigma'} &= J(1 - \delta_{\sigma\sigma'}), \\ V_{\sigma\sigma'} &= V. \end{aligned} \tag{17}$$

As before, rotational invariance enables us to label the eigenstates by the eigenvalue of the rotation operator,

$$\phi^{(m)}_{\hat{k}} = \phi^{(m)}_a \cos(m\theta), \tag{18}$$

where the complex amplitude $\phi^{(m)}_a$ depends only on the subband index associated with $\hat{k}$, θ is the angle between $\hat{k}$ and an arbitrarily defined x axis, and m must be an even integer in the singlet channel and an odd integer in the triplet channel. Consequently, for each integer m, rotational symmetry reduces the eigenvalue problem to a 2×2 problem,

$$\sum_{a,a'} \tilde{g}^{(m)}_{a,a'} \phi^{(m)}_{a'} = \lambda^{(m)} \phi^{(m)}_a, \tag{19}$$

where

$$\tilde{g}^{(m)}_{a,a'} \equiv \int_a \frac{d\hat{k}}{2\pi} \int_{a'} \frac{d\hat{k}}{2\pi} g^{(y)}_{\hat{k},\hat{k}'} e^{-im\theta} e^{im\theta'}, \tag{20}$$

where $y = s$ (singlet) for m even and $y = t$ (triplet) for m odd. The most negative eigenvalue for fixed m is

$$\begin{aligned} \lambda^{(m)} = &-\left(\frac{\tilde{g}^{(m)}_{1,1} + \tilde{g}^{(m)}_{2,2}}{2}\right) \\ &-\sqrt{\left(\frac{\tilde{g}^{(m)}_{1,1} - \tilde{g}^{(m)}_{2,2}}{2}\right)^2 + \left|\tilde{g}^{(m)}_{1,2}\right|^2}. \end{aligned} \tag{21}$$

We first consider the *spin-triplet channel* (m odd) which is only a slight extension of the result obtained for a partially polarized Fermi surface. As shown in the Appendix, for odd m, the effective interaction is diagonal in the subband index, and depends on U and V, but not J:

$$\begin{aligned} g^{(t)}_{1,1}(\hat{k},\hat{q}) &= -\rho U^2 \chi_{1,1}(\hat{k} - \hat{q}) - 2\rho V^2 \chi_{2,2}(\hat{k} - \hat{q}), \\ g^{(t)}_{2,2}(\hat{k},\hat{q}) &= -\rho U^2 \chi_{2,2}(\hat{k} - \hat{q}) - 2\rho V^2 \chi_{1,1}(\hat{k} - \hat{q}), \\ g^{(t)}_{1,2}(\hat{k},\hat{q}) &= 0, \end{aligned} \tag{22}$$

where

$$\chi_{a,b}(\boldsymbol{k}) = \int \frac{d^2p}{(2\pi)^2} \frac{f(\epsilon_{\boldsymbol{p}+\boldsymbol{k},a}) - f(\epsilon_{\boldsymbol{p},b})}{\epsilon_{\boldsymbol{p}+\boldsymbol{k},a} - \epsilon_{\boldsymbol{p},b}} \tag{23}$$

is the particle-hole susceptibility generalized to the two-band system. The intraband susceptibilities are precisely the same functions used before:

$$\chi_{a,a}(\vec{q}) = \frac{\rho}{2}\left[1 - \frac{\mathrm{Re}\sqrt{q^2 - (2k_{Fa})^2}}{q}\right] \tag{24}$$

with the subband index playing the role that the spin played in the previous section. Therefore, we may simply transcribe the results found in the previous section to the present context. The band which forms the smaller Fermi surface ($a = 1$) does not exhibit a superconducting instability to $\mathcal{O}(U^2)$. The larger Fermi surface exhibits a triplet p-wave instability with a pairing strength determined solely by V:

$$\lambda^{(1)}(\eta) = -4\rho^2 V^2 \eta(1-\eta), \tag{25}$$

where $\eta = (k_{F1}/k_{F2})$.

