Quantum spin Hall, triplet superconductor, and topological liquids on the honeycomb lattice

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We classify the order parameters on the honeycomb lattice using the SO(4) symmetry of the Hubbard model. We focus on the topologically nontrivial quantum spin Hall order and spin triplet superconductor, which together belong to the (**3**,**3**) representation of the SO(4) symmetry. Depending on the microscopic parameters, this (**3**,**3**) order parameter has two types of ground states with different symmetries: type A, with ground-state manifold $[S^2 \otimes S^2]/Z_2$, and type B, with ground-state manifold $SO(3) \otimes Z_2$. We demonstrate that phase A is adjacent to a $Z_2 \otimes Z_2$ topological phase with mutual semion statistics between spin and charge excitations, while phase B is adjacent to a nonabelian phase described by SU(2) Chern–Simons theory. Connections of our study to the recent quantum Monte Carlo simulation on the Hubbard model on the honeycomb lattice are also discussed.

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I. INTRODUCTION AND SYMMETRY

We consider a class of (extended) Hubbard models on a bipartite lattice at half filling, with the following form:

$$\begin{aligned} H &= \sum_{\langle i,j\rangle,\sigma} -t c^\dagger_{i,\sigma} c_{j,\sigma} + \text{H.c.} + U n_{i,\uparrow} n_{i,\downarrow} + H', \\ H' &= \sum_{i\in sA,\ j\in sB} t_{ij} c^\dagger_{i,\sigma} c_{j,\sigma} + \text{H.c.} \\ &\quad + \sum_{i,j} J_{ij} \vec{S}_i \cdot \vec{S}_j + V_{ij} \vec{T}_i \cdot \vec{T}_j + \cdots, \\ \vec{T}_i &= [(-1)^i \text{Re}[\Delta]_i, (-1)^i \text{Im}[\Delta]_i, n_i - 1], \end{aligned} \tag{1}$$

where $\vec{S}_i = \frac{1}{2} c^\dagger_i \vec{\sigma} c_j$ is the spin operator and $\Delta_i = c^t_i i\sigma^y c_i$ is the on-site spin singlet Cooper pair. This extended Hubbard model has a manifest SU(2) spin symmetry. However, if electrons only hop between two different sublattices (sublattice A and B, denoted as sA and sB), after a sublattice-dependent particle-hole transformation for spin down electrons,

$$c_{i,\downarrow} \to (-1)^i c^\dagger_{i,\downarrow}, \tag{2}$$

the Hamiltonian is almost unchanged except that U changes sign and J_{ij} switches with V_{ij}. This implies that in addition to the apparent $SU(2)_{\text{spin}}$ symmetry, this model also has an $SU(2)_{\text{charge}}$ symmetry that mixes $[c_{i,\uparrow},(-1)^i c^\dagger_{i,\downarrow}]$. Therefore the full symmetry of this extended Hubbard model is[1,2]

$$SO(4) \sim [SU(2)_{\text{spin}} \otimes SU(2)_{\text{charge}}]/Z_2, \tag{3}$$

for arbitrary parameters in Eq. (1). For instance, this SO(4) symmetry holds for the simplest Hubbard model with only on-site Hubbard interaction and nearest-neighbor electron hopping.

Since the SO(4) symmetry is the full symmetry of the extended Hubbard model, Eq. (1), on any bipartite lattice, all the order parameters should be classified in terms of the representations of the SO(4) Lie algebra. In this work we take the honeycomb lattice as an example. Using the notation introduced in Ref. 3, we expand the electron at two Dirac valleys by $d_{1,2} = e^{i\vec{Q}_{1,2}\cdot\vec{r}} c$ [where $\vec{Q}_{1,2} = \pm(4\pi/3\sqrt{3},0)$ are the wave vectors of the valleys] and introduce Pauli matrices τ^α and μ^α which act on the sublattice and valley spaces, respectively. Then, after introducing real Majorana fermions ζ_a as the real and imaginary parts of $e^{i\frac{\pi}{4}\tau^x} e^{i\frac{\pi}{4}\mu^x}(d_1, i\tau^y d_2)^t$, we obtain the continuum Lagrangian for the semimetal phase:

$$\mathcal{L}_0 = \sum_{a=1}^{8} \bar{\zeta}_a \gamma_\mu \partial_\mu \zeta_a. \tag{4}$$

Here μ is a 2 + 1-dimensional space-time index, and the Dirac γ matrices are $(\gamma_0,\gamma_1,\gamma_2) = \tau^y,\tau^z,\tau^x$, $\bar{\zeta} = \zeta^t\gamma^0$. Using this notation, in the low-energy field theory, the $SU(2)_{\text{spin}}$ and $SU(2)_{\text{charge}}$ symmetries are generated by the following matrices:[3]

$$\begin{aligned} S^x &= \sigma^x\rho^y, \quad S^y = \sigma^y, \quad S^z = \sigma^z\rho^y, \\ T^x &= \sigma^y\rho^z, \quad T^y = \sigma^y\rho^x, \quad T^z = \rho^y. \end{aligned} \tag{5}$$

Here σ^a are spin Pauli matrices, while ρ^a are Pauli matrices that mix the real and imaginary parts of the electron. Notice that $SU(2) \otimes SU(2)$ is a double covering of SO(4), which leads to the Z_2 in Eq. (3).

Based on the symmetry Eq. (3) and the Lie algebra Eq. (5), the spin and charge are dual to each other for a large class of the extended Hubbard model. This spin-charge duality leads to many interesting results in our paper. The structure of this paper is as follows: in Sec. II, we show that the two types of topological orders, the quantum spin Hall (QSH) order and the triplet superconductor (T-SC) are unified as one representation of the SO(4) group, and the Ginzburg–Landau theory gives two types of ground states with different symmetry breakings. Sections III and IV study the phase diagrams driven by proliferating the topological defects in the two types of orders described in Sec. II, respectively. In both Secs. III and IV, we first give an argument of the phase diagram based on the quantum numbers of the topological defects; then a more solid description based on the Majorana liquid formalism developed in Ref. 3 is presented, and the results from these two approaches match perfectly with each other. Section V discusses the situation with $SU(2)_{\text{charge}}$ broken down to $U(1)_{\text{charge}}$ symmetry.

II. SO(4) CLASSIFICATION AND GINZBURG–LANDAU FORMALISM

Using the SO(4) algebra in Eq. (5), we classify the order parameters which immediately open up a mass gap for the Dirac fermion in the semimetal phase. Some simple Dirac mass gap order parameters can be classified with these symmetries straightforwardly. For instance, the quantum Hall order parameter $\bar{\zeta}\zeta$ is a $(\mathbf{1},\mathbf{1})$ representation of SO(4); i.e., it is a singlet of both SU(2) symmetries. The two sublattice Néel order $N^a = \bar{\zeta} S^a \mu^y \zeta$ is a $(\mathbf{3},\mathbf{1})$ representation. The fermion bilinear $M^a = \bar{\zeta} T^a \mu^y \zeta$, which belongs to the $(\mathbf{1},\mathbf{3})$ representation, is a two sublattice charge-density wave (CDW) and s-wave superconductor: $M^z \sim (-1)^i (n_i - 1)$ and $M^x + iM^y \sim c_i^t i\sigma^y c_i$. M^a can be viewed as the spin-charge dual version of N^a. In the simplest Hubbard model, N^a and M^a orders can be realized in the two limits $U \gg |t|$ and $-U \gg |t|$, respectively.

