



Classification of gapped symmetric phases in one-dimensional spin systems

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Quantum many-body systems divide into a variety of phases with very different physical properties. The questions of what kinds of phases exist and how to identify them seem hard, especially for strongly interacting systems. Here we make an attempt to answer these questions for gapped interacting quantum spin systems whose ground states are short-range correlated. Based on the local unitary equivalence relation between short-range-correlated states in the same phase, we classify possible quantum phases for one-dimensional (1D) matrix product states, which represent well the class of 1D gapped ground states. We find that in the absence of any symmetry all states are equivalent to trivial product states, which means that there is no topological order in 1D. However, if a certain symmetry is required, many phases exist with different symmetry-protected topological orders. The symmetric local unitary equivalence relation also allows us to obtain some simple results for quantum phases in higher dimensions when some symmetries are present.

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I. INTRODUCTION

For a long time, we believed that Landau symmetry-breaking theory^{1,2} describes all possible orders in materials, and all possible (continuous) phase transitions. However, in the last 20 years, it has become more and more clear that Landau symmetry-breaking theory does not describe all possible orders. For example, different fractional quantum Hall (FQH) states^{3,4} all have the same symmetry. Thus it is impossible to use symmetry breaking to characterize different FQH states.

If Landau symmetry-breaking theory is not enough, then what should we use to describe those new states of matter? It turns out that we need to develop a totally new theory, to describe the new types of orders—topological or quantum order^{5,6}—that appear in the FQH states and the spin liquid states.

In Ref. 7, a systematic understanding of a large class of topological orders in strongly correlated bosonic systems without symmetry has been developed based on string-net condensations. In Ref. 8, the string-net classification of topological orders was generalized, based on local unitary transformations. In Refs. 6,9 and 10, topological orders with symmetry are studied using projective symmetry group and tensor network renormalization. But so far we still do not have a complete classification of topological orders for interacting systems.

Recently, for noninteracting gapped fermion systems with certain symmetries, a complete classification of topological phases has been developed based on K theory.^{11,12} A generalization of the free-fermion result to interacting cases has been obtained for one-dimensional (1D) systems.¹³

For 1D bosonic systems, the authors of Ref. 14 studied quantized Berry phases for spin-rotation-symmetric systems. In Ref. 15 the entanglement spectrum and the symmetry properties of matrix product states were studied. Using those tools, the authors obtained some interesting results which are special cases of the situation considered here.

In this paper, we will apply the approach used in Refs. 8 and 16 to 1D strongly correlated systems. Ground states of gapped 1D systems can be well described by finite-dimensional matrix product states.¹⁷ We assume that matrix product states capture all possible gapped phases in 1D systems. We will combine the local unitary transformation with the symmetry properties of matrix product states, and try to obtain a complete classification of all gapped phases in 1D quantum spin systems with certain simple symmetries. We find the following:

(a) If there is no symmetry (including translation symmetry), all gapped 1D spin systems belong to the same phase.

(b) For 1D non-translation-invariant (NTI) spin systems with *only* an on-site symmetry described by a group G , all the phases of gapped systems that do not break the symmetry are classified by the equivalence classes in the second cohomology group $H^2(G, \mathbb{C})$ of the group G , provided that the physical states on each site form a linear representation of the group G .

[Note that the equivalence classes in $H^2(G, \mathbb{C})$ classify the types of projective representations of G over the field \mathbb{C} of complex numbers. Appendix C gives a brief introduction to projective representation and the second cohomology group.] In certain cases where G has infinitely many 1D representations, for example, when $G = U(1)$, further classifications according to different 1D representations exist. The relation between projective representations and the symmetry of matrix product states has been noted before.^{18,19}

But quantum states are defined only up to global change of phases; therefore the symmetry operations of group G only need to be represented by operators $u(g)$ that satisfy

$$u(g_1)u(g_2) = e^{i\theta(g_1, g_2)}u(g_1g_2) \quad (1)$$

for any group elements g_1 and g_2 . Such operators form a projective representation of group G . When we consider this general case, we find that the classification result remains the same as with linear representations, that is,

(c) for 1D non-translation-invariant spin systems with *only* an on-site symmetry described by a group G , all the phases of

gapped systems that do not break the symmetry are classified by the equivalence classes in the second cohomology group $H^2(G, \mathbb{C})$ of the group G , provided that the physical states on each site form a projective representation of the group G .

In certain cases where G has infinitely many 1D representations, further classifications according to different 1D representations exist.

Applying these general results to specific cases allows us to reach the following conclusions: First, result (a) means that there is no nontrivial topological order in 1D systems without any symmetry. Using result (b), we find that NTI integer-spin chains with only on-site $SO(3)$ spin rotation symmetry can have two and only two different phases that do not break the $SO(3)$ symmetry. Result (c) implies that NTI half-integer-spin chains with only on-site $SO(3)$ spin rotation symmetry (which is represented projectively) also have two and only two gapped phases that do not break the $SO(3)$ symmetry. We note that the cyclic \mathbb{Z}_n group has no nontrivial projective representations; thus a NTI spin chain with only on-site \mathbb{Z}_n symmetry can have one and only one gapped phase that does not break the \mathbb{Z}_n symmetry. The $U(1)$ symmetry group has no nontrivial projective representation either; however, due to the special structure of the group of 1D representations of $U(1)$, NTI spin chains with only on-site $U(1)$ symmetry can have three and only three gapped phases that do not break the $U(1)$ symmetry.

We also considered systems with translation invariance (TI) and correspondingly many results have been obtained.

(a) If there is no other symmetry, all gapped TI systems belong to the same phase. (This has been discussed as the generic case in Ref. 16.)

(b) For 1D spin systems with *only* translation symmetry and an on-site symmetry described by a group G , all the phases of gapped systems that do not break the two symmetries are labeled by the equivalence classes in the second cohomology group $H^2(G, \mathbb{C})$ of the group G and different 1D representations $\alpha(G)$ of G , provided that the physical states on each site form a linear representation of the group G .

As in the NTI case, we should consider projective representations of G at each site. However, we find the following:

(c) There is no translation-invariant gapped ground state symmetric under on-site symmetry of group G that is represented projectively on the state space at each site.

In particular, we can show that the $SO(3)$ spin-rotation-symmetric integer-spin chain has two different gapped TI phases:^{20–22} the spin-0 trivial phase and the Haldane phase,²³ while a translation-invariant $SO(3)$ -symmetric half-integer spin chain must either be gapless or have degeneracy in the ground space due to broken discrete symmetries.^{18,19,24,25} On the other hand, the $SU(2)$ symmetric spin chains where on-site degrees of freedom contain both integer-spin and half-integer-spin representations have only one gapped TI phase. The crossover between two $SO(3)$ symmetric phases when $SO(3)$ symmetry breaks down to $SU(2)$ symmetry has been noticed before.²⁶ We also show that the spin chain with only translation and parity symmetry (defined as exchange of sites together with an on-site Z_2 operation) has four different gapped TI phases.^{20–22}

For systems with time-reversal symmetry, we find that NTI time-reversal-symmetric systems belong to two phases while TI time-reversal-symmetric phases in integer-spin systems have two phases and those in half-integer-spin systems are either gapless or have degeneracy in the ground space.

The paper is organized as follows: Sec. II gives a detailed definition of gapped quantum phases and explains how that gives rise to an equivalence relation between gapped ground states within the same phase. Section III shows that short-range-correlated matrix product states represent faithfully 1D gapped ground states and hence will be our object of study. Section IV discusses the situation where no symmetry is required and we found no topological order in 1D. Section V gives a classification of phases for 1D systems with certain symmetries, for example on-site symmetry and time-reversal symmetry. It classifies phases in translational-invariant systems, and furthermore in systems where translational invariance is present together with other symmetries such as on-site symmetry, parity symmetry, and time-reversal symmetry. Section VI generalizes some simple 1D results to higher dimensions. Finally in Sec. VII we summarize our results and conclude this paper.

II. DEFINITION OF QUANTUM PHASES

To obtain the above-stated results, we need to first briefly discuss the definition of quantum phases. A more detailed discussion can be found in Ref. 8. A quantum phase describes an equivalence relation between quantum systems. The systems we consider exist on an n -dimensional lattice and interactions are local (with finite range). A gapped quantum phase is usually defined as a class of gapped Hamiltonians which can smoothly deform into each other without closing the gap and hence without any singularity in the local properties of the ground state. Such an equivalence relation between Hamiltonians can be reinterpreted as an equivalence relation between ground states, as discussed in Refs. 8 and 16: Two gapped ground states belong to the same phase if and only if they are related by a local unitary (LU) transformation. (It might seem insufficient to discuss equivalence between Hamiltonians completely in terms of equivalence between ground states, as a local unitary transformation mapping $|\phi_1\rangle$ to $|\phi_2\rangle$ might not map the corresponding Hamiltonian H_1 to H_2 , but instead to some H'_2 . However, as H_2 and H'_2 are both gapped and have the same ground states, their equivalence is obvious.) As LU transformations can change local entanglement structure but not the global one, states in the same phase have the same long-range entanglement (LRE) and hence the same topological order.^{5,27} States equivalent to product states have only short-range entanglement (SRE) and hence trivial topological order. All the states with short-range entanglement belong to the same phase while states with long-range entanglement can belong to different phases. These considerations lead to the phase diagram as illustrated in Fig. 1(a), for the class of systems without any symmetry requirement.

A LU transformation U can take the form of finite time evolution with a local Hamiltonian

$$U = \mathcal{T} \left[e^{-i \int_0^1 dg \tilde{H}(g)} \right] \quad (2)$$

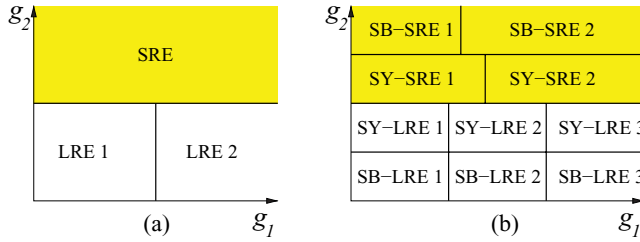


FIG. 1. (Color online) (a) The possible phases for the class of Hamiltonians $H(g_1, g_2)$ without any symmetry restriction. (b) The possible phases for the class of Hamiltonians $H_{\text{sym}}(g_1, g_2)$ with some symmetries. The shaded regions in (a) and (b) represent the phases with short-range entanglement.

where \mathcal{T} denotes a time-ordered integral and $\tilde{H}(g)$ is a sum of local Hermitian terms. Alternatively, U can take the form of a constant-depth quantum circuit,

$$U = \prod_{i_1} U_{i_1}^{(1)} \cdots \prod_{i_R} U_{i_R}^{(R)}, \quad (3)$$

which is composed of R layers of unitaries and the $U_{i_k}^{(k)}$'s within each layer k are local and commute with each other.⁸ These two forms of LU transformation are equivalent to each other and we will mainly take the quantum circuit form for the discussion in this paper (the time evolution form is used for discussion of translation-invariant systems). More generally, we will consider the equivalence relation between states defined on different Hilbert spaces and hence we allow a broader notion of unitarity [which is called a generalized LU transformation in Ref. 8 and corresponds to the disentangles and isometries in multiscale entanglement renormalization ansatz (MERA) (Ref. 28)]. Each LU operation $U_{i_k}^{(k)}$ we consider in the quantum circuit will act unitarily on the support space of the reduced density matrix of the region with $(U_{i_k}^{(k)})^\dagger U_{i_k}^{(k)} = I$ of the original Hilbert space and $U_{i_k}^{(k)} (U_{i_k}^{(k)})^\dagger = I$ of the support space of the reduced density matrix. While the total Hilbert space might change under such an operation, the entanglement structure of the state remains intact and hence the state remains in the same phase with the same topological order.

If the class of systems under consideration has further symmetry constraints, two Hamiltonians are in the same phase if they can be connected by a smooth path that stays within this symmetric region. Correspondingly, the equivalence relation between gapped ground states of the same phase needs to be modified:⁸ If the class of systems has a certain symmetry, two gapped ground states belong to the same phase if and only if they are related by a LU transformation which does not break the symmetry. Such a restricted equivalence relation leads to a phase diagram with more structure, as shown in Fig. 1(b). First, not all short-range-entangled states belong to the same phase. short-range-entangled states with different symmetry breaking will belong to different phases. Those symmetry-breaking phases with short-range entanglement (SB-SRE phases) are well described by Landau's symmetry-breaking theory.^{1,29}

Can all phases with short-range entanglement be described by symmetry breaking? Landau's symmetry-breaking theory suggests that states with the same symmetry always belong to

the same phase, which implies that all phases with short-range entanglement are described by symmetry breaking. However, this result turns out to be not quite correct. States that do not break any symmetry can still belong to different phases as well.⁶ We will refer to the order in symmetric short-range-entanglement (SY-SRE) states as *symmetry-protected topological order*.²¹ For example, in the presence of parity symmetry, the Haldane phase and the $S^z = 0$ phase of a spin-1 chain belong to two different phases even though both phases have short-range entanglement and do not break the parity symmetry. Also, in the presence of time-reversal symmetry, the topological insulators and the band insulators belong to two different phases. Again both phases have short-range entanglement.

For systems with long-range entanglement, the phase diagram similarly divides into symmetry-breaking (SB-LRE) and symmetric (SY-LRE) phases. The charge- $4e$ superconducting states³⁰ and the symmetric \mathbb{Z}_2 states^{6,31} are examples of the SB-LRE phases and the SY-LRE phases, respectively.

III. 1D GAPPED SPIN SYSTEMS AND MATRIX PRODUCT STATES

Having defined the universality classes of phases as the equivalence classes of states under (symmetric) local unitary transformations, we would then like to know which phases exist, or in other words, to classify all possible phases in strongly correlated systems. Some partial classifications have been discussed for strongly correlated systems through string-net states,⁷ and for free-fermion systems with certain symmetries through K theory.^{11,12} In this paper, we would like to consider 1D gapped strongly correlated spin systems both with and without symmetry, and try to classify all such systems whose ground states do not break any symmetry. (In other words, the ground state has the same symmetry as the Hamiltonian.)

Complete classification of strongly correlated spin systems seems to be a hard task as in general strongly interacting quantum many-body systems are very hard to solve. However, the recent insight about describing 1D gapped ground states of spin systems with matrix product state formalism^{32,33} provides us with a handle to deal with this problem. A matrix product state (MPS) is expressed as

$$|\phi\rangle = \sum_{i_1, i_2, \dots, i_N} \text{Tr} \left(A_{i_1}^{[1]} A_{i_2}^{[2]} \cdots A_{i_N}^{[N]} \right) |i_1 i_2 \cdots i_N\rangle, \quad (4)$$

where $i_k = 1, \dots, d$ with d being the physical dimension of a spin at each site, and the $A_{i_k}^{[k]}$'s are $D \times D$ matrices on site k with D being the inner dimension of the MPS. It has been shown that matrix product states capture the essential features of 1D gapped ground states, for example an entanglement area law³⁴ and finite correlation length,^{35,36} and provide an efficient description of such states.¹⁷ On the other hand, generic matrix product states satisfying a condition called "injectivity" are all gapped ground states of local Hamiltonians.^{32,33} Therefore, studying this class of MPSs will enable us to give a full classification of 1D gapped spin systems.

Now the question of what phases exist in 1D gapped spin systems can be restated as what equivalence classes of matrix product states exist under LU transformations. The authors of

Ref. 16 gave a specific way to apply such LU transformations, which realizes a renormalization group transformation on MPSs that removes local entanglement and takes the states to a simple fixed-point form. A partial classification of MPSs is also given in Ref. 16. In the following we will use this procedure to classify gapped phases of 1D spin systems, in particular 1D systems with various symmetries. We see that the possible phases in 1D strongly correlated systems depend on the symmetry of the class of systems.

