Conductivity of interacting massless Dirac particles in graphene: Collisionless regime

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We provide detailed calculation of the ac conductivity in the case of $1/r$ Coulomb interacting massless Dirac particles in graphene in the collisionless limit when $\omega \gg T$. The analysis of the electron self-energy, current vertex function, and polarization function, which enter into the calculation of physical quantities including the ac conductivity, is carried out by checking the Ward-Takahashi identities associated with the electrical charge conservation and making sure that they are satisfied at each step. We adopt a variant of the dimensional regularization of Veltman and 't Hooft by taking the spatial dimension $D = 2 - \epsilon$ for $\epsilon > 0$. The procedure adopted here yields a result for the conductivity correction which, while explicitly preserving charge conservation laws, is nevertheless different from the results reported previously in literature.

I. INTRODUCTION

The role of Coulomb electron-electron interactions in systems described by massless two-dimensional Dirac fermions has been a subject of interest for some time.[1–15] Discovery of graphene, a single atomic layer of sp^2 hybridized carbon, and more recently of topological insulators, both of which support such massless Dirac fermions, brought this issue into sharp focus. In particular, which physically measurable quantities are modified from their noninteracting values, and by how much, would allow deeper understanding of the physics governed by electron-electron interactions in these systems.

When weak, the unscreened $1/r$ Coulomb interactions are expected to modify the velocity of the Dirac fermions as $v_F \rightarrow v_F + \frac{e^2}{4} \ln \Lambda/k$, where k is the wavenumber measured from the Dirac point. This modification of the electronic dispersion is expected to lead to logarithmic suppression of the density of states near the Dirac point, an effect, in principle, observable in tunneling experiments. In addition, the low-temperature electronic contribution to the specific heat should be suppressed from T^2 to $T^2/\log^2 T$, as shown in Ref. 4, and the strength of this suppression is related to the strength of the Coulomb interaction.

The role of Coulomb interaction in ac electrical transport was investigated by Mishchenko in Ref. 7, who originally concluded that the ac conductivity $\sigma(\omega)$ vanishes as $\omega \rightarrow 0$ and the system is a (weak) *insulator*. Were this the case, the interactions would have dramatic effect on transport since the ac conductivity of the noninteracting system is finite,[16] i.e., $\sigma_0(\omega) = \pi e^2/2h$ for $\omega \gg T$. This was later argued to be incorrect by Sheehy and Schmalian,[8] and independently by the present authors[9] using renormalization-group (RG) scaling analysis. While the former presented only a scaling argument, without calculating the correction to transport, the latter reported on an explicit calculation where

$$\sigma(\omega) = \sigma_0 \left(1 + \mathcal{C} \frac{e^2}{v_F + \frac{e^2}{4} \ln \frac{\Lambda}{\omega}} \right) \quad (1)$$

with the coefficient found to be $\mathcal{C} = (25 - 6\pi)/12 \simeq 0.5125$. Note that, since e^2 does not renormalize,[9,17] any change in the cutoff in the above expression for the conductivity may be compensated by a redefinition of the Fermi velocity, v_F. At small ω the correction vanishes, and the noninteracting value of σ is recovered. At small but finite frequencies, the correction scales as $1/|\log \omega|$, with the *interaction independent* prefactor determined by $\mathcal{C}$. The numerical value of this correction, which can be understood as correction to scaling near the Gaussian fixed point and which is expected to be universal, has since been a subject of debate. In subsequent work, Mishchenko[10] recovered the functional form in Eq. (1), which gives metallic conductivity at small ω, but argued for a different value of $\mathcal{C} = (19 - 6\pi)/12 \simeq 0.01254$ which happens to be much smaller than the one found by us. Technically, the difference originated from different regularization adopted in the two approaches. The standard momentum space cutoff, motivated by the underlying discrete lattice structure and reported in Ref. 9 was questioned in Ref. 10, where the correction to conductivity was calculated using a cutoff on the $1/r$ interaction and argued to be the same regardless of whether it is calculated using Kubo formula (current-current correlator) or continuity equation and density-density correlator. The same calculational procedure was later advocated by Sheehy and Schmalian,[18] who argued that unlike hard cutoff in momentum space, cutoff on the interaction leads to expressions obeying Ward-Takahashi identity. In addition, they claimed the result obtained in such way is consistent with the experimentally measured optical conductivity, where, surprisingly, no discernible correction to the noninteracting value was reported.[19]

In quantum-field theories, it seems reasonable that if two ultraviolet (UV) regularization schemes give different results for physical quantities, then the regularization that is typi-

cally chosen is the one which respects charge $U(1)$ symmetry, as is the case for chiral anomaly in $(3+1)$-dimensional massless quantum electrodynamics (QED), for instance.[20] Here we argue that the regularization of the electron-electron interaction alone is incomplete and cannot serve as a consistent regularization of the theory. We also show by explicit calculation that the dimensional regularization used here preserves the Ward-Takahashi identity, i.e., that it is consistent with $U(1)$ gauge symmetry of the theory, and that, moreover, has the additional advantage of serving as an interaction-independent regularization scheme for the whole field theory. The interaction correction to the conductivity within this regularization scheme is calculated independently using the current-current and the density-density correlators, which both yield the same number $\mathcal{C}=(11-3\pi)/6 \simeq 0.2625$ in Eq. (1), precisely as a consequence of explicitly preserved $U(1)$ gauge symmetry. Furthermore, we show that while the hard-cut-off regularization, in principle, violates the Ward-Takahashi identity, the original integral expression[9] for the constant $\mathcal{C}$ is in fact UV convergent, and when computed with necessary care it unambiguously leads to the same value as quoted above.

A comparison with experiment which has been performed at *high* frequencies near the cutoff[19,21] (see also Ref. 22) may be misleading, since the result for the leading logarithmic correction to the conductivity in Eq. (1) is valid only at frequencies of the order of 1 meV, much *smaller* than the cutoff. As the Coulomb coupling constant in graphene e^2/v_F is believed to be of order 1, we expect that in this region the interaction corrections to different observables, relative to the values in the noninteracting theory, should be significant. Why the interaction correction to the conductivity, in particular, appears to be small even in the high-frequency regime is unclear at the moment.

Whereas the results in the collisionless limit $(\omega \gg T)$ discussed here at least, in principle, follow from a straightforward application of the perturbative renormalization group, transport in the collision-dominated regime $(\omega \ll T)$ requires resummation of an infinite series of Feynman diagrams. This is easily seen in the noninteracting limit where a finite temperature T produces a finite, linear in T, "Drude" δ-function response in conductivity, $\sim T\delta(\omega)$. Collisions due to the electron-electron scattering lead to broadening of the δ function and clearly the result must be nonanalytic in e^2/v_F as the interaction $V(\mathbf{r}) \to 0$. Alternative approach has been advanced in Refs. 23 and 24, where the leading correction is argued to be captured by the solution to the quantum Boltzman equation with the collision integral calculated perturbatively in the interaction strength. In the clean limit, the conductivity in the collision dominated regime is found to *increase* with decreasing T and proportional to $\ln^2(T/\Lambda)$. Interestingly, experiments on suspended samples at the neutrality point[25] find conductivity which *decreases* with decreasing T. Finally, T-linear increase of the dc conductivity observed in small devices,[26] has been argued to arise from purely ballistic transport,[27] where conductivity grows with the sample size L and temperature T as $\sigma \sim TL/\hbar v_F$.

The paper is organized as follows: in Sec. II we introduce the Lagrangian and the response functions, and in Sec. III we discuss different regularization schemes for massless Dirac fermions. In Sec. IV, we review some well-known results regarding the polarization tensor and the conductivity. In Sec. V, we explicitly construct polarization tensor for the noninteracting theory and in Sec. VI we consider the same problem for the contact interactions to first order in the interaction strength and to $\mathcal{O}(N)$. We do not discuss the (random-phase approximation-like) contribution to the order N^2 which while simple to calculate, does not contribute to transport. The main results of the paper are presented in Sec. VII, where we show that the Coulomb correction to the polarization tensor is transversal, as well as that the Coulomb vertex function obeys the Ward-Takahashi identity within the dimensional regularization. In this section, we also present calculations of the Coulomb correction to the ac conductivity using both the current-current correlator (Kubo formula) and the density-density correlator, within the dimensional regularization. Section VIII is reserved for further discussion of these results and comparison with previous results reported in the literature. Various technical details of the calculations are presented in the Appendices A–H.

II. HAMILTONIAN, LAGRANGIAN, AND THE RESPONSE FUNCTIONS

We start with the Hamiltonian

$$\hat{H} = \int d^D\mathbf{r}\,\psi^\dagger(\mathbf{r})v_F\sigma_a p_a\psi(\mathbf{r})$$

$$+ \frac{1}{2}\int d^D\mathbf{r}d^D\mathbf{r}'\,\psi^\dagger(\mathbf{r})\psi(\mathbf{r})V(|\mathbf{r}-\mathbf{r}'|)\psi^\dagger(\mathbf{r}')\psi(\mathbf{r}'), \quad (2)$$

where we consider N copies of two-component Fermi fields $\psi(\mathbf{r},\tau)$ (which therefore has $2N$ components), the momentum operator $p_a = -i\hbar\partial_a$ and σ_a are Pauli matrices. Operators in the interaction term are assumed normal ordered. Hereafter, the Latin letters a,b are used only for the spatial indices while the Greek letters μ,ν are reserved for the spacetime ones and summation over the repeated indices is assumed. $V(\mathbf{r})$ is the two-body interaction potential, which is left unspecified at the moment. Later we will consider two different cases: a short-range contact interaction $V(\mathbf{r})=u\delta(\mathbf{r})$ and the three-dimensional (3D) Coulomb potential $V(\mathbf{r})=e^2/r$ with e^2/v_F as the dimensionless Coulomb coupling constant. To simplify the notation, we will work in the natural units $\hbar = c = k_B = 1$. When the speed of light c does not appear, we will also set $v_F=1$. In our final results we will restore the physical units.

The corresponding imaginary time Lagrangian is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}, \quad (3)$$

where

$$\mathcal{L}_0 = \int d^D\mathbf{r}\,\psi^\dagger(\tau,\mathbf{r})\left[\frac{\partial}{\partial\tau} + v_F\sigma\cdot\mathbf{p}\right]\psi(\tau,\mathbf{r}) \quad (4)$$

and

$$\mathcal{L}_{int} = \frac{1}{2} \int d^D\mathbf{r} d^D\mathbf{r}' \rho(\tau,\mathbf{r}) V(|\mathbf{r}-\mathbf{r}'|) \rho(\tau,\mathbf{r}'), \qquad (5)$$

where $\rho(\tau,\mathbf{r}) \equiv \psi^\dagger(\tau,\mathbf{r})\psi(\tau,\mathbf{r})$ is the density of fermions. The quantum partition function can then be written as the imaginary time Grassman path integral[28]

$$Z = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp\left(-\int_0^\beta d\tau \mathcal{L}\right), \qquad (6)$$

where the inverse temperature factor $\beta = 1/(k_B T)$. In the sections which follow, the additional imaginary-time index on the Fermi field $\psi(\tau,\mathbf{r})$ inside a path integral automatically means that they are considered to be coherent state Grassman fields. We will take $T \to 0$ first and then perform the calculations. Note that in light of the discussion in the Introduction, taking $T \to 0$ first automatically sets the collisionless limit.

