PHYSICAL REVIEW B **83**, 094415 (2011)

Dynamic response function in Ising systems below T_c

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(Received 15 April 2010; revised manuscript received 17 January 2011; published 15 March 2011)

With the aid of Nelson's method we derive expressions for the frequency-, momentum-, and temperature-dependent response function below a critical point in the Ising-type system. The scaling function is given within the renormalization-group formalism at one-loop order for zero external field. The comparison with the corresponding expressions above T_c and with the mean-field approximation is made. We also discuss the dynamic correlation function, focusing on the deviations from the Gaussian expression. The comparison with field-theoretical calculations is also given.

DOI: 10.1103/PhysRevB.83.094415 PACS number(s): 75.40.Gb, 75.10.Hk, 64.60.Ht, 05.70.Jk

I. INTRODUCTION

Recently the dynamic critical behavior for many physically interesting systems has actively been studied both theoretically[1–6] and experimentally.[7–11] One of the most important dynamic universality classes is that of the three-dimensional Ising-like model with purely relaxational dynamics [model *A* (Ref. 12)]. The critical exponents, the equation of state, and the static scaling functions for susceptibility, etc., are relatively well known[13] for this model. A theoretical basis for understanding these critical phenomena was provided mainly by the renormalization-group theory.[14] Comparatively less attention has been devoted to an investigation of an important quantity, the dynamic response function, especially in the low-temperature phase. Another quantity that measures the dynamic correlations is the dynamic structure factor (dynamic correlation function). It is especially important from the experimental point of view as it can be measured in neutron, light, or x-ray scattering experiments. This paper is concerned with the dynamic susceptibility and structure factor of the one-component Ginzburg-Landau model below the transition temperature. We restrict ourselves to study the simplest model (purely relaxational) with no massless Goldstone modes present in isotropic n-component models for $n > 1$, which appear as a consequence of the spontaneously broken symmetry in the ordered phase. The so-called coexistence anomalies have been studied for the static model[15,16] as well for dynamics.[17,18] The results of Täuber and Schwabl can be even extended to $n = 1$. However, the authors focused their study on the crossover to an asymptotical Gaussian limit.[18] The dynamic structure factor in the disordered and ordered phases has been studied also by Calabrese *et al.*[5] on the basis of a perturbative field-theoretical RG approach both above and below the transition temperature, and also by Monte Carlo simulation in the high-temperature phase. It is well known that calculations of dynamical functions are rare and complicated because we need to exponentiate singularities at both large and small arguments.[19] Many alternative approaches can give different results depending on the exponentiation procedure and the region of the reduced wave vector or frequency. One of the best methods for computation of the scaling function with exponentiated singularities valid for all values of arguments is the method introduced by Nelson[19] for computation of the static correlation function above the critical temperature. It was a generalization of the technique developed by Rudnick and Nelson[20] of mapping the Hamiltonian out of the critical regime into the noncritical region with small correlation length, where the usual perturbation expansion can be employed. Subsequently, the quantity calculated in the noncritical regime is related by a renormalization-group (RG) transformation to the corresponding quantity in the critical region. This technique allowed the authors to construct the equation of state to first order in ϵ as well as to obtain the scaling functions for the susceptibility, specific heat, etc., both in the ordered and disordered phases.

The motivation for this paper is to provide a relatively simple analytical expression for dynamic susceptibility and dynamic structure factor for the three-dimensional Ising-like system with relaxational dynamics. These expressions should be valid in the whole range of temperature, wave vector, and frequency with a reasonably accuracy. To find such a reliable estimate we use the generalization of Nelson's[19] method of integration of the renormalization-group recursion relations. Apart from its simplicity, a special advantage of this method lies in the possibility to obtain exponentiated singularities for the scaling function both for large- and small-wave vectors. Later Nelson's method was generalized by Achiam and Kosterlitz[21] to calculate the static momentum-dependent correlation function for arbitrary temperature and magnetic field. Dengler *et al.*[22] were the first to generalize this method into a dynamic correlation function by using the modified matching condition. They investigated relaxational and diffusive dynamics in the high-temperature phase. In this paper we extend the results of Nelson,[19] Achiam and Kosterlitz,[21] and Dengler *et al.*[22] by obtaining to first order in $\epsilon = 4 - d$ the expressions for the dynamic susceptibility and correlation function for the nonconserved Ising order parameter with purely relaxational dynamics both above and below the critical temperature. This is achieved by postulating a modified matching condition suitable for the ordered phase. As we shall see, at variance with the paramagnetic phase, the deviations of the dynamic form factor from the mean-field result are quite substantial in the ordered phase. We believe that our approach leads to more transparent results than the previous attempts.

The paper is organized as follows. In Sec. II we define the model and recall the solutions of the recursion relations for the high-temperature phase. In Sec. III we discuss the solutions of the recursion relations in the ordered phase as well as the matching condition and calculate the dynamic susceptibility. In Sec. IV we present the comparison of the real part of the dynamic susceptibility with the mean-field results for several values of frequency and wave vector. We also discuss the dynamic correlation function and its deviations from the conventional result. Finally, we compare the dynamic correlation scaling function found in this work with the corresponding results obtained within the field-theoretic framework by Calabrese *et al.*[5] and by Täuber and Schwabl.[17,18]

II. MODEL AND THE SOLUTIONS OF THE RECURSION RELATIONS ABOVE T_c

We consider an Ising-like ($n = 1$) continuous order parameter with purely dissipative relaxational dynamics,

$$\frac{\partial S(\mathbf{x},t)}{\partial t} = -\Gamma \frac{\delta H}{\delta S(\mathbf{x},t)} + \eta(\mathbf{x},t) \tag{1}$$

with the Ginzburg-Landau Hamiltonian

$$H = \frac{1}{2}\int d^d x \{rS^2 + (\nabla S)^2 + 2uS^4\}, \tag{2}$$

and the Gaussian white noise obeying the Einstein relation,

$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = 2\Gamma\delta(t-t')\delta(\mathbf{x}-\mathbf{x}'). \tag{3}$$