First we will briefly review how the renormalization group (RG) transformation¹⁶ is done. For the identification and optimal removal of local entanglement, a particularly useful mathematical construction is the double tensor $\mathbb{E}_{\alpha\gamma,\beta\chi}^{[k]} = \sum_i A_{i,\alpha\beta}^{[k]} \times (A_{i,\gamma\chi}^{[k]})^*$ of the MPS. $\mathbb{E}^{[k]}$ uniquely determines the state up to a local change of basis on each site,^{33,37} that is, if

$$\mathbb{E}_{\alpha\gamma,\beta\chi}^{[k]} = \sum_i A_{i,\alpha\beta}^{[k]} \times (A_{i,\gamma\chi}^{[k]})^* = \sum_i B_{i,\alpha\beta}^{[k]} \times (B_{i,\gamma\chi}^{[k]})^* \quad (5)$$

then $A_{i,\alpha\beta}^{[k]}$ and $B_{i,\alpha\beta}^{[k]}$ are related by a unitary transformation $U^{[k]}$:

$$B_{i,\alpha\beta}^{[k]} = \sum_j U_{ij}^{[k]} A_{j,\alpha\beta}^{[k]}. \quad (6)$$

Therefore, states described by $A_i^{[k]}$ and $B_i^{[k]}$ have exactly the same entanglement structure, which is faithfully captured in $\mathbb{E}^{[k]}$. A proof of this fact can be found in Ref. 37. For clarity, we present the proof in Appendix A following the notation of this paper.

Local unitary operations on MPSs can be applied through manipulation of $\mathbb{E}^{[k]}$. Treat $\mathbb{E}^{[k]}$ as a $D^2 \times D^2$ matrix with row index $\alpha\gamma$ and column index $\beta\chi$. To apply a unitary operation on n consecutive sites, we combine the double tensor of the n sites together into

$$\tilde{\mathbb{E}} = \mathbb{E}^{[1]} \mathbb{E}^{[2]} \dots \mathbb{E}^{[n]} \quad (7)$$

and then decompose $\tilde{\mathbb{E}}$ into a set of matrices $\tilde{A}_{\tilde{i}}$:

$$\tilde{\mathbb{E}}_{\alpha\gamma,\beta\chi} = \sum_{\tilde{i}} \tilde{A}_{\tilde{i},\alpha\beta} \times \tilde{A}_{\tilde{i},\gamma\chi}^*. \quad (8)$$

Note that $\tilde{A}_{\tilde{i},\alpha\beta}$ is determined up to a unitary transformation on \tilde{i} . The index \tilde{i} of $\tilde{A}_{\tilde{i},\alpha\beta}$, up to a unitary transformation, can be viewed as the combination of i_1, i_2, \dots, i_n , the indices of $A_{i_1,\alpha\beta}^{[1]}, A_{i_2,\alpha\beta}^{[2]}, \dots, A_{i_n,\alpha\beta}^{[n]}$. Going from the original indices i_1, i_2, \dots, i_n to the effective index \tilde{i} corresponds to applying a unitary operation on the n block, and $\tilde{A}_{\tilde{i}}$ describes the new state after operation.

The unitary operation can be chosen so that local entanglement is maximally removed. $\tilde{\mathbb{E}}$ contains all the information about the entanglement of the block with the rest of the system but no details of entanglement structure within the block. Hence we can determine from $\tilde{\mathbb{E}}$ the optimal way of decomposition into \tilde{A} , which corresponds to the unitary operation that maximally removes local entanglement while preserving the global structure. To do so, think of $\tilde{\mathbb{E}}_{\alpha\gamma,\beta\chi}$ as a matrix with row index $\alpha\beta$ and column index $\gamma\chi$. It is easy to

see that with such a recombination, $\tilde{\mathbb{E}}$ is a positive matrix and can be diagonalized:

$$\tilde{\mathbb{E}}_{\alpha\gamma,\beta\chi} = \sum_{\tilde{i}} \lambda_{\tilde{i}} V_{\tilde{i},\alpha\beta} V_{\tilde{i},\gamma\delta}^*, \quad (9)$$

where we have kept only the nonzero eigenvalues $\lambda_{\tilde{i}} > 0$ and the corresponding eigenvectors $V_{\tilde{i},\alpha\beta}$. \tilde{A} is then given by

$$\tilde{A}_{\tilde{i},\alpha\beta} = \sqrt{\lambda_{\tilde{i}}} V_{\tilde{i},\alpha\beta}, \quad (10)$$

which are the matrices representing the new state. In retaining only the nonzero eigenvalues, we have reduced the physical dimensions within the block to only those necessary for describing the entanglement between this block and the rest of the system. Local entanglement within the block has been optimally removed.

Each renormalization step in the renormalization procedure hence works by grouping every n consecutive sites together and then applying the above transformation to map $A^{[1]}, A^{[2]}, \dots, A^{[n]}$ to \tilde{A} . So one renormalization step maps the original matrices $(A_{i_k}^{[k]})^{(0)}$ on each site to renormalized matrices $(A_{i_k}^{[k]})^{(1)}$ on each block. Repeating this procedure for a finite number of times corresponds to applying a finite depth quantum circuit to the original state. If the matrices reaches a simple fixed-point form $(A_{i_k}^{[k]})^{(\infty)}$ (up to local unitaries), we can determine from it the universal properties of the phase to which the original state belongs.

Such a renormalization procedure hence provides a way to classify matrix product states under LU transformations by studying the fixed point $(A_{i_k}^{[k]})^{(\infty)}$ to which a state flows. Two states are within the same phase if and only their corresponding fixed-point states can be transformed into each other by (symmetric) LU transformations. In the following we will apply this method to study short-range-correlated matrix product states which faithfully represent the class of 1D gapped ground states.

The short-range correlation is an extra constraint on the set of matrix product states that we will consider. Not all matrix product states describe gapped ground states of 1D spin systems. In particular, 1D gapped ground states all have finite correlation length³⁶ for equal-time correlators of any local operator, while matrix product states can be long-range correlated. The finite correlation length puts an extra constraint on MPSs that the eigenspectrum of \mathbb{E} should have a nondegenerate largest eigenvalue (set to be 1) (see Appendix B).^{32,33,38} Therefore, we will assume this property of \mathbb{E} in our following discussion and corresponding MPSs will be called short-range-correlated (SRC) MPSs.

This renormalization method is well suited for the study of systems without translational invariance, which we will discuss in detail. We will also make an attempt to study translational-invariant systems with this method. While the full translational symmetry is reduced to block translational symmetry in the RG process, by studying the resulting equivalence classes for different values of block size n , we expect to obtain a more complete classification of translational-invariant 1D systems. Indeed, the classification result is further confirmed by using a translational-invariant LU transformation in the time evolution form to study equivalence between TI systems.

In the following sections, we will present our analysis and results for different cases. First we will consider the situation where no specific symmetry is required for the system.

IV. NO TOPOLOGICAL ORDER IN 1D

When no symmetry is required for the class of system, we want to know what kind of long-range entanglement exists and hence classify topological orders in 1D gapped spin systems. We will show that *all gapped 1D spin systems belong to the same phase if there is no symmetry*.

In other words, there is no topological order in 1D. This is similar to the generic case discussed in Ref. 16.

To obtain such a result, we use the fact that gapped 1D spin states are described by short-range-correlated matrix product states. (A state is a gapped state if there exists a Hamiltonian H such that the state is the nondegenerate gapped ground state of H .) Then one can show that all SRC matrix product states can be mapped to product states with LU transformations and hence there is no topological order in 1D.

Consider a generic system without any symmetry (including translation symmetry) whose gapped ground state is described as a MPS with matrices $A_i^{[k]}$ that vary from site to site. Reference 33 gives a ‘‘canonical form’’ for the matrices so that the double tensor $\mathbb{E}_{\alpha\gamma,\beta\chi}^{[k]}$, when treated as a matrix with row index $\alpha\gamma$ and column index $\beta\chi$, has a left eigenvector $\Lambda_{\alpha\gamma}^{[k]} = \lambda_{\alpha}^{[k]} \delta_{\alpha\gamma}$ and corresponding right eigenvector $\tilde{\Lambda}_{\beta\chi}^{[k+1]} = \lambda_{\beta}^{[k+1]} \delta_{\beta\chi}$. Here the λ 's are positive numbers and $\sum_{\alpha} \lambda_{\alpha}^2 = 1$. $\delta_{\alpha\gamma} = 1$ when $\alpha = \gamma$ and $\delta_{\alpha\gamma} = 0$ otherwise. (The convention chosen here is different from that in Ref. 33, but equivalent up to an invertible transformation on the matrices $A_i^{[k]}$.) This eigenspace has the largest eigenvalue in $\mathbb{E}^{[k]}$,³⁹ and is usually set to be 1. Note that the right eigenvector on site k is the same as the left eigenvector on site $k + 1$ and has norm 1; therefore when the double tensors are multiplied together, this one-dimensional eigenspace will always be of eigenvalue 1.

There could be other eigenvectors of eigenvalue 1 in $\mathbb{E}^{[k]}$. However, this would lead to an infinite correlation length^{32,33,38} and hence is not possible in a 1D gapped state. Therefore, for short-range-correlated MPSs, $\mathbb{E}^{[k]}$ must have a nondegenerate largest eigenvalue 1. When the double tensors are multiplied together, the remaining block of $\mathbb{E}^{[k]}$ will decay exponentially with the number of sites. This consideration is essential for determining the fixed point of the renormalization procedure when applied to the MPS, as shown below.

Now we apply the renormalization procedure as discussed in the previous section to remove local entanglement from a general SRC MPS. Take block size n . The double tensors on the renormalized sites are given by $(\mathbb{E}^{[K]})^{(1)} = \prod_{k \in K} (\mathbb{E}^{[k]})^{(0)}$, where the k 's are the n sites in block K . (\mathbb{E} , again, is treated as a $D^2 \times D^2$ matrix with row index $\alpha\gamma$ and column index $\beta\chi$.) After the renormalization process is repeated a finite number of times, $(\mathbb{E}^{[k]})^{(R)}$ will be arbitrarily close to a fixed-point form $(\mathbb{E}^{[k]})^{(\infty)}$ with nondegenerate eigenvalue 1 and $(\mathbb{E}^{[k]})^{(\infty)}_{\alpha\gamma,\beta\chi} = \tilde{\Lambda}_{\alpha\gamma}^{[k]} \tilde{\Lambda}_{\beta\chi}^{[k+1]}$, where $\tilde{\Lambda}_{\alpha\gamma}^{[k]} = \tilde{\lambda}_{\alpha}^{[k]} \delta_{\alpha\gamma}$ and $\tilde{\Lambda}_{\beta\chi}^{[k+1]} = \tilde{\lambda}_{\beta}^{[k+1]} \delta_{\beta\chi}$.

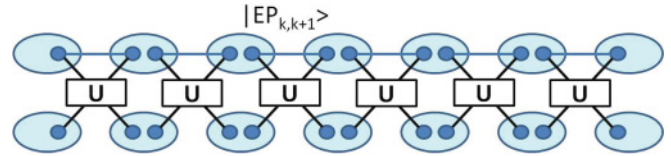


FIG. 2. (Color online) Disentangling a fixed-point state (upper layer, product of entangled pairs) into a direct product state (lower layer) with LU transformations.

Now we can decompose $(\mathbb{E}^{[k]})^{(\infty)}$ into matrices to find the fixed-point state. One set of matrices giving rise to this double tensor is given by

$$(A_{i^l i^r, \alpha\beta}^{[k]})^{(\infty)} = \sqrt{\tilde{\lambda}_{i^l}^{[k]}} \delta_{i^l \alpha} \cdot \sqrt{\tilde{\lambda}_{i^r}^{[k+1]}} \delta_{i^r \beta} \quad (11)$$

where $i^l, i^r = 1, \dots, D$. Here we use a pair of indices (i^l, i^r) to label the effective physical degrees of freedom on the renormalized site k , and $(A_{i^l i^r, \alpha\beta}^{[k]})^{(\infty)}$ is a set of matrices that defines the fixed-point MPS. It is clear from the form of the matrices that at a fixed point every site is composed of two virtual spins of dimension D . Every virtual spin is in an entangled pair with another virtual spin on the neighboring site $|EP_{k,k+1}\rangle = \sum_{i=1}^D \tilde{\lambda}_i^{[k+1]} |i, i\rangle$ and the full many-body state is a product of these pairs. An illustration of this state is given in Fig. 2 (upper layer). Obviously we can further disentangle these pairs by applying one layer of local unitary transformations between every pair of neighboring sites and map the state to a product state (Fig. 2, lower layer).

Therefore, through these steps we have shown that all SRC matrix product states can be mapped to product states with LU transformations and hence there is no topological order in a 1D NTI system.

V. SYMMETRY-PROTECTED TOPOLOGICAL ORDER IN 1D

If the class of systems under consideration has a certain symmetry, the equivalence classes of states are defined in terms of LU transformations that do not break the symmetry. Therefore, when applying the renormalization procedure, we should carefully keep track of the symmetry and make sure that the resulting state has the same symmetry at each step. Because of this constraint on local unitary equivalence, we will see that gapped ground states which do not break the symmetry of the system divide into different universality classes corresponding to different symmetry-protected topological orders. We will first discuss the case of on-site symmetries in detail for non-translational-invariant systems, i.e., the system has only an on-site symmetry and no translation symmetry. Then we shall make an attempt to study translational-invariant systems, with the possibility of having on-site symmetry or parity symmetry. Finally, we shall consider the case of time-reversal symmetry.

A. On-site symmetry

A large class of systems is invariant under on-site symmetry transformations. For example, the Ising model is symmetric under the Z_2 spin flip transformation and the Heisenberg model is symmetric under $SO(3)$ spin rotation transformations. In this section, we will consider the general case where the system

is symmetric under $u^{(0)}(g) \otimes \cdots \otimes u^{(0)}(g)$ with $u^{(0)}(g)$ being a unitary representation of a symmetry group G on each site. The representation can be linear or projective. That is, for any $g_1, g_2 \in G$,

$$u(g_1)u(g_2) = e^{i\theta(g_1, g_2)}u(g_1g_2), \quad (12)$$

where $\theta(g_1, g_2) = 0$ in a linear representation and $\theta(g_1, g_2)$ could take nontrivial values in a projective representation. A projective representation of a symmetry group is generally allowed in a quantum description of a system because the factor $e^{i\theta(g_1, g_2)}$ only changes the global phase of a quantum state but not any physically measurable quantity. Therefore, in our classification, we will consider not only the case of linear representation, but also projective representations in general.

The on-site symmetry is the only symmetry required for this class of system. In particular, we do not require translational symmetry for the systems. However, for a simple definition of phase, we will assume a certain uniformness in the state, which we will define explicitly in the following. We will classify possible phases for different G when the ground state is invariant (up to a total phase) under such on-site symmetry operations and is gapped (i.e., short-range correlated). Specifically, the ground state $|\phi_L\rangle$ on L sites satisfies

$$u^{(0)}(g) \otimes \cdots \otimes u^{(0)}(g)|\phi_L\rangle = \alpha_L(g)|\phi_L\rangle, \quad (13)$$

where $|\alpha_L(g)| = 1$ are g - and L -dependent phase factors.

1. On-site linear symmetry

First, let us consider the simpler case where the $u^{(0)}(g)$ form a linear representation of G . $\alpha_L(g)$ is then a one-dimensional linear representation of G . Now we will try to classify these symmetric ground states using symmetric LU transformations and we find the following: Consider 1D spin systems with *only* an on-site symmetry G which is realized linearly, all the gapped phases that do not break the symmetry are classified by $H^2(G, \mathbb{C})$, the second cohomology group of G , if $H^2(G, \mathbb{C})$ is finite and G has a finite number of 1D representations.