By the standard spectral representation theorems we can first calculate the correlation functions as imaginary time-ordered products, Fourier transform over time and then analytically continue to find the physical retarded (or advanced) response functions.[28] Specifically, for some bosonic operator $\hat{\mathcal{O}}_a(t,\mathbf{r})$ in the (real time) Heisenberg representation, the retarded correlation function

$$S_{ab}^{ret}(t-t',\mathbf{r},\mathbf{r}') = -i\theta(t-t')\langle[\hat{\mathcal{O}}_a(t,\mathbf{r}),\hat{\mathcal{O}}_b(t',\mathbf{r}')]\rangle \quad (7)$$

can be related to the imaginary time-ordered correlation function

$$S_{ab}(\tau-\tau',\mathbf{r},\mathbf{r}') = -\langle T_\tau \hat{\mathcal{O}}_a(\tau,\mathbf{r})\hat{\mathcal{O}}_b(\tau',\mathbf{r}')\rangle, \qquad (8)$$

where

$$\hat{\mathcal{O}}_a(\tau,\mathbf{r}) = e^{\beta\hat{H}}\hat{\mathcal{O}}_a(\mathbf{r})e^{-\beta\hat{H}}. \qquad (9)$$

In the Eqs. (7) and (8) the angular brackets denote thermal averaging

$$\langle\cdots\rangle = \frac{1}{Z}\text{Tr}(e^{-\beta\hat{H}}\cdots). \qquad (10)$$

Specifically, the frequency Fourier transforms

$$S_{ab}^{ret}(\omega;\mathbf{r},\mathbf{r}') = \int_{-\infty}^\infty dt e^{i\Omega t} S_{ab}^{ret}(t,\mathbf{r},\mathbf{r}'), \qquad (11)$$

$$S_{ab}(i\Omega_n;\mathbf{r},\mathbf{r}') = \int_0^\beta d\tau e^{i\Omega_n\tau} S_{ab}(\tau,\mathbf{r},\mathbf{r}') \qquad (12)$$

satisfy

$$S_{ab}^{ret}(\Omega;\mathbf{r},\mathbf{r}') = S_{ab}(i\Omega_n \to \Omega + i0^+;\mathbf{r},\mathbf{r}'), \qquad (13)$$

where the bosonic Matsubara frequency is $\Omega_n = 2\pi n/\beta$ for $n = 0, \pm 1, \pm 2,\ldots$. We will use the above relations in what follows when we focus on the electrical conductivity, in which case the bosonic operator $\hat{\mathcal{O}}$ of interest will be either charge density or charge current.

For completeness we note that for $V(\mathbf{r}) = 0$ the two-particle imaginary time Green's function is

$$\langle\psi(i\omega,\mathbf{k})\psi^\dagger(i\omega',\mathbf{k}')\rangle = \beta\delta_{\omega,\omega'}(2\pi)^2\delta(\mathbf{k}-\mathbf{k}')G_\mathbf{k}(i\omega), \qquad (14)$$

where

$$G_\mathbf{k}(i\omega) = \frac{i\omega + \sigma\cdot\mathbf{k}}{\omega^2 + \mathbf{k}^2}, \qquad (15)$$

which will be used extensively in the later sections. Strictly speaking, in any solid state system which supports massless Dirac particles, the above propagator is valid only for wavevectors smaller than some cutoff Λ, which depends on the physical situation. In the case of electrons on the honeycomb lattice, the order of magnitude of the cutoff, Å^{-1}, is determined by the requirement that the true electronic dispersion does not deviate appreciably from the conical (Dirac).

III. REGULARIZATION SCHEMES FOR MASSLESS DIRAC FERMIONS

Since we are interested in the long distance (low-frequency) behavior of physical quantities, we can use the above low-energy field theory, given by the above Lagrangian with the corresponding propagators, provided that divergent terms in the perturbation theory are properly regularized. In the context of high energy physics it is also well known that a quantum-field theory of Dirac fermions needs to be regularized[20] and typically there is no unique way of doing so. Additional requirements, usually based on the symmetries of the theory, determine what type of regularization should be employed.

In case of the theory of the Coulomb interacting Dirac fermions, we will require that the $U(1)$ gauge symmetry must be preserved, or equivalently, that the charge must be conserved. As we show below, dimensional regularization introduced by 't Hooft and Veltman[29] is consistent with this requirement. Before discussing this regularization scheme, let us briefly review the hard cutoff and the Pauli-Villars regularization schemes in the context of the fermionic field theory considered here.

A. Hard cutoff

The idea of the hard-cut-off regularization is to impose a cutoff in the upper limit of an otherwise divergent momentum integral. Physically, this is due to the **k**-space restriction on the modes which appear in the theory, a condition which appears naturally within Wilson formulation of the RG.[30] The singular part of the integral then appears dependent on the cut-off scale. Although very simple to implement, this regularization scheme is known to violate $U(1)$ gauge symmetry of QED, for instance.[20] Terms that violate the gauge symmetry appear as a power of the cut-off scale and must be subtracted in order to ensure that the gauge symmetry is preserved. On the other hand, the typical divergent terms appear as the logarithm of the cut-off scale. Of course, the cut-off scale must not appear explicitly in any observable quantity in order for the theory to be physically meaningful. The disappearance of the cut-off scale Λ indeed occurs in the calculation of the interaction correction to the ac conductiv-

ity within quantum-field theory of the Coulomb interacting Dirac fermions, as discussed below Eq. (1). However, as we show in Appendix D, and as was anticipated in Ref. 18, the hard-cut-off regularization violates the Ward-Takahashi identity. We are thus led to conclude that this regularization scheme is, in principle, not consistent with $U(1)$ gauge symmetry of the theory. This conclusion notwithstanding, the particular coefficient $\mathcal{C}$ from the introduction may be written as an integral which is unambiguous and perfectly convergent in the upper limit, provided the momentum cutoff is taken to infinity after *all* the integrals have been performed (see Appendix H).

B. Pauli-Villars regularization

Another way to regularize divergent self-energy and vertex diagrams in QED is to introduce an additional artificial "heavy photon."[20] In Euclidean spacetime this leads to the following replacement of the photon propagator:

$$\frac{1}{\Omega^2 + \mathbf{k}^2} \rightarrow \frac{1}{\Omega^2 + \mathbf{k}^2} - \frac{1}{\Omega^2 + \mathbf{k}^2 + M^2}$$

and the mass parameter M is sent to ∞ at the end of the calculation. Since the additional fictitious particle couples minimally to the fermions, the regularization preserves Ward-Takahashi identities which relate the self-energy to the current vertex. However, as such, this regularization is unable to render photon polarization diagrams finite. This can be avoided by introducing additional Pauli-Villars fermions,[31] at the expense of making the method complicated.[20]

In the context of the $(2+1)D$ massless Dirac fermions interacting with static (nonretarded) $1/r$ Coulomb interaction, the analog of the Pauli-Villars regularization is

$$\frac{1}{|\mathbf{k}|} \rightarrow \frac{1}{|\mathbf{k}|} - \frac{1}{\sqrt{\mathbf{k}^2 + M^2}}.$$

Physically, this corresponds to cutting off the short-distance divergence of the $1/r$ interaction, without affecting its long range tail. This modified interaction preserves Ward-Takahashi identities relating vertex and the self-energy,[18] but, just as in the case of QED, it fails to regularize the polarization function without introducing additional Pauli-Villars fermions. Therefore, as such it cannot serve as a complete regularization of the theory.

C. Dimensional regularization

Originally introduced in the context of relativistic quantum-field theory, the basic idea of the dimensional regularization is to regularize four-momentum integrals by lowering the number of spacetime dimensions over which the integral is performed. This procedure was introduced by 't Hooft and Veltman[29] to preserve the symmetries of gauge theories. It also bypasses the necessity to introduce Pauli-Villars fermions and bosons.

Here we employ a variant of the dimensional regularization scheme in that the frequency integrals are performed from $-\infty$ to $+\infty$ while the momentum integrals are analyti-

cally continued from $D=2$ to $D=2-\epsilon$ dimensions. Such separation of time from space is used because in the case considered here the Lorentz invariance is violated by the interaction terms. A momentum integral is therefore calculated for an arbitrary number of dimensions D, and expanded in the parameter ϵ. Singular parts of the integral then appear as the first-order poles in the Laurent expansion over the parameter ϵ, i.e., as terms of the form $1/\epsilon$, and the finite part is the term of order ϵ^0 in this expansion.

The following D-dimensional (Euclidean) integrals are frequently encountered in this regularization scheme[20]

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 + \Delta)^n} = \frac{\Gamma\left(n - \frac{D}{2}\right)}{(4\pi)^{D/2} \Gamma(n)} \frac{1}{\Delta^{n-(D/2)}} \qquad (16)$$

and

$$\int \frac{d^D \ell}{(2\pi)^D} \frac{\ell^2}{(\ell^2 + \Delta)^n} = \frac{1}{(4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma\left(n - \frac{D}{2} - 1\right)}{\Gamma(n)} \frac{1}{\Delta^{n-(D/2)-1}}, \qquad (17)$$

where $\Gamma(x)$ is the Euler gamma function and $\Delta \geq 0$.

Furthermore, Pauli matrices are also embedded in $D=2-\epsilon$-dimensional space. We thus use the following identity:

$$\sigma_a \sigma_\mu \sigma_a = D\delta_{0\mu} + (2 - D)\sigma_a \delta_{a\mu}, \qquad (18)$$

where the sum over the Latin letters a, b, used only for the spatial indices, is assumed. The Greek letters μ, ν are reserved for the spacetime indices. The last term on the right-hand side turns out to be crucial for the proof of the Ward-Takahashi identity, guaranteed by the $U(1)$ charge conservation. This is discussed in later sections. In short, the last term in Eq. (18) yields the last term in Eq. (A13). If the latter were omitted the Ward-Takahashi identity would be violated. As elaborated on in the discussion section, the same term also accounts for the discrepancy between the results found in this work, Eqs. (82), and the result we found previously [Eq. (G29)] for the Coulomb interaction correction to the conductivity, where the last term was omitted. Details of this calculation can be found in Appendix E.

IV. CONSERVATION LAWS, CONDUCTIVITY, AND THE STRUCTURE OF THE POLARIZATION TENSOR

In the interest of self-containment, in this section we review some well-known results regarding response functions and $U(1)$ conservation laws. Most of these results can be found (scattered) in many body—quantum-field theory textbooks.[20,28]

In order to calculate the response functions to external electromagnetic fields, it is useful to define the imaginary time-correlation function

$$\Pi_{\mu\nu}(\tau, \mathbf{r}) = \langle T_\tau j_\mu(\tau, \mathbf{r}) j_\nu(0,0) \rangle, \qquad (19)$$

where the current "three-vector" j_μ is composed of the imaginary time density and current as

$$j(\tau,\mathbf{r}) = [\rho(\tau,\mathbf{r}),\mathbf{j}(\tau,\mathbf{r})] = [\psi^\dagger(\tau,\mathbf{r})\psi(\tau,\mathbf{r}), v_F\psi^\dagger(\tau,\mathbf{r})\vec{\sigma}\psi(\tau,\mathbf{r})].$$
$$(20)$$

In this section we temporarily restore v_F to clearly distinguish it from the speed of light c used below.