In the *spin-singlet channel*, the matrix g has off-diagonal components:

$$\begin{aligned} g^{(s)}_{1,1}(\hat{k},\hat{q}) &= \rho U_1 - 2\rho V^2 \chi_{2,2}(\hat{k}-\hat{q}), \\ g^{(s)}_{2,2}(\hat{k},\hat{q}) &= \rho U_2 - 2\rho V^2 \chi_{1,1}(\hat{k}-\hat{q}), \\ g^{(s)}_{1,2}(\hat{k},\hat{q}) &= \rho U_{12} + \rho V J[\chi_{1,2}(\hat{k}+\hat{q}) + \chi_{1,2}(\hat{k}-\hat{q})], \end{aligned} \tag{26}$$

where U_{ab} are momentum-independent interactions,

$$\begin{aligned} U_1 &\equiv U + U^2 P_1(\Omega) + J^2 P_2(\Omega) + U^2 \chi_{1,1}(\hat{k}+\hat{q}), \\ U_2 &\equiv U + U^2 P_2(\Omega) + J^2 P_1(\Omega) + U^2 \chi_{2,2}(\hat{k}+\hat{q}), \\ U_{12} &\equiv UJ[P_1(\Omega) + P_2(\Omega)], \end{aligned} \tag{27}$$

where the particle-particle susceptibility

$$\begin{aligned} P_a(\Omega) &\equiv \int \frac{d^2q}{(2\pi)^2} \frac{2f(\epsilon_{\mathbf{q},a}) - 1}{i\Omega - 2\epsilon_{\mathbf{q},a}} \\ &\sim \rho \ln\left[\frac{E_F - \delta_a}{\Omega}\right] + \mathcal{O}(\Omega) \end{aligned} \tag{28}$$

is a momentum-independent constant which diverges logarithmically at low energies. Despite this divergence, the second-order contributions to U_α are unimportant, since they do not enter the gap equation for any $m \neq 0$, and any s-wave solution is already killed by the first-order terms proportional to U.

For $m > 0$ and even, the effective intraband interaction depends only on the interaction V and is nonzero only for the larger Fermi surface, whereas the interband interaction depends both on V and J:

$$\begin{aligned} \tilde{g}^{(m)}_{1,1} &= 0, \\ \tilde{g}^{(m)}_{2,2} &= -2V^2\rho \int \frac{d\theta}{2\pi} \chi_{1,1}[2k_{F2}|\sin(\theta/2)|]\cos(m\theta), \\ \tilde{g}^{(m)}_{1,2} &= 2VJ\rho \int \frac{d\theta}{2\pi} \chi_{1,2}(k_\theta)\cos(m\theta), \\ k_\theta &= k_{F2}\sqrt{(1-\eta)^2 + 4\eta\sin^2(\theta/2)}. \end{aligned} \tag{29}$$

The explicit expression for the interband susceptibility $\chi_{1,2}(\mathbf{q})$ is derived in the Appendix. Since the $m = 0$ eigenvalues are always positive, the dominant singlet instability is in the d-wave ($m = 2$) channel. The quantity $\tilde{g}^{(2)}_{2,2}$ is obtained by computing

$$\begin{aligned} \tilde{g}^{(2)}_{2,2} &= -2V^2\rho^2 \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} d\theta \frac{\mathrm{Re}\sqrt{\sin^2\frac{\theta}{2} - \eta^2}}{\sin\frac{\theta}{2}} \cos(2\theta) \\ &= -V^2\rho^2\eta(\eta-1)(\eta^2+\eta-1). \end{aligned} \tag{30}$$

The interband interaction $\tilde{g}^{(2)}_{1,2}$ is also obtained using Eq. (26):

$$\tilde{g}^{(2)}_{1,2} = -\frac{VJ\rho^2}{2\pi}\Phi(\eta), \tag{31}$$

where, for $0 \leqslant x < 1$,

$$\Phi(x) = \frac{\pi x^4 + 2\sin^{-1}x}{x^2} - 2\frac{\sqrt{1-x^2}}{x}. \tag{32}$$

This function is discussed in detail in the Appendix. An important property of $\Phi(x)$ is that it is a monotonically increasing function of x for $0 \leqslant x < 1$ (see Fig. 3). Therefore, the effective interband scattering grows with η. The pairing strength in the d-wave channel is obtained from these quantities via

$$\lambda^{(2)}(\eta) = \frac{\tilde{g}^{(2)}_{2,2}}{2} - \frac{1}{2}\sqrt{\left(\tilde{g}^{(2)}_{2,2}\right)^2 + 4\left(\tilde{g}^{(2)}_{1,2}\right)^2}. \tag{33}$$

Having derived closed-form expressions for the p-wave and d-wave pairing strengths, we can construct the phase diagram, shown in Fig. 1. The phases are labeled according to the symmetry of the most negative eigenvalue, so the phase boundaries are the lines at which $\lambda^{(2)} = \lambda^{(1)} < 0$. Since the d-wave and p-wave eigenvalues are both negative for all $0 \leqslant \eta \leqslant 1$, where one phase is stable, the other is metastable. It would require different methods of analysis to completely characterize the phase competition. However, in weak coupling, the phase with the larger $|\lambda_m|$ has an exponentially larger T_c, and so gaps the entire Fermi surface at temperatures far above the putative transition temperature of the subdominant phase. Thus, a BCS mean-field treatment of this problem would suggest that at low temperatures, there is a direct, first-order transition, or at most an exponentially narrow region of phase coexistence between the extremal pure d-wave and p-wave phases.