Now we discuss the following order parameters which belong to the $(\mathbf{3},\mathbf{3})$ representation of SO(4):

$$Q_{ab} = \bar{\zeta} A_{ab} \zeta,$$
$$A_{ab} = T^a S^b = \begin{pmatrix} -\sigma^z\rho^x & \rho^z & \sigma^x\rho^x \\ \sigma^z\rho^z & \rho^x & -\sigma^x\rho^z \\ \sigma^x & \sigma^y\rho^y & \sigma^z \end{pmatrix}. \tag{6}$$

This 3×3 matrix Q_{ab} has $\mathrm{SO(3)_{left}}$ and $\mathrm{SO(3)_{right}}$ transformations, which correspond to $\mathrm{SU(2)_{charge}}$ and $\mathrm{SU(2)_{spin}}$ symmetry, respectively. Q_{3b} corresponds to the QSH vector,[4,5] while $Q_{2b} + iQ_{1b}$ is the spin triplet pairing between next-nearest-neighbor sites:

$$Q_{3b} \sim \sum_{j \in sA,\ a=1,2,3} i c_j^\dagger \sigma^b c_{j+e_a} + \mathrm{H.c.} - (sA \to sB),$$
$$Q_{2b} + iQ_{1b} \sim \sum_{j \in sA,\ a=1,2,3} i c_j^t i\sigma^y \sigma^b c_{j+e_a} + (sA \to sB). \tag{7}$$

$e_1 = \sqrt{3}\hat{x}$, e_2, and $e_3 = -\frac{\sqrt{3}}{2}\hat{x} \pm \frac{3}{2}\hat{y}$ are three vectors on the honeycomb lattice that connect next-nearest-neighbor sites (Fig. 1). Therefore the two types of topological order parameters, QSH and T-SC, are unified through the SO(4) symmetry. Under time-reversal symmetry $\mathcal{T}$, Q_{3b} and Q_{2b} are even, while Q_{1b} is odd. Under reflection symmetry P_x: $y \to -y$, Q_{1b} and Q_{2b} are even, while Q_{3b} is odd; under P_y: $x \to -x$, all components of Q_{ab} are odd.

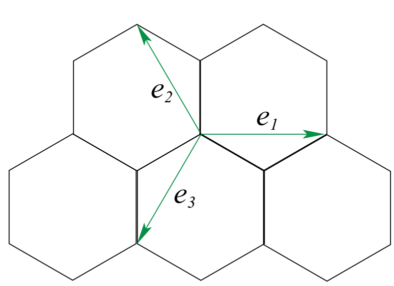

FIG. 1. (Color online) Honeycomb lattice and the vectors e_a.

The low-energy dynamics of Q_{ab} can be described by the following Ginzburg–Landau field theory:

$$\mathcal{L}_Q = \mathrm{tr}[\partial_\mu Q^t \partial_\mu Q] + r(\mathrm{tr}[Q^t Q]) + g(\mathrm{tr}[Q^t Q])^2 + u(\mathrm{tr}[Q^t Q Q^t Q]) + \cdots. \tag{8}$$

The first three terms have an enlarged SO(9) symmetry which corresponds to the rotation between the nine order parameters in matrix Q; while the last term, u, breaks this SO(9) symmetry down to the SO(4) symmetry. Another term $\mathrm{Det}[Q]$ is also invariant under SO(4) transformation, but $\mathrm{Det}[Q]$ breaks the time-reversal and reflection symmetry of the honeycomb lattice; therefore $\mathrm{Det}[Q]$ is forbidden in the Lagrangian equation, Eq. (8). However, if the system already breaks the time-reversal and reflection symmetry (for instance, $\langle \bar{\zeta}\zeta \rangle \neq 0$), $\mathrm{Det}[Q]$ would be allowed.

In Eq. (8), when $r < 0$, Q is ordered, and the SO(4) symmetry is broken down to its subgroups. Depending on the sign of u, there are two types of ground states as follows.

Type A, $u < 0$. One example state of this phase is $\langle Q_{33} \rangle \neq 0$, and all the other components $\langle Q_{ab} \rangle = 0$. In this phase the SO(4) symmetry is broken down to its following subgroup:

$$[\mathrm{U(1)_{spin}} \otimes \mathrm{U(1)_{charge}} \otimes Z_2]/Z_2. \tag{9}$$

The $\mathrm{U(1)_{spin}}$ and $\mathrm{U(1)_{charge}}$ symmetry are generated by matrices S^z and T^z in Eq. (5). The Z_2 in the numerator corresponds to reversing the direction of S^z and T^z simultaneously, while keeping Q_{33} invariant. The Z_2 in the denominator is the same Z_2 as in Eq. (3), which corresponds to changing the sign of electron operator. The ground-state manifold (GSM) of this phase is

$$\mathrm{GSM} \sim \left[S^2_{\mathrm{spin}} \otimes S^2_{\mathrm{charge}}\right]/Z_2. \tag{10}$$

The ground state can be described by two independent unit vectors $\vec{N}_s$ and $\vec{N}_c$ which belong to the $(\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{3})$ representation of SO(4), respectively, and $Q_{ab} = N_c^a N_s^b$. This phase has four independent Goldstone modes. The Z_2 in Eq. (10) is due to the fact that $\vec{N}_s$ and $\vec{N}_c$ can reverse direction simultaneously, and the ground state remains invariant.

Type B, $u > 0$. One example state of this phase is $\langle Q_{11} \rangle = \langle Q_{22} \rangle = \langle Q_{33} \rangle \neq 0$. The SO(4) group element $G_{\mathrm{so(4)}}$ can be written as $G_{\mathrm{su(2),spin}} \otimes G_{\mathrm{su(2),charge}}$, and the type B phase breaks the SO(4) down to its subgroup with $G_{\mathrm{su(2),spin}} = \pm G_{\mathrm{su(2),charge}}$. This implies that the residual symmetry group elements can be parametrized as $\pm R^s(\theta^s, \vec{n}^s)$ with $\theta^s \in (0, 2\pi)$, which is equivalent to the diagonal subgroup $\mathrm{SU(2)}_+$ generated by operators $G^a = S^a + T^a$. Here $R^s(\theta^s, \vec{n}^s)$ represents spin rotation by angle θ^s about axis $\vec{n}^s$. The GSM of phase B is

$$\mathrm{GSM} = (\mathrm{SO(4)} \otimes \mathcal{T})/\mathrm{SU(2)}_+ = \mathrm{SO(3)} \otimes Z_2, \tag{11}$$

with three Goldstone modes. $\mathcal{T}$ denotes the time-reversal symmetry, and type B phase spontaneously breaks $\mathcal{T}$. Therefore the GSM of phase B contains two disconnected sub-manifolds, with positive and negative $\mathrm{Det}[\langle Q \rangle]$, respectively.

