Dynamical polarization of graphene in a magnetic field

P. K. Pyatkovskiy[1] and V. P. Gusynin[2]
[1]*Department of Applied Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada*
[2]*Bogolyubov Institute for Theoretical Physics, 03680, Kiev, Ukraine*

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The one-loop dynamical polarization function of graphene in an external magnetic field is calculated as a function of wave vector and frequency at finite chemical potential, temperature, band gap, and width of Landau levels. The exact analytic result is given in terms of digamma functions and generalized Laguerre polynomials and has the form of a double sum over Landau levels. Various limits (static, clean, etc.) are discussed. The Thomas-Fermi inverse length q_F of screening of the Coulomb potential is found to be an oscillating function of a magnetic field and a chemical potential. At zero temperature and scattering rate, it vanishes when the Fermi level lies between the Landau levels.

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I. INTRODUCTION

The fabrication of graphene[1] initiated extensive theoretical and experimental studies of its remarkable electronic properties aimed at promised applications of this material in next-generation electronic devices. The noninteracting charge carriers in single layer graphene are described by the analog of the Dirac equation for the massless fermions with the relativistic-like linear spectrum[2] and a vanishing density of states at zero doping. In the presence of an external magnetic field the spectrum of these Dirac quasiparticles has the form of relativistic Landau levels, in contrast to the equidistantly spaced levels in a usual two-dimensional electron gas. These peculiar features of the noninteracting charge carriers in graphene result in several interesting physical phenomena such as the unconventional quantum Hall effect,[3–6] the universal optical conductivity,[7,8] and magneto-spectroscopy.[5,9,10]

Although these and other electronic and transport phenomena in graphene are well described in terms of free Dirac quasiparticles, the effects of interactions and, in particular, the Coulomb interaction, are not settled yet. The vanishing density of states at the Dirac point ensures that the Coulomb interaction between the electrons remains unscreened due to vanishing of the static polarization when the wave vector $q \to 0$.[11] The large value of the unscreened coupling constant $g = e^2/\hbar v_F$, where e is the electron charge, $v_F \approx 10^6$ m/s is the Fermi velocity, could lead to an instability in pristine graphene and the formation of an excitonic condensate and a quasiparticle gap, followed by a quantum phase transition to an insulating phase above some critical g_c. This possibility is studied in a series of theoretical works[12,13] (see also recent papers[14]), but experimental evidence for such an insulating phase is still absent.[15]

The screening of the Coulomb potential due to many-body interactions is determined by the polarization function, which is also an important physical quantity for the spectrum of collective excitations (plasmons). This function, in monolayer graphene without a magnetic field, has been studied in the one-loop approximation in Refs. 13,16–18. In the presence of an external magnetic field, the polarization function was calculated in[19] at zero temperature and impurity rate, with the result given by the double sum over the Landau levels. A similar expression was also obtained later in,[20] where it was employed to study the spectrum of collective excitations in a magnetic field. However, to the best of our knowledge, the most general expression for dynamical polarization at finite temperature, chemical potential, impurity rate, quasiparticle gap, and magnetic field is not given in the literature.

The present paper deals with this more general case. The paper is organized as follows: In Sec. II we describe the model used and present our main result for the polarization function. We consider the clean graphene limit of this function in Sec. III. In Sec. IV we focus on the static screening properties of graphene. Then, in Sec. V we discuss some other limits of the polarization function, and in Sec. VI we give a brief summary of our results. Finally, we provide the details of our calculations in appendices A and B. In Appendix A we derive the expression for the dynamical polarization as a double sum over the Landau levels while in Appendix B we employ the Schwinger proper time method to get a double-integral representation for the polarization.

II. MODEL AND GENERAL EXPRESSION FOR POLARIZATION FUNCTION

The Lagrangian describing the noninteracting Dirac quasiparticles confined to the graphene plane, in an external magnetic field, reads (we use the units $\hbar = c = 1$)

$$\mathcal{L} = \sum_{\sigma=1}^{N_f} \bar{\Psi}_\sigma [i\gamma^0(\partial_t - i\mu) + iv_F \boldsymbol{\gamma}(\nabla + ie\mathbf{A}^{\rm ext}) - \Delta]\Psi_\sigma, \tag{1}$$

where $\Psi_\sigma^T = (\psi_{KA}^\sigma, \psi_{KB}^\sigma, \psi_{K'B}^\sigma, \psi_{K'A}^\sigma)$ is the four-component wave function describing the Bloch states on the A and B sublattices and in the vicinity of K and K' points in momentum space. $\bar{\Psi}_\sigma = \Psi_\sigma^\dagger \gamma^0$ is the Dirac conjugated spinor, σ is the spin variable, and gamma-matrices $\gamma^\nu = \sigma_3 \otimes (\sigma_3, i\sigma_2, -i\sigma_1)$ form a reducible 4×4 representation in $2 + 1$ dimensions.

We will neglect the Zeeman splitting which in graphene is very small ($\sim 1.34B[T]$ K) compared to the distance between the zeroth and the first Landau levels ($\sim 424\sqrt{B[T]}$ K) [here and in what follows energy and a magnetic field are given in Kelvin's and Tesla, respectively]. Therefore, the electron spin results in only the degeneracy factor (number of flavors) $N_f = 2$. We have also included the gap term Δ which can be induced in graphene by placing it on a top of an appropriate substrate[21] that breaks the sublattice symmetry, or can be generated dynamically in a magnetic field (the phenomenon of magnetic catalysis).[12,13] An external magnetic field $\mathbf{B} = \nabla \times \mathbf{A}^{\text{ext}}$ is applied normally to the graphene plane and the vector potential is taken in the symmetric gauge $\mathbf{A}^{\text{ext}} = (-By/2, Bx/2)$. The chemical potential μ can be varied by applying a gate voltage.

The Green's function of Dirac quasiparticles described by this Lagrangian in an external magnetic field reads

$$G(t-t',\mathbf{r};\mathbf{r}') = \exp[-ie\mathbf{r}\mathbf{A}^{\text{ext}}(\mathbf{r}')]S(t-t',\mathbf{r}-\mathbf{r}'), \quad (2)$$

where $S(t-t',\mathbf{r}-\mathbf{r}')$ is the translation-invariant part of the propagator. Using the expression for $S(i\omega_s,\mathbf{q})$ from Refs. 22 and 13, we obtain for the propagator in the configuration space (in the Matsubara representation)

$$S(i\omega_m,\mathbf{r}) = \frac{i}{2\pi l^2}\exp\left(-\frac{\mathbf{r}^2}{4l^2}\right) \times \sum_{n=0}^{\infty}\frac{[\gamma^0(i\omega_m+\mu+i\Gamma_n\text{sgn}\omega_m)+\Delta]f_1^n(\mathbf{r})+f_2^n(\mathbf{r})}{(i\omega_m+\mu+i\Gamma_n\text{sgn}\omega_m)^2-M_n^2},$$

$$\omega_m = (2m+1)\pi T, \quad (3)$$

where T is the temperature (we use $k_B = 1$), $M_n = \sqrt{2nv_F^2/l^2+\Delta^2}$, $E_n = \pm M_n$ are the energies of the relativistic Landau levels, and $l = 1/\sqrt{|eB|}$ is the magnetic length. The functions $f_{1,2}^n(\mathbf{r})$ are defined as

$$f_1^n(\mathbf{r}) = P_- L_n\left(\frac{\mathbf{r}^2}{2l^2}\right) + P_+ L_{n-1}\left(\frac{\mathbf{r}^2}{2l^2}\right),$$
$$P_\pm = \tfrac{1}{2}[1 \pm i\gamma^1\gamma^2\,\text{sgn}(eB)], \quad (4)$$

$$f_2^n(\mathbf{r}) = -\frac{iv_F}{l^2}(\boldsymbol{\gamma}\cdot\mathbf{r})L_{n-1}^1\left(\frac{\mathbf{r}^2}{2l^2}\right), \quad (5)$$

where $L_n^\alpha(z)$ are the generalized Laguerre polynomials [by definition, $L_n(z) \equiv L_n^0(z)$ and $L_{-1}^\alpha(z) \equiv 0$].

The finite parameter Γ_n (>0) represents the width of Landau levels or, equivalently, the scattering rate of Dirac quasiparticles. It is expressed through the retarded fermion self energy and, in general, depends on the energy, temperature, magnetic field, and the Landau level index. In our calculations, we assume that the width is independent of the energy (frequency).