Since the pair-hopping term only affects spin-singlet superconductivity, and since $|\lambda^{(1)}| > |\tilde{g}^{(2)}_{2,2}|$, it follows that for

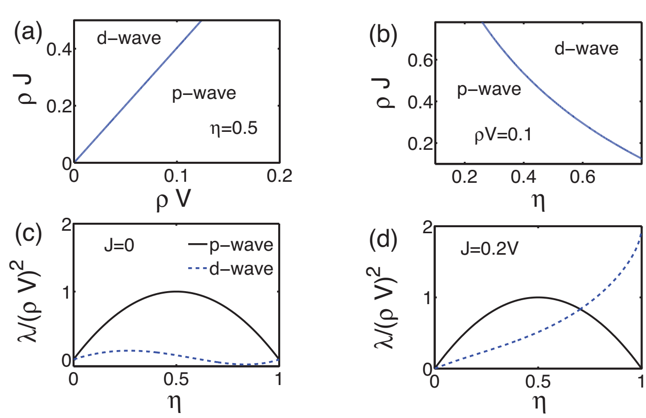

FIG. 1. (Color online) Phase diagram of a 2DEG having two subbands. (a) Phase diagram for fixed $\eta \equiv k_{F1}/k_{F2} = 0.5$ as a function of the dimensionless couplings ρV and ρJ. U does not enter the problem except in that it is responsible for the suppression of s-wave pairing. (b) Phase diagram for fixed $\rho V = 0.1$ as a function of η and ρJ. (c) The dimensionless strength of the pairing interaction in the p-wave (solid line) and d-wave (dashed line) channels for fixed $J = 0$. (d) Same as (c), but for $J = 0.2$V.

$J = 0$, the p-wave solution always remains the favored ground state, as can be seen from Fig. 1(a). However, as the interband scattering is enhanced, the d-wave pairing strength grows. Since the interband scattering increases monotonically as a function of both η and J, it is seen that for sufficiently large values of either parameter the p-wave superconductivity gives way to a d-wave ground state. In Figs. 1(c) and 1(d), we show how the magnitude of $\lambda^{(1)}$ and $\lambda^{(2)}$ depend on η, from which one can see that T_c is maximal in the p-wave channel when $\eta = 0.5$. However, when $J \neq 0$, the d-wave channel grows monotonically with η and ultimately overtakes the p-wave pairing strength as $\eta \to 1$. Note, however, that η can never equal unity in this context, since it is determined by the thickness of the quantum well.

V. DISCUSSION

We have obtained analytical expressions for various unconventional superconducting ground states of a clean 2DEG in the presence of weak, short-ranged repulsive interactions. Ultimately, to make contact with experiments involving real 2DEGs, we must take into account the Coulomb interactions. In the small r_s limit, the Coulomb interactions are sufficiently well screened that it may be reasonable to treat them as weak and short-ranged (a diagrammatic approach to the full Coulomb problem in 3D at small r_s was explored in Ref. 17, but there are many subtleties which make this hard to extend). We thus imagine we can relate the physical problem to a problem with short-ranged interactions and speculate on two ways in which unconventional superconductivity could be found in the 2DEG in physically realizable semiconductor heterostructures. (We shall present more complicated examples in a forthcoming publication.[11])

In the first scenario, an in-plane magnetic field is applied to partially polarize the 2DEG in a narrow quantum well. This system is predicted to exhibit p-wave pairing with a transition temperature which is nonmonotonic in the magnetic field. The optimal transition temperature is obtained for a magnetic field at which the ratio of the distinct spin Fermi momenta is $\eta = 1/2$. In the second scenario, the 2DEG is confined to a relatively broad quantum well, and the density is tuned to the range in which two transverse subbands are occupied. For fixed total electron density, the ratio, η^2, of densities in the two subbands increases with increasing thickness w of the quantum well. When this ratio is small, a p-wave ground state arises, with a T_c that rises sharply with increasing η so long as $\eta < 1/2$. However, this gives rise to a d-wave ground state above a certain critical thickness.