By fluctuation-dissipation theorem,[28] the expectation value of the electrical current-density operator $\mathbf{J}(t,\mathbf{r})$, in real time t, is related to the imaginary time correlator $\Pi_{\mu\nu}(i\Omega,\mathbf{q})$. The latter is the Fourier transform in Eq. (12) of the tensor defined in Eq. (19). The expectation value of the Fourier transform of the electrical current-density is then

$$\langle J_a(\Omega,\mathbf{q})\rangle = -\frac{e^2}{\hbar}\Pi_{a0}(i\Omega_n \to \Omega + i0, \mathbf{q})\Phi(\Omega,\mathbf{q})$$
$$+ \frac{e^2}{\hbar}\Pi_{ab}(i\Omega_n \to \Omega + i0, \mathbf{q})\frac{A_b(\Omega,\mathbf{q})}{c}. \quad (21)$$

The Fourier components of the electric and magnetic fields are related to the ones of the scalar and vector potentials as

$$E_a(\Omega,\mathbf{q}) = i\frac{\Omega}{c}A_a(\Omega,\mathbf{q}) - iq_a\Phi(\Omega,\mathbf{q}), \quad (22)$$

$$B(\Omega,\mathbf{q}) = i\epsilon_{ab}q_a A_b(\Omega,\mathbf{q}), \quad (23)$$

where ϵ_{ab} is completely antisymmetric (Levi-Civita) rank two tensor. Using Faraday's law of induction we can further relate the Fourier components of the electric and magnetic fields as

$$\epsilon_{ab}q_a E_b(\Omega,\mathbf{q}) = \frac{\Omega}{c}B(\Omega,\mathbf{q}). \quad (24)$$

In condensed matter systems with massless Dirac particles, propagating with velocity v_F, as the relevant low-energy degrees of freedom considered here, the (pseudo) Lorentz invariance is violated by interactions. If we were to consider finite temperature T the (pseudo) Lorentz invariance would be violated even in the noninteracting limit. Nevertheless, when spatial $O(2)$ rotational invariance is preserved, as is the case for problems studied here, the general structure of the imaginary time polarization tensor is[32]

$$\Pi_{\mu\nu}(i\Omega_n,\mathbf{q}) = \Pi_A(i\Omega_n,|\mathbf{q}|)A_{\mu\nu} + \Pi_B(i\Omega_n,|\mathbf{q}|)B_{\mu\nu}, \quad (25)$$

where the three tensors are

$$B_{\mu\nu} = \delta_{\mu a}\left(\delta_{ab} - \frac{q_a q_b}{\mathbf{q}^2}\right)\delta_{b\nu}, \quad (26)$$

$$A_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} - B_{\mu\nu}. \quad (27)$$

The Euclidean three-momenta appearing in the above tensors are

$$g_{\mu\nu} = \mathrm{diag}[-1,1,1]_{\mu\nu}, \quad (28)$$

$$q_\mu = g_{\mu\nu}(-i\Omega_n,\mathbf{q})_\nu = (i\Omega_n,\mathbf{q})_\mu, \quad (29)$$

$$q^2 = q_\mu g_{\mu\nu}q_\nu = \Omega_n^2 + \mathbf{q}^2. \quad (30)$$

The real-time continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{J} = 0 \quad (31)$$

requires that, with our choice of the imaginary time "three" current $j=(\rho,\vec{j})$, the transversality of the $\Pi_{\mu\nu}(i\Omega,\mathbf{q})$ is equivalent to the condition

$$(-i\Omega,\mathbf{q})_\mu\Pi_{\mu\nu}(i\Omega,\mathbf{q}) = \Pi_{\mu\nu}(i\Omega,\mathbf{q})(-i\Omega,\mathbf{q})_\nu = 0. \quad (32)$$

Note that this is explicitly satisfied by the expression (25). If, in addition, the Lorenz invariance is satisfied, $\Pi_A=\Pi_B$, and there is no need to separate out the spatially transverse component of the polarization tensor.

A. Ward-Takahashi identity and vertex functions

In addition to the condition (32), the continuity Eq. (31) constrains the form of the vertex function. If we define the four-point matrix function

$$\pi_\mu(\mathbf{r}'-\mathbf{r},\tau'-\tau;\mathbf{r}-\mathbf{r}'',\tau-\tau'')$$
$$= \langle T_\tau j_\mu(\tau,\mathbf{r})\psi(\tau',\mathbf{r}')\psi^\dagger(\tau'',\mathbf{r}'')\rangle, \quad (33)$$

where the imaginary time three current was defined in Eq. (20), then we must have[20]

$$\left(\frac{\partial}{\partial\tau},\frac{\nabla}{i}\right)_\mu \pi_\mu(\tau'-\tau,\mathbf{r}'-\mathbf{r};\tau-\tau'',\mathbf{r}-\mathbf{r}'')$$
$$= [\delta(\tau-\tau'')\delta^D(\mathbf{r}-\mathbf{r}'') - \delta(\tau'-\tau)\delta^D(\mathbf{r}'-\mathbf{r})]$$
$$\times \mathcal{G}(\tau'-\tau'',\mathbf{r}'-\mathbf{r}''). \quad (34)$$

The above expression relates the exact imaginary time four-point function to the *exact* imaginary time Green's function

$$\mathcal{G}(\tau,\mathbf{r}) = \langle T_\tau\psi(\tau,\mathbf{r})\psi^\dagger(0,0)\rangle. \quad (35)$$

If we rewrite the Fourier transform of π_μ in terms of the vertex function Λ_μ as

$$\pi_\mu(\mathbf{k},i\omega;\mathbf{k}+\mathbf{q},i\omega+i\Omega) = \mathcal{G}_\mathbf{k}(i\omega)\Lambda_\mu(\mathbf{k},i\omega;\mathbf{k}+\mathbf{q},i\omega$$
$$+ i\Omega)\mathcal{G}_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega) \quad (36)$$

then the Ward-Takahashi identity for the vertex function Λ_μ can be written as

$$(-i\Omega,\mathbf{q})_\mu\Lambda_\mu(\mathbf{k},i\omega;\mathbf{k}+\mathbf{q},i\omega+i\Omega) = \mathcal{G}_{\mathbf{k}+\mathbf{q}}^{-1}(i\omega+i\Omega) - \mathcal{G}_\mathbf{k}^{-1}(i\omega)$$
$$= \Sigma_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega) - \Sigma_\mathbf{k}(i\omega). \quad (37)$$

This identity has to be satisfied order by order in perturbation theory. In what follows, we show that this is indeed the case for the interacting theories studied here when we adopt the dimensional regularization.

B. Electrical conductivity

To relate the polarization tensor to the electrical conductivity, we simply need to relate the expectation value of the current to the electric field. Since we have the response to the

electromagnetic scalar and vector potentials, we just need to relate those to the electric and magnetic fields. Finally, magnetic field can be related to the electric field using Maxwell's equations. As is well known, at finite wave vector $\mathbf{q}$ and frequency Ω, one can define the longitudinal and transverse conductivity as the proportionality between the induced current and the longitudinal or transverse component of the electric field.

Using Eqs. (21) and (25)–(27), we find

$$
\begin{aligned}
\langle J_a(\Omega,\mathbf{q})\rangle &= \frac{e^2}{\hbar}\Pi_A(\Omega+i0,|\mathbf{q}|)\frac{\Omega q_a}{\mathbf{q}^2-\Omega^2}\Phi(\Omega,\mathbf{q}) \\
&\quad - \frac{e^2}{\hbar}\Pi_A(\Omega+i0,|\mathbf{q}|)\frac{\Omega^2 q_a q_b}{\mathbf{q}^2(\mathbf{q}^2-\Omega^2)}\frac{A_b(\Omega,\mathbf{q})}{c} \\
&\quad + \frac{e^2}{\hbar}\Pi_B(\Omega+i0,|\mathbf{q}|)\left(\delta_{ab}-\frac{q_a q_b}{\mathbf{q}^2}\right)\frac{A_b(\Omega,\mathbf{q})}{c}.
\end{aligned}
\tag{38}
$$

Furthermore, Eqs. (22)–(24) imply

$$
\begin{aligned}
\langle J_a(\Omega,\mathbf{q})\rangle &= \frac{e^2}{\hbar}\Pi_A(\Omega+i0,|\mathbf{q}|)\frac{i\Omega}{\mathbf{q}^2-\Omega^2}\frac{q_a q_b}{\mathbf{q}^2}E_b(\Omega,\mathbf{q}) \\
&\quad + \frac{e^2}{\hbar}\Pi_B(\Omega+i0,|\mathbf{q}|)\frac{1}{i\Omega}\left(\delta_{ab}-\frac{q_a q_b}{\mathbf{q}^2}\right)E_b(\Omega,\mathbf{q}).
\end{aligned}
\tag{39}
$$

From the above equations we can read off the longitudinal and transverse electrical conductivity

$$
\sigma_\parallel(\Omega,|\mathbf{q}|)=\frac{e^2}{\hbar}\frac{i\Omega}{\mathbf{q}^2-\Omega^2}\Pi_A(\Omega+i0,|\mathbf{q}|),
\tag{40}
$$

$$
\sigma_\perp(\Omega,|\mathbf{q}|)=\frac{e^2}{\hbar}\frac{\Pi_B(\Omega+i0,|\mathbf{q}|)}{i\Omega}.
\tag{41}
$$

For $\mathbf{q}\neq 0$ (and $\Omega\neq 0$), $\sigma_\parallel$ need not be equal $\sigma_\perp$. However, at $\mathbf{q}=0$, the ac conductivities

$$
\sigma_\parallel(\Omega,\mathbf{q}=0)=\sigma_\perp(\Omega,\mathbf{q}=0)
\tag{42}
$$

due to the $O(2)$ spatial rotational symmetry.

In the following, we will work solely in the imaginary time—Matsubara frequency space, and since we restrict ourselves to $T=0$, we will drop the subscript n on $i\Omega_n$.