The dynamical polarization determines many physically interesting properties, such as the effective electron-electron interaction, the Friedel oscillations, and the spectrum of collective modes. The retarded one-loop dynamical polarization function is given by the expression

$$\Pi(i\Omega_s,\mathbf{q}) = e^2TN_f\int d^2re^{-i\mathbf{q}\cdot\mathbf{r}} \times \sum_{m=-\infty}^{\infty}\text{tr}[\gamma^0S(i\omega_m,\mathbf{r})\gamma^0S(i\omega_m-i\Omega_s,-\mathbf{r})],$$

$$\Omega_s = 2\pi sT, \quad (6)$$

analytically continued from Matsubara frequencies to the real Ω axis. Note that our definition of the polarization function differs by a factor of $-e^2$ from that used in Refs. 19 and 20. Details of the calculation of this function are given in Appendix A; here we reproduce only the final expression and then analyze various limiting cases. Thus, our main result reads

$$\Pi(\Omega,\mathbf{q}) = \frac{e^2N_f}{4\pi l^2}\sum_{n,n'=0}^{\infty}\sum_{\lambda,\lambda'=\pm}Q_{nn'}^{\lambda\lambda'}(y,\Delta) \times\left[\frac{Z_{nn'}^{\lambda\lambda'}(\Omega,\Gamma,\mu,T)}{\lambda M_n-\lambda'M_{n'}-\Omega-i(\Gamma_n-\Gamma_{n'})} + \frac{Z_{n'n}^{-\lambda',-\lambda}(\Omega,\Gamma,-\mu,T)}{\lambda M_n-\lambda'M_{n'}-\Omega-i(\Gamma_{n'}-\Gamma_n)} - \frac{Z_{nn}^{\lambda\lambda}(\Omega,\Gamma,\mu,T)+Z_{n'n'}^{-\lambda',-\lambda'}(\Omega,\Gamma,-\mu,T)}{\lambda M_n-\lambda'M_{n'}-\Omega-i(\Gamma_n+\Gamma_{n'})}\right], \quad (7)$$

and we have introduced the following notations:

$$Z_{nn'}^{\lambda\lambda'}(\Omega,\Gamma,\mu,T) = \frac{1}{2\pi i}\left[\psi\left(\frac{1}{2}+\frac{\mu-\lambda M_n+\Omega+i\Gamma_n}{2i\pi T}\right) - \psi\left(\frac{1}{2}+\frac{\mu-\lambda'M_{n'}+i\Gamma_{n'}}{2i\pi T}\right)\right], \quad (8)$$

$$Q_{nn'}^{\lambda\lambda'}(y,\Delta) = e^{-y}y^{|n-n'|}\left\{\left(1+\frac{\lambda\lambda'\Delta^2}{M_nM_{n'}}\right)\left(\frac{n_<!}{n_>!}[L_{n_<}^{|n-n'|}(y)]^2 + (1-\delta_{0n_<})\frac{(n_<-1)!}{(n_>-1)!}[L_{n_<-1}^{|n-n'|}(y)]^2\right) + \frac{4\lambda\lambda'v_F^2}{l^2M_nM_{n'}}\frac{n_<!}{(n_>-1)!}L_{n_<-1}^{|n-n'|}(y)L_{n_<}^{|n-n'|}(y)\right\}, \quad (9)$$

where $y = l^2\mathbf{q}^2/2$, $n_> = \max(n,n')$, $n_< = \min(n,n')$, and $\psi(z)$ is the digamma function. Note the symmetry properties of the function $Q_{nn'}^{\lambda\lambda'}(y,\Delta)$ with respect to the exchange of indices λ,λ' and n,n' and $Q_{nn'}^{\lambda\lambda'}(y,\Delta) = Q_{nn'}^{-\lambda,-\lambda'}(y,\Delta)$.

For gapless graphene (with $\Delta = 0$) the function (9) reduces to

$$Q_{nn'}^{\lambda\lambda'}(y,0) = e^{-y}y^{|n-n'|}\left(\sqrt{\frac{(1+\lambda\lambda'\delta_{0n_>})n_<!}{n_>!}}L_{n_<}^{|n-n'|}(y) + \lambda\lambda'(1-\delta_{0n_<})\sqrt{\frac{(n_<-1)!}{(n_>-1)!}}L_{n_<-1}^{|n-n'|}(y)\right)^2. \quad (10)$$

Taking the limit of zero temperature, expression (8) simplifies to

$$Z_{nn'}^{\lambda\lambda'}(\Omega,\Gamma,\mu,0) = \frac{1}{2\pi i}\ln\left(\frac{\mu-\lambda M_n+\Omega+i\Gamma_n}{\mu-\lambda'M_{n'}+i\Gamma_{n'}}\right). \quad (11)$$

The polarization function (7) is an analytic function of Ω without singularities in the whole upper complex half-plane. It depends only on the absolute value of the chemical potential (this can be verified by the replacement $\lambda \leftrightarrow -\lambda', n \leftrightarrow n'$)

and obeys the relation $\Pi(-\Omega,\mathbf{q}) = [\Pi(\Omega,\mathbf{q})]^*$ [can be verified by the replacement $\lambda \leftrightarrow \lambda', n \leftrightarrow n'$ and taking into account $[Z^{\lambda\lambda'}_{nn'}(\Omega,\Gamma,\mu,T)]^* = -Z^{-\lambda,-\lambda'}_{nn'}(-\Omega,\Gamma,-\mu,T)$]. At a finite scattering rate, the polarization function (7) receives contributions both from the inter- (with $\lambda n \neq \lambda' n'$) and intra-Landau-level ($\lambda n = \lambda' n'$) transitions. Note that $Q^{\lambda,-\lambda}_{00}(y,\Delta) = 0$, which reflects the fact that levels with energies $\pm\Delta$ belong to different valleys, and intervalley transitions are not incorporated in our model.

III. CLEAN GRAPHENE

In the absence of scattering of Dirac quasiparticles ($\Gamma_n = 0$) the general expression (7) for the polarization function reduces by means of Eq. (A17) to the following form:

$$\Pi(\Omega,\mathbf{q}) = -\frac{e^2 N_f}{4\pi l^2} \sum_{n,n'=0}^{\infty} \sum_{\lambda,\lambda'=\pm} Q^{\lambda\lambda'}_{nn'}(y,\Delta) \times \frac{n_F(\lambda M_n) - n_F(\lambda' M_{n'})}{\lambda M_n - \lambda' M_{n'} + \Omega + i0}, \tag{12}$$

where $n_F(x) = [e^{(x-\mu)/T} + 1]^{-1}$ is the Fermi distribution function. One can easily see from the above expression that only the terms with $\lambda n \neq \lambda' n'$ (corresponding to the inter-Landau level transitions) survive in the clean limit. However, this is not the case when the limit $\Gamma \to 0$ is taken after setting $\Omega = 0$ [see Eq. (17) below]. When both scattering rate and temperature are zero, it simplifies further to (the order of taking limits $\Gamma_n \to 0$ and $T \to 0$ is not important)

$$\Pi(\Omega,\mathbf{q}) = \frac{e^2 N_f}{4\pi l^2} \sum_{n,n'=0}^{\infty} \sum_{\zeta=\pm} \frac{Q^{-+}_{nn'}(y,\Delta)}{M_n + M_{n'} + \zeta(\Omega + i0)} + \frac{e^2 N_f}{4\pi l^2}\theta(\mu^2 - \Delta^2) \sum_{n=0}^{\infty} \sum_{n'=0}^{N_F} \sum_{\lambda,\zeta=\pm} \times \frac{Q^{\lambda+}_{nn'}(y,\Delta)}{\lambda M_n - M_{n'} + \zeta(\Omega + i0)}, \tag{13}$$

where we used the symmetry of the function $Q^{\lambda\lambda'}_{nn'}(y,\Delta)$ with respect to upper indices, and

$$N_F = \left[\frac{(\mu^2 - \Delta^2) l^2}{2 v_F^2}\right] \tag{14}$$

is the number of the highest filled Landau level (square brackets here denote the integer part of the expression). For $\mu < 0$ it is a positive number denoting the highest empty Landau level in the valence band. The first term in Eq. (13) describes the vacuum contribution and takes into account only interband processes while the second term represents intraband and interband contributions when the chemical potential lies in the conduction or valence band. Notice that this second term does not receive contributions from terms with $n = n'$ and $\lambda = +1$.

In the gapless case ($\Delta = 0$) we have

$$\Pi(\Omega,\mathbf{q}) = \frac{e^2 N_f}{4\pi l^2} \sum_{n,n'=0}^{\infty} \sum_{\zeta=\pm} \frac{Q^{-+}_{nn'}(y,0)}{M_n + M_{n'} + \zeta(\Omega + i0)} + \frac{e^2 N_f}{4\pi l^2} \sum_{n=0}^{\infty} \sum_{n'=1}^{N_F} \sum_{\lambda,\zeta=\pm} \frac{Q^{\lambda+}_{nn'}(y,0)}{\lambda M_n - M_{n'} + \zeta(\Omega + i0)}. \tag{15}$$

Expressions (13) and (15) coincide with the polarization function calculated in Ref. 19. Reference 20 considered only the gapless case and obtained an expression similar to Eq. (15) but with a twice-larger contribution from the lowest Landau level ($n = 0$), while the results of Refs. 23 and 24 are completely different from ours.