Insight into the dependence of V/J on the thickness, w, is obtained by considering the Coulomb interactions. A simple estimate shows that for $k_F w \ll 1$, $V \sim e^2/k_F$ and $J \sim V(wk_F)$. Therefore, for thicker quantum wells, J becomes increasingly important and favors d-wave pairing whereas thinner quantum wells should exhibit p-wave pairing. Depending on the ratio of V/J, the optimal T_c occurs either for $\eta \approx 1/2$ (p wave) or for the largest possible η (d wave). In both cases, $T_c \sim E_F \exp[-\alpha/(\rho V)^2]$, where α is an $\mathcal{O}(1)$ constant. We have found that for d-wave superconductivity in the two-subband system, values as low as $\alpha \sim 1$ are within reach.

Three practical considerations warrant mention. Due to the unconventional nature of the superconductvity, it is very fragile to even weak quenched disorder. Therefore, the results presented here are likely to be realized only in the purest samples with mean free paths exceeding the Fermi wavelength by several orders of magnitude. Furthermore, for small r_s, the plasma frequency is small compared to the Fermi energy, i.e., $\omega_p \sim \sqrt{r_s} E_F$, so even if it is reasonable to treat the interactions as short ranged at low energies, this approximation is certainly not valid all the way to the Fermi energy. Finally, since the transition temperatures are exponentially low in the effective interactions, ultimately the superconductivity studied here is likely to be observable only in the regime $r_s \sim 1$, where the long-range character of the Coulomb interaction may not be negligible, and where, even for short-range interactions, a well-controlled solution to the problem is unfeasible. We therefore are forced to rely on the hope that the asymptotic results smoothly extrapolate to the intermediate coupling regime, where it is conceivable that these states can be observed in experiment.

With these caveats, we turn to the most uncertain part of the discussion, and make the following *crude* quantitative estimate of T_c based on our calculations: We identify V with the Fourier transform of the Coulomb interaction evaluated at k_F, i.e., $V \approx e^2\pi/k_F$, from which it follows that $\rho V \approx (r_s/4)$. Since we are going to extrapolate to $r_s \sim 1$ in any case, we simply ignore subtleties associated with the small value of ω_p. Then, $T_c \sim E_F \exp[-\alpha(4/r_s)^2]$, where, for optimal circumstances, $\alpha \approx 1$.

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APPENDIX: PERTURBATION THEORY

In this section, we derive the perturbative expansion of the effective interaction in the Cooper channel for the problems studied in the main text. Due to the presence of multiple bands, adopting a more compact notation enables us to treat all of the above problems in a unified fashion. We consider a Hamiltonian of the form

$$H = H_0 + H_1,$$
$$H_0 = \sum_{\boldsymbol{k},a} \epsilon_{\boldsymbol{k},a} c^\dagger_{\boldsymbol{k},a} c_{\boldsymbol{k},a}, \tag{A1}$$
$$H_1 = \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3} \sum_{a,b,c,d} V_{ab,cd} c^\dagger_{\boldsymbol{k}_1,a} c^\dagger_{\boldsymbol{k}_2,b} c_{\boldsymbol{k}_3,c} c_{\boldsymbol{k}_4,d},$$

where $\boldsymbol{k}_4 = \boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3$. The Latin subscripts denote a collective set of "band indices" which label the energy eigenstates. In the problem of partially polarized Fermi surfaces, they simply label the majority and minority spin bands. In the problem of multiple subbands in a quantum well, it indexes the subbands, and in the problem involving Rashba spin-orbit coupling, the Latin index refers to the positive and negative helicity subband. The bare interaction H_1 is to be interpreted as a matrix; its rows labels the outgoing states and its columns label the incoming states.

Higher-order scattering processes are derived using diagrammatic perturbation theory in the usual manner.[16] In addition to integrating over the internal momenta, the band indices of any internal line are also summed over, weighted by the appropriate component of the interaction vertex, as will be made clear from the examples below.