We will also discuss the case of the $U(1)$ group, which has an infinite number of 1D representations. We will again assume that all gapped states can be represented as short-range-correlated matrix product states. We will use the renormalization flow used before¹⁶ to simplify the matrix product states and use the fixed-point matrix product states to characterize different equivalent classes of LU transformations, as two symmetric states belong to the same class if and only if their corresponding fixed-point states can be mapped to each other with symmetric LU transformations.

In order to compare different equivalent classes under *symmetric* LU transformations, it is important to keep track of the symmetry while doing the renormalization. First, in the renormalization procedure we group n sites together into a new site. The on-site symmetry transformation becomes $\tilde{u}^{(0)}(g) = [\otimes u^{(0)}(g)]^n$, which is again a linear representation of G . The next step in RG transformation applies a unitary transformation $w_1^{[k]}$ to the support space of the new site k . This is actually itself composed of two steps. First we project onto the support space of the new site, which is the combination of n sites in the original chain. This is an allowed operation

compatible with symmetry G as the reduced density matrix ρ_n is invariant under $\tilde{u}^{(0)}(g)$, so the support space forms a linear representation of G . The projection of $\tilde{u}^{(0)}(g)$ onto the support space $P_n \tilde{u}^{(0)}(g) P_n$ hence remains a linear representation of G . In the next step, we do some unitary transformation $w_1^{[k]}$ within this support space which relabels different states in the space. The symmetry property of the state should not change under this relabeling. In order to keep track of the symmetry of the state, the symmetry operation needs to be redefined as $(u^{[k]})^{(1)}(g) = w_1^{[k]} P_n \tilde{u}^{(0)}(g) P_n (w_1^{[k]})^\dagger$. After this redefinition, the symmetry operations $(u^{[k]})^{(1)}(g)$ on each new site k form a new linear representation of G .

By redefining $(u^{[k]})^{(i)}(g)$ at each step of RG transformation, we keep track of the symmetry of the system. Finally at the fixed point (i.e., at a large RG step $i = R$), we obtain a state described by $(A_{i'ir}^{[k]})^{(R)}$ which is again given by the fixed-point form Eq. (11). To describe a state that does not break the on-site symmetry, here $(A_{i'ir}^{[k]})^{(R)}$ is invariant (up to a phase) under $(u^{[k]})^{(R)}(g)$ on each site k . Therefore,⁴⁰

$$\sum_{j'l} u_{i'ir}^{[k]}(g) A_{j'jr}^{[k]} = \alpha_{[k]}^{(R)}(g) N_{[k]}^{-1}(g) A_{i'ir}^{[k]} M_{[k]}(g), \quad (14)$$

$$N_{[k]}(g) = M_{[k-1]}(g)$$

must be satisfied with some invertible matrix $N_{[k]}(g)$ and $M_{[k]}(g)$. Here k labels the coarse-grained sites and we have dropped the RG step label R [except in $\alpha_{[k]}^{(R)}(g)$]. Each coarse-grained site is a combination of n^R original lattice sites and $\alpha_{[k]}^{(R)}(g)$ form a 1D (linear) representation of G .

Solving this equation we find the following results (see Appendix D):

(a) $N_{[k]}(g)$ and $M_{[k]}(g)$ are projective representations of G [see Eq. (7)]. Projective representations of G belong to different classes which form the second cohomology group $H^2(G, \mathbb{C})$ of G . (For a brief introduction on projective representation, see Appendix C). $M_{[k]}(g)$ and $N_{[k]}(g)$ correspond to the same element ω in $H^2(G, \mathbb{C})$.

(b) The linear symmetry operation $u^{[k]}(g)$ must be of the form $\alpha_{[k]}^{(R)}(g) u^{[k],l}(g) \otimes u^{[k],r}(g)$ where $u^{[k],l}$ and $u^{[k],r}$ are projective representations of G and correspond to inverse elements ω and $-\omega$ in $H^2(G, \mathbb{C})$, respectively. $\alpha_{[k]}^{(R)}(g)$ is a 1D (linear) representation of G . $u^{[k],l}$ and $u^{[k],r}$ act on the two virtual spins separately [see Eq. (D7)]. Therefore, the fixed-point state is formed by entangled pairs $|EP_{k,k+1}\rangle$ of virtual spins which are invariant, up to a phase [due to the nontrivial $\alpha_{[k]}^{(R)}(g)$], under the linear transformation $u^{[k],r}(g) \otimes u^{[k+1],l}(g)$.

Now we use the uniformness of the state and simplify our discussions. Specifically, we assume that $\alpha_{[k]}^{(R)}(g)$ does not depend on the site index k . Certainly, $\alpha_{[k]}^{(R)}(g)$ does not depend on k if the state has translation symmetry. If the 1D representations of G are discrete, then for weak randomness that slightly breaks the translation symmetry, $\alpha_{[k]}^{(R)}(g)$ still does not depend on k . So we can drop the k index and consider $\alpha^{(R)}(g)$.

Do different $\alpha^{(R)}(g)$ label different symmetric phases? First, the answer is no if the number of 1D representations of G is finite [as is the case for Z_n , $SO(3)$, etc.]. This is because we can always choose block size n properly so that

$\alpha^{(R)}(g) = 1$ and the difference between symmetric states due to $\alpha^{(R)}(g)$ disappears. In the case of the $U(1)$ group, there are infinitely many different 1D representations $e^{im\theta}$, labeled by integer m . For two states with positive m_1, m_2 , we can always choose the block size $m_2 n^R, m_1 n^R$, respectively, so that the 1D representations become the same. This is also true for negative m_1, m_2 . But if m_1, m_2 take different signs (or one of them is 0), the 1D representations will always be different no matter what blocking scheme we use. Therefore, due to their different 1D representations, $U(1)$ -symmetric states divide into three classes that can be labeled by $\{+, 0, -\}$. After these considerations, we will ignore the 1D representations $\alpha^{(R)}(g)$ in the following discussion.

We find that the entangled pairs $|EP_{k,k+1}\rangle$ of virtual spins in the fixed-point state are exactly invariant under the linear transformation $u^{[k],r}(g) \otimes u^{[k+1],l}(g)$. The left virtual spin of each site forms a projective representation of G corresponding to element ω in $H^2(G, \mathbb{C})$, while the right virtual spin corresponds to element $-\omega$. In Appendix E, we will show that fixed-point states with the same ω can be related by a symmetric LU transformation, while those with different ω cannot. Therefore, the phases of SRC MPSs that are invariant under linear on-site symmetry of group G are classified by the second cohomology group $H^2(G, \mathbb{C})$. [When $G = U(1)$, further divisions of classes due to different 1D representations of G exist. The equivalence classes are labeled by $\alpha \in \{+, 0, -\}$ and $\omega \in H^2(U(1), \mathbb{C})$.]

2. On-site projective symmetry

Due to the basic assumption of quantum mechanics that the global phase of a quantum state will not have any effect on the physical properties of the system, it is necessary to consider not only the linear representation of symmetry operations on the system, but also the projective representations. For example, on a half-integer spin, rotation by 2π is represented as $-I$, minus the identity operator instead of I . Hence, the rotation symmetry $SO(3)$ is represented projectively on half-integer spins. In order to cover situations like this, we discuss in this section systems with on-site projective symmetry of group G .

Again, we consider the case when the ground state does not break the symmetry, i.e., $u^{(0)}(g) \otimes \dots \otimes u^{(0)}(g)|\phi_L\rangle = \alpha_L(g)|\phi_L\rangle$, where $u^{(0)}(g)$ form a projective representation of group G corresponding to class ω . Assuming uniformness of the state, we require that ω does not vary from site to site.

But this can be reduced to the previous linear case. As long as $H^2(G, \mathbb{C})$ is finite and ω has a finite order n , we can take block size n so that after blocking, the symmetry operation on the renormalized site $\tilde{u}^{(0)}(g) = [\otimes u^{(0)}(g)]^n$ corresponds to $n\omega = \omega_0$ in $H^2(G, \mathbb{C})$. Therefore, the state after one blocking step is symmetric under an on-site linear representation of group G and all the reasoning in the previous section applies. We find that the classification with projective on-site symmetry is the same as for linear on-site symmetry. That is, considering 1D spin systems with *only* an on-site symmetry G which is realized projectively, all the gapped phases that do not break the symmetry are classified by $H^2(G, \mathbb{C})$, the second cohomology group of G , if $H^2(G, \mathbb{C})$ is finite and G has a finite number of 1D representations.

The $U(1)$ group does not have a nontrivial projective representation and will not introduce any complication here.

3. Examples

Since $G = \mathbb{Z}_n$ has no nontrivial projective representations, we find that all 1D gapped systems with only on-site \mathbb{Z}_n symmetry belong to the same phase.

For spin systems with only spin rotation symmetry, $G = SO(3)$. $SO(3)$ has two types of projective representation described by $H^2(SO(3), \mathbb{C}) = \{0, 1\}$, corresponding to integer- and half-integer-spin representations. We find that for integer-spin systems, all 1D gapped systems with only on-site $SO(3)$ spin rotation symmetry have two different phases.

Such a result has some relation to a well-known result⁴¹ for a NTI spin-1 Heisenberg chain,

$$H = \sum_i J_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}. \tag{15}$$

The model undergoes an impurity-driven second-order phase transition from the Haldane phase²³ to the random singlet phase^{42,43} as the randomness in J_i increases.

For half-integer-spin systems, $SO(3)$ is represented projectively on each site, yet the classification is the same as in the integer case. We find that for half-integer-spin systems, all 1D gapped states with only on-site $SO(3)$ spin rotation symmetry have two different phases. Representative states of the two phases are nearest-neighbor dimer states, but with the dimer between sites $2i$ and $2i + 1$ in the first phase and between sites $2i - 1$ and $2i$ in the second phase.

The projective representation of $SO(3)$ on half-integer spins forms a linear representation of $SU(2)$. If we think of the linear representation of $SO(3)$ on integer spins as a(n) (unfaithful) linear representation of $SU(2)$ and allow the mixture of integer and half-integer spins on one site, then the two phases of $SO(3)$ merge into one.²⁶ Therefore, systems with only on-site $SU(2)$ symmetry (which implies a mixture of integer and half-integer spins on each site) belong to one phase, as we can map integer-spin singlets into half-integer-spin singlets without breaking the $SU(2)$ symmetry (see Appendix E). Such a procedure breaks down if $SO(3)$ symmetry is required for each site as the direct sum of a linear representation (on integer spins) and a projective representation (on half-integer spins) is no longer a projective representation for $SO(3)$.

In this way, we have obtained a full classification of the phases of gapped NTI 1D spin systems with various on-site symmetries.

B. Translation invariance

We have discussed the gapped phases of 1D NTI systems that have some on-site symmetries. In this section, we would like to discuss translation-symmetric systems. We will consider those translation-symmetric systems whose ground states are gapped and translation invariant. (In general, a state which does not break the translational symmetry of the system is invariant under translation up to a phase. That is, the state carries a finite momentum. However, we will restrict ourselves only to the case where the ground state has zero momentum

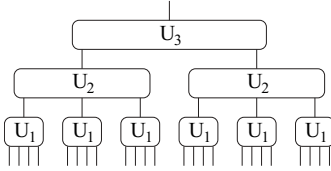


FIG. 3. An $\{n_i\}$ -block TI LU transformation described by a quantum circuit of three layers. Here $n_1 = 4$, $n_2 = 3$, $n_3 = 2$. The unitary transformations U_i on different blocks in the i th layer are the same.

and we will say the states are translationally invariant instead of translationally symmetric.)

1. $\{n_i\}$ -block TI LU transformations and b phases

To discuss the TI phases, we need to discuss the equivalence classes under TI LU transformations. However, it is hard to use quantum circuits to describe TI LU transformations. Thus, the quantum circuit formulation used in this paper is inconvenient to describe TI LU transformations. However, in this section, we will first try to use the quantum circuit formulation of LU transformations to discuss the phases of TI gapped states. While we are able to identify many different phases in this way, we cannot rigorously prove the equivalence of states within each phase using the non-translational-invariant circuit. In order to establish this equivalence, we later apply the other formulation of local unitary transformation—a finite time evolution with a local Hamiltonian [Eq. (2)] where TI can be preserved exactly, and we confirm the classification result obtained with quantum circuits (see Appendix G).

Let us consider the LU transformations represented by quantum circuits which are formed by the unitary operators on blocks of n_i sites in the i th layer (see Fig. 3). We will call such LU transformations $\{n_i\}$ -block LU transformations. If the unitary operators on different blocks in the same layer of the quantum circuit are the same, we will call the LU transformations $\{n_i\}$ -block TI LU transformations.

In this paper, we will try to use the quantum circuit formulation of the LU transformations to discuss gapped translation-symmetric phases. One way to do so is to use the equivalent classes of the $\{n_i\}$ -block TI LU transformations to classify the gapped translation-symmetric phases. However, since the $\{n_i\}$ -block TI LU transformations are different from the TI LU transformations, the equivalent classes of $\{n_i\}$ -block TI LU transformations are different from the equivalent classes of the TI LU transformations. But since the TI LU transformations are special cases of $\{n_i\}$ -block TI LU transformations, each equivalent class of $\{n_i\}$ -block TI LU transformations is formed by one or more equivalent classes of TI LU transformations.

Therefore, we can use the equivalent classes of $\{n_i\}$ -block TI LU transformations to describe the gapped TI phases, since different equivalent classes of $\{n_i\}$ -block TI LU transformations always represent different TI quantum phases. On the other hand, the equivalent classes of $\{n_i\}$ -block TI LU transformations may not separate all gapped TI phases. Sometimes, a single equivalent class of $\{n_i\}$ -block TI LU transformations may contain several different gapped TI phases.

To increase the resolution of the $\{n_i\}$ -block TI LU transformations, we would like to introduce the block-equivalent classes: two states belong to the same block-equivalent classes, if and only if, for all values of n_i , they can be mapped into each other through $\{n_i\}$ -block TI LU transformations.

Clearly, each block-equivalent class might still contain several different gapped TI phases. In this paper, we will call a block-equivalent class a block phase, or b phase. It is possible that the block-equivalent classes are the same as the universal classes that represent the gapped TI phases. In this case the gapped TI b phases are the same as the gapped TI phases, and we can study the gapped TI phases through block-equivalent classes. In the following, we will first study the b phases for 1D strongly correlated systems with translation and some other symmetries and then confirm that gapped TI b phases do coincide with gapped TI phases.

To describe TI gapped states we will use the TI MPS representation with site-independent matrices. But how do we know a TI gapped state can always be represented by a MPS with site-independent matrices? A nonuniform MPS in Eq. (4) can represent a TI state if the matrices $A_i^{[k]}$ satisfy, for example,

$$A_i^{[k+1]} = M^{-1} A_i^{[k]} M \quad (16)$$

for an invertible matrix M .

It is proven in Ref. 33 that every TI state does have a TI MPS representation. Specifically, in the example considered above, we can transform the matrices so that they become site independent. Let us introduce

$$\tilde{A}_i^{[k]} = N_{[k-1]}^{-1} A_i^{[k]} N_{[k]}. \quad (17)$$

By construction, $A_i^{[k]}$ and $\tilde{A}_i^{[k]}$ will represent the same MPS. We see that

$$\begin{aligned} \tilde{A}_i^{[k+1]} &= N_{[k]}^{-1} A_i^{[k+1]} N_{[k+1]} \\ &= N_{[k]}^{-1} M^{-1} A_i^{[k]} M N_{[k+1]} \\ &= N_{[k]}^{-1} M^{-1} N_{[k-1]} \tilde{A}_i^{[k]} N_{[k]}^{-1} M N_{[k+1]}. \end{aligned} \quad (18)$$

We see that if we choose

$$N_{[k]} = M^{-k}, \quad (19)$$

we will have

$$\tilde{A}_i^{[k+1]} = \tilde{A}_i^{[k]}, \quad (20)$$

hence reducing the original nonuniform representation to a uniform one.