The response (and correlation) function can be most suitably expressed using the functional representation of equations of motion,[23–27]

$$\frac{\delta\langle S(\mathbf{x},t)\rangle}{\delta h(\mathbf{x}',t')} = \Gamma\langle S(\mathbf{x},t)\tilde{S}(\mathbf{x}',t')\rangle \equiv G(\mathbf{x}-\mathbf{x}',t-t'), \tag{4}$$

where h is an external field coupled to the order parameter and the nonequilibrium averages are calculated with the aid of dynamic functional $\mathcal{J}\{\tilde{S},S\}$, which determines the probability of the whole trajectory $\{S(\mathbf{x})\}_{t\in[-t_0,t_0]}$ in some time interval,

$$\langle O(S,\tilde{S})\rangle = \frac{1}{Z}\int \mathcal{D}[i\tilde{S}]\mathcal{D}[S]\; O[S,\tilde{S}]\; \exp\mathcal{J}\{\tilde{S},S\}. \tag{5}$$

The new functional

$$\mathcal{J}\{\tilde{S},S\} = \int d^d x dt \left[\tilde{S}\Gamma\tilde{S} - \tilde{S}\left(\dot{S} + \Gamma\frac{\delta H}{\delta S}\right) + \frac{1}{2}\Gamma\frac{\delta^2 H}{\delta S^2}\right], \tag{6}$$

is a function of an artificial imaginary field $\tilde{S}$ known as the response field because an additional term related to the external magnetic field h in the Hamiltonian gives the contribution $\Gamma h\tilde{S}$ in the functional.

The scaling relation for the Fourier-transformed response function (dynamic susceptibility) at the lth stage of iteration is

$$\chi(q,\omega,r,u) = e^{(2l-\int_0^l \eta(l')dl')}\chi(qe^l,\omega e^{zl},r(l),u(l)), \tag{7}$$

where $\eta(l)$ reduces to critical exponent η at the critical point $u^* = \epsilon/36K_4$ (η can be neglected to first order in ϵ) and z is the dynamic critical exponent and $r(l)$, $u(l)$ are the renormalized couplings in the effective Hamiltonian $H(l)$.

Above T_c the renormalization-group flow equations for $r(l)$ and $u(l)$ are the same as in the static case and to order ϵ are given by[20]

$$\begin{aligned}\tau(l) &\equiv r(l) + 6K_4\{u(l) - u(l)r(l)\ln[1+r(l)]\}\\ &= \tau(0)e^{2l}/Q(l)^{1/3},\end{aligned} \tag{8}$$

$$u(l) = ue^{\epsilon l}/Q(l), \tag{9}$$

where $\tau(0) \equiv \tau \propto T - T_c$ is proportional to the reduced temperature, $K_4 = 1/8\pi^2$, and

$$Q(l) = 1 + 36K_4 u(e^{\epsilon l} - 1)/\epsilon. \tag{10}$$

At the fixed point $\tau(l)$ scales as $\tau(l) = \tau e^{l/\nu}$ with ν as the correlation length exponent. The right-hand side of Eq. (7) can be calculated for some value $l = l^*$ where not all the arguments: $qe^{l^*}, \omega e^{zl^*}, r(l^*)$ vanish simultaneously.

The original matching condition

$$\tau(l^*) + q^2 e^{2l^*} = 1, \tag{11}$$

introduced by Nelson[19] in order to evaluate the two-point correlation function $G(q,\tau)$ in the dynamic case was replaced by Dengler *et al.*[22] (for the symmetric phase) by

$$[(\omega/\Gamma)e^{zl^*}]^{4/z} + [(\tau e^{l^*/\nu})^{2\nu} + q^2 e^{2l^*}]^2 = 1, \tag{12}$$

which permits an analytic solution. Later this matching condition was generalized to a more flexible form in the context of ultrasonic attenuation calculations:[28]

$$[(\omega/\Gamma)e^{zl^*}]^{4/z} + [\chi(l^*)^{-2/(2-\eta)} + q^2 e^{2l^*}]^2 = 1, \tag{13}$$

where $\chi(l^*)$ is the static susceptibility at the lth stage of renormalization and to first order in ϵ the exponent $-2/(2-\eta)$ can be replaced by -1.

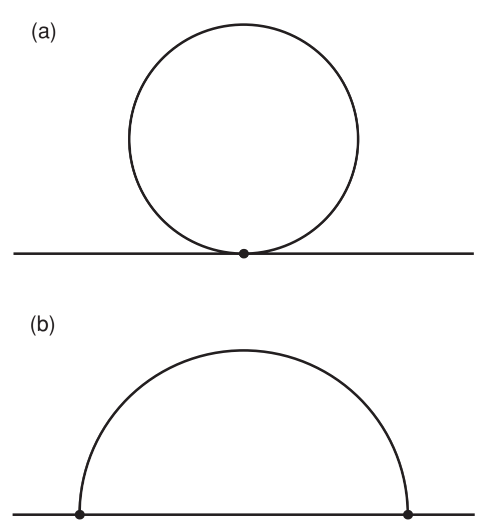

FIG. 1. The diagrams contributing to $O(\epsilon)$ to the self-energy below T_c.

With the matching condition (12) the dynamic susceptibility in the symmetric phase is given by[22]

$$\chi^{-1}(q,\omega,r,u) = e^{-2l^*}\{(-i\omega/\Gamma)e^{zl^*} + \tau(l^*) + q^2e^{2l^*} + p\tau(l^*)[1-\tau(l^*)^{-\alpha}]\}, \tag{14}$$

where $p = 1+\epsilon+O(\epsilon^2)$ and the second line of this equation represents the Hartree diagram from Fig. 1(a) proportional to the energy. α is the critical exponent of the specific heat.