The primary quantity of interest here is the two-particle scattering amplitude in the Cooper channel, denoted $\Gamma(\boldsymbol{k},\boldsymbol{q})$, which is the amplitude for scattering a pair of electrons with momenta $\pm\boldsymbol{k}$ into a pair with momenta $\pm\boldsymbol{q}$. If the system at hand possesses inversion symmetry (so that the kinetic energy consists of terms that are even powers of momentum), superconducting states can be classified as being even or odd parity states; the former include (e.g., s-wave, d-wave, etc.), and the latter include (p-wave, f-wave, etc.) instabilities are perfectly decoupled from one another. If, in addition to inversion symmetry, spin-rotation symmetry is also preserved, then the scattering amplitudes in the singlet channel consist of processes in which the incoming electrons have opposite spin polarizations, whereas in the triplet channel, they have identical spin polarizations. On the other hand, when inversion symmetry is broken by the presence of Rashba spin-orbit coupling, there is no sharp distinction between even- and odd-parity pairing. However, since the Rashba coupling breaks the twofold degeneracy of single-particle states at each momenta, there is pairing between states of opposite helicity (i.e., opposite momenta and opposite in-plane component of the spin).

Figure 2 shows the lowest-order Feynman diagrams which contribute to $\Gamma(\boldsymbol{k},\boldsymbol{q})$. Generally, all of these diagrams contribute both in the singlet and triplet channels. The diagrams

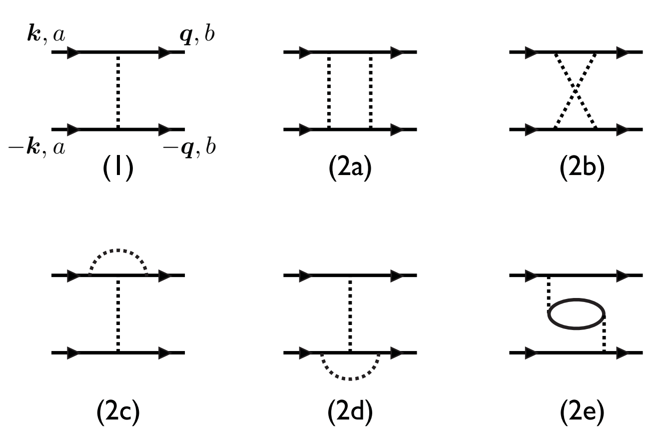

FIG. 2. Diagrams which contribute to $V(\boldsymbol{k},\boldsymbol{q})$, shown to quadratic order in the interactions. Each of the incoming quasiparticles has momentum $\pm\boldsymbol{k}$ and band index a. The outgoing electrons have momentum $\pm q$, and are in band b (with the exception of the first diagram, the momenta and band indices of each diagram are not shown). Both intraband and interband scattering processes contribute to the effective interaction.

are each equivalent to

$$\begin{aligned}
1:&\quad V_{bb,aa},\\
2a:&\quad \sum_c V_{bb,cc}V_{cc,aa}\int_p G_c(-p)G_c(p),\\
2b:&\quad \sum_{c,d} V_{bc,ad}V_{db,ca}\int_p G_c(p)G_d(p+k+q),\\
2c:&\quad \sum_{c,d} V_{bc,ad}V_{db,ac}\int_p G_c(p)G_d(p+q-k),\\
2d:&\quad \sum_{c,d} V_{bd,ca}V_{bc,ad}\int_p G_c(p)G_d(p+k-q),\\
2e:&\quad -\sum_{c,d} V_{bd,ca}V_{cb,ad}\int_p G_c(p)G_d(p+k-q),
\end{aligned} \tag{A2}$$

where

$$\int_p \equiv \int \frac{d\omega_p d^2p}{(2\pi)^3} \tag{A3}$$

and

$$G(p) = \frac{1}{i\omega_p - \epsilon_{\boldsymbol{p}}} \tag{A4}$$

is the single-particle Green function of the noninteracting system. We next apply this general formalism to the three problems studied in this paper.

1. Partially polarized Fermi surfaces

Although this problem is rather simple, and the Feynman rules for a single band system are sufficient, we will apply the notation above to this problem. This will certainly prove to be valuable in the more nontrivial examples studied thereafter. The interaction vertex, in the basis $(\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow)$, is

$$V_{ab,cd} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & U & 0\\ 0 & U & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A5}$$

Note that in this problem, there are no processes which scatter two electrons from one Fermi surface, to two electrons in a different Fermi surface. Therefore, $\Gamma(\boldsymbol{k},\boldsymbol{q})$ is diagonal in the band index—which is just a long-winded way of saying that only equal spin pairing (i.e., spin-triplet pairing) can occur. Indeed, the only diagram which contributes to the effective interaction is $2e$, which yields

$$\begin{aligned}
\Gamma_{\uparrow\uparrow}(\boldsymbol{k},\boldsymbol{q}) &= -U^2\chi_{\downarrow}(\boldsymbol{k}-\boldsymbol{q}) + \mathcal{O}(\Omega_0),\\
\Gamma_{\downarrow\downarrow}(\boldsymbol{k},\boldsymbol{q}) &= -U^2\chi_{\uparrow}(\boldsymbol{k}-\boldsymbol{q}) + \mathcal{O}(\Omega_0),
\end{aligned} \tag{A6}$$