We also wish to remark that by a ‘TI gapped state,’ we mean that there exists a TI gapped Hamiltonian H_L on a lattice of L sites such that the TI gapped state is the ground state of H_L . We would like to stress that we do not need the above condition to be true for all values of L . We only require H_L to be gapped for a sequence of lattice sizes $\{L_i\}$ such that $\lim_{i \rightarrow \infty} L_i = \infty$. In this paper, when we discuss a system of size L , we always assume that L belongs to such a sequence $\{L_i\}$. This consideration is important if we want to include in our discussion, for example, boson systems with $1/2$ particle per site, which can only be defined for even system size L .

2. Gapped TI b phases coincide with gapped TI phases

In the above discussion, we see that for a large block size, the related matrix [see (7)] $\tilde{\mathbb{E}} = \mathbb{E}^{[1]}\mathbb{E}^{[2]} \dots \mathbb{E}^{[n]}$ is dominated by its largest eigenvalue. For translation-invariant states, $\mathbb{E}^{[i]} = \mathbb{E}$ and $\tilde{\mathbb{E}} = \mathbb{E}^n$. So the fixed-point $\tilde{\mathbb{E}}$, and hence the gapped TI b phase, is directly determined by the largest eigenvalue and the corresponding left and right eigenvectors of \mathbb{E} .

Since a gapped TI b phase is directly determined by \mathbb{E} on each site, we do not need to do any blocking transformation to understand the gapped TI b phase. We can directly extract its fixed-point tensor from \mathbb{E} . This suggests that gapped TI b phases coincide with gapped TI phases. Indeed, we can directly deform \mathbb{E} to the fixed-point $\tilde{\mathbb{E}}$ by changing other non-largest eigenvalues to zero. Such a procedure allows us to deform the matrix $A_{i,\alpha\beta}$ of the initial MPS to the fixed-point matrix $\tilde{A}_{i,\alpha\beta}$ of the final MPS. Since the largest eigenvalue of \mathbb{E} is not degenerate, the state remains short range correlated (and gapped) during the deformation.^{32,33,38} (Recently, we learned that a similar method is used by Schuch *et al.*⁴⁴) Also the state does not break the translation symmetry (unlike the n -block transformations) and other symmetries during the deformation. This allows us to show that the gapped TI phases is characterized by the largest eigenvalue and the corresponding left and right eigenvectors of \mathbb{E} . Thus gapped TI b phases coincide with gapped TI phases. For details, see Appendix G.

In the following, we will use the $\{n_i\}$ -block TI LU transformations to discuss various gapped TI phases of 1D systems.

3. 1D systems with only translation symmetry

For 1D systems with only translation symmetry, there is only one gapped TI phase.

This result generalizes the earlier result that for 1D systems with no symmetry, there is only one gapped phase. To obtain the new result, we basically repeat what we did in Sec. IV. The only difference is that the matrices representing the state now are site independent, and in Sec. IV we use $\{n_i\}$ -block LU transformations to reduce the 1D NTI MPS, while to derive the new result, here we use $\{n_i\}$ -block TI LU transformations to reduce the 1D TI MPS.

4. 1D systems with translation and on-site unitary symmetries

Similarly, by repeating the discussions in Secs. VA 1 and VA 2 for $\{n_i\}$ -block TI LU transformations on the 1D TI MPS, we can show that, for a 1D spin system with translation and an on-site projective symmetry $u(g)$, the symmetric ground state cannot be short-range correlated, if the projective symmetry $u(g)$ corresponds to a nontrivial element in $H^2(G, \mathbb{C})$.

The reason is as follows. If a 1D state with translation symmetry is short-range correlated, it can be represented by a TI MPS. Its fixed-point MPS also has an on-site projective unitary symmetry $\tilde{u}(g)$. For a proper choice of block size n , we can make $u(g)$ and $\tilde{u}(g)$ be the same type of projective representation described by $\omega_{\text{sym}} \in H^2(G, \mathbb{C})$. For TI fixed-point MPSs, we have $\omega_{[k]} = \omega_{[k-1]}$ since $M_{[k]}(g) = M_{[k-1]}(g)$ (cf. Appendix D). Thus $\omega_{\text{sym}} = 0$, that is, the trivial element in $H^2(G, \mathbb{C})$. So, if $\omega_{\text{sym}} \neq 0$, the 1D TI state cannot be short-range correlated. In other words, 1D spin systems with

translation and an on-site projective symmetry are always gapless or have degenerate ground states that break the symmetries.

If the ground state of the 1D spin system does not break the on-site symmetry and the translation symmetry, then the ground state is not short-range correlated and is gapless. If the ground state of the 1D spin system breaks the on-site symmetry or the translation symmetry, then the ground state is degenerate. As an application of the above result, we find that 1D half-integer-spin systems with translation and $SO(3)$ spin rotation symmetry are always gapless or have degenerate ground states, which agrees with the well-known result of Ref. 24 and its generalizations.^{18,19}

To have a gapped TI 1D state with an on-site symmetry, the symmetry must act linearly (i.e., not projectively). In this case, we can show that the total phase factor of the state $\alpha_L(g)$ breaks up into L 1D representations $\alpha(g)$ (see Appendix F): For 1D spin systems of L sites with translation and on-site symmetry G , a gapped state that does not break the two symmetries must transform as

$$u^{(0)}(g) \otimes \dots \otimes u^{(0)}(g) |\phi_L\rangle = [\alpha(g)]^L |\phi_L\rangle \quad (21)$$

for all values of L that are large enough. Here $u^{(0)}(g)$ is the linear representation of G acting on the physical states in each site and $\alpha(g)$ is a one-dimensional linear representation of G .

Let us apply the above result to a boson system with p/q bosons per site. Here the boson number is conserved and there is a $U(1)$ symmetry. Certainly, the system is well defined only when the number of sites L has the form $L = Jq$ (assuming p and q have no common factors). For such L , we find that $\alpha_L(g) = \alpha_0(g)^J = \alpha_0(g)^{L/q}$, where $\alpha_0(g)$ is the generating 1D representation of the $U(1)$ symmetry group. So Eq. (21) is *not* satisfied for some large L . Therefore, a 1D state of conserved bosons with fractional bosons per site must be gapless, if the state does not break the $U(1)$ and the translation symmetry.

In higher dimensions, the situation is very different. A 2D state of conserved bosons with fractional bosons per site can be gapped, and, at the same time, does not break the $U(1)$ and the translation symmetry. 2D fractional quantum Hall states of bosons on a lattice provide examples of this kind of state.

Also, by repeating the discussions in Sec. VA 1 for $\{n_i\}$ -block TI LU transformations on the 1D TI MPS, we can show that for 1D spin systems with only translation and on-site linear symmetry G , all the phases of gapped states that do not break the two symmetries are classified by a pair (ω, α) where $\omega \in H^2(G, \mathbb{C})$ label different types of projective representations of G and α labels different 1D representations of G .

Here $\alpha(g)$ is a 1D representation of G that appears in Eq. (21). The symmetric LU transformations cannot change the 1D representation $\alpha(g)$. So the different phases are also distinguished by the 1D representations α of G .

Here are a few concrete examples. If we choose the symmetry group to be $G = \mathbb{Z}_n$, we find that, for 1D spin systems with only translation and on-site \mathbb{Z}_n symmetry, there are n phases for gapped states that do not break the two symmetries.

This is because \mathbb{Z}_n has no projective representations and has n different 1D representations. As an example, consider the model

$$H = \sum_i [-h\sigma_i^z - \sigma_{i-1}^x \sigma_i^y \sigma_{i+1}^z], \quad (22)$$

where $\sigma^{x,y,z}$ are the Pauli matrices. The model has a \mathbb{Z}_2 symmetry generated by σ^z . The two different \mathbb{Z}_2 -symmetric phases correspond to the $h \rightarrow \infty$ phase and the $h \rightarrow -\infty$ phase of the model.

If we choose the symmetry group to be $G = SO(3)$, we find that, for 1D integer-spin systems with only translation and $SO(3)$ spin rotation symmetry, there are two phases for gapped states that do not break the two symmetries. This is because $SO(3)$ has only one 1D representation and $H^2(SO(3), \mathbb{C}) = \mathbb{Z}_2$. Such a result agrees with the well-known result that the Affleck-Kennedy-Lieb-Tasaki (AKLT) state⁴⁵ of a spin-1 chain and the direct product state with spin 0 on each site represent two different $SO(3)$ -symmetric TI phases. The AKLT state (and the related Haldane phase²³) has gapless boundary spin-1/2 states⁴⁶⁻⁴⁸ and nontrivial string orders,^{49,50} which indicate that the AKLT state is really different from the spin-0 product state. Actually, the full symmetry of $SO(3)$ can be relaxed to only the dihedral group $D_2(\mathbb{Z}_2 \times \mathbb{Z}_2)$ by rotation by π around the x , y , and z axes. As explained in Appendix C, D_2 has one nontrivial projective representation, to which the AKLT state corresponds. The AKLT state is different from the spin-0 product state as long as on-site D_2 symmetry is preserved. This is consistent with the result in Refs. 22 and 50.

5. 1D systems with translation and parity symmetries

In this section, we will consider the case of parity symmetry for a translational-invariant system. We define the parity operation P for a spin chain to be in general composed of two parts: P_1 , exchange of sites n and $-n$; P_2 , on-site unitary operation $u^{(0)}$ where $(u^{(0)})^2 = I$. (The \mathbb{Z}_2 operation $u^{(0)}$ is necessary in the definition of parity if we want to consider, for example, a fixed-point state with $|EP\rangle = |00\rangle + |11\rangle$ to be parity symmetric. The state is not invariant after exchange of sites, and only maps back to itself if in addition the two virtual spins on each site are also exchanged.) As in the previous discussion, P gets redefined as we renormalize the state until at the fixed point P_1 becomes the exchange of renormalized sites and P_2 becomes $u^{(\infty)}$ on every site, $(u^{(\infty)})^2 = I$. The fixed-point matrices hence satisfy (the ∞ label is dropped)

$$\sum_{j^l j^r} u_{i^l i^r, j^l j^r} A_{j^l j^r}^T = \pm M^{-1} A_{i^l i^r} M \quad (23)$$

for some invertible matrix M , where we have used that the 1D representation of parity is either $(1, 1)$ or $(1, -1)$. We label the two 1D representations with $\alpha(P) = \pm 1$. Here M satisfies $M^{-1} M^T = e^{i\theta}$. But $M = (M^T)^T = e^{2i\theta} M$; therefore $e^{i\theta} = \pm 1$ and correspondingly M is either symmetric, $M = M^T$, or antisymmetric, $M = -M^T$. We will label this sign factor as $\beta(P) = \pm 1$.

Solution of this equation gives that $u = \alpha(P)v(u^l \otimes u^r)$, where v is the exchange operation of two virtual spins i^l and i^r and u^l, u^r act on i^l, i^r , respectively. $(u^l)^T = \beta(P)u^l$ and

$(u^r)^T = \beta(P)u^r$. It can then be shown that each entangled pair $|EP_{k,k+1}\rangle$ must be symmetric under parity operations and satisfies $u_k^r \otimes u_{k+1}^l |EP_{k,k+1}\rangle = \alpha(P)|EP_{k,k+1}\rangle$. There are hence four different symmetric phases corresponding to $\alpha(P) = \pm 1$ and $\beta(P) = \pm 1$. By enlarging the local Hilbert space, we can show as before that fixed points within each class can be mapped from one to the other with the $\{n_i\}$ -block TI LU transformation preserving the parity symmetry. On the other hand, fixed points in different classes cannot be connected without breaking the symmetries. Therefore, under the $\{n_i\}$ -block TI LU transformation, there are four block classes with parity symmetry and hence four parity-symmetric TI phases: For 1D spin systems with only translation and parity symmetry, there are four phases for gapped states that do not break the two symmetries.

As an example, consider the following model:

$$H = \sum_i [-BS_i^z + \mathbf{S}_i \cdot \mathbf{S}_{i+1}], \quad (24)$$

where \mathbf{S}_i are the spin-1 operators. The model has a parity symmetry. The $B = 0$ phase and the $B \rightarrow +\infty$ phase of the model correspond to two of the four phases discussed above. The $B = 0$ state²³ is in the same phase as the AKLT state. In the fixed-point state for such a phase, $|EP_{k,k+1}\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$. The parity transformation exchanges the first and the second spin, and induces a minus sign: $P : |EP_{k,k+1}\rangle \rightarrow -|EP_{k,k+1}\rangle$. The $B \rightarrow +\infty$ state is the $S^z = 1$ state. Its entangled pairs are $|EP_{k,k+1}\rangle = |\uparrow\uparrow\rangle$, which do not change sign under the parity transformation. Thus the stability of the Haldane or AKLT state is also protected by the parity symmetry.²⁰⁻²²

To understand why there are four parity-symmetric phases instead of two (parity even and parity odd), we give four representative states in Fig. 4, one for each phase. A connected pair of black dots denotes an entangled pair. + stands for a parity-even pair, for example, $|00\rangle + |11\rangle$, and - stands for a parity-odd pair, for example, $|01\rangle - |10\rangle$. Each rectangle corresponds to one site, with four virtual spins on each site. The four states are all translational invariant. If the parity operation is defined to be exchange of sites together with exchange of virtual spins 1 and 4, 2 and 3 on each site, then states (a) and (d) are parity even while (b) and (c) are parity odd. But (a) and (d) [or (b) and (c)] are different parity-even (-odd) states and cannot be mapped to each other through local unitary transformations without breaking parity symmetry.

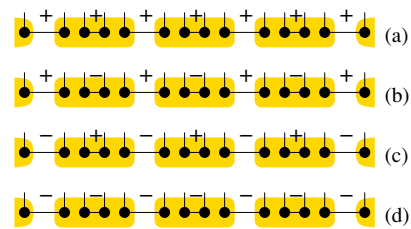


FIG. 4. (Color online) Representative states of the four parity-symmetric phases, corresponding to (a) $\alpha(P) = 1$, $\beta(P) = 1$, (b) $\alpha(P) = -1$, $\beta(P) = 1$ (c) $\alpha(P) = -1$, $\beta(P) = -1$, (d) $\alpha(P) = 1$, $\beta(P) = -1$. + stands for a parity-even entangled pair (e.g., $|00\rangle + |11\rangle$), and - stands for a parity-odd entangled pair (e.g., $|01\rangle - |10\rangle$). Each site contains four virtual spins.

Written in the matrix product representation, the matrices of the four states will transform with $\alpha(P) = \pm 1$ and $\beta(P) = \pm 1$, respectively. Therefore, the parity-even or parity-odd phase breaks into two smaller phases and there are in all four phases for parity-symmetric systems.

C. Time-reversal symmetry

Time reversal, unlike other symmetries, is represented by the antiunitary operator T , which is equivalent to the complex conjugate operator K followed by a unitary operator U . T has two projective representations: one on integer spins with $T^2 = I$ and the other on half-integer spins with $T^2 = -I$. The classification of gapped 1D time-reversal-invariant phases follows closely the cases discussed before. In this section, we will highlight the differences and give our conclusion.