The static clean limit of the polarization function essentially depends on the order of taking limits $\Omega \to 0$ and $\Gamma_n \to 0$. Indeed, first taking the limit $\Omega = 0$, the expression for the polarization function (7) reduces to

$$\Pi(0,\mathbf{q}) = \frac{e^2 N_f}{8\pi^3 l^2 T} \sum_{n=0}^{n_c} \sum_{\lambda=\pm} Q^{\lambda\lambda}_{nn}(y,\Delta) \times \mathrm{Re}\,\psi'\left(\frac{1}{2} + \frac{\mu - \lambda M_n + i\Gamma_n}{2i\pi T}\right) + \frac{e^2 N_f}{4\pi^2 l^2} \sum_{\substack{n,n'=0 \\ \lambda n \neq \lambda' n'}}^{n_c} \sum_{\lambda,\lambda'=\pm} Q^{\lambda\lambda'}_{nn'}(y,\Delta) \times \mathrm{Im}\frac{\psi\left(\frac{1}{2} + \frac{\mu - \lambda M_n + i\Gamma_n}{2i\pi T}\right) - \psi\left(\frac{1}{2} + \frac{\mu - \lambda' M_{n'} + i\Gamma_{n'}}{2i\pi T}\right)}{\lambda M_n - \lambda' M_{n'} - i(\Gamma_n - \Gamma_{n'})}, \tag{16}$$

where we took into account that the numerator of the third term in square brackets in Eq. (7) vanishes at $\Omega = 0$. Here we also introduced the ultraviolet cutoff n_c due to the divergence of the sum over the Landau levels at finite width Γ_n. This cutoff is estimated to be $n_c \sim 10^4/B[T]$ due to finiteness of the bandwidth.[20] The expression (16) for static polarization is obviously a real function. In the clean graphene limit $\Gamma_n = 0$, we get

$$\Pi(0,\mathbf{q}) = \frac{e^2 N_f}{16\pi l^2 T} \sum_{n=0}^{\infty} \sum_{\lambda=\pm} \frac{Q^{\lambda\lambda}_{nn}(y,\Delta)}{\cosh^2\left(\frac{\mu - \lambda M_n}{2T}\right)} - \frac{e^2 N_f}{4\pi l^2} \sum_{\substack{n,n'=0 \\ \lambda n \neq \lambda' n'}}^{\infty} \sum_{\lambda,\lambda'=\pm} Q^{\lambda\lambda'}_{nn'}(y,\Delta) \times \frac{n_F(\lambda M_n) - n_F(\lambda' M_{n'})}{\lambda M_n - \lambda' M_{n'}} \tag{17}$$

(the sum over the Landau levels is convergent). On the other hand, if we take limit $\Omega \to 0$ in (12) (i.e., after setting $\Gamma_n = 0$), we obtain expression (17) without the first term. This term gives the contribution from the intra-level transitions ($n \leftrightarrow n$) even at zero width of Landau levels. At zero temperature it turns into a sequence of delta functions $\delta(\mu \pm M_n)$ and does not contribute at integer filling factors of Landau levels which, for noninteracting quasiparticles in graphene, take the values $\nu = 0, \pm 2, \pm 6, \pm 10, \ldots$. Therefore, for $T = 0$, we arrive at the same expression (13) with $\Omega = 0$.

IV. STATIC SCREENING

The screening of the static Coulomb potential $\phi_0(r) = Ze/r$ is determined by the static polarization function

$$\phi(r) = \frac{Ze}{\varepsilon_0}\int \frac{d^2q}{2\pi}\frac{\exp(i\mathbf{q}\cdot\mathbf{r})}{q + (2\pi/\varepsilon_0)\Pi(0,\mathbf{q})} = \frac{Ze}{\varepsilon_0}\int_0^\infty \frac{dq\, q\, J_0(qr)}{q + (2\pi/\varepsilon_0)\Pi(0,q)}, \quad (18)$$

where $J_0(z)$ is the Bessel function and ε_0 is the background dielectric constant due to a substrate.

In what follows we assume that, even in the case of clean graphene, the limit $\Gamma_n \to 0$ of the polarization function is taken after the limit $\Omega \to 0$ when calculating the screened potential (18). This order of limits, which leads to expression (17) for $\Pi(0,\mathbf{q})$, seems to be more natural due to the fact that real graphene samples cannot be completely free from impurities and some broadening of the Landau levels always occurs.

In the general case, the static polarization function is given by expression (16), which does not have singularities (such as, for example, the discontinuity of the second derivative at $q = 2\mu/v_F$ in the absence of a magnetic field[17]). Therefore, the asymptotical behavior of the screened potential at small or large distances is determined solely by the asymptotics of $\Pi(0,\mathbf{q})$ at large or small wave vectors, respectively. At large momenta we have the zero-magnetic-field result

$$\Pi(0,\mathbf{q}) \simeq \frac{e^2 N_f|\mathbf{q}|}{8v_F}, \quad \mathbf{q} \to \infty, \quad (19)$$

and (18) implies

$$\phi(r) \simeq \frac{Ze}{\varepsilon_0^* r}, \quad r \to 0, \quad (20)$$

where

$$\varepsilon_0^* = \varepsilon_0 + \frac{\pi e^2 N_f}{4v_F} \approx \varepsilon_0 + 3.4 \quad (21)$$

is the "effective" background dielectric constant ($N_f = 2$). At small wave-vector values ($\mathbf{q} \to 0$), the static polarization function (16) behaves as

$$\Pi(0,\mathbf{q}) \simeq \frac{\varepsilon_0}{2\pi}(q_F + a\mathbf{q}^2), \quad \mathbf{q} \to 0, \quad (22)$$

with

$$q_F = \frac{2\pi}{\varepsilon_0}\Pi(0,0) = \frac{e^2 N_f}{2\pi^2\varepsilon_0 l^2 T}\sum_{n=0}^{n_c}\sum_{\lambda=\pm}(2-\delta_{0n}) \times \mathrm{Re}\psi'\left(\frac{1}{2} + \frac{\mu - \lambda M_n + i\Gamma_n}{2i\pi T}\right), \quad (23)$$

$$a = -\frac{e^2 N_f}{4\pi^2\varepsilon_0 T}\sum_{n=0}^{\infty}\sum_{\lambda=\pm}(4n+\delta_{0n}) \times \mathrm{Re}\psi'\left(\frac{1}{2} + \frac{\mu - \lambda M_n + i\Gamma_n}{2i\pi T}\right)$$
$$+ \frac{e^2 N_f v_F^2}{\varepsilon_0 l^2}\sum_{n=0}^{\infty}\sum_{\lambda,\lambda'=\pm}\frac{\frac{\lambda(n+1)}{M_{n+1}} + \frac{\lambda' n}{M_n}}{\lambda M_{n+1} - \lambda' M_n} \times \mathrm{Im}\frac{\psi\left(\frac{1}{2} + \frac{\mu - \lambda M_{n+1} + i\Gamma_{n+1}}{2i\pi T}\right) - \psi\left(\frac{1}{2} + \frac{\mu - \lambda' M_n + i\Gamma_n}{2i\pi T}\right)}{\lambda M_{n+1} - \lambda' M_n - i(\Gamma_{n+1} - \Gamma_n)}, \quad (24)$$

where we used the following asymptotics of the functions (9) at $y \to 0$:

$$Q_{nn}^{\lambda\lambda'}(y,\Delta) = 2\delta_{\lambda\lambda'}[2 - \delta_{0n} - (4n + \delta_{0n})y] + O(y^2), \quad (25)$$

$$Q_{n,n+1}^{\lambda\lambda'}(y,\Delta) = Q_{n+1,n}^{\lambda\lambda'}(y,\Delta) = y\left[2n + 1 + \lambda\lambda'\left(\frac{nM_{n+1}}{M_n} + \frac{(n+1)M_n}{M_{n+1}}\right)\right] + O(y^2), \quad (26)$$

$$Q_{nn'}^{\lambda\lambda'}(y,\Delta) = O(y^2), \qquad n \neq n \pm 1, \quad \lambda n \neq \lambda' n'. \quad (27)$$

It is clearly seen that q_F receives the contribution only from intra-Landau-level ($n \leftrightarrow n$) transitions while the parameter a contains also the contribution from the transitions between levels $n \leftrightarrow \pm n \pm 1$.

In the case $q_F \neq 0$, we find from (18) the following asymptotical behavior:

$$\phi(r) \simeq \frac{Ze}{\epsilon_0 q_F^2 r^3}, \quad r \to \infty, \quad (28)$$

that describes Thomas-Fermi screening in graphene.[17] In contrast to the three-dimensional case where for nonzero charge density, the Coulomb potential $1/r$ is replaced by an exponentially decreasing potential, in the two-dimensional case we have $1/r^3$ behavior at large r, which is a well-known fact.[25] The strength of the screening is determined by the magnitude of the Thomas-Fermi wave vector $q_F = (2\pi/\varepsilon_0)\Pi(0,0)$.