where

$$\begin{aligned}
\chi_\sigma(\boldsymbol{k}) &= \int_p G_\sigma(p)G_\sigma(p+k)\\
&= -\int \frac{d^2p}{(2\pi)^2}\frac{f(\epsilon_{\boldsymbol{p}+\boldsymbol{k},\sigma}) - f(\epsilon_{\boldsymbol{p},\sigma})}{\epsilon_{\boldsymbol{p}+\boldsymbol{k},\sigma} - \epsilon_{\boldsymbol{p},\sigma}}
\end{aligned} \tag{A7}$$

is the noninteracting susceptibility of each spin band.

2. Multiple subbands in a 2DEG quantum well

For the problem involving two subbands in a quantum well, we choose the basis to be $(1\sigma 1\sigma', 1\sigma 2\sigma', 2\sigma 1\sigma', 2\sigma 2\sigma')$ and

$$V_{\alpha\beta,\gamma\delta} = \begin{pmatrix} U_{\sigma\sigma'} & 0 & 0 & J_{\sigma\sigma'} \\ 0 & 0 & V_{\sigma\sigma'} & 0 \\ 0 & V_{\sigma\sigma'} & 0 & 0 \\ J_{\sigma\sigma'} & 0 & 0 & U_{\sigma\sigma'} \end{pmatrix}, \tag{A8}$$

where

$$\begin{aligned} U_{\sigma\sigma'} &= U(1-\delta_{\sigma\sigma'}), \\ J_{\sigma\sigma'} &= J(1-\delta_{\sigma\sigma'}), \\ V_{\sigma\sigma'} &= V. \end{aligned} \tag{A9}$$

In this basis σ and σ' refer to the spins of the incoming electron states. Since there is inversion and spin-rotational symmetry in this problem, we can study the effective interaction in the singlet ($\sigma = -\sigma'$) and triplet ($\sigma = \sigma'$) channels separately. We shall refer to the effective interaction as $\Gamma_{s(t)}$, where the subscript stands for singlet and triplet. Having specified the spin polarizations of the electrons, $\Gamma_{s(t)}$ will still be a 2×2 matrix due to the presence of two subbands. We will denote this as

$$\Gamma_{s(t)}(\boldsymbol{k}_a,\boldsymbol{q}_b) = \begin{pmatrix} \Gamma_{s(t)}(\boldsymbol{k}_1,\boldsymbol{q}_1) & \Gamma_{s(t)}(\boldsymbol{k}_1,\boldsymbol{q}_2) \\ \Gamma_{s(t)}(\boldsymbol{k}_2,\boldsymbol{q}_1) & \Gamma_{s(t)}(\boldsymbol{k}_2,\boldsymbol{q}_2) \end{pmatrix} \tag{A10}$$

with the understanding that $\boldsymbol{k}_a$ denotes momentum states associated with band a.

Next, we state the contributions from each of the diagrams in Fig. 2. First, in the singlet channel,

$$\begin{aligned} \Gamma_s(1) &= \begin{pmatrix} U & J \\ J & U \end{pmatrix}, \\ \Gamma_s(2a) &= \begin{pmatrix} U^2P_1 + J^2P_2 & UJ(P_1+P_2) \\ UJ(P_1+P_2) & U^2P_2 + J^2P_1 \end{pmatrix}, \\ \Gamma_s(2b) &= \begin{pmatrix} U^2\chi_{1,1}(\boldsymbol{k}_1+\boldsymbol{q}_1) & VJ\chi_{1,2}(\boldsymbol{k}_1+\boldsymbol{q}_2) \\ VJ\chi_{2,1}(\boldsymbol{k}_2+\boldsymbol{q}_1) & U^2\chi_{2,2}(\boldsymbol{k}_2+\boldsymbol{q}_2) \end{pmatrix}, \\ \Gamma_s(2c) &= \begin{pmatrix} 0 & VJ\chi_{1,2}(\boldsymbol{k}_1-\boldsymbol{q}_2) \\ VJ\chi_{2,1}(\boldsymbol{k}_2-\boldsymbol{q}_1) & 0 \end{pmatrix}, \\ \Gamma_s(2d) &= \begin{pmatrix} 0 & VJ\chi_{1,2}(\boldsymbol{k}_1-\boldsymbol{q}_2) \\ VJ\chi_{2,1}(\boldsymbol{k}_2-\boldsymbol{q}_1) & 0 \end{pmatrix}, \\ \Gamma_s(2e) &= -\begin{pmatrix} 2V^2\chi_{2,2}(\boldsymbol{k}_1-\boldsymbol{q}_1) & 0 \\ 0 & 2V^2\chi_{1,1}(\boldsymbol{k}_2-\boldsymbol{q}_2) \end{pmatrix}, \end{aligned} \tag{A11}$$