The order of Q can be obtained through the following SO(4) invariant interacting Lagrangian for Dirac fermions on the honeycomb lattice:

$$\mathcal{L} = \sum_{a=1}^{8} \bar{\zeta}_a \gamma_\mu \partial_\mu \zeta_a - g\,\mathrm{tr}[Q^t Q]. \tag{12}$$

The interaction $-g\mathrm{tr}[Q^t Q]$ can be generated with SO(4) invariant interaction on the lattice, for instance, $\sum_{\ll i,j\gg} \vec{S}_i \cdot \vec{S}_j \sim -\mathrm{tr}[Q^t Q]/8 + \cdots$. A simple mean-field calculation after the standard Hubbard–Stratonovich transformation of Eq. (12) shows that the type A phase has more favorable ground-state energy compared with the type B phase on the honeycomb lattice. In the following we mainly focus on the analysis on the type A phase.

III. PHASE DIAGRAM AROUND TYPE A PHASE

Now we hope to understand the topological defects and the phase transitions driven by the topological defects in phase A. We first give an argument about the phase diagram and phase transitions using the quantum numbers carried by the topological defects, and then a systematic description based on the Majorana liquid formalism developed in Ref. 3 is presented. We demonstrate that these two approaches match very well.

A. Topological defects and phase transitions

In phase A, since the GSM is $[S^2 \otimes S^2]/Z_2$, both spin and charge sectors can have skyrmion-like defects characterized by the homotopy group $\pi_2[S^2]$. Again, let us assume that Q_{33} is the only component that acquires a nonzero expectation value; then $\langle Q_{33}\rangle$ breaks the SO(4) symmetry down to residual symmetries generated by S^z and T^z in Eq. (5). According to Refs. 6–8, under our current assumption that $\vec{N}_c \parallel \hat{z}$ (Q_{ab} is the QSH vector), a skyrmion of the spin sector manifold S^2_{spin} carries charge $2e$, and a skyrmion current is identified as the charge current:

$$J^\mu = \frac{2e}{8\pi}\epsilon_{\mu\nu\rho}\int d^2x \epsilon_{abc}\hat{N}^a_s \partial_\nu \hat{N}^b_s \partial_\rho \hat{N}^c_s. \tag{13}$$

For the same reason, a skyrmion of the charge sector manifold will carry spin-1: $S^z = 1$. For a general state with $\langle Q_{ab}\rangle \neq 0$, the spin-skyrmion carries the quantum number of the $\mathrm{U}(1)_{\mathrm{charge}}$ residual symmetry, while the charge-skyrmion carries the $\mathrm{U}(1)_{\mathrm{spin}}$ quantum number; i.e., spin and charge view each other as topological defects.

The condensation of skyrmions with nontrivial quantum numbers can lead to unconventional quantum phase transitions. For instance, the proposal of deconfined quantum criticality is based on the observation that the skyrmion of the Néel order carries lattice momentum;[9–11] hence the condensate of the skyrmion is equivalent to the valence bond solid state. In our current case, since a charge-skyrmion carries spin-1, if the charge-skyrmion is condensed, then the $\mathrm{SU}(2)_{\mathrm{charge}}$ symmetry is fully restored, which implies that the condensate of the charge skyrmion is a Mott insulator. Meanwhile, the residual spin symmetry is further spontaneously broken down to $Z_2 \otimes Z_2$. One of these Z_2 corresponds to changing the sign of the electron; the other one corresponds to reversing the direction of $\vec{N}_s$. The GSM of the charge skyrmion condensate is

$$\mathrm{GSM} = \mathrm{SU}(2)_{\mathrm{spin}}/[Z_2 \otimes Z_2] = \mathrm{SO}(3)/Z_2. \tag{14}$$

We mod Z_2 from SO(3), because after the proliferation of charge skyrmion, $\vec{N}_c$ is completely disordered, and $\vec{N}_s$ becomes a headless vector, due to the Z_2 in Eq. (10).

How do we determine the order of the charge-skyrmion condensate unambiguously? As was pointed out in Ref. 12, the phase of the O(3) skyrmion condensate can be identified as order parameters that share an O(5) Wess–Zumino–Witten (WZW) term with the O(3) order parameter. Therefore, to unambiguously identify the order of spin-skyrmion condensate, we need to seek for order parameters that have an O(5) WZW term with vector $\varphi^a = Q_{ab}$. It turns out that the Néel order parameter $\vec{N}$ is the only candidate of the charge-skyrmion condensate. For arbitrary b, we obtain the following WZW term between $\varphi^a = Q_{ab}$ and $\vec{N} \sim \bar{\chi}\mu^y\vec{S}\chi$:

$$\begin{aligned}\mathcal{L} &= \sum_{a=1}^{5}\frac{1}{g}(\partial_\mu\phi^a)^2 - \frac{3i}{4\pi}\int dud^3x\epsilon_{abcde}\phi^a\partial_x \\ &\quad\times \phi^b\partial_y\phi^c\partial_\tau\phi^d\partial_u\phi^e, \\ \phi^a &= \varphi^a = \bar{\chi}T^aS^b\chi, \quad a = 1,2,3, \\ \phi^4 &= N^c \sim \bar{\chi}\mu^yS^c\chi, \\ \phi^5 &= N^d \sim \bar{\chi}\mu^yS^d\chi, \quad c,d \neq b.\end{aligned} \tag{15}$$

Therefore the charge-skyrmion condensate contains both headless vector $\vec{N}_s$ and Néel order $\vec{N}$, and $\vec{N}_s \perp \vec{N}$. Physically the headless vector $\vec{N}_s$ corresponds to the spin nematic order $\mathcal{S}_{ab} = 3N^a_s N^b_s - \delta^{ab}(\vec{N}_s)^2$, which is invariant under reversing the direction of $\vec{N}_s$. All these results are confirmed later with the Majorana liquid formalism.

Notice that manifold SO(3) is equivalent to the projected manifold S^3/Z_2, which gives us a convenient way of parametrizing SO(3). Let us introduce SU(2) spinon z_α with constraint $|z_1|^2 + |z_2|^2 = 1$. This constraint implies that the SU(2) spinon z_α parametrizes S^3. Then by coupling z_α to a Z_2 gauge field, the gauge invariant GSM of the condensate of z_α automatically becomes SO(3).[13] The SO(3) manifold can also be viewed as the manifold of all the configurations of three perpendicular unit vectors: $\vec{q}_1$, $\vec{q}_2$, and $\vec{q}_3$. These three vectors can be parametrized as

$$\vec{q}_1 = z^\dagger\sigma^b z, \quad \vec{q}_2 + i\vec{q}_3 = z^t i\sigma^y\sigma^b z, \tag{16}$$

which automatically guarantees the perpendicularity of these vectors. In our situation, the three perpendicular vectors that characterize the GSM are $\vec{N}_s$, $\vec{N}$, and $\vec{N}_s \times \vec{N}$. Since $\vec{N}_s$ is headless, the GSM is, in fact, $\mathrm{SO}(3)/Z_2$. And in Sec. III B we demonstrate that it is most convenient to describe this GSM by introducing a $Z_2 \otimes Z_2$ gauge field.