V. NONINTERACTING LIMIT: $V(\mathbf{r})=0$

For the sake of completeness, and in order to illustrate how the general results presented in the previous section appear in the specific solvable problem, we first examine $\Pi_{\mu\nu}(i\Omega,\mathbf{q})$ in the limit of vanishing $V(\mathbf{r})$. The Fourier transform in Eq. (12) of the polarization function in Eq. (19) in the noninteracting limit is easily shown to be

$$
\begin{aligned}
\Pi_{\mu\nu}^{(0)}(i\Omega,\mathbf{q}) = &-N\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^D\mathbf{k}}{(2\pi)^D} \\
&\times\mathrm{Tr}[G_\mathbf{k}(i\omega)\sigma_\mu G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega)\sigma_\nu],
\end{aligned}
\tag{43}
$$

where σ_0 is the 2×2 unit matrix. To this end it is useful to define the vertex function

$$
\mathcal{P}_\mu(\mathbf{q},i\Omega)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^D\mathbf{k}}{(2\pi)^D}G_\mathbf{k}(i\omega)\sigma_\mu G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega)
\tag{44}
$$

in terms of which

$$
\Pi_{\mu\nu}^{(0)}(i\Omega,\mathbf{q})=-N\,\mathrm{Tr}[\mathcal{P}_\mu(\mathbf{q},i\Omega)\sigma_\nu].
\tag{45}
$$

The above expression is divergent at large momenta (UV divergent) as is easily seen by counting powers. Note that this appears even in the *noninteracting theory* when calculating the response functions. As is well known in the context of relativistic field theories, this UV divergence is unphysical and to obtain the correct answer a regularization is necessary.[20,33] The regularization of choice here is dimensional regularization which leads to finite expressions and which is consistent with $U(1)$ gauge symmetry of the theory.

As shown in detail in the Appendix A, using dimensional regularization, we obtain

$$
\begin{aligned}
\mathcal{P}_\mu(\mathbf{q},i\Omega)=&\frac{\sqrt{\Omega^2+\mathbf{q}^2}}{64} \\
&\times\left[\sigma_\mu-2\delta_{\mu0}-\frac{(i\Omega+\sigma\cdot\mathbf{q})\sigma_\mu(i\Omega+\sigma\cdot\mathbf{q})}{\Omega^2+\mathbf{q}^2}\right].
\end{aligned}
\tag{46}
$$

Performing the trace in Eq. (45) we find

$$
\Pi_{\mu\nu}^{(0)}(i\Omega,\mathbf{q})=-\frac{N}{16\sqrt{\Omega^2+\mathbf{q}^2}}\begin{pmatrix}-\mathbf{q}^2 & -i\Omega q_x & -i\Omega q_y \\ -i\Omega q_x & q_y^2+\Omega^2 & -q_x q_y \\ -i\Omega q_y & -q_x q_y & q_x^2+\Omega^2\end{pmatrix}_{\mu\nu}.
\tag{47}
$$

We can write the above matrix more compactly as

$$
\Pi_{\mu\nu}^{(0)}(i\Omega,\mathbf{q})=-\frac{N}{16}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\sqrt{\Omega^2+\mathbf{q}^2},
\tag{48}
$$

where we used definitions from Eqs. (28)–(30). The correlation function in Eqs. (47) and (48) is explicitly transverse, as it should be, and

$$
(-i\Omega,\mathbf{q})_\mu\Pi_{\mu\nu}^{(0)}(i\Omega,\mathbf{q})=\Pi_{\mu\nu}^{(0)}(i\Omega,\mathbf{q})(-i\Omega,\mathbf{q})_\nu=0.
\tag{49}
$$

From the above equations we find

$$
\Pi_A^{(0)}(i\Omega,\mathbf{q})=\Pi_B^{(0)}(i\Omega,\mathbf{q})=-\frac{N}{16}\sqrt{\Omega^2+\mathbf{q}^2}.
\tag{50}
$$

Analytically continuing according to Eqs. (40) and (41), with the branch cut of the $\sqrt{z}$-function lying along negative real axis, we find the well-known expression for the (Gaussian) ac conductivity

$$\sigma_{\parallel}^{(0)}(\Omega) = \sigma_{\perp}^{(0)}(\Omega) = \frac{N}{16}\frac{e^2}{\hbar}. \qquad (51)$$

As a side remark, if we were to define $\tilde{\Pi}_{\mu\nu}^{(0)}$ as a correlation function of a slightly different three current $(-i\rho, \vec{j})$, the result obtained directly from Eqs. (47) and (48) transforms as tensor under Euclidean $O(3)$ transformations. In real frequencies this is equivalent to relativistic Lorentz transformations due to the invariance of the noninteracting Lagrangian $\mathcal{L}_0$.

Therefore, regulating the UV divergences via dimensional regularization implemented here leads to finite expressions which preserve the required $U(1)$ conservation laws. The necessary regularization of the "integration measure," as done here via dimensional regularization, is independent of the electron-electron interaction $V(\mathbf{r})$, as it must be if the noninteracting theory is to lead to finite-correlation functions. Therefore, as shown already by this example, regulating only the "momentum transfer" as advocated in Refs. 10 and 18 is clearly insufficient.

VI. SHORT-RANGE INTERACTIONS: $V(\mathbf{r}) = u\,\delta(\mathbf{r})$

While the problem of $(2+1)D$ massless Dirac fermions with the contact interactions is not exactly solvable, one can calculate the interaction corrections to the polarization tensor perturbatively in powers of the interaction strength u. Such contact interactions certainly constitute an idealized special case.[34] Nevertheless, this theory has the advantage that one can determine the first correction in u to the noninteracting (Gaussian) polarization tensor $\Pi_{\mu\nu}^{(0)}$, found in the previous section, explicitly for finite $\mathbf{q}$ and Ω. We can then test the general symmetry requirements listed before. The technique of choice is again the (variant of the) dimensional regularization of Veltman and 't Hooft introduced in Sec. III. Since

this interaction violates Lorentz invariance we can also use this example to study how the difference between Π_A and Π_B arises in such theory.

It is straightforward to use the Wick's theorem to show that in this case, the first order in u, and to $\mathcal{O}(N)$, correction to the polarization tensor is

$$\delta\Pi_{\mu\nu}(i\Omega, \mathbf{q}) = uN\int \frac{d^D\mathbf{k}}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int$$
$$\times \frac{d^D\mathbf{p}}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\{\text{Tr}[G_{\mathbf{k}}(i\omega)\sigma_\mu G_{\mathbf{k+q}}(i\omega$$
$$+ i\Omega)G_{\mathbf{p}}(i\omega')\sigma_\nu G_{\mathbf{p-q}}(i\omega' - i\Omega)]$$
$$+ \text{Tr}[G_{\mathbf{k}}(i\omega)\sigma_\mu G_{\mathbf{k+q}}(i\omega + i\Omega)G_{\mathbf{p}}(i\omega')G_{\mathbf{k+q}}(i\omega$$
$$+ i\Omega)\sigma_\nu] + \text{Tr}[G_{\mathbf{k}}(i\omega)\sigma_\mu G_{\mathbf{k+q}}(i\omega$$
$$+ i\Omega)\sigma_\nu G_{\mathbf{k}}(i\omega)G_{\mathbf{p}}(i\omega')]\}. \qquad (52)$$

The last two terms correspond to the self-energy correction while the first one is the vertex correction. Because the self-energy for the contact interaction vanishes

$$\int \frac{d^D\mathbf{k}}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G_{\mathbf{k}}(i\omega) = 0 \qquad (53)$$

the last two terms in the Eq. (52) vanish as well. The remaining term can be written rather succinctly in terms of $\mathcal{P}_\mu$ defined previously in Eq. (44) as

$$\delta\Pi_{\mu\nu}(i\Omega, \mathbf{q}) = uN\,\text{Tr}[\mathcal{P}_\mu(\mathbf{q}, i\Omega)\mathcal{P}_\nu(-\mathbf{q}, -i\Omega)]. \qquad (54)$$

The above expression is manifestly transverse, i.e., it satisfies Eq. (32), as can be seen from Eq. (46).

Namely, inserting Eq. (46) and performing the traces we find

$$\delta\Pi_{\mu\nu}(i\Omega, \mathbf{q}) = \frac{uN}{512(\Omega^2 + \mathbf{q}^2)}\begin{bmatrix} \mathbf{q}^2(\mathbf{q}^2 - \Omega^2) & i\Omega q_x(\mathbf{q}^2 - \Omega^2) & i\Omega q_y(\mathbf{q}^2 - \Omega^2) \\ i\Omega q_x(\mathbf{q}^2 - \Omega^2) & q_x^2(q_y^2 - \Omega^2) + (q_y^2 + \Omega^2)^2 & -q_xq_y(3\Omega^2 + \mathbf{q}^2) \\ i\Omega q_y(\mathbf{q}^2 - \Omega^2) & -q_xq_y(3\Omega^2 + \mathbf{q}^2) & q_x^4 - \Omega^2 q_y^2 + \Omega^4 + q_x^2(q_y^2 + 2\Omega^2) \end{bmatrix}_{\mu\nu}.$$

Finally, the above tensor can be factorized as given by Eqs. (25) and (26), and we find to first nontrivial order in the contact coupling u

$$\Pi_A(i\Omega, |\mathbf{q}|) = -\frac{N}{16}\sqrt{\Omega^2 + \mathbf{q}^2} + \frac{uN}{512}(\Omega^2 - \mathbf{q}^2) + \mathcal{O}(u^2),$$

$$\Pi_B(i\Omega, |\mathbf{q}|) = -\frac{N}{16}\sqrt{\Omega^2 + \mathbf{q}^2} + \frac{uN}{512}(\Omega^2 + \mathbf{q}^2) + \mathcal{O}(u^2). \qquad (55)$$

Expectedly, the above expression shows that the interaction

correction to the polarization functions Π_A and Π_B are *different* (note the sign difference in front of $\mathbf{q}^2$). As stated above, the reason for the difference is that the contact density-density interaction term $u[\psi^\dagger(\mathbf{r})\psi(\mathbf{r})]^2$ breaks the Lorentz invariance of the noninteracting part of the Lagrangian. Lorentz transformations, in general, rotate between density and current, and we have purposefully omitted any current-current interaction.

We can further test the Ward-Takahashi identity in Eq. (37) for the vertex function in Eq. (36) in this example with the short-range interactions. It can be readily seen that the first order in u correction to the vertex vector is

$$\delta\Lambda_\mu(\mathbf{k},i\omega;\mathbf{k}+\mathbf{q},i\omega+i\Omega) = -u\mathcal{P}_\mu(\mathbf{q},i\Omega). \quad (56)$$

It follows from the Eq. (46) that

$$-i\Omega\mathcal{P}_0(\mathbf{q},i\Omega) + \mathbf{q}_a\mathcal{P}_a(\mathbf{q},i\Omega) = 0. \quad (57)$$

Therefore the Ward-Takahashi identities in Eq. (37) are satisfied, since, as mentioned previously in this section, the self-energy correction vanishes to first order in u for the short-range interactions.