First, a state $|\phi\rangle$ is called time-reversal invariant if $T \otimes T \cdots \otimes T|\phi\rangle = \beta|\phi\rangle$, where $|\beta| = 1$. But for antiunitary T , the global phase β is arbitrary and in particular we can redefine $|\phi'\rangle = \sqrt{|\beta|}|\phi\rangle$, such that $T \otimes T \cdots \otimes T|\phi'\rangle = |\phi'\rangle$. Therefore, in the following discussion, we will assume without loss of generality that $\beta = 1$.

Now let us consider a system without translational invariance. $T^2 = I$ or $-I$ does not make a difference here as we can take block size 2 so that on the renormalized site, T^2 is always equal to I . Using arguments similar to those in the case of on-site unitary symmetry, we can keep track of and redefine the symmetry operations as we do the renormalization. Finally, at the fixed point we have a state described by matrices $(A_{i'j'r}^{[k]})^{(\infty)}$ which is invariant under the time-reversal operation $(T^{[k]})^{(\infty)} = (u^{[k]})^{(\infty)}K$, that is,

$$\sum_{j'l'} u_{i'j'r}^{[k]} (A_{j'l'r}^{[k]})^* = N_{[k]}^{-1} A_{i'j'r}^{[k]} M_{[k]}, \quad (25)$$

$$N_{[k]} = M_{[k-1]},$$

where the fixed-point label ∞ has been omitted. Solving this equation, we find the following:

- (a) $M_{[k]}M_{[k]}^* = e^{i\theta}I$. As $M_{[k]}$ is invertible, $e^{i\theta} = \pm 1$.
- (b) $u^{[k]} = u^{[k],l} \otimes u^{[k],r}$, where $u^{[k],l}(u^{[k],l})^* = \pm I$ and $u^{[k],r}(u^{[k],r})^* = \pm I$. Therefore, each entangled pair is time-reversal invariant,

$$(u^{[k],r} \otimes u^{[k+1],l})K|EP_{k,k+1}\rangle = |EP_{k,k+1}\rangle. \quad (26)$$

As in previous sections, we can show that $uu^* = I$ and $uu^* = -I$ correspond to two equivalence classes and two time-reversal-invariant fixed-point states can be mapped into each other if and only if they belong to the same class. Therefore, our classification result for time-reversal symmetry is that, for 1D gapped spin systems with *only* time-reversal symmetry, there are two phases that do not break the symmetry.

If the system has additional translation symmetry, we can similarly classify the TI phases and find that, for 1D systems with only translation and time-reversal symmetry T , there are two gapped phases that do not break the two symmetries, if on each site the time-reversal transformation satisfies $T^2 = I$. 1D integer-spin systems are examples of this case. The Haldane or AKLT state and the spin-0 product state are representatives of the two phases respectively.^{15,21,22} We also have the following result: 1D systems with translation and

time-reversal symmetry are always gapless or have degenerate ground states, if on each site the time-reversal transformation satisfies $T^2 = -I$. 1D half-integer-spin systems are examples of this case.

VI. GENERALIZATION TO HIGHER DIMENSIONS

In the last few sections, we classified symmetry-protected topological orders in one dimension, using (symmetric) LU transformations. Can we use (symmetric) LU transformations to classify (symmetry-protected) topological orders in higher dimensions?

In higher dimensions, the situation is much more complicated. First, infinitely many kinds of nontrivial topological orders exist for class of systems without any symmetries.^{7,51} A partial classification is given in Ref. 8 for such a case in 2D. In the presence of symmetry, the phase diagram is even more complicated. 2D topological orders protected by time-reversal and point-group symmetry were studied in Ref. 52. So far, we do not have a detailed understanding of topological phases in the presence of symmetry.

However, using similar arguments as those used for 1D systems, we can obtain some simple partial results for higher dimensions. For example, we have that, for d -dimensional spin systems with only translation and an on-site symmetry G which is realized linearly, the object $(\alpha, \omega_1, \omega_2, \dots, \omega_d)$ labels distinct gapped quantum phases that do not break the two symmetries. Here α labels the different 1D representations of G and $\omega_i \in H^2(G, \mathbb{C})$ label the different types of projective representations of G .

Let us illustrate the above result in 2D by considering a tensor network state (TNS) on a square lattice, where the physical states existing on each vertex \mathbf{i} are labeled by $m_{\mathbf{i}}$. A translation-invariant TNS is defined by the following expression for the many-body wave function $\Psi(\{m_{\mathbf{i}}\})$:

$$\Psi(\{m_{\mathbf{i}}\}) = \sum_{ijkl\dots} A_{e_j i f}^{m_1} A_{j h k g}^{m_2} A_{l q r k}^{m_3} A_{t l s i}^{m_4} \cdots \quad (27)$$

Here $A_{ijkl}^{m_{\mathbf{i}}}$ is a complex tensor with one physical index $m_{\mathbf{i}}$ and four inner indices i, j, k, l . The physical index runs over the number of physical states d on each vertex, and the inner indices run over D values. The TNS can be represented graphically as in Fig. 5. If the tensor A satisfies

$$A_{lrud}^m = \alpha(g) \sum_{l'r'u'd'} u_{mm'}(g) [M^{-1}(g)]_{ll'} M_{r'r}(g) \times [N^{-1}(g)]_{dd'} N_{u'u}(g) A_{l'r'u'd'}^{m'}, \quad (28)$$

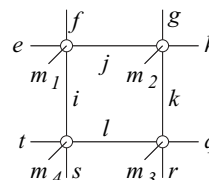


FIG. 5. Tensor network: A graphic representation of the tensor-product wave function (27) on a 2D square lattice. The indices on the links are summed over.

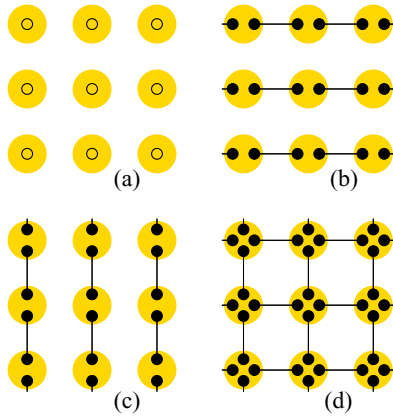


FIG. 6. (Color online) $(\omega_1 = 0, 1; \omega_2 = 0, 1)$ label four distinct states in integer-spin systems with translation and spin rotation symmetries: (a) $(\omega_1, \omega_2) = (0, 0)$, (b) $(\omega_1, \omega_2) = (0, 1)$, (c) $(\omega_1, \omega_2) = (1, 0)$, and (d) $(\omega_1, \omega_2) = (1, 1)$. The open dots represent 0 spins. The solid dots represent $1/2$ spins. Two solid dots connected by a line represent a spin singlet formed by two $1/2$ spins. The state in (a) has a spin 0 on each site. The state in (b) and (c) has two spins $1/2$ on each site [or four states per site that form the spin-(0+1) representation of $SO(3)$]. The state in (d) has four spins $1/2$ on each site [or 16 states per site that form the spin-(0 + 0 + 1 + 1 + 1 + 2) representation of $SO(3)$].

for some invertible matrices $M(g)$ and $N(g)$ then the many-body wave function $\Psi(\{m_i\})$ is symmetric under the on-site symmetry transformation g in the on-site symmetry group G . Here $\alpha(g)$ is a one-dimensional representation of G , the $D \times D$ matrices $M(g)$ form a projective representation represented by $\omega_1 \in H^2(G, \mathbb{C})$, and the $D \times D$ matrices $N(g)$ form a projective representation represented by $\omega_2 \in H^2(G, \mathbb{C})$. Since a symmetric LU transformation cannot change $(\alpha, \omega_1, \omega_2)$, $(\alpha, \omega_1, \omega_2)$ label distinct quantum phases.

In fact $(\alpha, \omega_1, \omega_2)$ are all measurable, so they indeed label distinct quantum phases. On a torus of size $L_x \times L_y$, the symmetric many-body wave function $\Psi(\{m_i\})$ transforms as

the 1D representation $\alpha^{L_x L_y}(g)$ under the on-site symmetry transformation g . If G has only a finite number of 1D representations, we can always choose L_x and L_y such that $\alpha^{L_x L_y}(g) = \alpha(g)$.

On a cylinder of size $L_x \times L_y$ with an open boundary in the x direction, the states on one boundary will form a projective representation which is represented by $L_y \omega_1 \in H^2(G, \mathbb{C})$. Similarly, if the open boundary is in the y direction, the states on one boundary will form a projective representation which is represented by $L_x \omega_2 \in H^2(G, \mathbb{C})$. If we choose L_x and L_y properly (for example, to make $L_x \omega_2 = \omega_2$ and $L_y \omega_1 = \omega_1$), we can detect both ω_1 and ω_2 .

A system with integer on-site spins gives us an example with $G = SO(3)$, if the system has translation and spin rotation symmetry. For $G = SO(3)$ there is no nontrivial 1D representation. So we can drop the consideration of α . Also $G = SO(3)$ has two types of projective representation: $H^2(SO(3), \mathbb{C}) = \{0, 1\}$, where $\omega = 0$ is the trivial projective representation, which corresponds to linear representations of $SO(3)$ (on integer spins), and $\omega = 1$ is the nontrivial projective representation, which corresponds to half-integer spins. Thus $(\omega_1 = 0, 1; \omega_2 = 0, 1)$ label four distinct states in 2D (see Fig. 6).

VII. CONCLUSION

Using the (symmetric) local unitary equivalence relation between gapped ground states in the same phase and the matrix product representation of 1D states, we classify possible quantum phases for strongly interacting 1D spin systems with a certain symmetry when the ground state of the system does not break the symmetry. Our results are summarized in Table I.

Many well-known results are rederived using a quite different approach, for example the existence of the Haldane phase²³ and the gaplessness of the spin- $1/2$ Heisenberg model.²⁴ Those results are also greatly generalized to other situations, for example to the cases of time-reversal symmetry

TABLE I. Summary of classification results for 1D gapped spin system with symmetric ground states. $H^2(G, \mathbb{C})$ is the second cohomology group of group G over complex number \mathbb{C} . $\alpha(G)$ is a 1D representation of G .

Symmetry	No. or label of different phases	Example system
None	1	
On-site linear symmetry of group G	$\omega \in H^2(G, \mathbb{C})^a$	On-site Z_n or $SU(2)$: One phase On-site $SO(3)$ or D_2 on integer spin: Two phases
On-site projective symmetry of group G	$\omega \in H^2(G, \mathbb{C})$	On-site $SO(3)$ or D_2 on half-integer spin: Two phases
Time reversal (TR)	2	
Translational invariance (TI)	1	
TI + on-site linear symmetry of group G	$\omega \in H^2(G, \mathbb{C})$ and $\alpha(G)$	TI + on-site Z_n : n phases TI + on-site $SO(3)$ on integer spin: Two phases
TI + on-site projective symmetry of group G	0	TI + on-site $SO(3)$ or D_2 on half-integer spin: No gapped phase
TI + parity	4	
TI + TR	2 if $T^2 = I$ 0 if $T^2 = -I$	TI + TR on integer spin: Two phases TI + TR on half-integer spin: No gapped phase

^aThis result applies when $\alpha(G)$ form a finite group; when $G = U(1)$, further classifications according to different $\alpha(U(1))$ exist.

and D_2 symmetry. We find that the projective representations play a very important role in understanding and formulating those generalized results.

In higher dimensions, things are more complicated. Nevertheless, similar considerations allow us to obtain some interesting examples of symmetry-protected topological orders. A complete classification of higher-dimensional phases, however, requires at least a full understanding of topological orders, an element that is absent in the 1D phase diagram.

Results similar to ours on the classification of integer and half-integer 1D spin chains with $SO(3)$ symmetry have been obtained independently by Kitaev recently.⁵³

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APPENDIX A: LOCAL UNITARY TRANSFORMATION ON MATRIX PRODUCT STATE AND INVARIANCE OF DOUBLE TENSOR

In this section we present the proof for the theorem that two sets of matrices $A_{i,\alpha\beta}$ and $B_{j,\alpha\beta}$, where i, j label different matrices and $\alpha\beta$ are the row and column indices of the matrices, give rise to the same double tensor $\mathbb{E}_{\alpha\gamma,\beta\chi} = \sum_i A_{i,\alpha\beta} \times A_{i,\gamma\chi}^* = \sum_j B_{j,\alpha\beta} \times B_{j,\gamma\chi}^*$, if and only if they are related by a unitary transformation $B_{j,\alpha\beta} = \sum_i U_{ji} A_{i,\alpha\beta}$. The dimension M of i and the dimension N of j can be different. Suppose that $M < N$. We can always append the list of matrices $A_{i,\alpha\beta}$ with zero matrices and make the dimension of i equal to N . U_{ji} is then a unitary operator defined on N -dimensional Hilbert space. The complete proof of this theorem can be found in Ref. 37, where this property is discussed in terms of ‘‘the unitary degree of freedom in the operator sum representation of quantum channels.’’ Here we re-present this proof following the notation and terminology of the current paper for simplicity of understanding.

First we prove the ‘‘if’’ part of the theorem. Suppose that $B_{i,\alpha\beta} = \sum_j U_{ij} A_{j,\alpha\beta}$; then

$$\begin{aligned} \mathbb{E}_{\alpha\gamma,\beta\chi}^B &= \sum_i B_{i,\alpha\beta} \times B_{i,\gamma\chi}^* \\ &= \sum_i \sum_{j_1} \sum_{j_2} U_{ij_1} A_{j_1,\alpha\beta} \times U_{ij_2}^* A_{j_2,\gamma\chi}^* \\ &= \sum_{j_1, j_2} \sum_i U_{ij_1} U_{ij_2}^\dagger A_{j_1,\alpha\beta} \times A_{j_2,\gamma\chi}^* \\ &= \sum_{j_1} A_{j_1,\alpha\beta} \times A_{j_1,\gamma\chi}^* = \mathbb{E}_{\alpha\gamma,\beta\chi}^A \end{aligned} \quad (\text{A1})$$

Therefore the two double tensors are the same. This proves the first part of the theorem.

On the other hand, suppose that the two double tensors are the same, $\mathbb{E}_{\alpha\gamma,\beta\chi}^A = \mathbb{E}_{\alpha\gamma,\beta\chi}^B = \mathbb{E}_{\alpha\gamma,\beta\chi}$. Reorder the indices of

\mathbb{E} and treat it as a matrix with row indices $\alpha\beta$ and column indices $\gamma\chi$. We will denote the double tensor after this reordering as $\hat{\mathbb{E}}_{\alpha\beta,\gamma\chi}$. It is easy to see that $\hat{\mathbb{E}}$ is a positive semidefinite matrix, as

$$\begin{aligned} &\sum_{\alpha\beta,\gamma\chi} v_{\alpha\beta}^* \hat{\mathbb{E}}_{\alpha\beta,\gamma\chi} v_{\gamma\chi} \\ &= \sum_i \left(\sum_{\alpha\beta} v_{\alpha\beta}^* B_{i,\alpha\beta} \right) \times \left(\sum_{\gamma\chi} B_{i,\gamma\chi}^* v_{\gamma\chi} \right) \geq 0 \end{aligned} \quad (\text{A2})$$

for any vector $v_{\alpha\beta}$.