Manifold SO(3) has the homotopy group $\pi_1[\mathrm{SO}(3)] = Z_2$; therefore phase B has a topologically stable half-vortex. Using the CP(1) spinon description introduced in Eq. (16), this half-vortex can also be viewed as the vison (a dynamical π flux) of the Z_2 gauge field coupled to z_α. Pictorially, a vison can be viewed as a configuration with (for instance) $\vec{q}_1$ being uniform in space, while $\vec{q}_2$ and $\vec{q}_3$ have a vortex. Now since $\vec{N}_s$ and $\vec{N}_s \times \vec{N}$ are both headless vectors, this state also supports a "half-vison," where $\vec{N}$ is uniform, while $\vec{N}_s$ has a half-vortex in space. In fact, this half-vison has a counterpart in phase A. Since phase A has GSM $[S^2 \otimes S^2]/Z_2$, there exists a "double half-vortex," and both $\vec{N}_s$ and $\vec{N}_c$ reverse direction after encircling this double half-vortex. After phase A is destroyed by proliferating the charge-skyrmion, this double half-vortex becomes the half-vison of $\mathrm{SO}(3)/Z_2$.

In the spin-charge dual side of the theory, all the conclusions can be obtained by straightforward generalization. Once the spin-skyrmion is condensed, the system will also enter a phase with GSM $SO(3)/Z_2$, and the $SU(2)_{\text{spin}}$ symmetry is fully restored, which implies that the condensate of the spin-skyrmion is a spin singlet. In Ref. 6, the authors proposed that after the proliferation of skyrmions of the QSH vector, the system enters a spin singlet s-wave superconductor. In our situation, since there is a generic $SU(2)_{\text{charge}}$ symmetry, the s-wave superconductor is promoted to a phase with GSM $SO(3)/Z_2$. If $\vec{N}_c \parallel \hat{z}$ (Q_{ab} is the QSH vector), the $SO(3)/Z_2$ manifold is characterized with the headless vector $\vec{N}_c$ and an s-wave superconductor. The order of the headless vector $\vec{N}_c$ implies that the degeneracy between the CDW and the s-wave superconductor is spontaneously lifted. In general, the order after spin-skyrmion proliferation can also be determined with the same WZW term analysis as in Eq. (15). A full list of order parameters with WZW terms can be found in Ref. 14.

This skyrmion condensation transition is described by the same CP(1) field theory as the deconfined quantum criticality.[10,11] How do we see the CP(1) transition directly? The CP(1) model $\mathcal{L} = \frac{1}{g}|(\partial_\mu - iA_\mu)z|^2$ describes a transition between a condensate of spinon z_α and a photon phase. The spinon condensate has GSM S^2, while the photon phase has GSM S^1, as it is a condensate of the U(1) gauge flux. In our case the skyrmion condensation is a transition between $[S^2 \otimes S^2]/Z_2$ and $SO(3)/Z_2$, while SO(3) can be roughly viewed as $S^2 \otimes S^1$; therefore, effectively the skyrmion condensation is more or less also a transition between S^2 and S^1, so it is equivalent to the CP(1) transition. This hand-waving argument is made precise in the next subsection by the Majorana liquid formalism.

Since a spin-skyrmion (charge-skyrmion) carries charge-$2e$ (spin-1), then the corresponding half-skyrmion (vortex) will carry charge-e and spin-1/2, respectively. If a charge-e excitation encircles around a spin-1/2 excitation bound with a charge-vortex, the charge-e excitation will acquire a π phase shift; on the other hand, if a spin-1/2 excitation encircles a charge-e excitation bound with a spin-vortex, the spin-1/2 excitation will also gain a π Berry phase. This implies that in phase A charge-e and spin-1/2 excitations have mutual semion statistics.

In phase A, a vortex is not a local excitation, and the gapless Goldstone mode of phase A makes the adiabatic braiding between two excitations impossible; therefore the semion statistics in phase A is not well defined. However, later we will see that phase A is adjacent to a liquid phase where the spin-charge mutual semion statistics persists, and it becomes a well-defined property.

B. Phase diagram with Majorana liquid formalism

From now on we hope to understand the phase diagrams discussed above with a more solid formalism. In Ref. 3, we discussed a fractionalized phase of electrons by decomposing ζ as follows:

$$\zeta = Z_s Z_c \chi,$$
$$Z_s = \phi_0^s + i\phi_1^s S^x + i\phi_2^s S^y + i\phi_3^s S^z, \quad (17)$$
$$Z_c = \phi_0^c + i\phi_1^c T^x + i\phi_2^c T^y + i\phi_3^c T^z.$$

The electron ζ decomposes into the bosonic fields Z_s and Z_c carrying its spin and charge, respectively, and into the Majorana fermion χ carrying the Fermi statistics. The resulting theory has a $SO(4)_g = SU(2)_{s,g} \otimes SU(2)_{c,g}$ gauge invariance: Z_s and χ carry $SU(2)_{s,g}$ charges, and Z_c and χ carry $SU(2)_{c,g}$ charges.

After the operator decomposition, when both Z_s and Z_c are gapped out, one obtains the parent state, i.e., the algebraic Majorana liquid (AML) state with Lagrangian

$$\mathcal{L}_{\text{AML}} = \bar{\chi}\gamma_\mu\left(\partial_\mu - iA^a_{s,\mu}S^a - iA^a_{c,\mu}T^a\right)\chi. \quad (18)$$

The fractionalized Majorana fermion χ fills the same mean-field band structure as the physical Majorana fermion ζ. The gauge fields $A^a_{s,\mu}$ and $A^a_{c,\mu}$ also couple to the spin and charge SU(2) rotors Z_s and Z_c as well. Since χ no longer carries physical spin and charge quantum numbers, the fermion bilinears of χ can only break the gauge symmetry, but not physical symmetry. If Z_s or Z_c condense, the formalism reduces to the two standard slave particle formalisms, with fermionic[15] or bosonic spinons,[9] respectively.