Finally, from Eqs. (40) and (41), we can infer that the above terms correct only the imaginary part of the ac conductivity but not the real part. At $\mathbf{q}=0$, correction is the same for the longitudinal and the transverse components, and to this order in u we have

$$\sigma_{\parallel,\perp}(\Omega) = \frac{e^2}{\hbar}\frac{N}{16}\left(1 - i\frac{u}{32}\Omega\right). \quad (58)$$

Again, the equality between $\sigma_\parallel(\Omega)$ and $\sigma_\perp(\Omega)$ is guaranteed due to the $O(2)$ rotational invariance of this theory. Note also that the fact that the interaction correction is proportional to the frequency is implied by the power counting at the Gaussian fixed point of the theory and is characteristic for any finite-range interaction.[35]

VII. COULOMB INTERACTION: $V(\mathbf{r}) = e^2/|\mathbf{r}|$

Armed with the above results we now focus on the main part of the paper where we study the effects of the Coulomb interaction. Unlike in the previous cases, we have been unable to find the explicit expression for the first-order correction to the polarization tensor at finite $\mathbf{q}$ and Ω. Nevertheless, we have been able to show explicitly that the first-order correction to the polarization tensor is transverse, i.e., it satisfies Eq. (32). This is shown using dimensional regularization in $D=2-\epsilon$ introduced in Sec. III. Next, we study the first correction to the Coulomb vertex function which must also satisfy the Ward-Takahashi identity in Eq. (37). Since in this case the first order self-energy is known to diverge logarithmically, the first-order correction to the vertex function should also diverge as $\epsilon\rightarrow0$. This can be found explicitly in terms of elliptic integrals to order ϵ^{-1} and ϵ^0 and the identity (37) is also explicitly confirmed. Finally, we proceed with the calculation of the electrical conductivity, first by using the spatial component of the polarization tensor at $\mathbf{q}=0$ but finite Ω (current-current correlation function) and then by using time component of the polarization tensor at finite but small $\mathbf{q}$ and finite Ω. The final results for the conductivity calculated in both ways are found to be the same. Specifically, we find $\mathcal{C}=(11-3\pi)/6$ in Eq. (1).

For unscreened 3D Coulomb interactions $V(\mathbf{r})=e^2/r$ the effect of screening due to dielectric medium is easily taken into account by rescaling e^2 in the above formula. The $\mathcal{O}(e^2)$ and $\mathcal{O}(N)$ correction to the polarization function is then

$$\delta\Pi_{\mu\nu}^{(c)}(i\Omega,\mathbf{q}) = N\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\{V_{\mathbf{p}-\mathbf{k}}$$
$$\times\text{Tr}[G_\mathbf{k}(i\omega)\sigma_\mu G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega)G_{\mathbf{p}+\mathbf{q}}(i\omega'$$
$$+i\Omega)\sigma_\nu G_\mathbf{p}(i\omega')] + V_{\mathbf{k}-\mathbf{p}}$$

$$\times\text{Tr}[G_\mathbf{k}(i\omega)\sigma_\mu G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega)G_{\mathbf{p}+\mathbf{q}}(i\omega'+i\Omega)$$
$$\times G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega)\sigma_\nu] + V_{\mathbf{k}-\mathbf{p}}$$
$$\times\text{Tr}[G_\mathbf{k}(i\omega)\sigma_\mu G_{\mathbf{k}+\mathbf{q}}(i\omega$$
$$+i\Omega)\sigma_\nu G_\mathbf{k}(i\omega)G_\mathbf{p}(i\omega')]\} \quad (59)$$

where

$$V_\mathbf{k} = \int d^2\mathbf{r}\,e^{i\mathbf{k}\cdot\mathbf{r}}V(\mathbf{r}) = \frac{2\pi e^2}{|\mathbf{k}|}. \quad (60)$$

Just as in the case of contact interactions, the first term in the expression for $\delta\Pi_{\mu\nu}^{(c)}$ corresponds to the vertex correction and the last two terms to the self-energy corrections. Unlike in the case of contact interactions, however, the self-energy correction does not vanish. The expression (59) will be used in later sections as a starting point in the calculation of the Coulomb interaction correction to the ac conductivity in the collisionless regime.

A. Proof of the transversality of $\delta\Pi_{\mu\nu}^{(c)}$ within dimensional regularization

Because, as mentioned above, the explicit evaluation of Eq. (59) at finite $\mathbf{q}$ and Ω yields intractable expressions, we proceed by first showing that Eq. (59) is transverse, i.e., that it satisfies the condition (32), when dimensional regularization employed in this paper is used. As such it therefore *does not* lead to any violation of the charge conservation, a virtue questioned in Ref. 18. By two-dimensional rotational invariance, this in turn implies that the Coulomb polarization tensor can be written in the form of Eq. (25).

To prove Eq. (32) we follow Ref. 18 and use

$$-i\Omega\sigma_0 + \mathbf{q}\cdot\sigma = G_{\mathbf{k}+\mathbf{q}}^{-1}(i\omega+i\Omega) - G_\mathbf{k}^{-1}(i\omega) \quad (61)$$

to find

$$(-i\Omega,\mathbf{q})_\mu\delta\Pi_{\mu\nu}^{(c)}(i\Omega,\mathbf{q}) = N\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{d^D\mathbf{p}}{(2\pi)^D}\int_{-\infty}^{\infty}$$

$$\times\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}V_{\mathbf{p}-\mathbf{k}}\{\text{Tr}[G_\mathbf{k}(i\omega)\sigma_\nu$$
$$\times G_\mathbf{k}(i\omega)G_\mathbf{p}(i\omega')] - \text{Tr}[G_{\mathbf{k}+\mathbf{q}}(i\omega$$
$$+i\Omega)\sigma_\nu G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega)G_{\mathbf{p}+\mathbf{q}}(i\omega'$$
$$+i\Omega)]\}.$$

At this point it is not immediately obvious that we can shift the integration variables $\mathbf{k}$ and $\mathbf{p}$ in the second term by $\mathbf{q}$, which if true would readily yield the desired relation (32) since the frequency integral can be shifted. We therefore define a function of frequency and *two* momentum variables

$$\Sigma_{\mathbf{p},\mathbf{q}}(i\Omega) = \int\frac{d^D\mathbf{k}}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}V_{\mathbf{k}-\mathbf{p}}G_{\mathbf{k}+\mathbf{q}}(i\omega'+i\Omega) \quad (62)$$

in terms of which we have unambiguously

$$(-i\Omega,\mathbf{q})_\mu \delta\Pi^{(c)}_{\mu\nu}(i\Omega,\mathbf{q})$$

$$= N \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \{ \mathrm{Tr}[G_\mathbf{k}(i\omega)\sigma_\nu G_\mathbf{k}(i\omega)\Sigma_{\mathbf{k},0}(0)]$$

$$- \mathrm{Tr}[G_{\mathbf{k+q}}(i\omega+i\Omega)\sigma_\nu G_{\mathbf{k+q}}(i\omega+i\Omega)\Sigma_{\mathbf{k},\mathbf{q}}(i\Omega)].$$

To continue, we need to find an explicit expression for $\Sigma_{\mathbf{p},\mathbf{q}}(i\Omega)$. Using the identity (16), Feynman parametrization

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dy \frac{y^{\alpha-1}(1-y)^{\beta-1}}{[yA+(1-y)B]^{\alpha+\beta}} \quad (63)$$

for $\alpha=\beta=1/2$ and

$$\int_0^1 dy\, y^{\alpha-1}(1-y)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (64)$$

we find

$$\Sigma_{\mathbf{p},\mathbf{q}}(i\Omega)$$

$$= \frac{e^2}{(4\pi)^{D/2}} \frac{\sigma\cdot(\mathbf{p+q})}{|\mathbf{p+q}|^{2-D}} \frac{\Gamma\left(1-\frac{D}{2}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\Gamma(D)}, \quad (65)$$

which agrees with Eq. (12) of Ref. 36. Note that this identity shows that within dimensional regularization, $\Sigma_{\mathbf{p},\mathbf{q}}(i\Omega) = \Sigma_{\mathbf{p+q},0}(i\Omega)$. Moreover, in what follows, there is no need to shift the integration variable. Rather, since the commutator of the self-energy and the Green's function vanishes

$$[G_{\mathbf{k+q}}(i\omega+i\Omega), \Sigma_{\mathbf{p},\mathbf{q}}(i\Omega)] = 0 \quad (66)$$

after a straightforward use of the cyclic property of the trace and the identity

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_{\mathbf{k+q}}(i\omega+i\Omega) G_{\mathbf{k+q}}(i\omega+i\Omega) = 0 \quad (67)$$

we prove that

$$(-i\Omega,\mathbf{q})_\mu \delta\Pi^{(c)}_{\mu\nu}(i\Omega,\mathbf{q}) = 0. \quad (68)$$

The same procedure as the one used above also leads to

$$\delta\Pi^{(c)}_{\mu\nu}(i\Omega,\mathbf{q})(-i\Omega,\mathbf{q})_\nu = 0. \quad (69)$$

This proof holds to all orders of ϵ. The regularization technique implemented here is therefore perfectly adequate and does not lead to violation of the charge conservation.

B. Coulomb vertex and the proof of the Ward-Takahashi identity

Next, we will demonstrate that the dimensional regularization used here preserves the Ward-Takahashi identity for the Coulomb vertex function. This proof is technically more involved than the proof in the previous section, but nevertheless, we find it important to present its details since our technique is not widely used in the community. We show the desired identity to order ϵ^{-1} and ϵ^0. Most of the technical details are presented in the Appendices A–C and in this sec-

tion we just present the main steps of the derivation.

The Coulomb vertex function to the first order in the coupling constant is

$$\delta\Lambda_\mu(\mathbf{p},i\nu;\mathbf{p+q},i\nu+i\Omega)$$

$$= \mathcal{P}^c_\mu(\mathbf{q},\mathbf{p},i\Omega)$$

$$= -\int \frac{d^D\mathbf{k}}{(2\pi)^D} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} V_{\mathbf{p-k}} G_\mathbf{k}(i\omega)\sigma_\mu G_{\mathbf{k+q}}(i\omega+i\Omega). \quad (70)$$

The integrals on the right are logarithmically divergent in $D=2$ as can be easily seen by powercounting. This divergence is related to the divergence of the electron self-energy, calculated in the previous section

$$\Sigma_\mathbf{k}(i\omega) \equiv \Sigma_{\mathbf{k},0}(i\omega)$$

$$= \frac{e^2}{8}\sigma\cdot\mathbf{k}\left[\frac{2}{\epsilon} - \gamma + \ln 64\pi - \ln\mathbf{k}^2 + \mathcal{O}(\epsilon)\right], \quad (71)$$

where the Euler-Mascheroni constant $\gamma=0.577$ and, as before, $\epsilon=2-D$. Indeed, if the Ward-Takahashi identity

$$(-i\Omega,\mathbf{q})_\mu \mathcal{P}^c_\mu(\mathbf{q},\mathbf{p},i\Omega) = \Sigma_{\mathbf{p+q}}(i\nu+i\Omega) - \Sigma_\mathbf{p}(i\nu) \quad (72)$$

is to be satisfied, the vertex function must diverge logarithmically, which manifests in the dimensional regularization as the first-order pole in Laurent expansion in the parameter ϵ.

In the second part of the Appendix A we use dimensional regularization to determine $\mathcal{P}^c(\mathbf{q},\mathbf{p},i\Omega)$ to orders ϵ^{-1} and ϵ^0. Our final expression for finite $\mathbf{q}$, $\mathbf{p}$, and $i\Omega$, Eq. (A18), is left as an integral over a Feynman parameter x. We wish to stress that all of the integrals in this equation can be performed in the closed form in terms of elliptic integrals. However, we found that doing so leads to intractable and unrevealing expressions. We therefore chose to work with the expression (A18) and in effect manipulate the integral representation of the elliptic integrals. In the limiting case of $\mathbf{q}=0$, the vertex function is determined in the closed form in Appendix A up to, and including, ϵ^0.