The polarization function in Eq. (23) obeys the following relation:[26]

$$\Pi(0,0) = e^2\frac{\partial}{\partial\mu}\rho(\mu,T) = e^2\int_{-\infty}^{\infty}\frac{d\epsilon\, D(\epsilon)}{4T\cosh^2\left(\frac{\epsilon-\mu}{2T}\right)}, \quad (29)$$

where $D(\epsilon)$ is the density of states in graphene with impurities in a magnetic field,[27]

$$D(\epsilon) = \frac{N_f}{2\pi^2 l^2}\sum_{n=0}^{n_c}\sum_{\lambda=\pm}\frac{(2-\delta_{0n})\Gamma_n}{(\epsilon - \lambda M_n)^2 + \Gamma_n^2}, \quad (30)$$

and $\rho(\mu,T)$ is the density of Dirac quasiparticles:

$$\rho(\mu,T) = \int_{-\infty}^{\infty} d\epsilon\, D(\epsilon)\,[n_F(\epsilon) - \theta(-\epsilon)]. \quad (31)$$

At zero temperature and finite scattering rate the quantity $\Pi(0,0)$ is proportional to the density of states at the Fermi surface:

$$\Pi(0,0) = \frac{e^2 N_f}{2\pi^2 l^2}\sum_{n=0}^{n_c}\sum_{\lambda=\pm}\frac{(2-\delta_{0n})\Gamma_n}{(\mu - \lambda M_n)^2 + \Gamma_n^2} = e^2 D(\mu). \quad (32)$$

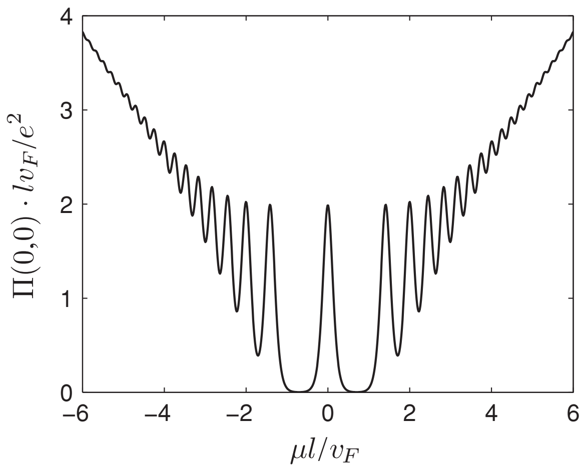

FIG. 1. Long-wavelength limit of the static polarization function (33) at $\Gamma = 0$, $\Delta = 0$, $T = 0.08 v_F/l$.

It is an oscillating function of chemical potential and magnetic field[27] and, therefore, the screened potential at large distances oscillates with changing μ at a fixed magnetic field, or with changing B at fixed μ.

For $\Gamma_n = 0$ and finite temperature, Eq. (23) reduces to the expression

$$\Pi(0,0) = \frac{e^2 N_f}{8\pi l^2 T} \sum_{n=0}^{n_c} \sum_{\lambda=\pm} \frac{2-\delta_{0n}}{\cosh^2\left(\frac{\mu-\lambda M_n}{2T}\right)}, \tag{33}$$

which has qualitatively similar oscillatory behavior; see Fig. 1. The weak-magnetic-field limit ($l \to \infty$) of the above expression can be obtained by replacing $n \to k^2 l^2/2$, with the sum turning into an integral over k, resulting in

$$\Pi(0,0) = \frac{e^2 N_f T}{\pi v_F^2}\left[\ln\left[2\cosh\left(\frac{\Delta+\mu}{2T}\right)\right] - \frac{\Delta}{2T}\tanh\left(\frac{\Delta+\mu}{2T}\right) + (\mu \to -\mu)\right], \tag{34}$$

which agrees with Ref. 13.

Some numerical results for the screened Coulomb potential in the case of clean gapless graphene at finite temperature are shown in Figs. 2 and 3 (we used $\varepsilon_0 = 1$). Figure 2 shows the Fourier transform of the potential (18):

$$\tilde{\phi}(q) = \frac{2\pi Ze}{\varepsilon_0 q + 2\pi \Pi(0,q)}, \tag{35}$$

and illustrates the long-wavelength screening behavior in two different cases corresponding to large ($q_F \gg l^{-1}$) and small ($q_F \ll l^{-1}$) values of the Thomas-Fermi wave vector. Figure 3 represents the screened potential (18) itself. While the asymptotics of $\phi(r)$ are always given by Eqs. (20) and (28), its behavior at intermediate distances can be qualitatively different, depending on the values of the parameters lT and $l\mu$. If the temperature is sufficiently low ($T \lesssim 0.1 v_F/l$) and

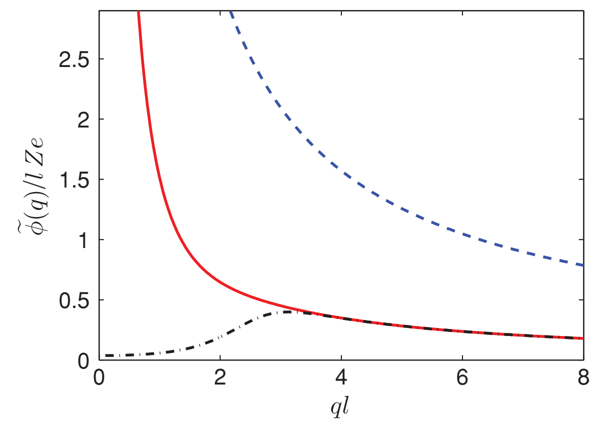

FIG. 2. (Color online) Fourier transform of the screened Coulomb potential at $\Gamma = 0$, $\Delta = 0$, $T = 0.01 v_F/l$. Dot-dashed (black) line: $\mu = 0.01 v_F/l$, $q_F \simeq 170/(l\varepsilon_0)$, solid (red) line: $\mu = 0.1 v_F/l$, $q_F \simeq 0.04/(l\varepsilon_0)$. Dashed (blue) line shows the unscreened case.

the chemical potential lies in the vicinity of one of the Landau levels ($|\mu - E_n| \lesssim T$), the coefficient a in (22) is negative and $1/q_F \ll l \ll |a|$. In this case the screened potential (18) oscillates at intermediate distances $1/q_F < r < |a|$, as shown in Figs. 3(a) and 3(b). When the chemical potential lies away from the Landau levels ($|\mu - E_n| \gg T$) or the temperature is larger than $0.4 v_F/l$, the coefficient a is positive and $\phi(r)$ does not oscillate [Figs. 3(c) and 3(d)]. In this case the asymptotic behavior of the screened potential for $r \gg l$ is given by

$$\begin{aligned}\phi(r) &\simeq \frac{Ze}{\varepsilon_0}\int \frac{d^2q}{2\pi}\frac{\exp(i\mathbf{q}\cdot\mathbf{r})}{q+q_F+aq^2} \\ &= \frac{\pi Ze}{2\varepsilon_0 a(q_1-q_2)}\{q_1[H_0(q_1 r) - Y_0(q_1 r)] \\ &\quad - q_2[H_0(q_2 r) - Y_0(q_2 r)]\},\end{aligned} \tag{36}$$

where $q_{1,2} = (1 \pm \sqrt{1-4aq_F})/(2a)$, $H_0(z)$ is the Struve function, and $Y_0(z)$ is the Bessel function of the second kind.

Now let us consider the case when both temperature and scattering rate are zero. In this case,

$$\Pi(0,0) = \frac{e^2 N_f}{2\pi l^2}\sum_{n=0}^{n_c}\sum_{\lambda=\pm}(2-\delta_{0n})\delta(\mu - \lambda M_n) = e^2 D_0(\mu), \tag{37}$$

where $D_0(\mu)$ is the density of states at the Fermi surface for clean graphene.[27] When the Fermi level lies between Landau levels (which corresponds to integer fillings) the above expression vanishes (i.e., $q_F = 0$). Restricting ourselves to these integer fillings and setting $\Omega = 0$ in (13) or, equivalently, setting $T = 0$, $\Gamma_n = 0$ in (16),