Now let us assume χ enters a type A phase; i.e., the matrix field $\tilde{Q}_{ab} = \bar{\chi}A_{ab}\chi$ condenses. For instance let us take

$$\langle\tilde{Q}_{33}\rangle = \langle\bar{\chi}\sigma^z\chi\rangle \neq 0. \quad (19)$$

Although the fractionalized Majorana fermion χ fills the same mean-field band structure as ζ, unlike the physical QSH vector, nonzero $\langle\tilde{Q}_{3b}\rangle$ breaks no discrete symmetries (time-reversal, refection) when rotor fields Z_s and Z_c are gapped. This is because the gauge symmetry released from gapping out the rotor fields can always reverse the sign of $\langle\tilde{Q}_{3b}\rangle$. This condensate of $\tilde{Q}_{33}$ breaks the $SU(2)_{s,g} \otimes SU(2)_{c,g}$ gauge invariance down to $U(1)_{s,g} \otimes U(1)_{c,g}$ gauge symmetry generated by S^z and T^z. Sometimes it will be convenient to use the following spin and charge CP(1) field:

$$z^s = \left(z_1^s, z_2^s\right)^t = \left(\phi_0^s + i\phi_3^s, \ -\phi_2^s + i\phi_1^s\right)^t,$$
$$z^c = \left(z_1^c, z_2^c\right)^t = \left(\phi_0^c + i\phi_3^c, \ -\phi_2^c + i\phi_1^c\right)^t. \quad (20)$$

It was discussed in our previous work[3] that, after integrating out χ, we obtain the low-energy theory when $\langle\tilde{Q}_{33}\rangle \neq 0$, which is a mutual Chern–Simons theory:

$$\mathcal{L}_{\text{cs}} = \frac{2i}{2\pi}\epsilon_{\mu\nu\rho}A^z_{c,\mu}\partial_\nu A^z_{s,\rho} + \left|\left(\partial_\mu - iA^z_{s,\mu}\right)z^s_\alpha\right|^2 + r_s\left|z^s_\alpha\right|^2$$
$$+ \left|\left(\partial_\mu - iA^z_{c,\mu}\right)z^c_\alpha\right|^2 + r_c\left|z^c_\alpha\right|^2 + \cdots. \quad (21)$$

The CP(1) fields z^s_α and z^c_α carry spin and charge, respectively, and we have chosen the notation to make both spin and charge SU(2) physical global symmetries manifest. Equation (21) implies that there is a mutual semion statistics between the charge and spin CP(1) fields z^c_α and z^s_α, which verifies the observation in Sec. III A.

This field theory is similar to the one on the triangular lattice,[16] with mutual semion statistics between charge and vison, except there the SU(2) symmetry of the vison is broken down to discrete symmetry by higher-order terms,[17] while here the SU(2) charge symmetry is exact. Based on this analogy, we can propose a global phase diagram, with tuning parameters r_s and r_c (Fig. 2), similar to that in Ref. 16, with a different interpretation of the phases.

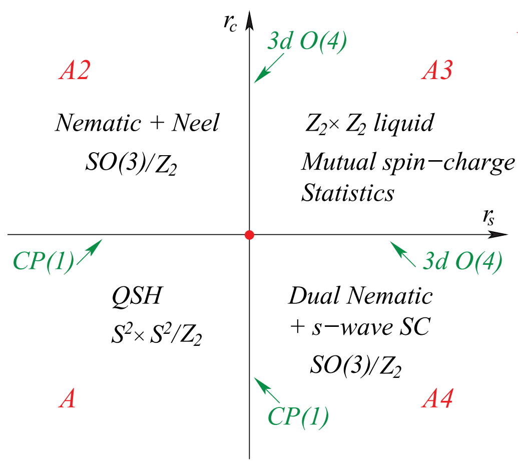

FIG. 2. (Color online) Phase diagram around type A phase with $\langle Q_{33}\rangle \neq 0$.

1. Phase A

Phase A is the phase with both z^s_α and z^c_α condensed, and the $\mathrm{SU}(2)_{s,g}$ and $\mathrm{SU}(2)_{c,g}$ gauge fields are both Higgsed and gapped out from the spectrum. This phase is characterized by the SU(2) vectors,

$$N^a_s = z^{s\dagger}\sigma^a z^s \sim \mathrm{tr}\big[Z^t_s S^a Z_s S^z\big], \quad N^a_c = z^{c\dagger}\sigma^a z^c \sim \mathrm{tr}\big[Z^t_c T^a Z_c T^z\big], \tag{22}$$

and the gauge invariant physical order parameter is

$$Q_{ab} = \bar{\zeta} A_{ab}\zeta = \bar{\chi} Z^t_s Z^t_c A_{ab} Z_s Z_c \chi \sim \mathrm{tr}\big[Z^t_c T^a Z_c T^f\big]\,\mathrm{tr}\big[Z^t_s S^b Z_s S^g\big]\,\langle \bar{\chi} A_{fg}\chi\rangle \sim \langle \tilde{Q}_{33}\rangle N^a_s N^b_c; \tag{23}$$

therefore the physical GSM is $[S^2 \otimes S^2]/Z_2$. This phase is precisely the phase A obtained in the GL formalism in Sec. II. The skyrmion of vector $\vec{N}_s$ is equivalent to the flux of gauge field $A^z_{s,\mu}$, and due to the mutual CS interaction, the gauge flux of $A^z_{s,\mu}$ (skyrmion of $\vec{N}_s$) carries charge excitation z^c_α, which confirms our analysis in Sec. III A.

2. Phase A2 and A4

Phase $A2$ has $r_s < 0$ and $r_c > 0$; hence it is a phase with z^s_α condensed while z^c_α is gapped. In Sec. III A we conclude that the GSM of this phase is $\mathrm{SO}(3)/Z_2$, and it has both nematic order and Néel order by directly calculating the WZW term. How do we understand the physical orders using the Majorana liquid formalism? Since z^c_α is gapped, in phase $A2$ there are no charge degrees of freedom; therefore the "QSH" vector $\tilde{Q}_{3b}$ should correspond to a pure spin operator. In fact, since z^c_α is gapped, $\langle Q_{ab}\rangle$ in Eq. (23) vanishes; hence the only gauge invariant operator which acquires a nonzero expectation value is

$$3\tilde{Q}^t_{cd}\tilde{Q}_{de}\,\mathrm{tr}\big[Z^t_s S^a Z_s S^c\big]\,\mathrm{tr}\big[Z^t_s S^b Z_s S^e\big] \sim \langle \tilde{Q}_{33}\rangle^2 N^a_s N^b_s. \tag{24}$$

Therefore we can define the physical order parameter as

$$\mathcal{S}_{ab} \sim 3N^a_s N^b_s - \delta^{ab}(\vec{N}_s)^2 \sim 3S^a_i S^b_j - \delta^{ab}\vec{S}_i\cdot\vec{S}_j. \tag{25}$$

Hence $\mathcal{S}_{ab}$ is the spin-2 nematic order parameter which breaks the spin rotation symmetry down to $\mathrm{U}(1)\otimes Z_2$, but preserves the discrete symmetries. Notice that the physical order parameter is always a bilinear of $\vec{N}_s$.