In Appendix B we in turn find that the vertex function in Eq. (A18) satisfies

$$(-i\Omega,\mathbf{q})_\mu \mathcal{P}^c_\mu(\mathbf{q},\mathbf{p},i\Omega) = N(\mathbf{p},\mathbf{q}) - \frac{e^2}{4}\sqrt{\Omega^2+\mathbf{q}^2}$$

$$\times \sigma\cdot[\mathbf{p}(\Omega^2+\mathbf{q}^2)L(\mathbf{p},\mathbf{q},\Omega)$$

$$+ \mathbf{q}M(\mathbf{p},\mathbf{q},\Omega)]. \quad (73)$$

Using the dimensional regularization, we then show that the function

$$N(\mathbf{p},\mathbf{q}) = \frac{e^2}{8}\left(\frac{2}{\epsilon}\sigma\cdot\mathbf{q} + (\ln 64\pi - \gamma)\sigma\cdot\mathbf{q}\right.$$

$$\left. - \sigma\cdot(\mathbf{p+q})\ln(\mathbf{p+q})^2 + \sigma\cdot\mathbf{p}\ln\mathbf{p}^2\right)$$

$$= \Sigma_{\mathbf{p+q}}(i\nu+i\Omega) - \Sigma_\mathbf{p}(i\nu) \quad (74)$$

and, in Appendix C, that $L(\mathbf{p},\mathbf{q},\Omega)=M(\mathbf{p},\mathbf{q},\Omega)=0$. This

proves the Ward-Takahashi identity to the first order in perturbation theory.

C. Calculation of the ac conductivity from the current-current correlator (Kubo formula)

In this section we calculate the diagonal spatial component of the Coulomb interaction correction of the polarization tensor, $\delta\Pi_{xx}^{(c)}$, at $\mathbf{q}=0$ and finite $i\Omega$. We then use this result to calculate the corresponding correction to the electrical conductivity. Given the decomposition in Eq. (25), one should, in principle, specify the direction in the $\mathbf{q}$ plane along which the limit $\mathbf{q}\to 0$ is taken. For example, if q_x is taken to 0 before q_y, $\delta\Pi_{xx}^{(c)}(i\Omega,0)$ is proportional to $\Pi_B(i\Omega,0)$. On the other hand, if q_y is taken to 0 before q_x, then $\delta\Pi_{xx}^{(c)}(i\Omega,0)$ is proportional to $\Pi_A(i\Omega,0)$. Similarly, if the limit is taken along a line that forms an angle θ with the q_x axis, then $\delta\Pi_{xx}^{(c)}(i\Omega)$ is proportional to $\cos^2\theta\Pi_A(i\Omega,0)$ $+\sin^2\theta\Pi_B(i\Omega,0)$. However, due to the $O(2)$ rotational invariance, $\Pi_A(i\Omega,0)=\Pi_B(i\Omega,0)$ and the result is independent of θ. We can therefore use either Eq. (40) and (41) along with diagonal spatial part of the polarization tensor in Eq. (59) to calculate the ac conductivity.

We start by showing that

$$\delta\Pi_{\mu\nu}^{(c)}(\mathbf{q}=0,i\Omega=0)=0. \tag{75}$$

This is expected, since a space and time-independent vector and scalar potential correspond to a pure gauge, and as such have no effect on the physics of the problem. Within our formalism, this identity can be shown to the first order in the Coulomb interaction by first performing the integral over the frequencies in Eq. (59), which, as can be easily seen, yields

$$\begin{aligned}
\delta\Pi_{\mu\nu}^{(c)}(i\Omega=0,\mathbf{q}=0) &= \frac{N}{4}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{1}{|\mathbf{p}|} \\
&\quad\times\mathrm{Tr}\left[\mathcal{P}_\mu^{(c)}(0,\mathbf{p},0)\left(\sigma_\nu-\frac{\sigma\cdot\mathbf{p}\,\sigma_\nu\,\sigma\cdot\mathbf{p}}{\mathbf{p}^2}\right)\right] \\
&\quad+\frac{N}{8}\times\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{1}{|\mathbf{k}|^3} \\
&\quad\times\mathrm{Tr}[(\sigma\cdot\mathbf{k}\,\sigma_\mu-\sigma_\mu\sigma\cdot\mathbf{k}) \\
&\quad\times(\Sigma_\mathbf{k}\sigma_\nu-\sigma_\nu\Sigma_\mathbf{k})].
\end{aligned}$$

Substituting the expression for the self-energy in Eq. (71) and the static vertex in Eq. (A20), performing the traces and using $k_a k_b\to\delta_{ab}\mathbf{k}^2/D$, we conclude that Eq. (75) holds, as it should. We are therefore free to subtract it from the expression for $\delta\Pi_{\mu\nu}^{(c)}$ at either finite Ω and/or finite $\mathbf{q}$.

Next, we set $\mathbf{q}=0$ and consider finite Ω in Eq. (59). The polarization tensor can be written as sum of the contributions from the self-energy correction and the vertex correction

$$\delta\Pi_{\mu\nu}^{(c)}(i\Omega,0)=\delta\Pi_{\mu\nu}^{(a)}(i\Omega,0)+\delta\Pi_{\mu\nu}^{(b)}(i\Omega,0), \tag{76}$$

where the self-energy is given by

$$\delta\Pi_{\mu\nu}^{(a)}(i\Omega,0)=2N\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}},$$

$$\mathrm{Tr}[G_\mathbf{k}(i\omega)\sigma_\mu G_\mathbf{k}(i\omega+i\Omega)\sigma_\nu G_\mathbf{k}(i\omega)G_\mathbf{p}(i\omega')] \tag{77}$$

and the vertex correction reads

$$\delta\Pi_{\mu\nu}^{(b)}(i\Omega,0)=N\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}},$$

$$\mathrm{Tr}[G_\mathbf{k}(i\omega)\sigma_\mu G_\mathbf{k}(i\omega+i\Omega)G_\mathbf{p}(i\omega'+i\Omega)\sigma_\nu G_\mathbf{p}(i\omega')]. \tag{78}$$

Both of these expressions need to be regulated due to the UV divergence.

In Appendix E we calculate both of these contributions to the electrical conductivity. The contribution to the conductivity coming from the self-energy part expanded up to the order ϵ^0 is found to be

$$\sigma_a=\frac{\sigma_0 e^2}{2}\left[-\frac{1}{\epsilon}+\frac{3}{2}+\gamma-\ln(64\pi)\right]. \tag{79}$$

The corresponding vertex part is found to be

$$\sigma_b=\frac{\sigma_0 e^2}{2}\left[\left(\frac{1}{\epsilon}-\frac{1}{2}-\gamma+\ln 64\pi\right)+\frac{8-3\pi}{3}\right]. \tag{80}$$

Adding these two terms we obtain the first order correction to the ac conductivity due to the Coulomb interaction

$$\delta\sigma^{(c)}=\sigma_a+\sigma_b=\frac{11-3\pi}{6}\sigma_0 e^2, \tag{81}$$

which corresponds to the value

$$\mathcal{C}=\frac{11-3\pi}{6} \tag{82}$$

in Eq. (1). We discuss this result in light of previous work as well as present day experiments in the concluding section.

D. Calculation of the ac conductivity using the density-density correlator

To show that our previous result for the conductivity is consistent, we now calculate the longitudinal conductivity given by Eq. (40) and show that it yields the same value of the constant $\mathcal{C}$ as in Eq. (82). This must be the case if the Ward-Takahashi identity and the $O(2)$ rotational invariance hold. The longitudinal correction to the conductivity can be calculated by focusing on $\delta\Pi_{00}^{(c)}$ since $B_{00}=0$. Unlike for Kubo formula, this component of the polarization tensor must be calculated at finite Ω and finite $\mathbf{q}$ since at $\mathbf{q}=0$ it vanishes. Fortunately, we need only the leading order term in the expansion in small $\mathbf{q}^2$, from which we can extract the conductivity.

According to Eq. (59), the Coulomb interaction correction to the density-density correlator reads

$$\delta\Pi_{00}^{(c)}(i\Omega,\mathbf{q})=\delta\Pi_{00}^{(a)}(i\Omega,\mathbf{q})+\delta\Pi_{00}^{(b)}(i\Omega,\mathbf{q}), \tag{83}$$

where, just as before, we have separated the self-energy contribution

$$\delta\Pi_{00}^{(a)}(i\Omega,\mathbf{q}) = N\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d^2\mathbf{p}}{(2\pi)^2}V_{\mathbf{k}-\mathbf{p}}$$
$$\times\{\text{Tr}[G_{\mathbf{k}}(i\omega)G_{\mathbf{p}}(i\omega')G_{\mathbf{k}}(i\omega)G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega)]$$
$$+\text{Tr}[G_{\mathbf{k}}(i\omega)G_{\mathbf{p}}(i\omega')G_{\mathbf{k}}(i\omega)G_{\mathbf{k}-\mathbf{q}}(i\omega-i\Omega)]\}$$
(84)

and the vertex contribution

$$\delta\Pi_{00}^{(b)}(i\Omega,\mathbf{q}) = N\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d^2\mathbf{p}}{(2\pi)^2}V_{\mathbf{k}-\mathbf{p}}$$
$$\times\text{Tr}[G_{\mathbf{k}}(i\omega)G_{\mathbf{p}}(i\omega')G_{\mathbf{p}-\mathbf{q}}(i\omega'-i\Omega)$$
$$\times G_{\mathbf{k}-\mathbf{q}}(i\omega-i\Omega)].$$
(85)

The details of the calculations are presented in the Appendix F. Here we just state the final result

$$\delta\sigma_{\parallel}^{(c)} = \frac{11-3\pi}{6}\sigma_0 e^2$$
(86)

in agreement with the result in Eq. (81) obtained from the current-current correlator. Such agreement is expected since, as we have shown to this order in the Coulomb interaction, the dimensional regularization explicitly preserves the $U(1)$ gauge symmetry of the theory of the Coulomb interacting Dirac fermions.