III. RECURSION RELATIONS, MATCHING CONDITION, AND SELF-ENERGY BELOW T_c

Below T_c the solution of the recursion relations can be written in a very similar way,[20]

$$r(l) + 12u(l)M(l)^2 = T(l) + 6K_4\{u(l) - u(l)T(l) \times \ln[1+T(l)]\}, \tag{15}$$

$$T(l) = \tau(l) + 12u(l)M(l)^2, \tag{16}$$

$$\tau(l) = \tau(0)e^{2l}/Q(l)^{1/3}, \tag{17}$$

with $M(l) = Me^{\lambda_m l^*}$ being the system's magnetization at the lth stage of the renormalization group and $\lambda_m = \beta/\nu = 1-\frac{\epsilon}{2}+O(\epsilon^2)$. The functions $Q(l)$ and $u(l)$ are given by Eqs. (10) and (9), respectively. Because in the low-temperature phase and at the critical point $[u = u^* = \epsilon/36K_4 + O(\epsilon^2)]$ we have (for $h=0$) $\chi(l^*) = \chi e^{-2l^*} = (1-18u^*K_4)|2\tau|^{-\gamma}e^{-2l^*} + O(\epsilon^2)$[20] so we choose the matching condition (13) for zero external field in the form

$$[a(\omega/\Gamma)e^{zl^*}]^{4/z} + [b|2\tau|^{2\nu}e^{2l^*} + cq^2e^{2l^*}]^2 = 1, \tag{18}$$

where the factors $a = 1+\epsilon/8$, $b = 1+\epsilon/2$, and $c = 1+\epsilon/12$ follow from the universal amplitude ratios for the static susceptibility, correlation length,[29–31] and the characteristic frequency, correct to $O(\epsilon)$:

$$C_+/C_- = 2^\gamma(1+\epsilon/2) + O(\epsilon^2), \tag{19}$$

$$f_+/f_- = 2^\nu(1+5\epsilon/24) + O(\epsilon^2), \tag{20}$$

$$\omega_c^+/\omega_c^- = (1+\epsilon/8)C_-/C_+ = 2^{-z\nu}(1-3\epsilon/8) + O(\epsilon^2). \tag{21}$$

The last estimate agrees qualitatively with the result for the ratio of the relaxation times $\tau_c^+/\tau_c^- = 3.35$ obtained by Wansleben and Landau[32] in their Monte Carlo investigation of critical dynamics in the three-dimensional Ising model. Equation (18) permits the explicit solution

$$e^{l^*} = |2\tau|^{-\nu}F(x,y), \tag{22}$$

with

$$F(x,y) = (1+\epsilon/2)^{-1/2}[y^{4/z} + (1+x^2)^2]^{-1/4}, \tag{23}$$

where $y = \frac{1+\epsilon/8}{1+\epsilon/2}(\omega/\Gamma)|2\tau|^{-z\nu} \equiv \omega/\omega_c^-$ / is the reduced frequency and $x = \sqrt{\frac{1+\epsilon/12}{1+\epsilon/2}}q|2\tau|^{-\nu} \equiv q\xi_-$ is the reduced wave vector. Different normalization conditions in Eq. (18) as that without the factors $(1+\epsilon/8)$, etc., are also possible. At one-loop order the self-energy of $\chi(qe^{l^*},\omega e^{zl^*},r(l^*),u(l^*))$ is represented by the two diagrams shown in Fig 1. The first term in the self-energy represented by Fig. 1(a) equals $12u\int_p\langle\sigma_\mathbf{p}\sigma_{-\mathbf{p}}\rangle$, where $\sigma = S - M$. It is the static Hartree diagram which is related to the internal energy by[21]

$$\int_p\langle\sigma_\mathbf{p}\sigma_{-\mathbf{p}}\rangle = 2\frac{\partial F(l)}{\partial\tau(l)} - M^2(l).$$

The energy $-\frac{\partial F}{\partial\tau}$ has been calculated to $O(\epsilon)$ by Nelson and Rudnick[20] and by Achiam and Kosterlitz.[21] Below T_c it is proportional to $\frac{1}{24u}\tau[4|2\tau|^{-\alpha} - 1]$. The coefficient in front of $|\tau|^{-\alpha}$ is consistent with the universal amplitude ratio for the specific heat $A_+/A_- = 2^\alpha(1+\epsilon)/4$.[33]

The second diagram has been evaluated in the static limit $(\omega = 0)$ by Achiam and Kosterlitz[21] in their calculations of the static correlation function, and also for the $q\to 0$ limit.[28,34] Thus the general expression for the response function can be written as

$$\chi^{-1}(q,\omega,r,u) = e^{-2l^*}\{(-i\omega/\Gamma)e^{zl^*} + \tau(l^*) + q^2e^{2l^*} + \tau(l^*)[1 - 4|2\tau(l^*)|^{-\alpha}] - 288u^2K_4M^2(l)e^{2\lambda_m l^*}C(l^*)\}. \tag{24}$$

Here we present the general expression for $C(l^*)$ valid for all reduced temperatures, wave vectors, and frequencies

$$C(l^*) = K_4^{-1}\int\frac{2d^dp}{(2\pi)^d(\{[-i\omega(l^*)/\Gamma] + 2T(l^*) + p^2 + [\mathbf{p}+\mathbf{q}(\mathbf{l}^*)]^2\}[T(l^*)+p^2])}$$
$$= \frac{-2^{-1-\epsilon/2}}{q(l^*)^2}\pi T(l^*)^{-\epsilon}\csc\left(\frac{\pi\epsilon}{2}\right)N^{-\epsilon/2}\Big\{2^\epsilon T(l^*)^\epsilon(1+iW)^{\epsilon/2}[R - i\omega(l^*)/\Gamma]\,{}_2F_1$$
$$\times\left[1-\frac{\epsilon}{2},\frac{\epsilon}{2},2-\frac{\epsilon}{2},\frac{1-iW}{2}\right] - N^{\epsilon/2}[V+R]Z_1^{\epsilon/2}\,{}_2F_1\left[1-\frac{\epsilon}{2},\frac{\epsilon}{2},2-\frac{\epsilon}{2},\frac{Z_2}{2}\right]\Big\} - \frac{1}{\epsilon} + I_c(l^*), \tag{25}$$