where

$$\begin{aligned} P_a &= \int_p G_a(p)G_a(-p) \\ &= \rho_a \ln[A/\Omega_0] + \mathcal{O}(\Omega_0), \\ \chi_{a,b}(\boldsymbol{k}) &= \int_p G_a(p+k)G_b(p) \\ &= \int \frac{d^2p}{(2\pi)^2} \frac{f(\epsilon_{\boldsymbol{p}+\boldsymbol{k},a}) - f(\epsilon_{\boldsymbol{p},b})}{\epsilon_{\boldsymbol{p}+\boldsymbol{k},a} - \epsilon_{\boldsymbol{p},b}} + \mathcal{O}(\Omega_0) \end{aligned} \tag{A12}$$

are the particle-particle and particle-hole susceptibilities, respectively.

In the triplet channel, only diagram $2e$ has a contribution and $\Gamma_t(\boldsymbol{k}_a,\boldsymbol{q}_b)$ is diagonal in the subband index:

$$\begin{aligned} \Gamma_t(\boldsymbol{k}_1,\boldsymbol{q}_1) &= -U^2\chi_{1,1}(\boldsymbol{k}_1-\boldsymbol{q}_1) - 2V^2\chi_{2,2}(\boldsymbol{k}_1-\boldsymbol{q}_1), \\ \Gamma_t(\boldsymbol{k}_2,\boldsymbol{q}_2) &= -U^2\chi_{2,2}(\boldsymbol{k}_2-\boldsymbol{q}_2) - 2V^2\chi_{1,1}(\boldsymbol{k}_2-\boldsymbol{q}_2), \\ \Gamma_t(\boldsymbol{k}_1,\boldsymbol{q}_2) &= 0. \end{aligned} \tag{A13}$$

Having computed the effective interaction $\Gamma_{s,t}$, we define the quantity

$$g_{s,t}(\boldsymbol{k}_a,\boldsymbol{q}_b) \equiv \sqrt{\frac{\bar{v}_f}{v_f(\hat{k}_a)}}\,\Gamma(\hat{k}_a,\hat{q}_b)\sqrt{\frac{\bar{v}_f}{v_f(\hat{q}_b)}}, \tag{A14}$$

which is also a matrix whose row and column indices are the set of momentum states on the Fermi surface.

3. Some relevant integrals

In this section, we compute the particle-hole susceptibility matrix of the two-subband problem:

$$\chi_{a,b}(\boldsymbol{k}) = -\int \frac{d^2p}{(2\pi)^2} \frac{f(\epsilon_{\boldsymbol{p}+\boldsymbol{k},a}) - f(\epsilon_{\boldsymbol{p},b})}{\epsilon_{\boldsymbol{p}+\boldsymbol{k},a} - \epsilon_{\boldsymbol{p},b}}. \tag{A15}$$

We let

$$\begin{aligned} \epsilon_{\boldsymbol{k},1} &= \epsilon_{\boldsymbol{k}}, \\ \epsilon_{\boldsymbol{k},2} &= \epsilon_{\boldsymbol{k}} + \Delta \end{aligned} \tag{A16}$$

and set $\Delta = (k_{F1}^2 - k_{F2}^2)/2m > 0$ without loss of generality. It follows that the intraband susceptibilities are

$$\chi_{aa}(\boldsymbol{q}) = 2\int_0^{k_{Fa}} \frac{kdk}{(2\pi)^2} \int_0^{2\pi} \frac{d\theta}{\epsilon_{\boldsymbol{q}} + kq\cos\theta/m}, \tag{A17}$$

where $k_{F1} = (2m\mu)^{1/2}$ and $k_{F2} = [2m(\mu+\Delta)]^{1/2}$. The integrals are standard, resulting in the following:

$$\chi_{a,a} = \frac{m}{2\pi}\left[1 - \frac{\mathrm{Re}\sqrt{q^2 - (2k_{Fa})^2}}{q}\right] \tag{A18}$$