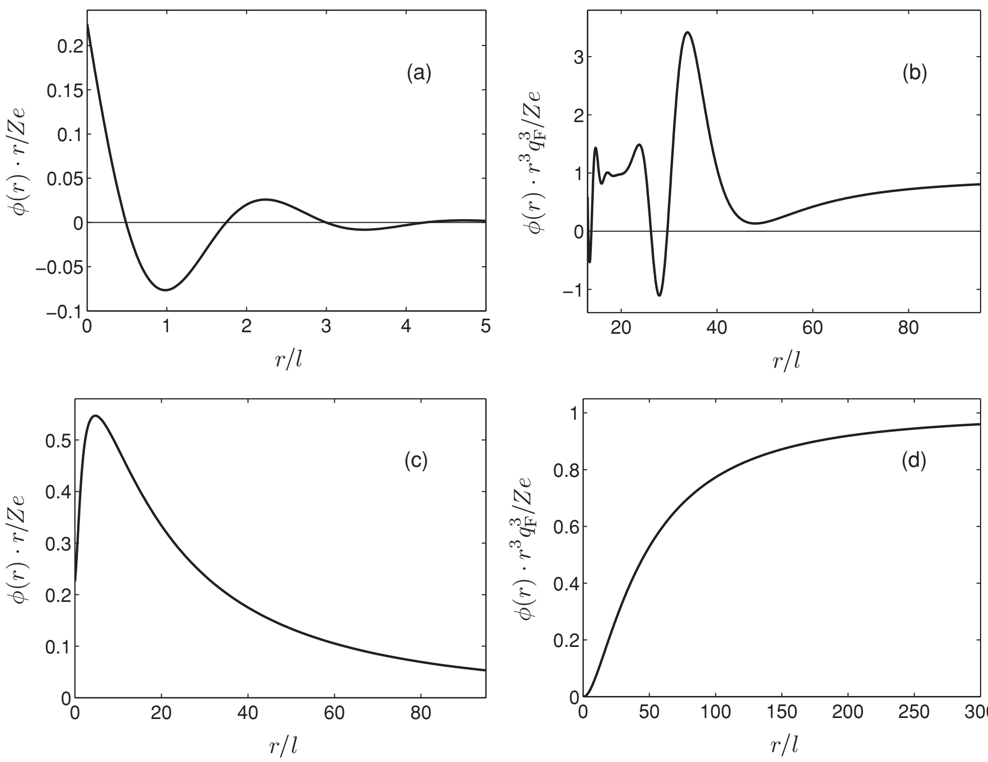

FIG. 3. Screened Coulomb potential at small [(a), (c)] and large [(b), (d)] distances. Here, $\Gamma = 0$, $\Delta = 0$, $T = 0.01 v_F/l$, and the value of the chemical potential is $\mu = 0.01 v_F/l$ for (a) and(b) and $\mu = 0.1 v_F/l$ for (c) and (d). Note that the scales used in the different plots are not the same.

we obtain

$$\Pi(0,\mathbf{q}) = \frac{e^2 N_f}{2\pi l^2}\sum_{n,n'=0}^{n_c}\frac{Q_{nn'}^{-+}(y,\Delta)}{M_n + M_{n'}} - \frac{e^2 N_f}{2\pi l^2}\theta(\mu^2-\Delta^2) \times \sum_{n'=0}^{n_c}\sum_{\substack{n=0\\ n\neq\lambda' n'}}^{N_F}\sum_{\lambda'=\pm}\frac{Q_{nn'}^{\lambda'+}(y,\Delta)}{M_n - \lambda' M_{n'}}. \tag{38}$$

Now the transitions between levels $n \leftrightarrow \pm n \pm 1$ give the main contribution at long wavelengths, because of asymptotics (25)–(27) of the functions $Q_{nn'}^{\lambda\lambda'}(y,\Delta)$. This leads to the behavior

$$\Pi(0,\mathbf{q}) \simeq \frac{\varepsilon_0}{2\pi} a \mathbf{q}^2, \qquad |\mathbf{q}| \ll 1/l. \tag{39}$$

The coefficient a at zero temperature and scattering rate is always positive and depends on the number N_F of filled Landau levels and the gap Δ. It is evaluated to be

$$a(N_F,\Delta) = \frac{e^2 N_f l}{\sqrt{2}\varepsilon_0 v_F}\left(F(d) + \theta(\mu^2-\Delta^2) \times \sum_{n=0}^{N_F}\frac{(2-\delta_{0n})(3n+2d)}{\sqrt{n+d}}\right), \tag{40}$$

where $d = l^2\Delta^2/(2v_F^2)$ is the dimensionless gap parameter, and we define the function $F(d)$ as

$$F(d) = \sum_{n=1}^{n_c}(\sqrt{n+d} - \sqrt{n-1+d})^3 \times \left(1 + \frac{d}{\sqrt{n+d}\sqrt{n-1+d}}\right). \tag{41}$$

At zero gap and $N_F = 0$, we obtain, in agreement with Refs. 13,19,28,

$$a(0,0) = \frac{e^2 N_f l}{\sqrt{2}\varepsilon_0 v_F}F(0),$$
$$F(0) = -6\zeta(-1/2) - \frac{1}{4\sqrt{n_c}} + O\left(n_c^{-3/2}\right) \simeq 1.247, \tag{42}$$

where $\zeta(z)$ is the Riemann zeta function.

From Eq. (18) we obtain that, at long distances, screening is absent (the correction to the bare Coulomb potential is of smaller order):

$$\phi(r) \simeq \frac{Ze}{\varepsilon_0 r}\left(1 - \frac{a^2}{r^2}\right), \qquad r \gg l. \tag{43}$$

V. OTHER LIMITING CASES

At zero momentum, only terms with $\lambda n = \lambda' n'$ survive in (7), and the general expression for the polarization function simplifies to

$$\Pi(\Omega,0) = \frac{ie^2 N_f}{\pi l^2 \Omega} \sum_{n=0}^{n_c} \frac{(2-\delta_{0n})\Gamma_n}{\Omega + 2i\Gamma_n} \sum_{\lambda=\pm} Z_{nn}^{\lambda\lambda}(\Omega,\Gamma,\mu,T) + (\mu \to -\mu). \quad (44)$$

At zero scattering rate and finite Ω the above expression vanishes, while its limit $\Omega \to 0$ at finite Γ_n is given by Eq. (23). Therefore, the static long-wavelength polarization function $\Pi(0,0)$ does not depend on the order of taking limits $\Omega \to 0$ and $\mathbf{q} \to 0$, unlike in the absence of a magnetic field.

The strong-magnetic-field limit ($l \to 0$) of the polarization function (7) also depends on the ratio between the scattering rate and the frequency. For $\Gamma_n/\Omega \neq 0$ the main contribution comes from the lowest Landau level ($n = 0$) and is given by the expression

$$\Pi(\Omega,\mathbf{q}) \simeq -\frac{e^2 N_f}{2\pi^2 l^2} \frac{\Gamma_0}{\Omega(\Omega + 2i\Gamma_0)} \times \sum_{\lambda,\lambda'=\pm} \left\{ \psi\left(\frac{1}{2} + \frac{\lambda\mu + \lambda'\Delta + \Omega + i\Gamma_0}{2i\pi T}\right) - \psi\left(\frac{1}{2} + \frac{\lambda\mu + \lambda'\Delta + i\Gamma_0}{2i\pi T}\right)\right\}. \quad (45)$$

However, this contribution vanishes in the clean-graphene limit (more precisely, for $\Gamma_0 = 0$ and nonzero Ω). In this case the transitions $n \leftrightarrow -n \pm 1$ in (7) dominate at high magnetic field, resulting in

$$\Pi(\Omega,\mathbf{q}) \simeq \frac{\varepsilon_0}{2\pi} a(0,0)\mathbf{q}^2, \quad (46)$$

which is equivalent to the static long-wavelength limit of the polarization function for clean gapless graphene at zero temperature in the case when only the lowest Landau level is filled.

VI. SUMMARY

In this paper we have derived the exact analytical expression for the one-loop dynamical polarization function in graphene, as a function of wave vector and frequency, at finite chemical potential, temperature, band gap, and taking into account the finite scattering rate of Dirac quasiparticles due to the presence of impurities. The most general result is given in terms of the digamma function and generalized Laguerre polynomials and has the form of a double sum over Landau levels [Eq. (7)]. In clean graphene at zero temperature, for integer fillings of Landau levels, this function correctly reproduces previously obtained results. The derived expression for dynamical polarization can be used to calculate the dispersion relation and the decay rate of magnetoplasmons depending on temperature and impurity rate.

The long-range behavior of the screened static Coulomb potential in graphene in a magnetic field is found to be essentially affected by the presence of impurities or finite temperature. When either the scattering rate or the temperature is nonzero, the usual Thomas-Fermi screening is present, and the resulting potential decays as $\sim 1/r^3$, which is typical for two-dimensional systems. The strength of the screening oscillates as a function of chemical potential or magnetic field. If both scattering rate and temperature are zero, these oscillations turn into a sequence of delta functions and, for integer fillings, the screening is absent.

In conclusion, we note that, while the present paper deals with a derivation of the most general analytical expression for dynamical polarization in the one-loop approximation with noninteracting quasiparticles, the interaction effects are very important to take into account, too. Because of the Kohn theorem,[29] the electron-electron interaction effects do not affect the bare cyclotron energy in a translational invariant system with a parabolic dispersion law. However, this theorem does not apply for quasiparticles in graphene, which have a linear dispersion law. The interaction correlations may produce significant changes in the exciton dispersion relations and, thus, be observable in experiments. The important studies in this direction were undertaken in recent papers.[30,31] The first step toward including electron-electron interactions in our approach would be to add the exchange corrections to the chemical potential along the lines described in Ref. 31 that we leave for future studies.