where $\omega(l^*) = \omega e^{zl^*}$, $q(l^*) = qe^{l^*}$, and abbreviations have been used:

$$N = q(l^*)^2 + 4T(l^*) - 2i\omega(l^*)/\Gamma,\quad R = \sqrt{q(l^*)^2N - \omega(l^*)^2/\Gamma^2},$$
$$V = q(l^*)^2 - i\omega(l^*)/\Gamma,\quad W = i\omega(l^*)/\Gamma\sqrt{R},$$
$$U = V + R,\quad Z_1 = (R-V)/R,\quad Z_2 = (V+R)/R.$$

The symbol $I_c(l^*)$ in Eq. (25) denotes a cutoff dependent part which is canceled by terms in Eq. (15) and the less singular logarithmic terms which have been omitted in the Dyson equation (24).[35] ${}_2F_1$ is the hypergeometric function.[36]

The singular part of the function $C(l^*)$ can be expanded in ϵ giving

$$\begin{aligned} C(l^*) = \frac{1}{2q(l^*)^2}\{&q(l^*)^2 + 2i\omega(l^*)/\Gamma \ln 2 - R\ln 4 \\ &-U\ln T(l^*) + [R - i\omega(l^*)/\Gamma]\ln N \\ &+2R\,[\ln Z_1 - \ln(1+iW)] + O(\epsilon)\}. \end{aligned} \tag{26}$$

It is interesting to take the $\omega \to 0$ limit of Eq. (26):

$$\begin{aligned} C(l^*,\omega = 0) = -\tfrac{1}{2}\{&\ln[T(l^*) + q(l^*)^2/4] \\ &+[1 - 1/P(l^*)]\ln[1 - P(l^*)] \\ &+[1 + 1/P(l^*)]\ln[1 + P(l^*)] - 1\} + O(\epsilon), \end{aligned} \tag{27}$$

with $P(l^*) = q(l^*)^2/[4T(l^*) + q(l^*)^2]$. It is equal to the function $I(l)$ calculated by Achiam and Kosterlitz[21] for the static case. The second limit of the function $C(l^*)$, which has been already known,[28,34] is for $q \to 0$ [$C(l^*)$ corresponds to the frequency-dependent specific heat, investigated thoroughly in the study of ultrasonic attenuation and dispersion, where the wavelength of the acoustic wave is much larger than the correlation length of the critical fluctuations]. Equation (26) can then be written as

$$\begin{aligned} C(l^*,q = 0) = -\frac{1}{2}\Bigg\{&\ln[T(l^*) - i\omega(l^*)/2\Gamma] \\ &+ i\,\frac{\ln[1 - i\omega(l^*)/2\Gamma T(l^*)]}{\omega(l^*)/2\Gamma T(l^*)}\Bigg\} + O(\epsilon). \end{aligned} \tag{28}$$

Thus the function $C(l^*)$ has the correct $q \to 0$ and $\omega \to 0$ limits and after substituting e^{l^*} from Eqs. (22) to (24) we obtain our final result:

$$\begin{aligned} \chi(q,\omega,\tau) = |2\tau|^{-\gamma}\,F^2\Bigg\{&-iyF^z a^{-1} + x^2F^2c^{-1} - F^{1/\nu} + 2F^{(1-\alpha)/\nu} \\ &-\frac{\epsilon F^{2\beta/\nu}}{2}\Bigg[(1 - \ln T) + \frac{c}{x^2F^2}\left(2R\ln\frac{R + iyF^z a^{-1} - x^2F^2c^{-1}}{R + iyF^z a^{-1}} + (R - iyF^z a^{-1})\ln\frac{N}{4T}\right)\Bigg]\Bigg\}^{-1} \end{aligned} \tag{29}$$

with $T(x,y) = \frac{3}{2}F^{2\beta/\nu} - \frac{1}{2}F^{1/\nu}$, $N(x,y) = x^2F^2c^{-1} + 4(\frac{3}{2}F^{2\beta/\nu} - \frac{1}{2}F^{1/\nu}) - 2iyF^z a^{-1}$, and $R(x,y) = \sqrt{x^2F^2c^{-1}N - y^2F^{2z}a^{-2}}$.

IV. RESULTS AND DISCUSSION

In Figs. 2–4 we show the comparison of the real part of the response function with the mean-field (Gaussian) result (dashed curves):

$$\begin{aligned} \chi_{\mathrm{MFA}}(q,\omega,\tau) &= (\tau + q^2 - i\omega/\Gamma)^{-1}, \quad \tau > 0, \\ \chi_{\mathrm{MFA}}(q,\omega,\tau) &= (|2\tau| + q^2 - i\omega/\Gamma)^{-1}, \quad \tau < 0, \end{aligned} \tag{30}$$

for several values of the wave vector and frequency. The position and the height of the maxima in Figs. 2 and 3 (small momentum) can be easily estimated from the hydrodynamic form of the dynamic susceptibility $C_\pm\,|\tau|^{-\gamma}\,(1 + x_\pm^2 - iy_\pm)^{-1}$:

$$\frac{|\tau^+_{\max}|}{|\tau^-_{\max}|} \simeq \left(\frac{\omega_c^-}{\omega_c^+}\right)^{1/z\upsilon}, \tag{31}$$

$$\frac{\chi^+_{\max}}{\chi^-_{\max}} \simeq \frac{C_+}{C_-}\left(\frac{\omega_c^+}{\omega_c^-}\right) \simeq 1 + \frac{\epsilon}{8} + O(\epsilon). \tag{32}$$

In Fig. 5 the comparison of a few curves for various frequencies is shown with $q = 0.01$ fixed.