VIII. DISCUSSION AND CONNECTION WITH PREVIOUS WORK

Let us now discuss the result in Eq. (82) for the correction to the ac conductivity due to the long-range Coulomb interaction in light of the ones previously reported in the literature.[9,10,18]

In Ref. 9, the Coulomb correction to the conductivity is shown to have the form given by Eq. (1), consistent with the renormalizability of the quantum-field theory of the Coulomb interacting Dirac fermions. Moreover, the value of the constant $\mathcal{C}=(25-6\pi)/12$ has been calculated from the current-current correlator and using hard-cut-off regularization. In Appendix D we show that in general hard cutoff violates the Ward-Takahashi identity. Nevertheless, it can be shown that the integral for $\mathcal{C}$ is, despite appearances, in fact UV convergent, but sensible to the order of integration. A correct way of performing the integral is to integrate both momenta up to finite cutoffs and take the cutoff to infinity after all the integrals are done first. This, as shown in the Appendix H, corrects the value of the constant precisely down to the $\mathcal{C}=(11-3\pi)/6$. As we showed in Appendix G, the previous result $\mathcal{C}=(25-6\pi)/12$ is also obtained when using a version of the dimensional regularization in which the Pauli matrices are treated as embedded in strictly two spatial dimensions and which also violates the Ward-Takahashi identity, as we argued in Appendix D. Technically, the origin of the discrepancy between the results for the Coulomb correction to conductivity within the two versions of the dimensional regularization may be traced if we consider the self-energy correction to the conductivity in Eq. (G2) and

its counterpart with Pauli matrices in $D=2-\epsilon$, given by Eq. (E5). The difference arises from the factor $D-1=1-\epsilon$ which is a consequence of the different treatment of Pauli matrices within the two schemes. The self-energy piece has a singular part proportional to $1/\epsilon$, and when multiplied by a term linear in ϵ coming from $D-1$, it gives rise to a finite contribution to the self-energy correction. Analogous situation occurs in the vertex part and in that case the last three terms in the integrand of Eq. (E13) account for the difference. Namely, when the trace over spatial indices of Pauli matrices is taken in $D=2$, these three terms cancel out, as it may be seen from the term proportional to $(\mathbf{k}\cdot\mathbf{p})^2$ in the integrand in Eq. (G6), but, in fact, when Pauli matrices are embedded in $D=2-\epsilon$, these terms yield a finite contribution to the conductivity, which may be directly checked following the steps in Eqs. (E18)–(E23).

On the other hand, in Ref. 10, the result for the constant $\mathcal{C}=(19-6\pi)/12$ has been calculated using three different methods, namely, the density-density correlator, the current-current correlator and the kinetic equation, and it has been argued that in order to obtain the unique value for the constant $\mathcal{C}$ a short-distance cutoff on the long-range Coulomb interaction has to be imposed. This regularization is an analog of the Pauli-Villars regularization in QED, but without the additional Pauli-Villars fermions introduced, that are, in fact, necessary to render it consistent.[31] This value for the constant $\mathcal{C}$ has also been obtained in Ref. 18 by regulating the short-distance behavior of the Coulomb interaction in the same manner as in Ref. 10. Although it has been shown that the same regularization preserves the Ward-Takahashi identity, besides lacking the Pauli-Villars fermions, this regularization cannot be applied to the theory of free Dirac fermions. Namely, the latter needs to be regularized when calculating the polarization bubble. Clearly, this cannot be achieved by imposing a short-distance cutoff on the long-range Coulomb interaction. Therefore, this regularization cannot serve as a consistent regularization of the entire field theory.

APPENDIX A: VERTEX INTEGRALS

The quantity of interest, which enters into the evaluation of the bare bubble and the leading order correction to the polarization tensor for short-range interactions u is

$$\mathcal{P}_\mu(\mathbf{q},i\Omega) = \int\frac{d^D\mathbf{k}}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G_{\mathbf{k}}(i\omega)\sigma_\mu G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega).$$
(A1)

Substituting Eq. (15), using Feynman parametrization in Eq. (63) for $\alpha=\beta=1$, and interchanging the order of integrations, we obtain

$$\mathcal{P}_\mu(\mathbf{q},i\Omega) = \int_0^1 dx$$

$$\times \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int_{-\infty}^\infty \frac{d\omega}{2\pi} \frac{1}{[(\omega+x\Omega)^2+(\mathbf{k}+x\mathbf{q})^2+\Delta]^2}$$

$$\times [i\omega + \sigma\cdot\mathbf{k}]\sigma_\mu[i\omega+i\Omega+\sigma\cdot(\mathbf{k}+\mathbf{q})], \quad \text{(A2)}$$

where

$$\Delta = x(1-x)(\Omega^2+\mathbf{q}^2). \quad \text{(A3)}$$

The standard next step when working in dimensional regularization is to define new integration variables $\ell_\omega = \omega + x\Omega$ and $\ell = \mathbf{k} + x\mathbf{q}$. We then perform the integral over ℓ_ω to obtain

$$\mathcal{P}_\mu(\mathbf{q},i\Omega) = \int_0^1 dx \int \frac{d^D\ell}{(2\pi)^D}\left[\frac{-\sigma_\mu}{4\sqrt{\ell^2+\Delta}}\right.$$

$$\left. + \frac{(\sigma\cdot\ell-xS)\sigma_\mu(\sigma\cdot\ell+(1-x)S)}{4(\ell^2+\Delta)^{3/2}}\right], \quad \text{(A4)}$$

where we defined

$$S \equiv i\Omega + \sigma\cdot\mathbf{q}. \quad \text{(A5)}$$

Since the integration measure is $O(2)$ symmetric, only the terms even in ℓ in numerator give a nontrivial contribution and we find

$$\mathcal{P}_\mu(\mathbf{q},i\Omega) = \frac{1}{4}\int_0^1 dx \int \frac{d^D\ell}{(2\pi)^D}\left[\frac{-\sigma_\mu}{\sqrt{\ell^2+\Delta}}\right.$$

$$\left. + \sigma_a\sigma_\mu\sigma_a \frac{\ell^2}{D(\ell^2+\Delta)^{3/2}} - \frac{x(1-x)S\sigma_\mu S}{(\ell^2+\Delta)^{3/2}}\right]. \quad \text{(A6)}$$

We next use the dimensional regularization integrals in Eqs.

(16) and (17), as well as the identity (18) to find

$$\mathcal{P}_\mu(\mathbf{q},i\Omega) = \frac{1}{8\pi}\int_0^1 dx\left[\sigma_\mu\sqrt{\Delta} - 2\delta_{\mu0}\sqrt{\Delta} - (i\Omega+\sigma\cdot\mathbf{q})\sigma_\mu(i\Omega\right.$$

$$\left. + \sigma\cdot\mathbf{q})\frac{x(1-x)}{\sqrt{\Delta}}\right]. \quad \text{(A7)}$$

Using Eq. (A3), we finally have

$$\mathcal{P}_\mu(\mathbf{q},i\Omega) = \frac{\sqrt{\Omega^2+\mathbf{q}^2}}{64}$$

$$\times\left[\sigma_\mu - 2\delta_{\mu0} - \frac{(i\Omega+\sigma\cdot\mathbf{q})\sigma_\mu(i\Omega+\sigma\cdot\mathbf{q})}{\Omega^2+\mathbf{q}^2}\right]. \quad \text{(A8)}$$

1. Coulomb vertex

The Coulomb vertex function to the first order in the coupling constant is

$$\mathcal{P}_\mu^c(\mathbf{q},\mathbf{p},i\Omega) = -\int \frac{d^D\mathbf{k}}{(2\pi)^D}\int_{-\infty}^\infty \frac{d\omega}{2\pi}\frac{2\pi e^2}{|\mathbf{p}-\mathbf{k}|}$$

$$\times G_\mathbf{k}(i\omega)\sigma_\mu G_{\mathbf{k}+\mathbf{q}}(i\omega+i\Omega) \quad \text{(A9)}$$

with the free fermion Green's function given by Eq. (15). After introducing Feynman parameters using Eq. (63), we obtain

$$\mathcal{P}_\mu^c(\mathbf{q},\mathbf{p},i\Omega) = -\int_0^1 dx \int \frac{d^D\ell}{(2\pi)^D}\frac{2\pi e^2}{|\mathbf{p}+x\mathbf{q}-\ell|}\left\{\frac{-\sigma_\mu}{4\sqrt{\ell^2+\Delta}} + \frac{[\sigma\cdot\ell-x(i\Omega+\sigma\cdot\mathbf{q})]\sigma_\mu[\sigma\cdot\ell+(1-x)(i\Omega+\sigma\cdot\mathbf{q})]}{4(\ell^2+\Delta)^{3/2}}\right\}. \quad \text{(A10)}$$

Now, we consider two terms in the above form of the Coulomb vertex separately. Using the Feynman parametrization in Eq. (63) and the D-dimensional integral in Eq. (16), the first term in the last equation acquires the form

$$\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{|\mathbf{p}+x\mathbf{q}-\ell|}\frac{1}{\sqrt{\ell^2+\Delta}} = \frac{1}{\pi}\int_0^1 \frac{dy}{\sqrt{y(1-y)}}\int \frac{d^D\ell}{(2\pi)^D}\frac{1}{\ell^2+(1-y)[y(\mathbf{p}+x\mathbf{q})^2+\Delta]}$$

$$= \frac{1}{\pi}\frac{\Gamma\left(1-\dfrac{D}{2}\right)}{(4\pi)^{D/2}}\int_0^1 \frac{dy}{\sqrt{y}(1-y)^{(3-D)/2}}\frac{1}{[y(\mathbf{p}+x\mathbf{q})^2+\Delta]^{1-D/2}}. \quad \text{(A11)}$$

After expanding the integrand to the first order in the parameter $\epsilon = 2-D$ and integrating over y, we have

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{|\mathbf{p}+x\mathbf{q}-\ell|} \frac{1}{\sqrt{\ell^2+\Delta}}$$

$$= \frac{\Gamma\left(1-\dfrac{D}{2}\right)}{(4\pi)^{D/2}} \left[1 - \frac{\epsilon}{2}\ln\frac{(\mathbf{p}+x\mathbf{q})^2}{16} - \epsilon \tanh^{-1}\sqrt{\frac{x(1-x)(\Omega^2+\mathbf{q}^2)}{(\mathbf{p}+x\mathbf{q})^2+x(1-x)(\Omega^2+\mathbf{q}^2)}} \right]. \tag{A12}$$

The second term in Eq. (A10), after introducing the Feynman parameter, shifting the variable $\ell - y(\mathbf{p}+x\mathbf{q}) \to \ell$, and integrating over ℓ, becomes

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{(\sigma\cdot\ell - xS)\sigma_\mu[\sigma\cdot\ell+(1-x)S]}{|\mathbf{p}+x\mathbf{q}-\ell|(\ell^2+\Delta)^{3/2}} = \frac{2}{\pi} \frac{\Gamma\left(1-\dfrac{D}{2}\right)}{(4\pi)^{D/2}} \int_0^1 dy \frac{(1-y)^{(D-1)/2}}{\sqrt{y}} \frac{\dfrac{D}{2}\delta_{\mu 0}}{[y(\mathbf{p}+x\mathbf{q})^2+\Delta]^{1-D/2}}$$

$$+ \frac{2}{\pi} \frac{\Gamma\left(2-\dfrac{D}{2}\right)}{(4\pi)^{D/2}} \int_0^1 dy \frac{(1-y)^{(D-3)/2}}{\sqrt{y}} \frac{[y\sigma\cdot(\mathbf{p}+x\mathbf{q})-xS]\sigma_\mu[y\sigma\cdot(\mathbf{p}+x\mathbf{q})+(1-x)S]}{[y(\mathbf{p}+x\mathbf{q})^2+\Delta]^{(2-D)/2}}$$

$$+ \frac{2}{\pi} \frac{\Gamma\left(2-\dfrac{D}{2}\right)}{(4\pi)^{D/2}} \int_0^1 dy \frac{(1-y)^{(D-1)/2}}{\sqrt{y}} \frac{\delta_{\mu a}\sigma_a}{[y(\mathbf{p}+x\mathbf{q})^2+\Delta]^{(1-D)/2}}, \tag{A13}$$

where $S = i\Omega + \sigma\cdot\mathbf{q}$, and we also used the identity (18) for the trace of the Pauli matrices over spatial indices. Note that the last term in the previous equation arises from the term proportional to $\epsilon = 2-D$ in Eq. (18). If we treated Pauli matrices strictly in $D=2$, this term would be omitted what would thus lead to the violation of the Ward identity, as it may be directly checked in Eq. (B16). Therefore, in order to preserve the gauge invariance of the theory, it is crucial to treat Pauli matrices, as well as the momentum integrals, in a general spatial dimension, and only at the end of the calculation to take $D=2-\epsilon$, and expand in ϵ. After expanding in ϵ, keeping terms up to order ϵ^0, and performing the integral over y in the first term, we have