ACKNOWLEDGMENTS

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APPENDIX A: CALCULATION OF POLARIZATION FUNCTION

After evaluation of the trace and using

$$\frac{1}{A_m^2 - M_n^2} = \frac{1}{2M_n} \sum_{\lambda=\pm} \frac{\lambda}{A_m - \lambda M_n}, \qquad \frac{A_m}{A_m^2 - M_n^2} = \frac{1}{2} \sum_{\lambda=\pm} \frac{1}{A_n - \lambda M_n}, \quad (A1)$$

with $A_m = i\omega_m + \mu + i\Gamma_n \,\mathrm{sgn}\omega_m$, Eq. (6) can be written in the following form:

$$\Pi(i\Omega_s,\mathbf{q}) = -\frac{e^2 T N_f}{8\pi^2 l^4} \sum_{n,n'=0}^{n_c} \sum_{\lambda,\lambda'=\pm} \sum_{m=-\infty}^{\infty} \frac{\left(1 + \frac{\lambda\lambda'\Delta^2}{M_n M_{n'}}\right)\left[I_{nn'}^0(y) + I_{n-1,n'-1}^0(y)\right] + \frac{4\lambda\lambda' v_F^2}{l^2 M_n M_{n'}} I_{n-1,n'-1}^1(y)}{(i\omega_m + \mu + i\Gamma_n \,\mathrm{sgn}\omega_m - \lambda M_n)(i\omega_{m-s} + \mu + i\Gamma_{n'} \,\mathrm{sgn}\omega_{m-s} - \lambda' M_{n'})}, \quad (A2)$$

where $y = \mathbf{q}^2 l^2/2$, and

$$I^{\alpha}_{nn'}(y) = \int d^2r\, e^{-i\mathbf{q}\cdot\mathbf{r}} \left(\frac{\mathbf{r}^2}{2l^2}\right)^{\alpha} \exp\left(-\frac{\mathbf{r}^2}{2l^2}\right) \times L^{\alpha}_n\left(\frac{\mathbf{r}^2}{2l^2}\right) L^{\alpha}_{n'}\left(\frac{\mathbf{r}^2}{2l^2}\right), \quad \alpha = 0,1. \quad \text{(A3)}$$

The above expression is nonzero only for $n,n' \geqslant 0$. Integrating over the angle and making the change of variable $\mathbf{r}^2 = 2l^2 t$, we get

$$\begin{aligned} I^{\alpha}_{nn'}(y) &= 2\pi l^2 \int_0^{\infty} dt\, e^{-t} t^{\alpha} J_0(2\sqrt{yt}) L^{\alpha}_n(t) L^{\alpha}_{n'}(t) \\ &= 2\pi l^2 (-n'-1)^{\alpha} \int_0^{\infty} dt\, e^{-t} J_0(2\sqrt{yt}) L^{\alpha}_n(t) L^{-\alpha}_{n'+\alpha}(t), \\ &\quad \alpha = 0,1, \end{aligned} \quad \text{(A4)}$$

where we have used

$$L^k_l(x) = (-x)^{-k} \frac{(l+k)!}{l!} L^{-k}_{l+k}(x), \quad l \geqslant 0, \quad k+l \geqslant 0. \quad \text{(A5)}$$

Now, using formula 7.422.2 of Ref. 32,

$$\begin{aligned} &\int_0^{\infty} dx\, x^{\nu+1} e^{-\alpha x^2} J_{\nu}(bx) L^{\nu-\sigma}_m(\alpha x^2) L^{\sigma}_n(\alpha x^2) \\ &= (-1)^{m+n} (2\alpha)^{-\nu-1} b^{\nu} e^{-\frac{b^2}{4\alpha}} L^{\sigma-m+n}_m\left(\frac{b^2}{4\alpha}\right) L^{\nu-\sigma+m-n}_n\left(\frac{b^2}{4\alpha}\right), \end{aligned} \quad \text{(A6)}$$

we obtain from (A4):

$$\begin{aligned} I^{\alpha}_{nn'}(y) &= 2\pi l^2 (-1)^{n-n'} (n'+1)^{\alpha} e^{-y} L^{n'-n}_n(y) L^{n-n'}_{n'+\alpha}(y) \\ &= 2\pi l^2 \frac{(n_< + \alpha)!}{n_>!} e^{-y} y^{|n-n'|} L^{|n-n'|}_{n_<}(y) L^{|n-n'|}_{n_<+\alpha}(y), \\ &\quad \alpha = 0,\, 1, \end{aligned} \quad \text{(A7)}$$

where we again used formula (A5) and the symmetry $I^{\alpha}_{nn'}(y) = I^{\alpha}_{n'n}(y)$, which follows from (A3). Now we can rewrite (A2) as

$$\Pi(i\Omega_s,\mathbf{q}) = -\frac{e^2 T N_f}{4\pi l^2} \sum_{n,n'=0}^{n_c} \sum_{\lambda,\lambda'=\pm} Q^{\lambda\lambda'}_{nn'}(y,\Delta)\mathcal{I}, \quad \text{(A8)}$$

where the functions $Q^{\lambda\lambda'}_{nn'}(y,\Delta)$ are defined in (9) and

$$\mathcal{I} = \sum_{m=-\infty}^{\infty} (i\omega_m + \mu + i\Gamma_n\, \mathrm{sgn}\omega_m - \lambda M_n)^{-1} \times (i\omega_{m-s} + \mu + i\Gamma_{n'}\, \mathrm{sgn}\omega_{m-s} - \lambda' M_{n'})^{-1}. \quad \text{(A9)}$$

To evaluate this sum, we expand it in terms of partial fractions and, using the summation formula

$$\sum_{n=0}^{\infty}\left(\frac{1}{n+a} - \frac{1}{n+b}\right) = \psi(b) - \psi(a), \quad \text{(A10)}$$

we obtain

$$\begin{aligned} -T\mathcal{I} = &\frac{Z^{\lambda\lambda'}_{nn'}(i\Omega_s,\Gamma,\mu,T)}{\lambda M_n - \lambda' M_{n'} - i\Omega_s - i(\Gamma_n - \Gamma_{n'})} \\ &+ \frac{Z^{-\lambda',-\lambda}_{n'n}(i\Omega_s,\Gamma,-\mu,T)}{\lambda M_n - \lambda' M_{n'} - i\Omega_s - i(\Gamma_{n'} - \Gamma_n)} \\ &- \frac{Z^{\lambda\lambda}_{nn}(i\Omega_s,\Gamma,\mu,T) + Z^{\lambda'\lambda'}_{n'n'}(-i\Omega_s,-\Gamma,\mu,T)}{\lambda M_n - \lambda' M_{n'} - i\Omega_s - i(\Gamma_n + \Gamma_{n'})}, \end{aligned} \quad \text{(A11)}$$

where the functions $Z^{\lambda\lambda'}_{nn'}(\Omega,\Gamma,\mu,T)$ are defined in (8). Using the above equation and the relation

$$Z^{\lambda'\lambda'}_{n'n'}(-i\Omega_s,-\Gamma,\mu,T) = Z^{-\lambda',-\lambda'}_{n'n'}(i\Omega_s,\Gamma,-\mu,T), \quad \text{(A12)}$$

which follows from the formula

$$\psi(1-z) = \psi(z) + \pi\cot(\pi z), \quad \text{(A13)}$$

we can rewrite (A8) as

$$\begin{aligned} \Pi(i\Omega_s,\mathbf{q}) = &\frac{e^2 N_f}{4\pi l^2} \sum_{n,n'=0}^{n_c} \sum_{\lambda,\lambda'=\pm} Q^{\lambda\lambda'}_{nn'}(y,\Delta) \\ &\times \left[\frac{Z^{\lambda\lambda'}_{nn'}(i\Omega_s,\Gamma,\mu,T)}{\lambda M_n - \lambda' M_{n'} - i\Omega_s - i(\Gamma_n - \Gamma_{n'})}\right. \\ &+ \frac{Z^{-\lambda',-\lambda}_{n'n}(i\Omega_s,\Gamma,-\mu,T)}{\lambda M_n - \lambda' M_{n'} - i\Omega_s - i(\Gamma_{n'} - \Gamma_n)} \\ &\left.- \frac{Z^{\lambda\lambda}_{nn}(i\Omega_s,\Gamma,\mu,T) + Z^{-\lambda',-\lambda'}_{n'n'}(i\Omega_s,\Gamma,-\mu,T)}{\lambda M_n - \lambda' M_{n'} - i\Omega_s - i(\Gamma_n + \Gamma_{n'})}\right]. \end{aligned} \quad \text{(A14)}$$