In many applications such as, for instance, the scattering experiments the dynamical correlation function is used instead of the response function. They are connected by the fluctuation-dissipation theorem ($k_{\mathrm{B}}T_c = 1$),

$$C(q,\omega,\tau) = \frac{2}{\omega}\,\mathrm{Im}\,\chi(q,\omega,\tau). \tag{33}$$

The scaling function $\widehat{C}(x,y)$ is usually introduced by the relation

$$C(q,\omega,\tau) = \frac{\chi}{\omega_c}\widehat{C}(x,y). \tag{34}$$

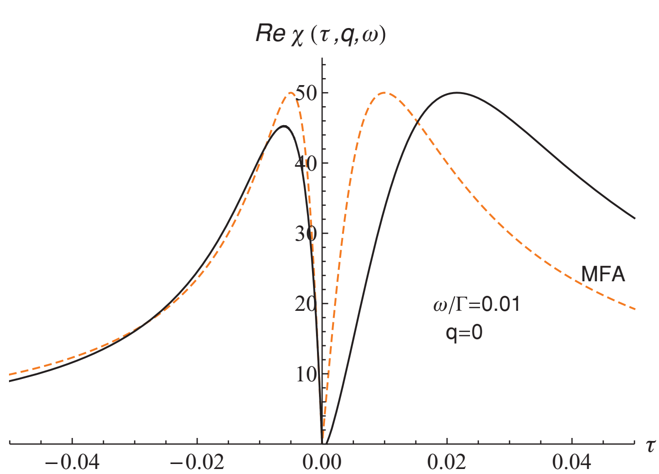

FIG. 2. (Color online) The real part of the susceptibility for $q = 0$ and $\omega/\Gamma = 0.01$. The dashed line is the mean-field approximation (MFA).

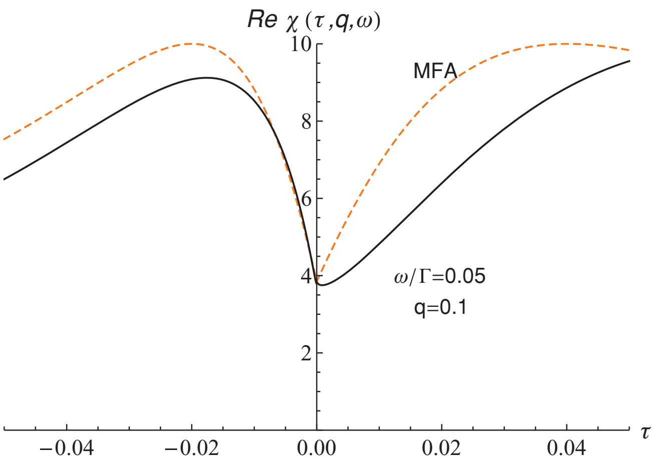

FIG. 3. (Color online) The real part of the susceptibility for $q = 0.1$ and $\omega/\Gamma = 0.05$. The dashed line is the mean-field approximation (MFA).

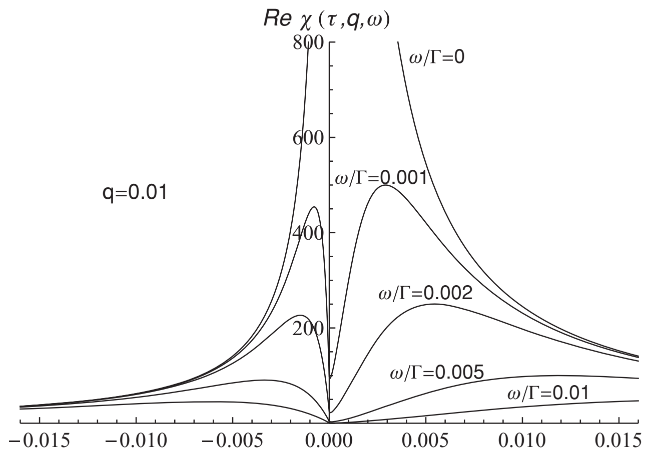

FIG. 5. The real part of the susceptibility for $q = 0.01$ and $\omega/\Gamma = 0$, 0.001, 0.002, 0.005, and 0.01.

Nelson's scaling function[19] yields correctly the Fisher-Langer corrections[37,38] giving rise to the maximum in the correlation function. Dengler *et al.*[22] have shown that this maximum occurs not only in the static limit ($\omega \to 0$) but also in the opposite, $q \to 0$, limit where the following behavior can be found: $\chi(q,\omega,\tau) = C_\pm \left|\tau^{-\gamma}\right| \widehat{D}_\pm(x,y)$, where the scaling function $\widehat{D}_\pm(0,y)$ for large y can be written as

$$\widehat{D}_\pm(0,y) = y^{-(2-\eta)/z}\left[D_0^\pm + \frac{D_1^\pm}{y^{(1-\alpha)/\nu}} + \frac{D_2^\pm}{y^{1/\nu}} + \cdots\right], \quad (35)$$

with $D_0^+ = i$, $D_1^+ = -p$, $D_2^+ = 1 + p$, and $D_0^- = i(1 + \epsilon/8)$, $D_1^- = 2 - \epsilon/2$, $D_2^- = -(1 + \epsilon/4)$ to first order in ϵ. The maximum occurs only in the high-temperature phase as it is confirmed by Figs. 6 and 7 where the dynamical correlation function is shown for various values of ω and q.