When z^s_α is condensed, the $\mathrm{SU}(2)_{s,g}$ gauge field is Higgsed; then the $\mathrm{SU}(2)_{s,g}$ gauge charge of χ becomes equivalent to the physical spin quantum number of ζ after a SU(2) gauge transformation. If we take $\langle \tilde{Q}_{33}\rangle \neq 0$, the low-energy field theory for fermions in phase $A2$ reads

$$\mathcal{L} = \bar{\psi}\gamma_\mu\big(\partial_\mu - iA^z_{c,\mu}\big)\psi + m\langle\tilde{Q}_{33}\rangle\cdot\bar{\psi}\sigma^z\psi, \tag{26}$$

where $\psi = \chi_1 + i\chi_2$. Based on the QSH physics, the flux of gauge field $A^z_{c,\mu}$ carries spin:

$$\nabla\times\vec{A}^z_c \sim \psi^\dagger\sigma^z\psi\,\mathrm{tr}\big[Z^t_s S^a Z_s S^z\big]. \tag{27}$$

This equivalence between the flux and spin implies that the photon phase of gauge field $A^z_{c,\mu}$, which is the condensate of the flux, is a spin XY order. This effect was studied in Refs. 18 and 19 with a projected wave-function calculation, and the photon phase of the U(1) gauge field is precisely the Néel order:

$$N^a \sim \bar{\chi}\mu^y S^c\chi\,\mathrm{tr}\big[Z^t_s S^a Z_s S^c\big]. \tag{28}$$

Based on these analyses, we conclude that phase $A2$ is a phase with both nematic vector $\vec{N}_s$ in Eq. (22) and AF Néel order $\vec{N}$ in Eq. (28). Equations (22) and (28) guarantee that $\vec{N}_s \perp \vec{N}$:

$$\vec{N}_s\cdot\vec{N} \sim \sum_a \mathrm{tr}\big[Z^t_s S^a Z_s S^z\big]\,\mathrm{tr}\big[Z^t_s S^a Z_s S^x\big] = 0. \tag{29}$$

As we mentioned before, since the nematic vector is headless, the GSM should be $\mathrm{SO}(3)/Z_2$. This analysis again confirms our prediction in Sec. III A with the WZW term.

Phase $A4$ is the spin-charge dual phase of phase $A2$, the GSM is also $\mathrm{SO}(3)/Z_2$ with three branches of Goldstone modes. The spin-charge dual of the in-plane Néel order is precisely an s-wave superconductor. The spin-charge dual of the nematic order $\mathcal{S}_{ab}$ will break the $\mathrm{SU}(2)_{\mathrm{charge}}$; for instance,

$$\mathcal{S}_{zz} \sim 2T^z_i T^z_j - T^x_i T^x_j - T^y_i T^y_j, \tag{30}$$

with $\vec{T}_i$ given by Eq. (1). Therefore, if we turn on an extra density repulsion between next-nearest-neighbor sites in Eq. (1), it corresponds to turning on $\mathcal{S}_{zz}$ and breaks the $\mathrm{SU}(2)_{\mathrm{charge}}$ down to $\mathrm{U}(1)\otimes Z_2$. This U(1) corresponds to the ordinary electron charge conservation, and Z_2 corresponds to the discrete particle-hole symmetry.

3. Phase A3

Phase $A3$ is a liquid state with $r_s > 0$ and $r_c > 0$; both z^s_α and z^c_α are gapped out. When z^c_α is gapped, $A^z_{c,\mu}$ is in the photon phase. Since the photon phase of $2+1$-dimensional U(1) gauge field is also the condensate of gauge flux based on the standard QED-superfluid duality, the mutual CS coupling in Eq. (21) implies that the photon phase of $A^z_{c,\mu}$ breaks the $\mathrm{U}(1)_{s,g}$ down to Z_2 gauge symmetry. For the same reason, $\mathrm{U}(1)_{c,g}$ is also broken down to Z_2. The mutual CS theory in Eq. (21) has the same topological degeneracy as the standard Z_2 gauge field on the torus,[16,20] and the mutual statistics between charge and spin is an analog of the mutual statistics between charge and vison of the well-known toric code model.[21]

In addition to the Z_2 gauge field coming from the mutual CS coupling, there is one extra residual Z_2 gauge symmetry which corresponds to reversing the sign of gauge symmetry generators S^z and T^z simultaneously, while leaving $\tilde{Q}_{33}$ invariant. This extra discrete Z_2 gauge symmetry contains group elements:

$$G^{(z2)} = I_{4\times4}, \text{ or } S^x T^x. \tag{31}$$

This Z_2 gauge field couples to both spin and charge SU(2) rotors Z_s and Z_c, but it was not explicit in our continuum limit field theory. Therefore phase $A3$ is characterized by $Z_2 \otimes Z_2$ gauge fields. Under Z_2 gauge symmetry $G^{(z2)}$, the fractionalized particles transform as

$$\begin{aligned} Z_{s,j} &\to Z_{s,j}\tfrac{1}{2}[(1+\mu_j)I_{4\times4} + (1-\mu_j)iS^x], \\ Z_{c,j} &\to Z_{c,j}\tfrac{1}{2}[(1+\mu_j)I_{4\times4} + (1-\mu_j)iT^x], \\ \chi_j &\to \tfrac{1}{2}[(1+\mu_j)I_{4\times4} + (1-\mu_j)S^xT^x]\chi_j, \\ &\mu_j = \pm 1. \end{aligned} \tag{32}$$

If the matter fields are ignored, these two Z_2 gauge fields are similar to a Z_4 gauge field with group elements $G^{(z4)} = \exp[i\theta S^x T^x]$, $\theta = 0, \pi/2, \pi, 3\pi/2$.

These two Z_2 gauge fields being together again implies that the GSM of $A2$ (the condensate of z^s_α) is SO(3)/Z_2, as we already concluded. If we approach phase $A2$ from phase $A3$, we can interpret phase $A2$ as the condensate of SU(2) spin rotor Z_s which couples to the two Z_2 gauge groups discussed above. With the condensate of Z_s we can again define three perpendicular vectors:

$$\begin{aligned} \vec{N}_s &= \mathrm{tr}\big[Z_s^t \vec{S} Z_s S^z\big] \sim z^{s\dagger}\vec{\sigma} z^s, \\ \vec{N}_1 &= \mathrm{tr}\big[Z_s^t \vec{S} Z_s S^x\big] \sim \mathrm{Re}\big[(z^s)^t i\sigma^y \vec{\sigma} z^s\big], \\ \vec{N}_2 &= \mathrm{tr}\big[Z_s^t \vec{S} Z_s S^y\big] \sim \mathrm{Im}\big[(z^s)^t i\sigma^y \vec{\sigma} z^s\big]. \end{aligned} \tag{33}$$

$\vec{N}_s$ and $\vec{N}_2$ change sign under gauge transformation $Z_s \to Z_s i S^x$ (i.e., $\mu_j = -1$), while $\vec{N}_1$ never changes sign; therefore, $\vec{N}_s$ and $\vec{N}_2$ become headless nematic vectors by coupling to the $Z_2 \otimes Z_2$ gauge group. Hence the manifold formed with $\vec{N}_s$, $\vec{N}_1$, and $\vec{N}_2$ is SO(3)/Z_2. This is completely consistent with our description of phase $A2$ in Secs. III A and III B 2.