Making the analytic continuation from Matsubara frequencies by replacing $i\Omega_s \to \Omega + i0$, we finally arrive at (7). At constant scattering rate $\Gamma_n = \Gamma$ the result simplifies to

$$\begin{aligned} \Pi(\Omega,\mathbf{q}) = &\frac{e^2 N_f}{4\pi l^2} \sum_{n,n'=0}^{\infty} \sum_{\lambda,\lambda'=\pm} Q^{\lambda\lambda'}_{nn'}(y,\Delta) \\ &\times \left[\frac{Z^{\lambda\lambda'}_{nn'}(\Omega,\Gamma,\mu,T) + Z^{-\lambda',-\lambda}_{n'n}(\Omega,\Gamma,-\mu,T)}{\lambda M_n - \lambda' M_{n'} - \Omega}\right. \\ &\left.- \frac{Z^{\lambda\lambda}_{nn}(\Omega,\Gamma,\mu,T) + Z^{-\lambda',-\lambda'}_{n'n'}(\Omega,\Gamma,-\mu,T)}{\lambda M_n - \lambda' M_{n'} - \Omega - 2i\Gamma}\right]. \end{aligned} \quad \text{(A15)}$$

One can check that the first term in square brackets does not have poles at $\Omega = \lambda M_n - \lambda' M_{n'}$ since the numerator vanishes at this point,

$$\begin{aligned} &Z^{\lambda\lambda'}_{nn'}(\Omega,\Gamma,\mu,T) + Z^{-\lambda',-\lambda}_{n'n}(\Omega,\Gamma,-\mu,T) \\ &= -\frac{\epsilon}{4\pi^2 T}\left[\psi'\left(\frac{1}{2} + \frac{\mu - \lambda' M_{n'} + i\Gamma}{2i\pi T}\right)\right. \\ &\left.+ \psi'\left(\frac{1}{2} - \frac{\mu - \lambda M_n - i\Gamma}{2i\pi T}\right)\right], \\ &\Omega = \lambda M_n - \lambda' M_{n'} + \epsilon, \quad \epsilon \to 0. \end{aligned} \quad \text{(A16)}$$

At $\Gamma \to 0$ the denominators in Eq. (A15) become equal, and the overall numerator reads

$$\begin{aligned} &Z_{nn'}^{\lambda\lambda'}(\Omega,0,\mu,T) + Z_{n'n}^{-\lambda',-\lambda}(\Omega,0,-\mu,T) \\ &\quad - Z_{nn}^{\lambda\lambda}(\Omega,0,\mu,T) - Z_{n'n'}^{-\lambda',-\lambda'}(\Omega,0,-\mu,T) \\ &= \frac{1}{2\pi i}\left\{-\left[\psi\left(\frac{1}{2}-\frac{\mu-\lambda M_n}{2i\pi T}\right)-\psi\left(\frac{1}{2}+\frac{\mu-\lambda M_n}{2i\pi T}\right)\right]\right. \\ &\quad \left.+\left[\psi\left(\frac{1}{2}-\frac{\mu-\lambda' M_{n'}}{2i\pi T}\right)-\psi\left(\frac{1}{2}+\frac{\mu-\lambda' M_{n'}}{2i\pi T}\right)\right]\right\} \\ &= n_F(\lambda' M_{n'}) - n_F(\lambda M_n), \end{aligned} \tag{A17}$$

where we used property (A13) of the digamma function.

APPENDIX B: SCHWINGER PROPER-TIME CALCULATION OF POLARIZATION FUNCTION IN A MAGNETIC FIELD

The general expression (7) for the polarization function as a double sum over the Landau levels is useful for high magnetic fields. Clearly, for weak fields, Eq. (7) is not convenient since we need to keep many terms in the double sum. In general, when Γ depends on the Landau index n it is impossible even to get a closed expression for the quasiparticle propagator, not to mention the polarization function itself. In principle, it is possible to perform the summation in Eq. (3) for $\Gamma =$ constant and $\mu \neq 0$, but the expression obtained looks rather cumbersome for further work with it. Therefore, we consider in this section only the case $\Gamma = \mu = 0$. Using the identity $1/a = \int_0^\infty dt\, e^{-at}$, $a > 0$ for introducing the proper-time coordinate t and the formula[33]

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = (1-z)^{-\alpha-1}\exp\left(\frac{xz}{z-1}\right), \quad |z|<1, \tag{B1}$$

we get a closed expression for the fermion propagator:

$$\begin{aligned} S(i\omega_m,\mathbf{r}) &= \frac{1}{4\pi i v_F^2}\int_0^\infty dt\, \exp\left[-t\frac{l^2(\omega_m^2+\Delta^2)}{v_F^2} - \frac{\mathbf{r}^2}{4l^2}\coth t\right] \\ &\quad \times\left\{(\gamma_0 i\omega_m + \Delta)\,[P_-(1+\coth t)\right. \\ &\quad \left. - P_+(1-\coth t)] - i\frac{v_F}{2l^2}\frac{\boldsymbol{\gamma}\cdot\mathbf{r}}{\sinh^2 t}\right\}. \end{aligned} \tag{B2}$$

The integrals can be evaluated through confluent hypergeometric functions,

$$\begin{aligned} I_1(a,b) &= \int_0^\infty dt\, e^{-at-b\coth t} = \frac{1}{2}e^{-b}\Gamma\left(\frac{a}{2}\right)\Psi\left(\frac{a}{2},0,2b\right), \\ I_2(a,b) &= \int_0^\infty dt\, e^{-at-b\coth t}\coth t = -\frac{dI_1(a,b)}{db}, \\ I_3(a,b) &= \int_0^\infty dt\, e^{-at-b\coth t}\coth^2 t = \frac{d^2 I_1(a,b)}{db^2}, \\ a &= \frac{l^2(\omega_m^2+\Delta^2)}{v_F^2}, \quad b = \frac{\mathbf{r}^2}{4l^2}. \end{aligned} \tag{B3}$$

Hence, we have

$$\begin{aligned} S(i\omega_m;\mathbf{r}) &= \frac{e^{-\mathbf{r}^2/4l^2}}{4\pi i v_F^2}\left\{(\gamma_0 i\omega_m+\Delta)\left[P_-\Gamma\left(\frac{a}{2}\right)\Psi\left(\frac{a}{2},1,\frac{\mathbf{r}^2}{2l^2}\right)\right.\right. \\ &\quad \left. + P_+\Gamma\left(1+\frac{a}{2}\right)\Psi\left(1+\frac{a}{2},1,\frac{\mathbf{r}^2}{2l^2}\right)\right] \\ &\quad \left. + i v_F\frac{\boldsymbol{\gamma}\cdot\mathbf{r}}{l^2}\Gamma\left(1+\frac{a}{2}\right)\Psi\left(1+\frac{a}{2},2,\frac{\mathbf{r}^2}{2l^2}\right)\right\}. \end{aligned} \tag{B4}$$

Using the integral representation (B2) for the propagator, we get from (6), taking the trace and performing a Gaussian integration over coordinates,

$$\begin{aligned} \Pi(i\Omega_s,\mathbf{q}) &= -\frac{e^2 T l^2 N_f}{\pi v_F^4}\sum_{m=-\infty}^{\infty}\int_0^\infty \frac{dt\,dx}{\coth t+\coth x}\exp\left[-t\frac{l^2(\omega_m^2+\Delta^2)}{v_F^2} - x\frac{l^2(\omega_m'^2+\Delta^2)}{v_F^2} - \frac{\mathbf{q}^2 l^2}{\coth t+\coth x}\right] \\ &\quad \times\left[(\Delta^2-\omega_m\omega_m')(1+\coth t\coth x) + \frac{v_F^2(\coth t+\coth x-\mathbf{q}^2 l^2)}{l^2\sinh^2(t+x)}\right], \\ \omega_m' &= \omega_m - \Omega_s. \end{aligned} \tag{B5}$$

Introducing new variables $t = z(1+v)/2$, $x = z(1-v)/2$, we obtain

$$\begin{aligned} \Pi(i\Omega_s,\mathbf{q}) &= -\frac{Te^2 N_f}{\pi l^2}\int_0^\infty du\int_{-1}^1\frac{dv}{2}\exp\left(-u\Delta^2 - \frac{\cosh z-\cosh zv}{2\sinh z}\mathbf{q}^2 l^2\right)\left[\frac{z}{\sinh^2 z}\left(1-\frac{\cosh z-\cosh zv}{2\sinh z}\mathbf{q}^2 l^2\right)\right. \\ &\quad \left. + u\coth z\left(\Delta^2+\frac{\Omega_s^2}{2}+\frac{\partial}{\partial u}-\frac{v}{u}\frac{\partial}{\partial v}\right)\right]R(u,v,\Omega_s), \end{aligned} \tag{B6}$$

where $u \equiv l^2 z/v_F^2$ and the sum

$$R(u,v,\Omega_m) = e^{-u(1-v^2)\Omega_s^2/4}\sum_{m=-\infty}^{\infty}\exp\left[-4\pi^2 T^2 u\left(m+\frac{1-s+sv}{2}\right)^2\right] \tag{B7}$$

can be written through the Jacobi elliptic function $\theta_3(v,q)$ or $\theta_4(v,q)$. For that we use the formula

$$\sum_{m=-\infty}^{\infty} q^{(m+c)^2} = q^{c^2}\theta_3\left(\frac{ic\ln q}{\pi},q\right) = e^{i\pi c^2\tau}\theta_3(c\tau|\tau) = (-i\tau)^{-1/2}\theta_3(c|-1/\tau), \quad q = e^{i\pi\tau}, \quad \mathrm{Im}\tau > 0, \tag{B8}$$

where for the third equality we used the Jacobi imaginary transformation, and the θ functions are defined as

$$\theta_3(v,q) \equiv \theta_3(v|\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2\pi n v), \tag{B9}$$

$$\theta_4(v,q) \equiv \theta_4(v|\tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2\pi n v). \tag{B10}$$