It is also worthwhile to compare the deviations of the scaling function $\widehat{C}(x,y)$ from the conventional Van Hove (Gaussian) function $\widehat{C_0}(x,y) = 2/[(1 + x^2)^2 + y^2]$. It should be noted that Monte Carlo simulations in the high-temperature phase for the dynamic structure factor are very well approximated by the Gaussian form up to moderately large values of reduced frequency ($y \leqslant 10$) and momentum ($x \leqslant 5$) as was shown in the work of Calabrese *et al.*[5,39] The function $\widehat{C}(x,y) - \widehat{C_0}(x,y)$ is presented in Figs. 8 and 9 where the deviations both for the high- and low-temperature phases are shown as a function of reduced frequency for several values of reduced momentum. As one can see, the deviations from the conventional Van Hove shape are much larger in the low-temperature phase where they can reach even 12% whereas in the high-temperature phase it is less than 1%. It should be noted, however, that for the low-temperature phase the function $\chi(q,\omega,\tau)$ in the hydrodynamic regime reduces to the form $C_-|\tau|^{-\gamma}\{1 + [1 + O(\epsilon^2)]x^2 - i[1 + O(\epsilon^2)]y\}^{-1}$ so in order to obtain a more precise form for $\widehat{C}(x,y)$, additional rescaling of x and y should be made, the effect of which results in replacing the factors $1 + O(\epsilon^2)$ by 1.

In all the above figures we have taken the values of the critical exponents and parameter p correct to $O(\epsilon)$: $z = 2$, $\eta = 0$, $1/\nu = 2 - \epsilon/3$, $\alpha/\nu = \epsilon/3$, $\beta/\nu = 1 - \epsilon/2$, and $p =$

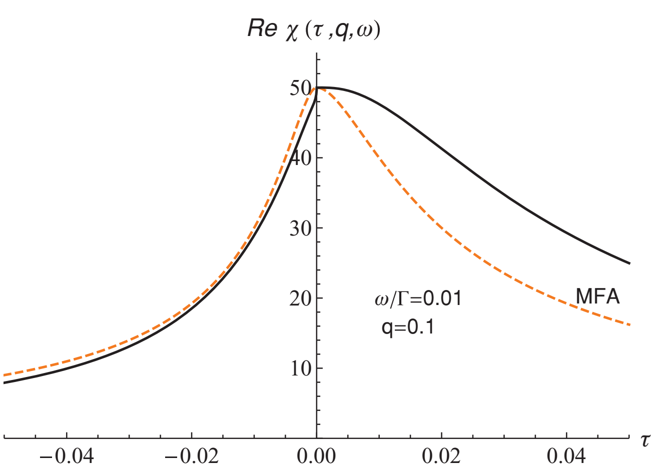

FIG. 4. (Color online) The real part of the susceptibility for $q = 0.1$ and $\omega/\Gamma = 0.01$. The dashed line is the mean-field approximation (MFA).

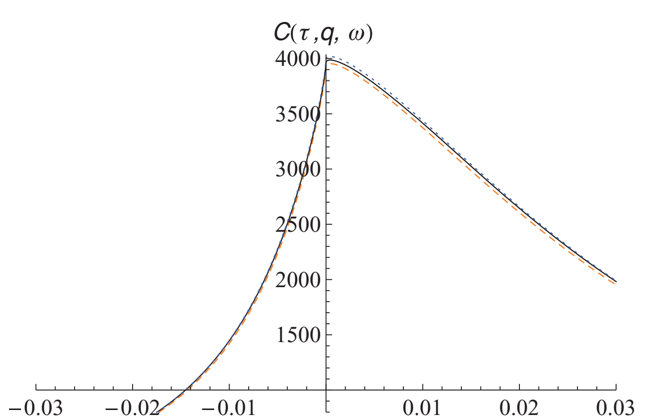

FIG. 6. (Color online) The temperature dependence of the dynamic correlation function $C(q,\omega,\tau)$ for $\epsilon = 1$ and values of q and ω/Γ shown in Fig. 7.

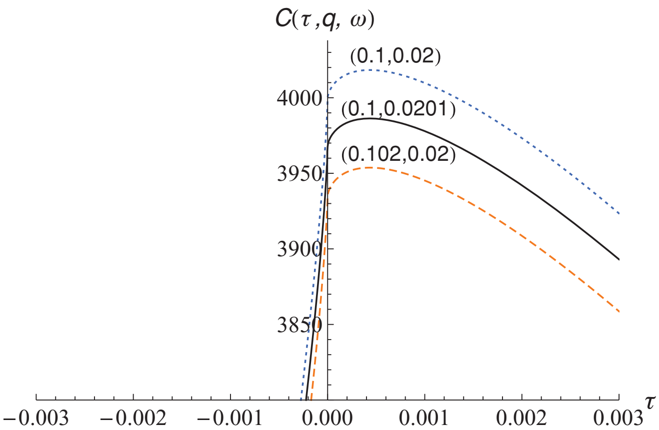

FIG. 7. (Color online) An enlargement of the previous figure near the critical point, showing small maxima above T_c.

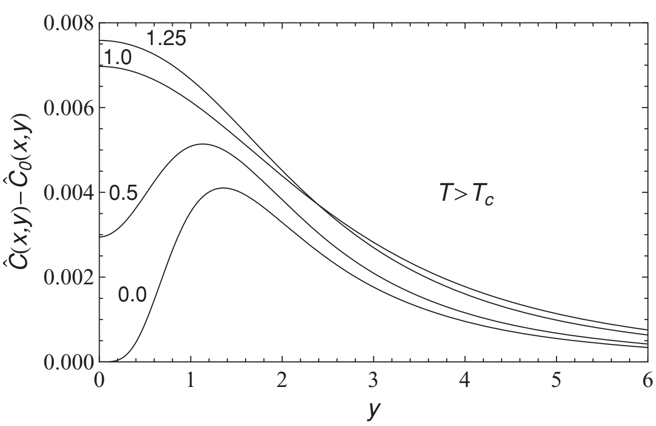

FIG. 8. Deviations of the scaling function $\widehat{C}(x,y)$ from the Van Hove shape $\widehat{C}_0(x,y)$ for the high-temperature phase shown as a function of the reduced frequency y for the reduced momentum x equal 0, 0.5, 1, and 1.25.