In addition to the mutual statistics between Z_s and Z_c, there is one more topological defect in phase $A3$ with

$$\prod_{\mathcal{C}} G^{(z2)} = S^x T^x, \tag{34}$$

where $\mathcal{C}$ is a closed loop on the lattice. This defect carries a gauge flux $S^x T^s$. After encircling this defect, $Z_s \to Z_s i S^x$ and $Z_c \to Z_c i T^x$. The vectors $\vec{N}_s$ and $\vec{N}_2$ always acquire a minus sign after encircling this defect. Therefore this defect is a counterpart of the "double half-vortex" and "half-vison" discussed in Sec. III A.

4. Discussion

The universality class of the phase transitions in Fig. 2 can also be analyzed with field theory Eq. (21), in the same way as in Ref. 16. Quoting the results in Ref. 16, the transition between phases $A3$ and $A2$, and the transition between phases $A3$ and $A4$ are three-dimensional O(4) transitions, because spinon z^s_α and z^c_α are O(4) vectors, and the fully gapped discrete gauge fields coupled to the O(4) vector do not change the O(4) universality class.[13] The transition $(A, A3)$ and the transition $(A, A4)$ are CP(1) transitions, which become manifest with the CP(1) fields z^s_α and z^c_α and the U(1) gauge fields $A^z_{s,\mu}$ and $A^z_{c,\mu}$.

A recent quantum Monte Carlo simulation of the Hubbard model on the honeycomb lattice suggests that there is a fully gapped spin liquid phase[22] sandwiched between the ordinary Néel order and semimetal phase, which has motivated spin liquid analysis on the honeycomb lattice using either slave boson or slave fermion techniques.[23,24] In our formalism, phase $A3$ in phase diagram Fig. 2 is a candidate of this gapped spin liquid. However, based on our analysis, phase $A3$ is not directly adjacent to a pure Néel order; instead phase $A3$ is adjacent to phase $A2$ with both Néel order and nematic order. Starting with phase $A2$, we need to go through one more transition which suppresses the nematic order and enters the final Néel order in the large Hubbard U limit. If our proposal is correct, then topological order parameters are indeed crucial for correctly understanding the phase diagram of the Hubbard model.

The spin-2 nematic order is a natural candidate of the ground state of spin-1 systems, with biquadratic interactions.[25] For spin-$1/2$ models, nematic order can exist when there is a considerable ring exchange or multi-spin interaction, which can be generated in the weak Mott insulator phase of the simplest Hubbard model with high-order perturbation of t/U. Our prediction of a phase with coexistence of nematic and Néel order can be checked numerically in the future. We will present a general classification about nematic orders and their adjacent spin liquid phases in a future presentation.[26]

A similar analysis can be applied to the Z_2 liquid phase obtained from the standard Schwinger boson formalism. Since the spinon z_α always couples to a Z_2 gauge field, the condensate of z_α is not an ordinary Néel order, because there always exists three perpendicular gauge invariant vectors like Eq. (16). In Ref. 27, the authors proposed that there is an intermediate chiral antiferromagnetic order between a fully gapped Z_2 liquid phase and a Néel order. This chiral AF state has GSM SO(3), which is different from the phase predicted in our paper with both nematic and Néel order.

IV. PHASE DIAGRAM AROUND TYPE B PHASE

Now let us move on to the phase B with GSM SO(3) $\otimes$ Z_2. All the phases discussed in this section break time-reversal symmetry $\mathcal{T}$; therefore we only focus on one of the two disconnected submanifolds SO(3). Phases with GSM SO(3) have been studied extensively with noncollinear spin density wave.[13] After disordering the state, both SU(2)$_{\text{spin}}$ and SU(2)$_{\text{charge}}$ are restored, while the vison of the SO(3) manifold is still locally conserved, and the system most naturally enters a Z_2 liquid phase.

Again, we hope to understand the phase diagram with the Majorana liquid formalism. Let us assume χ in the parent

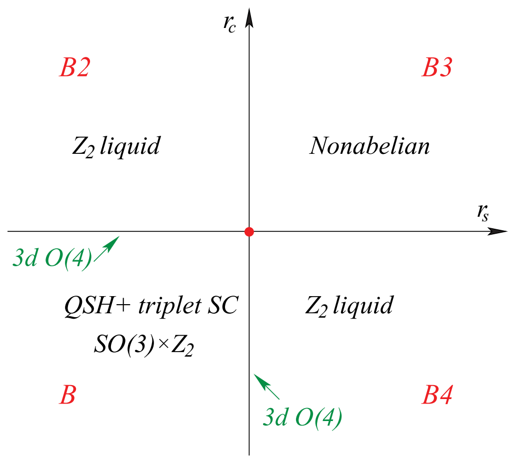

FIG. 3. (Color online) Phase diagram around type B phase with $\langle Q_{11}\rangle = \langle Q_{22}\rangle = \langle Q_{33}\rangle \neq 0$.

state Eq. (18) enters the type B phase, for instance, $\langle \tilde{Q}_{11}\rangle = \langle \tilde{Q}_{22}\rangle = \langle \tilde{Q}_{33}\rangle \neq 0$. It would be convenient to introduce the following CP(1) fields:

$$\begin{aligned} z^s &= \left(z_1^s,\, z_2^s\right)^t = \left(\phi_0^s - i\phi_3^s,\quad \phi_2^s - i\phi_1^s\right)^t, \\ z^c &= \left(z_1^c,\, z_2^c\right)^t = \left(\phi_0^c - i\phi_3^c,\quad \phi_2^c - i\phi_1^c\right)^t. \end{aligned} \tag{35}$$

The phase diagram around type B order is depicted in Fig. 3.

Phase B in Fig. 3 with both z_α^s and z_α^c condensed is precisely the phase B in the Ginzburg-Landau description in Sec. II, with GSM SO(3) $\otimes$ Z_2. In phase $B2$, the $\mathrm{SU}(2)_{s,g}$ gauge symmetry is Higgsed by the condensation of z_α^s, while the $\mathrm{SU}(2)_{c,g}$ is broken down to Z_2 gauge symmetry by $\langle \tilde{Q}_{ab}\rangle$; this is the Z_2 liquid phase we discussed previously. The residual Z_2 gauge symmetry is the subgroup of $\mathrm{SU}(2)_{c,g}$ that changes the sign of χ. Phase $B4$ is the same Z_2 liquid phase as phase $B2$.