Hence the sum (B7) takes the form

$$R(u,v,\Omega_m) = \frac{e^{-u(1-v^2)\Omega_s^2/4}}{2T\sqrt{\pi u}}\theta_3\left[\frac{1}{2} - \frac{(1-v)\Omega_m}{4\pi T}, e^{-1/(4uT^2)}\right] = \frac{e^{-u(1-v^2)\Omega_s^2/4}}{2T\sqrt{\pi u}}\theta_4\left[\frac{(1+v)\Omega_m}{4\pi T}, e^{-1/(4uT^2)}\right]. \tag{B11}$$

Since

$$\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)R(u,v,\Omega_s) = \frac{e^{-u(1-v^2)\Omega_s^2/4}}{2T\sqrt{\pi u}}\left(-\frac{1}{2} - \frac{(1+v^2)u\Omega_m^2}{4} + u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right)\theta_4\left[\frac{(1+v)\Omega_m}{4\pi T}, e^{-1/(4uT^2)}\right], \tag{B12}$$

we write

$$\Pi(i\Omega_s,\mathbf{q}) = -\frac{e^2N_f}{2\pi^{3/2}l^2}\int_0^\infty\frac{du}{\sqrt{u}}\int_{-1}^1\frac{dv}{2}\exp\left[-u\left(\Delta^2 + \frac{(1-v^2)\Omega_s^2}{4}\right) - \frac{\cosh z - \cosh zv}{2\sinh z}\mathbf{q}^2l^2\right] \times\left\{\frac{z}{\sinh^2 z}\left[1 - \frac{\cosh z - \cosh zv}{2\sinh z}\mathbf{q}^2l^2\right] + u\coth z\left[\Delta^2 + \frac{(1-v^2)\Omega_s^2}{4} - \frac{1}{2u} + \frac{\partial}{\partial u} - \frac{v}{u}\frac{\partial}{\partial v}\right]\right\} \times\theta_4\left[\frac{(1+v)\Omega_m}{4T}, e^{-1/(4uT^2)}\right]. \tag{B13}$$

The above integral is divergent at $u=0$, reflecting the primitive divergence of the polarization function. Therefore, in order to get a finite result one should regularize the initial expression, for example, by subtracting the same expression with Δ replaced by $M\to\infty$ (the Pauli-Villars regularization), which means that we write

$$\Pi(i\Omega_s,\mathbf{q}) = \lim_{M\to\infty}\frac{-e^2N_f}{2\pi^{3/2}l^2}\int_0^\infty\frac{du}{\sqrt{u}}\int_{-1}^1\frac{dv}{2} \times\{\cdots - (\Delta^2\to M^2)\}. \tag{B14}$$

Carefully separating the part with M^2 and taking into account that

$$\lim_{M\to\infty}\int_0^\infty\frac{du}{\sqrt{u}}\left[\exp(-uM^2)\left(\frac{1}{2u} + M^2\right) - \frac{1}{2u}\right] = 0, \tag{B15}$$

we finally get the following expression for the polarization function at finite temperature in a magnetic field,

$$\Pi(i\Omega_s,\mathbf{q}) = -\frac{e^2N_f}{2\pi^{3/2}l^2}\int_0^\infty\frac{du}{\sqrt{u}}\int_{-1}^1\frac{dv}{2}\left\{\frac{\exp(-u\Delta^2)}{\sinh z}\left\{z\exp\left[-u\left(\frac{1-v^2}{4}\Omega_s^2 + \frac{\cosh z - \cosh zv}{2z\sinh z}\mathbf{q}^2v_F^2\right)\right]\right.\right. \times\left[\frac{1}{\sinh z}\left(1 - \frac{\cosh z - \cosh zv}{2\sinh z}\mathbf{q}^2l^2\right) + \cosh z\left(\frac{2\Delta^2l^2}{v_F^2} + \frac{\Omega_s^2l^2}{2v_F^2} + \frac{2}{\sinh 2z} + \mathbf{q}^2l^2\frac{\cosh z\cosh zv - 1}{2\sinh^2 z}\right)\right] \left.\left.\times\theta_4\left[\frac{(1+v)\Omega_s}{4\pi T}, e^{-1/(4uT^2)}\right] - \cosh z\,\theta_4\left[0, e^{-1/(4uT^2)}\right]\right\} - \frac{1}{z}\right\}, \tag{B16}$$

where we also performed integration by parts of terms with derivatives over u,v.

Now we consider several limiting cases of Eq. (B16) and compare them with expressions existing in the literature. Taking the limit $T\to 0$ is very easy since θ functions take the value unity. After some transformations the zero-temperature limit can be recast in the form

$$\Pi(i\Omega_s,\mathbf{q}) = \frac{e^2 N_f \mathbf{q}^2}{4\pi^{3/2}} \int_0^\infty \frac{du}{\sqrt{u}} \int_{-1}^{1} \frac{dv}{2} \frac{z\cosh zv - zv\coth z\sinh zv}{\sinh z} \exp\left[-u\left(\Delta^2 + \frac{1-v^2}{4}\Omega_s^2 + \frac{\cosh z - \cosh zv}{2z\sinh z}\mathbf{q}^2 v_F^2\right)\right], \tag{B17}$$

the result first obtained in Ref. 28.

On the other hand, taking the limit of zero field, $l \to \infty$, in Eq. (B16) we get

$$\Pi(i\Omega_s,\mathbf{q}) = -\frac{e^2 N_f}{2\pi^{3/2} v_F^2} \int_0^\infty \frac{du}{u^{3/2}} \int_{-1}^{1} \frac{dv}{2} \left\{ \exp\left[-u\left(\Delta^2 + \frac{1-v^2}{4}\left(\Omega_s^2 + \mathbf{q}^2 v_F^2\right)\right)\right]\left[2 + u\left(2\Delta^2 + \frac{\Omega_s^2 + v^2\mathbf{q}^2 v_F^2}{2}\right)\right] \right.$$
$$\left. \times\, \theta_4\left[\frac{(1+v)\Omega_s}{4\pi T}, e^{-1/(4uT^2)}\right] - \theta_4[0, e^{-1/(4uT^2)}]e^{-u\Delta^2} - 1 \right\}. \tag{B18}$$

The integration over u in (B18) can be performed explicitly using a series representation for theta functions; we get in terms of the integration variable $x = (1+v)/2$:

$$\Pi(i\Omega_s,\mathbf{q}) = -\frac{e^2 N_f}{2\pi v_F^2} \int_0^1 dx \left[\frac{\Omega_s^2 + \mathbf{q}^2 v_F^2 + 4\left[\Delta^2 - x(1-x)\mathbf{q}^2 v_F^2\right]}{4a(x)} \frac{\sinh[a(x)/T]}{D(x)} + 4T \log \frac{\cosh[\Delta/(2T)]}{2D(x)} \right], \tag{B19}$$

where

$$a(x) = \sqrt{\Delta^2 + x(1-x)\left(\Omega_s^2 + \mathbf{q}^2 v_F^2\right)},$$
$$D(x) = \cosh^2[a(x)/(2T)] - \sin^2(\pi s x).$$

This expression can be rewritten in a somewhat different form if we integrate the last term in square brackets by parts and then use the identity among the integrals,

$$4T\Omega_s \int_0^1 dx \ln[4D(x)]$$
$$= 2\Omega_s \int_0^1 dx \frac{a(x)\sinh[a(x)/T]}{D(x)}$$
$$+ \left(\Omega_s^2 + \mathbf{q}^2 v_F^2\right) \int_0^1 dx (1-2x) \frac{\sin(2\pi s x)}{D(x)}, \tag{B20}$$

which can be obtained following the method described in the Appendix A of Ref. 34. Finally, we have

$$\Pi(i\Omega_s,\mathbf{q}) = \frac{e^2 N_f}{2\pi} \frac{\mathbf{q}^2}{\Omega_s^2 + \mathbf{q}^2 v_F^2} \int_0^1 dx$$
$$\times \left[2T \log[4D(x)] - \frac{\Delta^2}{a(x)} \frac{\sinh[a(x)/T]}{D(x)}\right]. \tag{B21}$$

For $\Delta = 0$, Eq. (B21) is in agreement with Eq. (A20) [together with (A23) and (A26)] in Ref. 34 while, for $T = 0$, it reduces to the well-known expression for the vacuum polarization operator in QED3.[35]

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