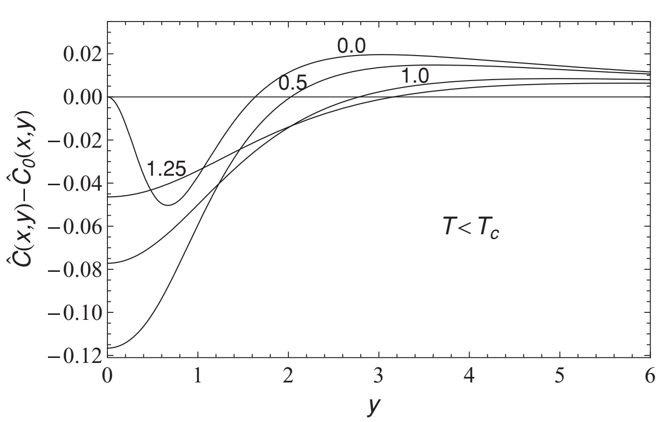

FIG. 9. Deviations of the scaling function $\widehat{C}(x,y)$ from the Van Hove shape $\widehat{C}_0(x,y)$ for the low-temperature phase shown as a function of the reduced frequency y for the reduced momentum x equal to 0, 0.5, 1, and 1.25.

1 with $\epsilon = 1$. As was suggested by Dengler *et al.*[22] in practice one may extend Eqs. (14) and (29) beyond the ϵ expansion taking for exponents the best numerical estimates such as, for example, those summarized in the review paper of Pelissetto and Vicari[13] and by requiring that $p = (2\nu - 1)/\alpha + O(\epsilon^2)$ [this condition comes from the requirement that Eq. (14) gives the correct hydrodynamic limit]. In the low-temperature phase we should also require that the universal amplitude ratios C_+/C_-, f_+/f_-, ω_c^+/ω^-, and A_+/A_- are replaced by their best numerical estimates from Ref. 13. With these replacements and with the mentioned addition rescaling of the reduced frequency and wave vector (giving the proper hydrodynamic behavior) Eqs. (14) and (29) are supposed to be a quite good approximation of the dynamic response function in the whole reduced temperature, wave-vector, and frequency range. We have used the best estimates for exponents in order to compare our low-temperature result with the scaling function $\widehat{C}(x,y)$ obtained by Calabrese *et al.*[5] in their perturbative calculation in the ϵ expansion within a field-theoretical framework as well as with the one-loop result of Täuber and Schwabl[17,18] obtained as a "byproduct" of their field-theoretical study of the coexistence limit supplemented by Amit and Goldsmidt's generalized minimal substraction procedure[40] in order to describe the entire crossover from the critical behavior to an asymptotically uncritical theory for $n = 1$.[18,41]

In Fig. 9 we present the comparison of the scaling function $\widehat{C}(x,y)$ obtained in this work with the results of Calabrese *et al.* [see Eq. (A10) in Ref. 5] and with the corresponding Täuber and Schwabl function. Also the conventional function $\widehat{C_0}(x,y) = 2/[(1+x^2)^2 + y^2]$ is shown.

Given that accurate computation of scaling functions is very difficult within any perturbative approach, it is rather satisfactory how well the different approaches agree with one another. In Figs. 11–14 the deviations from the Van Hove result are shown for different x. It can be seen that the

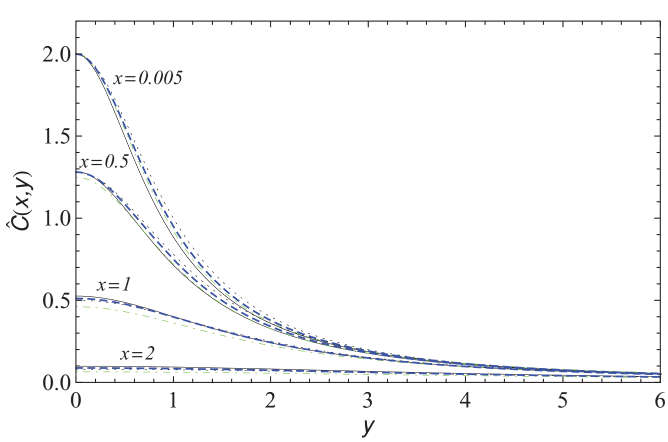

FIG. 10. (Color online) The scaling function $\widehat{C}(x,y)$ as a function of the reduced frequency y for several values of the reduced momentum x in the low-temperature phase. The solid line is the result obtained in this paper (PE); dot-dashed line corresponds to the Calabrese *et al.* (Ref. 5) approach (CMPV); the dashed curve is the result based on Täuber and Schwabl's work (Refs. 18,41) (TS). The dotted line is the Gaussian approximation.

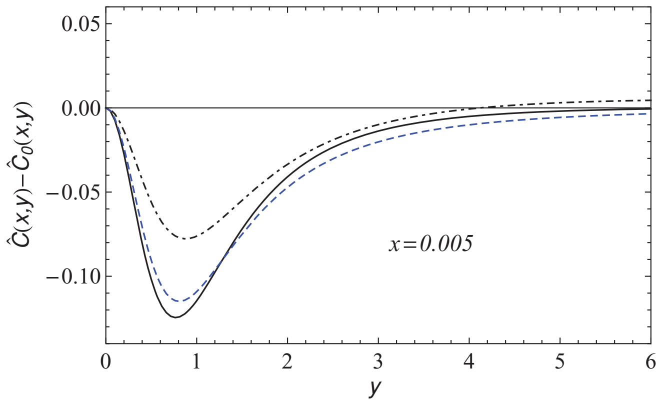

FIG. 11. (Color online) Deviations of the scaling function $\widehat{C}(x,y)$ from the Van Hove shape $\widehat{C}_0(x,y)$ for different approaches in the low-temperature phase shown as a function of the reduced frequency y for $x = 0.005$. Solid line: PE; dashed: TS; dot-dashed: CMPV approximation.