Diagonalize $\hat{\mathbb{E}}$ into

$$\hat{\mathbb{E}}_{\alpha\beta,\gamma\chi} = \sum_k \lambda_k e_{k,\alpha\beta} \times e_{k,\gamma\chi}^* \quad (\text{A3})$$

with $\lambda_k \geq 0$. Define vectors $\tilde{e}_{k,\alpha\beta} = \sqrt{\lambda_k} e_{k,\alpha\beta}$, so that $\hat{\mathbb{E}}_{\alpha\beta,\gamma\chi} = \sum_k \tilde{e}_{k,\alpha\beta} \times \tilde{e}_{k,\gamma\chi}^*$. $\tilde{e}_{k,\alpha\beta}$ form a complete orthogonal set and hence we can expand $A_{i,\alpha\beta}$ and $B_{i,\alpha\beta}$ in terms of them:

$$\begin{aligned} A_{i,\alpha\beta} &= \sum_k P_{ik} \tilde{e}_{k,\alpha\beta}, \\ B_{i,\alpha\beta} &= \sum_k Q_{ik} \tilde{e}_{k,\alpha\beta}. \end{aligned} \quad (\text{A4})$$

Then

$$\begin{aligned} \hat{\mathbb{E}}_{\alpha\beta,\gamma\chi} &= \sum_i A_{i,\alpha\beta} \times A_{i,\gamma\chi}^* \\ &= \sum_{k_1 k_2} \sum_i P_{ik_1} P_{ik_2}^* \tilde{e}_{k_1,\alpha\beta} \times \tilde{e}_{k_2,\gamma\chi}^*. \end{aligned} \quad (\text{A5})$$

But we know that $\hat{\mathbb{E}}_{\alpha\beta,\gamma\chi} = \sum_k \tilde{e}_{k,\alpha\beta} \times \tilde{e}_{k,\gamma\chi}^*$. Therefore, $\sum_i P_{ik_1} P_{ik_2}^* = \delta_{k_1 k_2}$ and P is a unitary matrix.

Similarly we can show that Q is a unitary matrix. Therefore $A_{i,\alpha\beta}$ and $B_{j,\alpha\beta}$ are related by a unitary transformation U_{ij} where $U = P Q^\dagger$. We have thus proved both parts of the theorem.

APPENDIX B: DEGENERACY OF LARGEST EIGENVALUE OF DOUBLE TENSOR AND CORRELATION LENGTH OF MATRIX PRODUCT STATE

In Sec. III, we cited the property of matrix product states that the finite correlation length of the state is closely related to the nondegeneracy of the largest eigenvalue of the double tensor, as discussed in Refs. 32,33, and 38. In this section, we give a brief illustration of why this is so. For simplicity of notation, we focus on the translational-invariant case first. Generalization to matrix product states without translational invariance is straightforward and similar conclusions can be reached.

For a matrix product state $|\phi\rangle$ described by matrices $A_{i,\alpha\beta}$ with double tensor $\mathbb{E}_{\alpha\gamma,\beta\chi} = \sum_i A_{i,\alpha\beta} \times A_{i,\gamma\chi}^*$, define $\mathbb{E}[O]_{\alpha\gamma,\beta\chi} = \sum_{ij} O_{ij} A_{i,\alpha\beta} \times A_{j,\gamma\chi}^*$ for arbitrary operator O_{ij} . Follow the previous convention and treat \mathbb{E} and $\mathbb{E}[O]$ as matrices with row index $\alpha\gamma$ and column index $\beta\chi$. The norm of the wave function is given by $\langle\phi|\phi\rangle = \text{Tr}(\mathbb{E}^N)$, where N is the total length of the chain. Without loss of generality, we can set the largest eigenvalue of \mathbb{E} to be 1 and hence the

norm goes to a finite value (dimension of the eigenspace) as N goes to infinity. The expectation value of any local operator O is $\langle O \rangle = \text{Tr}(\mathbb{E}^{N-1} \mathbb{E}[O]) / \text{Tr}(\mathbb{E}^N)$ and the correlation between two operators O_1 and O_2 becomes

$$\begin{aligned} & \langle O_1 O_2 \rangle - \langle O_1 \rangle \langle O_2 \rangle \\ &= \text{Tr}(\mathbb{E}^{N-L-2} \mathbb{E}[O_1] \mathbb{E}^L \mathbb{E}[O_2]) / \text{Tr}(\mathbb{E}^N) \\ & \quad - \text{Tr}(\mathbb{E}^{N-1} \mathbb{E}[O_1]) \text{Tr}(\mathbb{E}^{N-1} \mathbb{E}[O_2]) / \text{Tr}^2(\mathbb{E}^N). \end{aligned} \quad (\text{B1})$$

The physical constraints on the expectation value and correlation functions of local operators require that the double tensor \mathbb{E} has certain properties. First, put \mathbb{E} into its Jordan normal form and decompose it as $\mathbb{E} = \sum_{\lambda} \lambda P_{\lambda} + R_{\lambda}$, where P_{λ} is the diagonal part and R_{λ} the nilpotent part. But for the largest eigenvalue 1, R_1 must be 0, as otherwise for large system size N $\langle O \rangle = \text{Tr}(\mathbb{E}^{N-1} \mathbb{E}[O]) / \text{Tr}(\mathbb{E}^N) \sim \text{Tr}\{(P_1 + R_1)^N \mathbb{E}[O]\} / \text{Tr}(\mathbb{E}^N)$ will be unbounded for any $\mathbb{E}[O]$ that satisfies $\text{Tr}(R_1 \mathbb{E}[O]) \neq 0$. The physical requirement that any local operator has bounded norm requires that R_1 must be 0.

Next we will show that the dimension of P_1 is closely related to the correlation length of the state. At large system size N , the correlator $\langle O_1 O_2 \rangle - \langle O_1 \rangle \langle O_2 \rangle = \text{Tr}\{P_1 \mathbb{E}[O_1] (\sum_{\lambda} \lambda P_{\lambda} + R_{\lambda})^L \mathbb{E}[O_2]\} / \text{Tr}(P_1) - \text{Tr}(P_1 \mathbb{E}[O_1]) \text{Tr}(P_1 \mathbb{E}[O_2]) / \text{Tr}^2(P_1)$. When L is large, we keep only the first-order term in $(\sum_{\lambda} \lambda P_{\lambda} + R_{\lambda})^L$ and the correlator goes to $\text{Tr}(P_1 \mathbb{E}[O_1] P_1 \mathbb{E}[O_2]) / \text{Tr}(P_1) - \text{Tr}(P_1 \mathbb{E}[O_1]) \text{Tr}(P_1 \mathbb{E}[O_2]) / \text{Tr}^2(P_1)$. If P_1 is one dimensional, the two terms both become $\langle v_1 | \mathbb{E}[O_1] | v_1 \rangle \langle v_1 | \mathbb{E}[O_2] | v_1 \rangle$ and cancel each other for any O_1, O_2 , and the second-order term in $(\sum_{\lambda} \lambda P_{\lambda} + R_{\lambda})^L$ dominates, which decays as λ^L . For $\lambda < 1$, the correlator goes to zero exponentially and the matrix product state has a finite correlation length. On the other hand, if P_1 is more than one dimensional, the first-order term has a finite contribution independent of L , $\sum_{i,j} \langle v_i | \mathbb{E}[O_1] | v_j \rangle \langle v_j | \mathbb{E}[O_2] | v_i \rangle / \text{Tr}(P_1) - \langle v_i | \mathbb{E}[O_1] | v_i \rangle \langle v_j | \mathbb{E}[O_2] | v_j \rangle / \text{Tr}^2(P_1)$, where v_i, v_j are eigenbases for P_1 . Therefore, degeneracy of the largest eigenvalue of the double tensor implies a nondecaying correlation. To describe quantum states with finite correlation length, the double tensor must have a largest eigenvalue that is nondegenerate.

If the system is not translational invariant and $\mathbb{E}^{[k]}$ vary from site to site, we cannot diagonalize all the double tensors at the same time. However, as shown in the ‘‘canonical form’’ of Ref. 33, there is a largest eigenspace of $\mathbb{E}^{[k]}$ (with eigenvalue 1) such that the right eigenvector on site k is the same as the left eigenvector on site $k + 1$. Therefore, when they are multiplied together this eigenspace will always have eigenvalue 1. There could be other eigenspaces with eigenvalue 1 and matching eigenvectors from site to site. However, then we can show, as in the TI case, that this leads to an infinite correlation length. On the other hand, other eigenspaces could have eigenvalues smaller than 1 or they have mismatched eigenvectors. If this is the case, all other eigenspaces decay exponentially with the number of sites multiplied together which gives rise to a finite correlation length. We will say that $\mathbb{E}^{[k]}$ has a nondegenerate largest eigenvalue 1 for this case in general.

APPENDIX C: PROJECTIVE REPRESENTATION

The operators $u(g)$ form a projective representation of symmetry group G if

$$u(g_1)u(g_2) = \omega(g_1, g_2)u(g_1 g_2), \quad g_1, g_2 \in G. \quad (\text{C1})$$

Here $\omega(g_1, g_2) \in \mathbb{C}$, the factor system of the projective representation, satisfies

$$\omega(g_2, g_3)\omega(g_1, g_2 g_3) = \omega(g_1, g_2)\omega(g_1 g_2, g_3), \quad (\text{C2})$$

for all $g_1, g_2, g_3 \in G$. If $\omega(g_1, g_2) = 1$, this reduces to the usual linear representation of G .

A different choice of prefactor for the representation matrices $u'(g) = \beta(g)u(g)$ will lead to a different factor system $\omega'(g_1, g_2)$:

$$\omega'(g_1, g_2) = \frac{\beta(g_1 g_2)}{\beta(g_1)\beta(g_2)}\omega(g_1, g_2). \quad (\text{C3})$$

We regard $u'(g)$ and $u(g)$, which differ only by a prefactor, as equivalent projective representations and the corresponding factor systems $\omega'(g_1, g_2)$ and $\omega(g_1, g_2)$ as belonging to the same class ω .

Suppose that we have one projective representation $u_1(g)$ with factor system $\omega_1(g_1, g_2)$ of class ω_1 and another $u_2(g)$ with factor system $\omega_2(g_1, g_2)$ of class ω_2 ; obviously $u_1(g) \otimes u_2(g)$ is a projective presentation with factor group $\omega_1(g_1, g_2)\omega_2(g_1, g_2)$. The corresponding class ω can be written as a sum $\omega_1 + \omega_2$. Under such an addition rule, the equivalence classes of factor systems form an Abelian group, which is called the second cohomology group of G and denoted as $H^2(G, \mathbb{C})$. The identity element ω_0 of the group is the class that contains the linear representation of the group.

Here are some simple examples:

(a) Cyclic groups Z_n do not have a nontrivial projective representation. Hence for $G = Z_n$, $H^2(G, \mathbb{C})$ contains only the identity element.

(b) A simple group with nontrivial projective representation is the Abelian dihedral group $D_2 = Z_2 \times Z_2$. For the four elements of the group $(0/1, 0/1)$, consider representation with Pauli matrices $g(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $g(0, 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $g(1, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $g(1, 1) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. It can be checked that this gives a nontrivial projective representation of D_2 .

(c) When $G = SO(3)$, $H^2(G, \mathbb{C}) = Z_2$. The two elements correspond to integer and half-integer representations of $SO(3)$, respectively.

(d) When $G = U(1)$, $H^2(G, \mathbb{C})$ is trivial: $H^2(U(1), \mathbb{C}) = Z_1$. We note that $\{e^{im\theta}\}$ form a representation of $U(1) = \{e^{i\theta}\}$ when m is an integer, but $\{e^{im\theta}\}$ will form a projective representation of $U(1)$ when m is not an integer. But under the equivalence relation (C3), $\{e^{im\theta}\}$ correspond to the trivial projective representation, if we choose $\beta(g) = e^{-im\theta}$. Note that $\beta(g)$ can be a discontinuous function over the group manifold.

APPENDIX D: SOLVING THE SYMMETRY CONDITION FOR A FIXED POINT

In this Appendix, we explicitly solve the symmetry condition Eq. (14). The goals are (1) to classify possible symmetry

operations at a fixed point and (2) to find the corresponding symmetric fixed-point state. For simplicity, we drop the site index $[k]$ and rewrite Eq. (14) as

$$\sum_{j^l j^r} \frac{u_{i^l i^r, j^l j^r}(g)}{\alpha^{(R)}(g)} A_{j^l j^r} = N^{-1}(g) A_{i^l i^r} M(g), \quad (\text{D1})$$

where $u(g)$ is a projective or linear unitary representation of G , the matrix $A_{i^l i^r}$ is given by its matrix elements $A_{i^l i^r, \alpha\beta} = \sqrt{\lambda_{i^l}^r} \delta_{i^l \alpha} \cdot \sqrt{\lambda_{i^r}^l} \delta_{i^r \beta}$ with $i^l, \alpha = 1, \dots, D_l$, $i^r, \beta = 1, \dots, D_r$, and $M(g), N(g)$ are sets of invertible matrices labeled by g . Since $\frac{u(g)}{\alpha^{(R)}(g)}$ is also a projective or linear unitary representation of G , we can absorb $\alpha^{(R)}(g)$ into $u(g)$ and rewrite Eq. (D1) as

$$\sum_{j^l j^r} u_{i^l i^r, j^l j^r}(g) A_{j^l j^r} = N^{-1}(g) A_{i^l i^r} M(g). \quad (\text{D2})$$

We note that matrix elements $A_{i^l i^r, \alpha\beta}$ are nonzero only when $\alpha = i^l$, $\beta = i^r$, and the full set of $\{A_{i^l i^r}\}$ form a complete basis in the space of $(D_l \times D_r)$ -dimensional matrices.

$M(g), N(g)$ do not necessarily form a representation of G . But the fixed-point form of the matrices requires that $M(g), N(g)$ be a projective representation, as on the one hand

$$\begin{aligned} & \sum_{j^l j^r} u_{i^l i^r, j^l j^r}(g_1 g_2) A_{j^l j^r} \\ &= \sum_{j^l j^r k^l k^r} \omega_{\text{sym}}(g_1, g_2) u_{i^l i^r, k^l k^r}(g_1) u_{k^l k^r, j^l j^r}(g_2) A_{j^l j^r} \\ &= \sum_{k^l k^r} \omega_{\text{sym}}(g_1, g_2) u_{i^l i^r, k^l k^r}(g_1) N^{-1}(g_2) A_{k^l k^r} M(g_2) \\ &= \omega_{\text{sym}}(g_1, g_2) N^{-1}(g_2) N^{-1}(g_1) A_{i^l i^r} M(g_1) M(g_2), \end{aligned} \quad (\text{D3})$$

and on the other hand

$$\sum_{j^l j^r} u_{i^l i^r, j^l j^r}(g_1 g_2) A_{j^l j^r} = N^{-1}(g_1 g_2) A_{i^l i^r} M(g_1 g_2). \quad (\text{D4})$$

Therefore

$$\begin{aligned} & \omega_{\text{sym}}(g_1, g_2) N^{-1}(g_2) N^{-1}(g_1) A_{i^l i^r} M(g_1) M(g_2) \\ &= N^{-1}(g_1 g_2) A_{i^l i^r} M(g_1 g_2) \end{aligned} \quad (\text{D5})$$

for all $i^l i^r$. However, the set of matrices $\{A_{i^l i^r}\}$ form a complete basis in the space of $(D_l \times D_r)$ -dimensional matrices. Therefore,

$$\begin{aligned} & \omega_{\text{sym}}(g_1, g_2) N^{-1}(g_2) N^{-1}(g_1) \otimes M(g_1) M(g_2) \\ &= N^{-1}(g_1 g_2) \otimes M(g_1 g_2), \end{aligned} \quad (\text{D6})$$

$N(g)$ and $M(g)$ form two projective representations

$$\begin{aligned} N(gh) &= \omega_N(g, h) N(g) N(h), \\ M(gh) &= \omega_M(g, h) M(g) M(h), \end{aligned} \quad (\text{D7})$$

with $|\omega_N(g_1, g_2)| = |\omega_M(g_1, g_2)| = 1$, and

$$\omega_{\text{sym}}(g_1, g_2) = \frac{\omega_M(g_1, g_2)}{\omega_N(g_1, g_2)}. \quad (\text{D8})$$