The interband susceptibility is

$$\begin{aligned} \chi_{1,2}(\boldsymbol{q}) &= \int_0^{k_{F1}} \frac{kdk}{(2\pi)^2} \int_0^{2\pi} \frac{d\theta}{\epsilon_{\boldsymbol{q}} - \Delta + kq\cos\theta/m} \\ &= \int_0^{k_{F2}} \frac{kdk}{(2\pi)^2} \int_0^{2\pi} \frac{d\theta}{\epsilon_{\boldsymbol{q}} + \Delta + kq\cos\theta/m}. \end{aligned} \tag{A19}$$

These integrals are straightforward and they produce the final result:

$$\begin{aligned} \chi_{1,2}(q) = -\frac{\rho}{2}\Bigg[& \frac{\mathrm{Re}\sqrt{q^2[1-\lambda(q)]^2 - (2\eta k_{F2})^2}}{q} \\ & + \frac{\mathrm{Re}\sqrt{q^2[1+\lambda(q)]^2 - (2k_{F2})^2}}{q} \\ & - [1+\lambda(q)] - |1-\lambda(q)| \Bigg], \end{aligned} \tag{A20}$$

where

$$\eta = \frac{k_{F1}}{k_{F2}},\qquad \lambda(q) = \frac{k_{F2}^2}{q^2}(1-\eta^2). \tag{A21}$$

The effective interband attraction $\tilde{g}_{1,2}^{s,m}$ in the singlet channel is related to this susceptibility, as discussed in Sec. IV:

$$\tilde{g}_{1,2}^{(s,m)} = 2VJ\rho\int\frac{d\theta}{2\pi}\chi_{1,2}(k_\theta)\cos(m\theta),\qquad k_\theta = k_{F2}\sqrt{(1-\eta)^2+4\eta\sin^2(\theta/2)}, \tag{A22}$$

It is easy to show that the first two terms in Eq. (A20) do not contribute to $\chi_{1,2}(k_\theta)$ since

$$\frac{\sqrt{k_\theta^2[1-\lambda(k_\theta)]^2-(2\eta k_{F2})^2}}{k_\theta} = \frac{2\eta}{k_\theta}\sqrt{-\sin^2\theta} \tag{A23}$$

is purely imaginary for $0 \leqslant \eta \leqslant 1$. Thus,

$$\chi_{1,2}(k_\theta) = \frac{\rho}{2}[1+\lambda(k_\theta)+|1-\lambda(k_\theta)|]. \tag{A24}$$

This in turn can be rewritten as

$$\chi_{1,2}(k_\theta) = \rho\begin{cases}1, & \cos\theta < \eta\\ \frac{(1-\eta^2)}{\ell_\theta^2}, & \cos\theta > \eta,\end{cases} \tag{A25}$$

where $\ell_\theta = k_\theta/k_{F2}$. The integration in Eq. (A22) is performed in the complex plane defining $z = -i\theta$. We find, for the d-wave case ($m = 2$),

$$\tilde{g}_{1,2}^{(s,2)} = -\frac{VJ\rho^2}{2\pi}\Phi(\eta), \tag{A26}$$

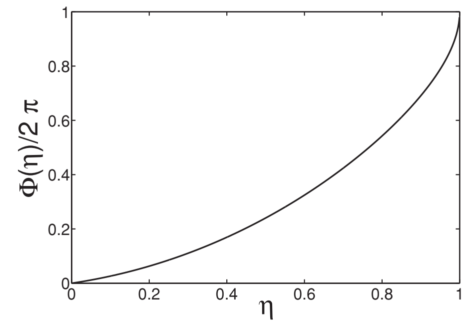

FIG. 3. The scaling function $\Phi(\eta)$ which determines the effective interband interaction $\tilde{g}_{1,2}^{s,2}$.

where, for $0 \leqslant x < 1$,

$$\Phi(x) = \frac{\pi x^4 + 2\sin^{-1}x}{x^2} - 2\frac{\sqrt{1-x^2}}{x} \tag{A27}$$

and $\Phi(1) = 0$. The function $\Phi(x)$ is shown in Fig. 3.

It should be noted that $\Phi(x)$ is discontinuous at $x = 1$, which results from the singluar behavior of the interband susceptibility in the limit where the bands become degenerate:

$$\lim_{\eta\to 1}\lim_{q\to 0}\chi_{1,2}(q) = \rho\frac{1+\eta^2}{1-\eta^2}. \tag{A28}$$

However, this feature does not have a physical consequence since there is always a nonzero splitting between the bands caused by the finite thickness of the semiconductor heterostructure.

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