Now we turn to phase $B3$ in Fig. 2. In this phase both z_α^s and z_α^c are gapped, while $\langle \tilde{Q}_{ab}\rangle$ breaks the SO(4) gauge symmetry down to $\mathrm{SU}(2)_+ = \mathrm{SU}(2)_{s,g} + \mathrm{SU}(2)_{c,g}$; hence in this phase there is only one SU(2) gauge field $A_\mu^a G^a$, with $G^a = S^a + T^a$. After integrating out the fermion χ, the following Chern–Simons term is induced for the residual $\mathrm{SU}(2)_+$ gauge field:

$$\begin{aligned} \mathcal{L} &= \frac{2}{4\pi}\mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \\ &\quad + \left|\left(\partial_\mu - \sum_a A_\mu^a\sigma^a\right)z^s\right|^2 + r_s|z^s|^2 \\ &\quad + \left|\left(\partial_\mu - \sum_a A_\mu^a\sigma^a\right)z^c\right|^2 + r_c|z^c|^2 + \cdots, \\ A &= A_\mu^a\sigma^a dx^\mu. \end{aligned} \tag{36}$$

Therefore phase $B3$ is characterized by SU(2) CS theory at level 2, which is a nonabelian theory.[28]

A different way of obtaining the same theory is by turning on another order parameter $\bar{\chi}\chi$ in addition to $\tilde{Q}_{ab}$. The order parameter $\bar{\chi}\chi$ will drive the system into a quantum Hall state and lead to the $\mathrm{SU}(2)_1$ CS theory for both $A_{s,\mu}^a$ and $A_{c,\mu}^a$, which is similar to the CS effective theory of the chiral spin liquid state.[15] The order $\langle \tilde{Q}_{ab}\rangle$ requires $A_{s,\mu}^a = A_{c,\mu}^a = A_\mu^a$; therefore the final theory becomes the SU(2) CS theory at level 2 in Eq. (36). Now by reducing the order $\langle\bar{\chi}\chi\rangle$ to zero, the SU(2) CS theory is unchanged. More detailed properties of this phase will be further discussed in a future presentation.[26]

In phase $B3$, the residual gauge symmetry $G_{s,g}$ is either $G_{c,g}$ or $-G_{c,g}$. This Z_2 structure does not show up in the Lie algebra of the gauge group, but it implies that there is one extra Z_2 gauge field that couples to χ and either one of z_α^s or z_α^c. Hence when z_α^s or z_α^c condenses, the gauge field A_μ^a is Higgsed, but the system still has a Z_2 gauge symmetry, which characterizes the Z_2 liquid phases $B2$ and $B4$.

The transition ($B2$, $B3$) and the transition ($B3$, $B4$) are Higgs transitions, described by spinon z_α^s or z_α^c coupled with SU(2) CS theory in Eq. (36). The universality class of these transitions is not understood yet.

V. SITUATION WITH $\mathrm{SU}(2)_{\mathrm{charge}}$ BROKEN TO $\mathrm{U}(1)_{\mathrm{charge}} \otimes Z_2$

When the $\mathrm{SU}(2)_{\mathrm{charge}}$ symmetry is broken down to $\mathrm{U}(1)_{\mathrm{charge}} \otimes Z_2$ symmetry, which corresponds to charge conservation and particle-hole transformation, the degeneracy between Q_{3b} and Q_{1b}, Q_{2b} is lifted. For instance, if an extra repulsive next-nearest-neighbor density interaction [linear with $\mathcal{S}_{zz}$ in Eq. (30)] is turned on,[29] the system favors developing Q_{3b}; i.e., the system only has QSH order with GSM S^2. Then according to Ref. 6, the skyrmion of the QSH vector carries charge-$2e$, and skyrmion condensate is an s-wave superconductor.

If the system favors having T-SC Q_{1b}, Q_{2b} rather than Q_{3b}, then depending on the microscopic parameters the T-SC can have orders with either $Q_{1b} \parallel Q_{2b}$ (type A) or $Q_{1b} \perp Q_{2b}$ (type B). The type A phase has a fully gapped fermion spectrum, with GSM$\sim [S^2 \otimes S^1]/Z_2$. Here S^2 corresponds to the spin direction of the triplet Cooper pair, while S^1 corresponds to the pairing phase angle. Again, there are spin and charge topological defects. For instance, the charge vortex (defect of the S^1 part of the GSM) carries spin-1/2 quantum number (quantum number of the residual $\mathrm{U}(1)_{\mathrm{spin}}$ symmetry). The proliferation of the charge vortex leads to the phase $A2$ in phase diagram Fig. 2, and the transition is a CP(1) theory with easy plane anisotropy on the charge CP(1) field z_α^c introduced in Eq. (20), which is consistent with the conclusion in Ref. 19.

The proliferation of the spin-skyrmion restores the $\mathrm{SU}(2)_{\mathrm{spin}}$ symmetry, but the $\mathrm{U}(1)_{\mathrm{charge}}$ symmetry is still broken. However, since the spin and charge sectors can change sign simultaneously without modifying the ground state, after the proliferation of the spin-skyrmion there is still a Z_2 gauge symmetry for the charge manifold. Therefore the GSM of this phase is S^1/Z_2, which corresponds to a charge-$4e$ superconductor, instead of a charge-$2e$ superconductor. A similar scenario was discussed for the polar state of the ultracold spin-1 spinor condensate,[30] which also has the GSM $[S^2 \otimes S^1]/Z_2$.

Unlike the type A order, the type B phase with $Q_{1b} \perp Q_{2b}$ does not have a fully gapped fermion spectrum. For instance, with $\langle Q_{11}\rangle = \langle Q_{22}\rangle \neq 0$, only spin-up is paired and gapped out, while spin-down is not gapped. Since the fermion spectrum is gapless, the quantum number of defects is no longer topologically stable.

VI. SUMMARY AND DISCUSSION

In this work we have classified the QSH and T-SC states on the honeycomb lattice using the SO(4) symmetry for a large class of extended Hubbard models. By analyzing the quantum numbers of topological defects, we obtained two different phase diagrams, which were also confirmed by the Majorana liquid formalism. The results of this paper can be straightforwardly generalized to other bipartite lattices. Our formalism also predicts a phase with both spin nematic and Néel order, sandwiched between a fully gapped $Z_2 \otimes Z_2$ liquid phase and the ordinary Néel order, which can be checked in the future using a method similar to that in Ref. 22.

The results we obtained in this work explicitly demonstrates the spin-charge duality of the Hubbard model. For instance, in the phase diagram in Fig. 2, spin and charge view each other as topological defects. A similar spin-charge duality was applied to the cuprates high-temperature superconductor,[31,32] and a global phase diagram with both spin and charge excitations was studied recently in Ref. 33. We also note that other authors[19,27] have also studied the duality between spin and charge with the presence of QSH order parameters; for instance, an easy plane version of spin-charge duality was identified as the self-duality of the easy plane noncompact CP(1) theory, in a model with inplane spin anisotropy. This duality led to a direct transition between the in-plane Néel order and the d-wave superconductor. In our current work we have shown that the generic symmetry of the Hubbard model and the condensate of matrix order parameter Q in Eq. (6) give us a complete and explicit duality between spin and charge in interacting electrons.

In both Figs. 2 and 3 there is a multicritical point with $r_s = r_c = 0$. The multicritical point in Fig. 2 was analyzed in Ref. 16, and for large enough spinon number this multicritical point is stable. Also, it has been proposed that a similar multicritical point is responsible for the spin liquid behavior in material $\kappa - (ET)_2Cu_2(CN)_3$ on the triangular lattice.[34] The multicritical point in Fig. 3 is more complicated, we will leave this multicritical point to future study.

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