Let us rewrite Eq. (D2) as

$$N(g) \left(\sum_{j^l j^r} u_{i^l i^r, j^l j^r}(g) A_{j^l j^r} \right) M^{-1}(g) = A_{i^l i^r}. \quad (\text{D9})$$

We note that

$$N(g) \left(\sum_{j^l j^r} (\tilde{N}^{-1})_{j^l, i^l} \tilde{M}_{i^r, j^r} A_{j^l j^r} \right) M^{-1}(g) = A_{i^l i^r}, \quad (\text{D10})$$

where the matrices \tilde{M} and \tilde{N} are given by

$$\tilde{M}_{\alpha\beta} = M_{\alpha\beta} \frac{\sqrt{\lambda_{\alpha}^r}}{\sqrt{\lambda_{\beta}^l}}, \quad \tilde{N}_{\alpha\beta} = N_{\alpha\beta} \frac{\sqrt{\lambda_{\beta}^l}}{\sqrt{\lambda_{\alpha}^r}}. \quad (\text{D11})$$

Since the set of matrices $\{A_{i^l i^r}\}$ forms a complete basis in the space of $(D_l \times D_r)$ -dimensional matrices, we find

$$u_{i^l i^r, j^l j^r}(g) = (\tilde{N}^{-1})_{j^l, i^l}(g) \tilde{M}_{i^r, j^r}(g). \quad (\text{D12})$$

Putting back the factor of $\alpha^{(R)}(g)$, we find that

$$u_{i^l i^r, j^l j^r}(g) = \alpha^{(R)}(g) (\tilde{N}^{-1})_{j^l, i^l}(g) \tilde{M}_{i^r, j^r}(g). \quad (\text{D13})$$

APPENDIX E: EQUIVALENCE BETWEEN SYMMETRIC FIXED-POINT STATES

From the solution in Appendix D, we know that the fixed-point state symmetric under linear on-site symmetry of group G takes the form

$$|\phi\rangle^{(\infty)} = |EP_{1,2}\rangle |EP_{2,3}\rangle \cdots |EP_{k,k+1}\rangle \cdots, \quad (\text{E1})$$

where $|EP_{k,k+1}\rangle$ is an entangled pair between the right virtual qubit on site k and the left virtual qubit on site $k+1$ (see Fig. 2, upper layer). Each entangled pair is invariant under a linear symmetry transformation of the form $u^{[k],r}(g) \otimes u^{[k+1],l}(g)$,

$$u^{[k],r}(g) \otimes u^{[k+1],l}(g) |EP_{k,k+1}\rangle = |EP_{k,k+1}\rangle. \quad (\text{E2})$$

But $u^{[k],r}(g)$ or $u^{[k+1],l}(g)$ alone might not form a linear representation of G . They could in general be projective representations of G . If $u^{[k],r}(g)$ is a projective representation corresponding to class ω in $H^2(G, \mathbb{C})$, then $u^{[k+1],l}$ must correspond to class $-\omega$. ω does not vary from site to site and labels a particular symmetric fixed-point state.

Now we will show that symmetric fixed-point states with the same ω can be connected through symmetric LU transformations and hence belong to the same phase while those with different ω cannot and belong to different phases. First, suppose that two symmetric fixed-point states $|\phi_1\rangle$ and $|\phi_2\rangle$ are related to the same ω , i.e.,

$$\begin{aligned} u_1^{[k],r}(g) \otimes u_1^{[k+1],l}(g) |EP_{k,k+1}\rangle_1 &= |EP_{k,k+1}\rangle_1, \\ u_2^{[k],r}(g) \otimes u_2^{[k+1],l}(g) |EP_{k,k+1}\rangle_2 &= |EP_{k,k+1}\rangle_2, \end{aligned} \quad (\text{E3})$$

where $|EP_{k,k+1}\rangle_{1(2)}$ is an entangled pair of virtual spins on the Hilbert space $\mathcal{H}_{1(2)}^{[k],r} \otimes \mathcal{H}_{1(2)}^{[k+1],l}$. $u_{1(2)}^{[k],r}(g)$ is a projective representation of G corresponding to ω on $\mathcal{H}_{1(2)}^{[k],r}$ and $u_{1(2)}^{[k+1],l}(g)$ a projective representation corresponding to $-\omega$ on $\mathcal{H}_{1(2)}^{[k+1],l}$.

We can think of $|EP_{k,k+1}\rangle_1$ and $|EP_{k,k+1}\rangle_2$ as existing together in a joint Hilbert space $(\mathcal{H}_1^{[k],r} \oplus \mathcal{H}_2^{[k],r}) \otimes (\mathcal{H}_1^{[k+1],l} \oplus \mathcal{H}_2^{[k+1],l})$. The symmetry representation on this joint Hilbert

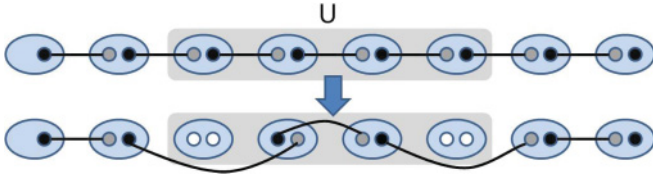


FIG. 7. (Color online) Fixed-point state related to projective representation of class ω before (upper) and after (lower) a local unitary operation on the shaded region that does not break the symmetry. White dots correspond to ω_0 , the identity element in $H^2(G, \mathbb{C})$, black dots correspond to ω , and gray dots correspond to $-\omega$.

space can be defined as

$$\begin{aligned} & u^{[k],r}(g) \otimes u^{[k+1],l}(g) \\ &= [u_1^{[k],r}(g) \oplus u_2^{[k],r}(g)] \otimes [u_1^{[k+1],l}(g) \oplus u_2^{[k+1],l}(g)]. \end{aligned} \quad (\text{E4})$$

As $u_1^{[k],r}(g)$ and $u_2^{[k],r}(g)$ [also $u_1^{[k+1],l}(g)$ and $u_2^{[k+1],l}(g)$] both correspond to ω ($-\omega$), their direct sum $u^{[k],r}(g)[u^{[k+1],l}(g)]$ is also a projective representation corresponding to ω ($-\omega$). Therefore, we have a linear representation of G on each site k , $u^{[k],l}(g) \otimes u^{[k],r}(g)$, and both $|EP_{k,k+1}\rangle_1$ and $|EP_{k,k+1}\rangle_2$ are symmetric under $u^{[k],r}(g) \otimes u^{[k+1],l}(g)$.

Now we can perform a LU transformation on the joint Hilbert space and rotate continuously between $|EP_{k,k+1}\rangle_1$ and $|EP_{k,k+1}\rangle_2$. That is,

$$U(\theta) = \cos\left(\frac{\theta}{2}\right) I - i \sin\left(\frac{\theta}{2}\right) (|a\rangle\langle b| + |b\rangle\langle a|), \quad (\text{E5})$$

where $|a\rangle = |EP_{k,k+1}\rangle_1$, $|b\rangle = |EP_{k,k+1}\rangle_2$, and θ goes from 0 to π . By doing this locally to each pair, we can map $|\phi_1\rangle$ to $|\phi_2\rangle$ (and vice versa) with LU transformations without breaking the on-site symmetry of group G . Therefore, $|\phi_1\rangle$ and $|\phi_2\rangle$ belong to the same phase if they are related to the same ω .

On the other hand, if $|\phi_1\rangle$ and $|\phi_2\rangle$ are related to ω_1 and ω_2 , respectively, we will show that they cannot be connected by any LU transformation that does not break the symmetry. Suppose that ω_1 is nontrivial, we start with $|\phi_1\rangle$, and apply a local unitary operation U to a finite region (shaded in Fig. 7). $|\phi_1\rangle$ is composed of invariant singlets of symmetry group G . If U does not break the symmetry, the resulting state should still be composed of singlets. The singlet pairs outside of the shaded region are not changed while those overlapping with the shaded region can take any possible structure after the operation U .

No matter what the change is, the right virtual spin on the site to the left of the region corresponding to ω_1 should form a singlet with some degrees of freedom in the region. As the singlet is invariant under a linear representation of G , these degrees of freedom in the region must form a projective representation of G corresponding to $-\omega_1$. These degrees of freedom could exist on one site or be distributed over several sites. However, the sites only support linear representations. Therefore, there must be some remaining degrees of freedom on the same sites that correspond to ω_1 . These remaining degrees of freedom must form singlets again with other degrees of freedom in the region that corresponds to $-\omega_1$. We can continue this argument until finally some degree of freedom in

the region corresponding to $-\omega_1$ connects with the left virtual spin on the site to the right of the region corresponding to ω_1 and forms a singlet.

In Fig. 7, we illustrate one possible structure of singlets after operation U . White dots correspond to ω_0 , the identity element in $H^2(G, \mathbb{C})$, black dots correspond to ω_1 , and gray dots correspond to $-\omega_1$.

Therefore, we can see that no matter what the symmetric LU operation on $|\phi_1\rangle$ might be, singlet entangled pairs related to ω_1 must connect head to tail and cover the whole length of the chain. In other words, we cannot shrink a chain of singlet entangled pairs related to nontrivial ω_1 continuously to a point or change it to ω_2 by acting on it locally and without breaking the symmetry. Hence fixed-point states with different ω cannot be related to each other by a symmetric LU transformation and hence belong to different classes.

APPENDIX F: A PROOF OF EQ. (21)

A gapped TI state can be represented by a uniform MPS. After R steps of $\{n_i\}$ -block RG transformation, we obtain a MPS described by matrices $(A_{i'ir})^{(R)}$ which is given by Eq. (11). To describe a state that does not break the on-site linear symmetry, here $(A_{i'ir})^{(R)}$ is invariant (up to a phase) under $u^{(R)}(g)$ on each site. Therefore,⁴⁰

$$\sum_{j'j''} u_{i'ir} u_{j'j''}^{(R)}(g) A_{j'j''} = \alpha^{(R)}(g) M^{-1}(g) A_{i'ir} M(g) \quad (\text{F1})$$

must be satisfied with some invertible matrix $M(g)$. Here we have dropped the RG step label R [except in $\alpha^{(R)}(g)$]. Each coarse-grained site is a combination of $\prod_{i=1}^R n_i$ original lattice sites and the $\alpha^{(R)}(g)$ form a 1D representation of G .

So if the number of sites has the form $L = Q \prod_{i=1}^R n_i$, then $\alpha_L(g)$ in Eq. (13) will have the form

$$\alpha_L(g) = [\alpha^{(R)}(g)]^Q \quad (\text{F2})$$

for any value of Q . Now let us choose $Q = \prod_{i=1}^{R'} n'_i$ where $\prod_{i=1}^R n_i$ and $\prod_{i=1}^{R'} n'_i$ have no common factors. The total system size becomes $L = \prod_{i=1}^R n_i \prod_{i=1}^{R'} n'_i$. We can perform, instead, an R' step of the $\{n'_i\}$ -block RG transformation, which leads to a 1D representation $\alpha^{(R')}(g)$. We find that $\alpha_L(g)$ in Eq. (13) will have the form

$$\alpha_L(g) = [\alpha^{(R')}(g)]^{Q'}, \quad (\text{F3})$$

where $Q' = L / \prod_{i=1}^{R'} n'_i = \prod_{i=1}^R n_i$. Thus

$$\alpha_L(g) = [\alpha^{(R)}(g)]^{\prod_{i=1}^{R'} n'_i} = [\alpha^{(R')}(g)]^{\prod_{i=1}^R n_i}. \quad (\text{F4})$$

Since $\prod_{i=1}^R n_i$ and $\prod_{i=1}^{R'} n'_i$ have no common factors, there must exist a 1D representation $\alpha(g)$ of G , such that

$$\alpha^{(R)}(g) = [\alpha(g)]^{\prod_{i=1}^R n_i}, \quad \alpha^{(R')}(g) = [\alpha(g)]^{\prod_{i=1}^{R'} n'_i}. \quad (\text{F5})$$

Now Eq. (F2) becomes

$$\alpha_L(g) = [\alpha(g)]^{Q \prod_{i=1}^R n_i} = [\alpha(g)]^L \quad (\text{F6})$$

which gives us Eq. (21).

APPENDIX G: EQUIVALENCE BETWEEN GAPPED TI b PHASES AND GAPPED TI PHASES

We have been using the quantum circuit formulation to study TI systems and classify b phases. As the quantum circuit explicitly breaks translational symmetry, it is possible that each b phase contains several different TI phases. (On the other hand, states in different b phases must belong to different TI phases.) In this Appendix, we will show that each b phase actually corresponds to a single TI phase by establishing a TI LU transformation between states in the same b phase. We will use the time evolution formulation of LU transformation (2) and find a smooth path of the gapped TI Hamiltonian whose adiabatic evolution connects two states within the same b phase.

First, as an example, we consider the case of TI only and show that there is only one gapped TI phase. Each translational-invariant MPS is described (up to a local change of basis) by a double tensor \mathbb{E} (see Fig. 8),

$$\mathbb{E}_{\alpha\gamma,\beta\chi} = \sum_i A_{i,\alpha\beta} \otimes A_{i,\gamma\chi}^*. \quad (\text{G1})$$

The MPS is short-range correlated if \mathbb{E} has a nondegenerate largest eigenvalue 1. \mathbb{E} can be written as

$$\mathbb{E}_{\alpha\gamma,\beta\chi} = \mathbb{E}_{\alpha\gamma,\beta\chi}^0 + \mathbb{E}'_{\alpha\gamma,\beta\chi} = \Lambda_{\alpha\gamma} \Lambda_{\beta\chi} + \mathbb{E}'_{\alpha\gamma,\beta\chi}, \quad (\text{G2})$$

where Λ is the eigenvector of eigenvalue 1 and \mathbb{E}' is of eigenvalue less than 1. In the canonical form,³³ $\Lambda_{\alpha\gamma} = \lambda_\alpha \delta_{\alpha\gamma}$, $\lambda_\alpha > 0$. Obviously, \mathbb{E}^0 is a valid double tensor and represents a state in fixed-point form. We can smoothly change \mathbb{E} to \mathbb{E}^0 by reducing the \mathbb{E}' term to 0 from $t = 0$ to $t = T$ as

$$\mathbb{E}(t) = \mathbb{E}^0 + \left(1 - \frac{t}{T}\right) \mathbb{E}'. \quad (\text{G3})$$

Every $\mathbb{E}(t)$ represents a TI SRC MPS. To see this, note that if we recombine the indices $\alpha\beta$ as row indices and $\gamma\chi$ as column indices and denote the new matrix as $\hat{\mathbb{E}}$ (see Fig. 8), then both $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}^0$ are positive semidefinite matrices. But then every $\hat{\mathbb{E}}(t)$ is also positive semidefinite, as for any vector $|v\rangle$

$$\begin{aligned} \langle v | \hat{\mathbb{E}}(t) | v \rangle &= \langle v | \hat{\mathbb{E}}^0 | v \rangle + \left(1 - \frac{t}{T}\right) \langle v | \mathbb{E}' | v \rangle \\ &= \left(1 - \frac{t}{T}\right) \langle v | \hat{\mathbb{E}} | v \rangle + \frac{t}{T} \langle v | \hat{\mathbb{E}}^0 | v \rangle > 0. \end{aligned}$$

$\mathbb{E}(t)$ is hence a valid double tensor, and the state represented can be determined by decomposing $\mathbb{E}(t)$ back into matrices $A_i(t)$. Such a decomposition is not unique. $A_i(t)$ at different times is determined only up to a local unitary on the physical

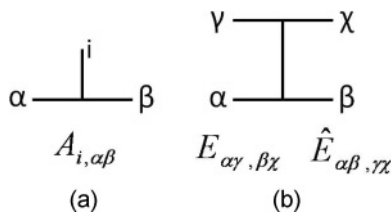


FIG. 8. Graphical representation of (a) matrices labeled by physical index i and (b) double tensor denoted by \mathbb{E} or $\hat{\mathbb{E}}$. \mathbb{E} is a matrix with $\alpha\gamma$ as row index and $\beta\chi$ as column index. $\hat{\mathbb{E}}$ is a matrix with $\alpha\beta$ as row index and $\gamma\chi$ as column index.

index i . But without loss of generality, we can choose the local unitary to be continuous in time, so that the $A_i(t)$ vary continuously with time and reach the fixed-point form at $t = T$ (up to a local change of basis). The state represented by $|\phi(t)\rangle$ hence also changes smoothly with t and is a pure state with a finite correlation length as all eigenvalues of $\mathbb{E}(t)$ except for 1 are diminishing with t .^{32,54} Therefore, $\mathbb{E}(t)$ represents a smooth path in the TI SRC MPS that connects any state to a fixed-point state (up to a local change of basis). Note that the number of matrices $A_i(t)$ necessary to compose $\mathbb{E}(t)$ may change with t , corresponding to a change in the local Hilbert space dimension as we do the deformation. We allow such changes in general as we can imagine the states to be embedded in very large (but finite) local Hilbert spaces. At each time point t , the state might be supported only on a subspace of the total Hilbert space.