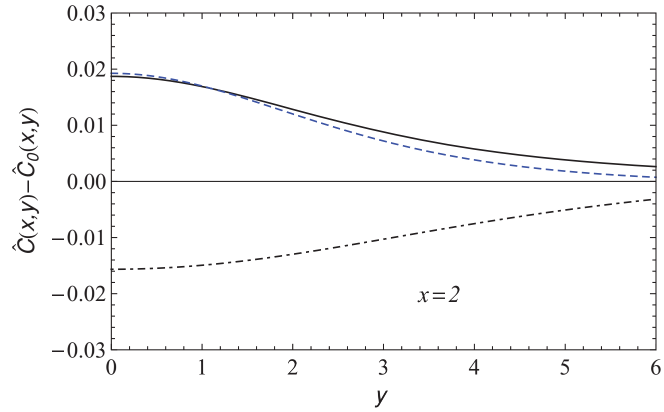

FIG. 13. (Color online) Deviations of the scaling function $\widehat{C}(x,y)$ from the Van Hove shape for the different approaches in the low-temperature phase shown for $x = 2$. Solid line: PE; dashed: TS; dot-dashed: CMPV approximation.

deviations are strongest as $x \to 0$ and they seem to agree on the direction and almost magnitude of deviation from the Gaussian expression in this regime. For intermediate and large x the deviations are much smaller and the present work results coincide rather with Täuber and Schwabl's approach but differ in sign from the result of Calabrese *et al.* as shown in Figs. 13 and 14. It seems that Nelson's method, which is used in this paper, should give a better result in this limit, as it tries to incorporate the Fisher-Langer corrections,[37] which matter in the large-x region. These next-to-leading singularities are unexponentiated in the function B [see Eq. (A10) in Ref. 5] in the Calabrese *et al.* investigation.

In this paper we have extended Nelson's technique of evaluation of the correlation function to $O(\epsilon)$ to include dynamics in the ordered phase. Because of the presence of the Goldstone modes in rotationally invariant systems the results obtained here are limited only to the simplest case of a system with a one-component order parameter. Using a modified matching condition we were able to obtain an analytic expression for the dynamic susceptibility for arbitrary momentum and frequency at temperatures both above and below T_c in zero external field. We have carried our analysis with exponents at their $\epsilon = 1$ values and with u set to its fixed-point value. The dependence of the real part of the dynamic susceptibility on the temperature and frequency is in qualitative agreement with the mean-field expression and the results are displayed in Figs. 2–5. The position and the height of the maxima in the dynamic susceptibility are estimated for a small reduced wave vector. We have also calculated the dynamic correlation function. The deviations of the dynamic form factor from the conventional result are much larger (about one order of magnitude) in the ordered phase than the ones observed in the paramagnetic phase as was illustrated in Figs. 8 and 9. The comparison of three one-loop methods is presented in Figs. 10–14. It has been shown that the correlation function obtained in the present work and that from the Calabrese *et al.* approach[5] as well as Täuber and Schwabl's function[17,18] all seem to agree on the direction and almost magnitude of deviation from the Van Hove results, at least for not very large values of the reduced wave vector.

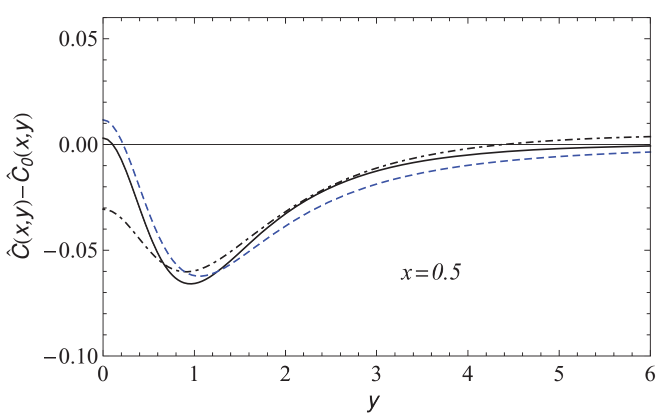

FIG. 12. (Color online) Deviations of the scaling function $\widehat{C}(x,y)$ from the Van Hove shape for different approaches in the low-temperature phase shown as a function of the reduced frequency y for $x = 0.5$. Solid line: PE; dashed: TS; dot-dashed: CMPV approximation.

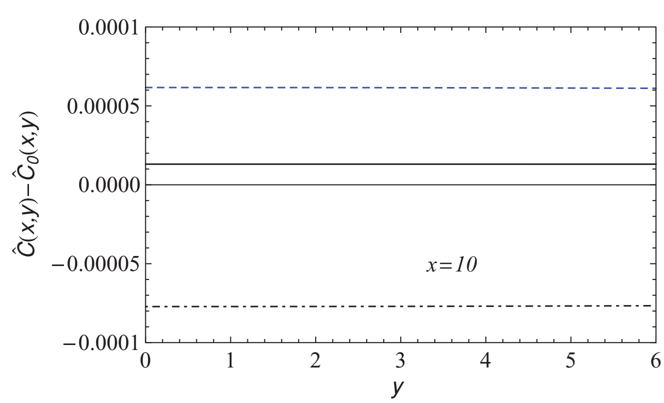

FIG. 14. (Color online) Deviations of the scaling function $\widehat{C}(x,y)$ from the conventional shape for the different approaches in the low-temperature phase shown for $x = 10$. Solid line: PE; dashed: TS; dot-dashed: CMPV approximation.

ACKNOWLEDGMENTS

We would like to thank Uwe C. Täuber for most useful discussions. One of us (R.E.) would like to express his gratitude to the Turkish Scientific and Technological Research Council of Turkey (TÜBİTAK) for financial support (Grant No. 106T579) during the first stages of this work.

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