How do we know that no phase transition happens along the path? This is because for every state $|\phi(t)\rangle$, we can find a parent Hamiltonian that changes smoothly with t and has the state as a unique gapped ground state.⁴⁴ Following the construction in Refs. 32 and 33, we choose a sufficiently large but finite l and set the parent Hamiltonian to be $H(t) = -\sum_k h(t)_{k,k+l}$, where $h(t)_{k,k+l}$ is the projection onto the support space of the reduced density matrix on site k to $k+l$ at time t . Note that this Hamiltonian is translation invariant. For large enough l , $h(t)_{k,k+l}$ will always be $(D \times D)$ dimensional. As the state changes continuously, its reduced density matrices from site k to $k+l$ change smoothly. Because the dimension of the space does not change, $h(t)_{k,k+l}$ also changes smoothly with time. Moreover, it can be shown that $H(t)$ is always gapped as the second largest eigenvalue of $\mathbb{E}(t)$ never approaches 1.^{32,33} Therefore, by evolving the Hamiltonian adiabatically from $t = 0$ to $t = T$, we obtain a local unitary transformation⁵⁵ connecting any state to the fixed-point form, and in particular without breaking the translation symmetry.

Because any TI fixed-point state can be disentangled into product states in a TI way, we find that *all TI 1D gapped ground states are in the same phase, if no other symmetries are required.*

If the system is TI and has on-site symmetry, we need to maintain the on-site symmetry while doing the smooth deformation. A TI SRC MPS that is symmetric under on-site symmetry of group G is described by matrices that satisfy

$$\sum_j u_{ij}(g) A_j = \alpha(g) M^{-1}(g) A_i M(g) \quad (\text{G4})$$

for some invertible projective representation $M(g)$. The double tensor \mathbb{E} hence satisfies

$$\mathbb{E}_{\alpha\gamma,\beta\chi} = \sum_{\alpha'\beta'\gamma'\chi'} M_{\alpha\alpha'}^{-1} M_{\beta\beta'} (M^*)_{\gamma\gamma'}^{-1} M_{\chi\chi'}^* \mathbb{E}_{\alpha'\gamma',\beta'\chi'}, \quad (\text{G5})$$

where the group element label g has been omitted. Because \mathbb{E}^0 as the nondegenerate one-dimensional eigenspace of \mathbb{E} must be invariant under the same transformation, so must \mathbb{E}' . Therefore we have

$$\begin{aligned} \mathbb{E}_{\alpha\gamma,\beta\chi}^0 &= \sum_{\alpha'\beta'\gamma'\chi'} M_{\alpha\alpha'}^{-1} M_{\beta\beta'} (M^*)_{\gamma\gamma'}^{-1} M_{\chi\chi'}^* \mathbb{E}_{\alpha'\gamma',\beta'\chi'}^0, \\ \mathbb{E}'_{\alpha\gamma,\beta\chi} &= \sum_{\alpha'\beta'\gamma'\chi'} M_{\alpha\alpha'}^{-1} M_{\beta\beta'} (M^*)_{\gamma\gamma'}^{-1} M_{\chi\chi'}^* \mathbb{E}'_{\alpha'\gamma',\beta'\chi'}. \end{aligned}$$

Now we smoothly change the double tensor as in Eq. (G3). Evidently, the symmetry condition Eq. (G5) is satisfied for all t .

Decompose $\mathbb{E}(t)$ back to matrices $A_i(t)$ so that the represented state $|\phi(t)\rangle$ changes smoothly with time. Denote the symmetry-transformed double tensor as $\mathbb{E}_{M(g)}$. As $\mathbb{E}_{M(g)}(t) = \mathbb{E}(t)$, there must exist a unitary operator $\tilde{u}(g)(t)$, such that

$$\sum_j \tilde{u}_{ij}(g)(t) A_j(t) = M^{-1}(g) A_i(t) M(g), \quad (\text{G6})$$

where $\tilde{u}(g)(t)$ is a linear representation of G . Redefine $u(g)(t) = \tilde{u}(g)(t) \times \alpha(g)$; then

$$\sum_j u_{ij}(g)(t) A_j(t) = \alpha(g) M^{-1}(g) A_i(t) M(g). \quad (\text{G7})$$

As $A_i(t)$ is chosen to be continuous with time, from the above equation we can see that $u(g)(t)$ is also continuous in time. On the other hand, $u(g)(t)$ forms a linear representation of G . For all the cases we are interested in, the linear representations of G are discrete. Therefore, as $u(g)(t)$ evolves smoothly with time, it cannot change from one representation to another but only from one equivalent form to another, which differ by a unitary conjugation. That is, $u(g)(t) = V(t) u(g) V^\dagger(t)$, with a continuous $V(t)$. We can incorporate $V(t)$ into the matrices $A_i(t)$ and define $\tilde{A}_i(t) = \sum_j V_{ij}^\dagger(t) A_j(t)$, so that $\tilde{A}_i(t)$ is symmetric under $u(g)$ for all t . In the following discussion, we will assume that such a redefinition is made and the symmetry operation of the system will always be $u(g) \otimes \cdots \otimes u(g)$. Therefore, the continuous evolution of $\mathbb{E}(t)$ from $t=0$ to $t=T$ corresponds to a continuous evolution of short-range-correlated states $|\phi(t)\rangle$ which is always symmetric under the same on-site symmetry $u(g)$, with the same phase factor $[\alpha(g)]^L$ and related to the same projective representation ω .

Such a smooth path in symmetric state space corresponds to a smooth path in symmetric Hamiltonian space. Construct the

parent Hamiltonian as discussed previously. Because the state is symmetric under on-site $u(g)$, the support space on sites k to $k+l$ must then form a representation space for $[\otimes u(g)]^l$. Therefore, it is easy to see that the parent Hamiltonian, being a summation of projections onto such spaces, is also a symmetry under on-site $u(g)$. Moreover, the Hamiltonian remains gapped and TI. In this way, we have found a smooth path of a symmetric, in particular TI, Hamiltonian whose adiabatic evolution connects any symmetric state labeled by $\alpha(g)$ and ω to the corresponding fixed-point state (up to a local change of basis), hence establishing the symmetric TI LU equivalence between them.

As we show in Appendix E that fixed-point states with the same $\alpha(g)$ and ω can be related by symmetric local unitary transformations to each other, we now complete the proof that for 1D spin systems with only translation and an on-site linear symmetry G , all gapped phases that do not break the two symmetries are classified by a pair (ω, α) where $\omega \in H^2(G, \mathbb{C})$ labels different types of projective representations of G and α labels different 1D representations of G .

Similarly, if the system has translation and parity symmetry, we can establish the equivalence between states labeled by the same $\alpha(P)$ and $\beta(P)$ in a translational-invariant way [see the discussion below Eq. (23)]. The procedure is totally analogous to that in the on-site symmetry case, with the only difference that the symmetry conditions for the matrices and double tensors become

$$\sum_j u_{ij} A_j^T = \pm M^{-1} A_i M, \\ \mathbb{E}_{\beta\chi, \alpha\gamma} = \sum_{\alpha'\beta'\gamma'\chi'} M_{\alpha\alpha'}^{-1} M_{\beta\beta'} M_{\gamma\gamma'}^{-1} M_{\chi\chi'}^* \mathbb{E}_{\alpha'\gamma', \beta'\chi'}.$$

We find that for 1D spin systems with only translational and parity symmetry, there are four gapped phases that do not break the two symmetries.

¹L. D. Landau, *Phys. Z. Sowjetunion* **11**, 26 (1937).

²L. D. Landau and E. M. Lifschitz, *Statistical Physics*, Course of Theoretical Physics Vol. 5 (Pergamon, London, 1958).

³D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982).

⁴R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).

⁵X.-G. Wen, *Adv. Phys.* **44**, 405 (1995).

⁶X.-G. Wen, *Phys. Rev. B* **65**, 165113 (2002).

⁷M. Levin and X.-G. Wen, *Phys. Rev. B* **71**, 045110 (2005).

⁸X. Chen, Z.-C. Gu, and X.-G. Wen, *Phys. Rev. B* **82**, 155138 (2010).

⁹S.-P. Kou, M. Levin, and X.-G. Wen, *Phys. Rev. B* **78**, 155134 (2008).

¹⁰Z.-C. Gu and X.-G. Wen, *Phys. Rev. B* **80**, 155131, (2009).

¹¹A. Kitaev, e-print [arXiv:0901.2686](https://arxiv.org/abs/0901.2686).

¹²S. Ryu, A. Schnyder, A. Furusaki, and A. Ludwig, *New J. Phys.* **12**, 065010 (2010).

¹³L. Fidkowski and A. Kitaev, *Phys. Rev. B* **81**, 134509 (2010).

¹⁴T. Hirano, H. Katsura, and Y. Hatsugai, *Phys. Rev. B* **77**, 094431 (2008).

¹⁵F. Pollmann, E. Berg, A. M. Turner and M. Oshikawa, *Phys. Rev. B* **81**, 064439 (2010).

¹⁶F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, *Phys. Rev. Lett.* **94**, 140601 (2005).

¹⁷N. Schuch, M. M. Wolf, F. Verstraete, and J. I. Cirac, *Phys. Rev. Lett.* **100**, 030504 (2008).

¹⁸R. F. Werner, in *Finitely Correlated Pure States*, edited by M. Fannes, Ch. Maes, and A. Verbeure, On Three Levels, NATO ASI Series B Vol. 324 (Plenum Press, New York, 1994), pp. 193–202.

¹⁹Taku Matsui, *Commun. Math. Phys.* **218**, 393 (2001).

²⁰E. Berg, E. G. D. Torre, T. Giamarchi, and E. Altman, *Phys. Rev. B* **77**, 245119 (2008).

²¹Z.-C. Gu and X.-G. Wen, *Phys. Rev. B* **80**, 155131 (2009).

²²F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, e-print [arXiv:0909.4059](https://arxiv.org/abs/0909.4059).

²³F. D. M. Haldane, *Phys. Lett. A* **93**, 464 (1983).

²⁴E. Lieb, T. Schultz, and D. Mattis, *Ann. Phys. (NY)* **16**, 407 (1961).

²⁵T. Hirano, H. Katsura, Y. Hatsugai, *Phys. Rev. B* **78**, 054431 (2008).

²⁶F. Anfuso and A. Rosch, *Phys. Rev. B* **75**, 144420 (2007).

- ²⁷X.-G. Wen, *Quantum Field Theory of Many-Body Systems—From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2004).
- ²⁸G. Vidal, *Phys. Rev. Lett.* **99**, 220405 (2007); in *Understanding Quantum Phase Transitions*, edited by Lincoln D. Carr (Taylor & Francis, Boca Raton, 2010).
- ²⁹V. L. Ginzburg and L. D. Landau, *Zh. Eksp. Teor. Fiz.* **20**, 1064 (1950).
- ³⁰W.-H. Ko, P. A. Lee, and X.-G. Wen, *Phys. Rev. B* **79**, 214502 (2009).
- ³¹S.-P. Kou and X.-G. Wen, *Phys. Rev. B* **80**, 224406 (2009); e-print [arXiv:0907.4537](https://arxiv.org/abs/0907.4537).
- ³²M. Fannes, B. Nachtergaele, and R. F. Werner, *Commun. Math. Phys.* **144**, 443 (1992).
- ³³D. Pérez-García, F. Verstraete, M. Wolf, and J. I. Cirac, *Quantum Inf. Comput.* **7**, 401 (2007).
- ³⁴M. B. Hastings, *J. Stat. Mech.* P08024 (2007).
- ³⁵M. B. Hastings, *Phys. Rev. Lett.* **93**, 140402 (2004).
- ³⁶M. B. Hastings and T. Koma, *Commun. Math. Phys.* **265**, 781 (2006).
- ³⁷M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- ³⁸M. Fannes, B. Nachtergaele, and R. F. Werner, *J. Funct. Anal.* **120**, 511 (1994).
- ³⁹D. E. Evans and E. Hoegh-Krohn, *J. London Math. Soc.* **17**, 345 (1978).
- ⁴⁰D. Perez-Garcia, M. M. Wolf, M. Sanz, F. Verstraete and J. I. Cirac, *Phys. Rev. Lett.* **100**, 167202 (2008).
- ⁴¹R. A. Hyman and K. Yang, *Phys. Rev. Lett.* **78**, 1783 (1997).
- ⁴²S. K. Ma, C. Dasgupta, and C. K. Hu, *Phys. Rev. Lett.* **43**, 1434 (1979).
- ⁴³D. S. Fisher, *Phys. Rev. B* **50**, 3799 (1994).
- ⁴⁴N. Schuch, D. Pérez-García and J. I. Cirac, [<http://physics.ruc.edu.cn/meeting/20090915/3109.pdf>].
- ⁴⁵I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, *Phys. Rev. Lett.* **59**, 799 (1987).
- ⁴⁶M. Hagiwara, K. Katsumata, I. Affleck, B. I. Halperin, and J. P. Renard, *Phys. Rev. Lett.* **65**, 3181 (1990).
- ⁴⁷S. H. Glarum, S. Geschwind, K. M. Lee, M. L. Kaplan, and J. Michel, *Phys. Rev. Lett.* **67**, 1614 (1991).
- ⁴⁸T.-K. Ng, *Phys. Rev. B* **50**, 555 (1994).
- ⁴⁹M. den Nijs and K. Rommelse, *Phys. Rev. B* **40**, 4709 (1989).
- ⁵⁰T. Kennedy and H. Tasaki, *Phys. Rev. B* **45**, 304 (1992).
- ⁵¹X.-G. Wen, *Int. J. Mod. Phys. B* **4**, 239 (1990).
- ⁵²H. Yao and S. A. Kivelson, *Phys. Rev. Lett.* **105**, 166402 (2010).
- ⁵³A. Kitaev (private communication).
- ⁵⁴M. Fannes, B. Nachtergaele, and R. F. Werner, *Lett. Math. Phys.* **25**, 249 (1992).
- ⁵⁵M. B. Hastings, X.-G. Wen, *Phys. Rev. B* **72**, 045141 (2005).