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{(\sigma\cdot\ell - xS)\sigma_\mu[\sigma\cdot\ell+(1-x)S]}{|\mathbf{p}+x\mathbf{q}-\ell|(\ell^2+\Delta)^{3/2}} = \frac{\Gamma\left(1-\dfrac{D}{2}\right)}{(4\pi)^{D/2}} \delta_{\mu 0} \left\{ 1 - \frac{\epsilon}{2}\ln\frac{(\mathbf{p}+x\mathbf{q})^2}{16} - \epsilon\tanh^{-1}\sqrt{\frac{\Delta}{(\mathbf{p}+x\mathbf{q})^2+\Delta}} \right.$$

$$\left. - \epsilon\frac{\sqrt{\Delta[(\mathbf{p}+x\mathbf{q})^2+\Delta]}-\Delta}{(\mathbf{p}+x\mathbf{q})^2} - \frac{\epsilon}{2} \right\} + \frac{1}{4\pi}\delta_{\mu a}\sigma_a$$

$$+ \frac{1}{2\pi^2}\int_0^1 dy \frac{1}{\sqrt{y(1-y)}} \frac{[y\sigma\cdot(\mathbf{p}+x\mathbf{q})-xS]\sigma_\mu[y\sigma\cdot(\mathbf{p}+x\mathbf{q})+(1-x)S]}{y(\mathbf{p}+x\mathbf{q})^2+\Delta}$$

$$= \frac{\Gamma\left(1-\dfrac{D}{2}\right)}{(4\pi)^{D/2}} \delta_{\mu 0} \left\{ 1 - \frac{\epsilon}{2}\ln\frac{(\mathbf{p}+x\mathbf{q})^2}{16} - \epsilon\tanh^{-1}\sqrt{\frac{\Delta}{(\mathbf{p}+x\mathbf{q})^2+\Delta}} \right.$$

$$\left. - \epsilon\frac{\sqrt{\Delta[(\mathbf{p}+x\mathbf{q})^2+\Delta]}-\Delta}{(\mathbf{p}+x\mathbf{q})^2} - \frac{\epsilon}{2} \right\} + \frac{1}{4\pi}\delta_{\mu a}\sigma_a + \frac{1}{2\pi^2}\{-x(1-x)S\sigma_\mu S\mathcal{I}_0[(\mathbf{p}+x\mathbf{q})^2,\Delta]$$

$$+ [-xS\sigma_\mu\sigma\cdot(\mathbf{p}+x\mathbf{q})+(1-x)(\mathbf{p}+x\mathbf{q})\cdot\sigma\sigma_\mu S]\mathcal{I}_1[(\mathbf{p}+x\mathbf{q})^2,\Delta] + \sigma\cdot(\mathbf{p}$$

$$+ x\mathbf{q})\sigma_\mu\sigma\cdot(\mathbf{p}+x\mathbf{q})\mathcal{I}_2[(\mathbf{p}+x\mathbf{q})^2,\Delta]\} \tag{A14}$$

with Δ defined in Eq. (A3). The remaining integrals over y read

$$\mathcal{I}_0(a,\Delta) = \int_0^1 \frac{dy}{\sqrt{y(1-y)}} \frac{1}{ya+\Delta} = \frac{\pi}{\sqrt{\Delta(a+\Delta)}}, \tag{A15}$$

$$\mathcal{I}_1(a,\Delta) = \int_0^1 \frac{dy}{\sqrt{y(1-y)}} \frac{y}{ya+\Delta} = \frac{\pi}{a}\left(1-\sqrt{\frac{\Delta}{a+\Delta}}\right), \tag{A16}$$

$$\mathcal{I}_2(a,\Delta) = \int_0^1 \frac{dy}{\sqrt{y(1-y)}} \frac{y^2}{ya+\Delta} = \frac{\pi}{2a^2}\left[a - 2\Delta\left(1-\sqrt{\frac{\Delta}{a+\Delta}}\right)\right]. \tag{A17}$$

Therefore, using Eqs. (A11) and (A14), we can write the Coulomb vertex in the form

$$
\begin{aligned}
\mathcal{P}_\mu^c(\mathbf{q},\mathbf{p},i\Omega) = &-\frac{\pi e^2}{2}\int_0^1 dx \left[\frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}} \epsilon\delta_{\mu 0}\left\{ \frac{\Delta - \sqrt{\Delta[\Delta+(\mathbf{p}+x\mathbf{q})^2]}}{(\mathbf{p}+x\mathbf{q})^2} - \frac{1}{2} \right\} + \frac{1}{4\pi}\sigma_a\delta_{\mu a} - \sigma_a\delta_{\mu a}\frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}}\left[1 - \frac{\epsilon}{2}\ln\frac{(\mathbf{p}+x\mathbf{q})^2}{16} \right.\right. \\
&\left. - \epsilon\tanh^{-1}\sqrt{\frac{x(1-x)(\Omega^2+\mathbf{q}^2)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}} \right] + \frac{1}{2\pi}\left(\frac{-x(1-x)(i\Omega+\sigma\cdot\mathbf{q})\sigma_\mu(i\Omega+\sigma\cdot\mathbf{q})}{\sqrt{\Delta((\mathbf{p}+x\mathbf{q})^2+\Delta)}} \right. \\
&+ \frac{[-x(i\Omega+\sigma\cdot\mathbf{q})\sigma_\mu\sigma\cdot(\mathbf{p}+x\mathbf{q}) + (1-x)(\mathbf{p}+x\mathbf{q})\cdot\sigma\sigma_\mu(i\Omega+\sigma\cdot\mathbf{q})]}{(\mathbf{p}+x\mathbf{q})^2}\left[1 - \sqrt{\frac{\Delta}{(\mathbf{p}+x\mathbf{q})^2+\Delta}} \right] \\
&\left.\left. + \frac{\sigma\cdot(\mathbf{p}+x\mathbf{q})\sigma_\mu\sigma\cdot(\mathbf{p}+x\mathbf{q})}{2(\mathbf{p}+x\mathbf{q})^4}\left\{ (\mathbf{p}+x\mathbf{q})^2 - 2\Delta\left[1 - \sqrt{\frac{\Delta}{(\mathbf{p}+x\mathbf{q})^2+\Delta}} \right] \right\} \right) \right],
\end{aligned}
\tag{A18}
$$

where again $\Delta = x(1-x)(\Omega^2+\mathbf{q}^2)$, and we explicitly wrote the functions $\mathcal{I}_0$, $\mathcal{I}_1$, and $\mathcal{I}_2$, defined in Eqs. (A15)–(A17). The above expression diverges as $D \to 2$ from below, or equivalently as $\epsilon \to 0^+$. As discussed in the main text, this divergence is tied to the divergence of the self-energy and is a consequence of the Ward-Takahashi identity proved below. All the integrals over x in the above expression can be performed in the closed form in terms of elliptic integrals. (The expressions involving $\tanh^{-1}$ need to be integrated by parts

first to bring them to the form easily expressible in terms of the elliptic integrals.) However, we found that doing so leads to intractable expressions and we thus chose to work with the above form of the Coulomb vertex, in which the integrals over the variable x can be thought of as the integral representation of the elliptic integrals.

At $\mathbf{q}=0$ the above expressions simplify significantly and we have

$$
\begin{aligned}
\mathcal{P}_\mu^c(0,\mathbf{p},i\Omega) = &-\delta_{\mu a}\frac{e^2}{8}\left\{ \sigma_a\left[1 - \left(\frac{2}{\epsilon} - \gamma + \ln 64\pi - \ln\mathbf{p}^2\right) + \frac{2|\Omega|}{\sqrt{\Omega^2+4\mathbf{p}^2}}K \right] + [\mathbf{p}\cdot\sigma,\sigma_a]\frac{i\Omega}{\mathbf{p}^2}\left[1 - \frac{|\Omega|}{\sqrt{\Omega^2+4\mathbf{p}^2}}K - \frac{\sqrt{\Omega^2+4\mathbf{p}^2}}{|\Omega|} \right.\right. \\
&\left.\left. \times(E-K) \right] + \frac{\sigma\cdot\mathbf{p}\sigma_a\sigma\cdot\mathbf{p}}{\mathbf{p}^2}\left[1 - \frac{\Omega^2}{3\mathbf{p}^2} + \frac{(\Omega^4-16\mathbf{p}^4)E + (16\mathbf{p}^4-2\Omega^2\mathbf{p}^2)K}{3|\Omega|\mathbf{p}^2\sqrt{\Omega^2+4\mathbf{p}^2}} \right] \right\},
\end{aligned}
\tag{A19}
$$

where the arguments of the complete elliptic integrals of the first and second kinds, respectively, are

$$K \equiv K\left(\frac{|\Omega|}{\sqrt{\Omega^2+4\mathbf{p}^2}}\right); \quad E \equiv E\left(\frac{|\Omega|}{\sqrt{\Omega^2+4\mathbf{p}^2}}\right).$$

Note that to this order, at $\mathbf{q}=0$ (and at any Ω), only the spatial components of $\mathcal{P}_\mu^c$ are finite. This is a consequence of the Ward-Takahashi identity since to this order the self-energy is frequency independent.

Finally, at $\mathbf{q}=0$ and $\Omega=0$ the integrals in Eq. (A10) can be performed for arbitrary D without the necessity of expanding in powers of ϵ. The Coulomb vertex then becomes

$$\mathcal{P}_\mu^c(0,\mathbf{p},0) = \frac{e^2}{(4\pi)^{D/2}} \frac{\delta_{\mu a}}{|\mathbf{p}|^{2-D}} \frac{\Gamma\left[\frac{D}{2}+\frac{1}{2}\right]\Gamma\left[\frac{D}{2}-\frac{1}{2}\right]}{\Gamma[D]}$$

$$\times\left(\sigma_a \Gamma\left[1-\frac{D}{2}\right] - \left(\sigma_a + \frac{\sigma\cdot\mathbf{p}\,\sigma_a\,\sigma\cdot\mathbf{p}}{\mathbf{p}^2}\right)\right.$$

$$\left.\times\Gamma\left[2-\frac{D}{2}\right]\right). \tag{A20}$$

This form of the vertex function is used to show that $\delta\Pi_{\mu\nu}^{(c)}(0,0)=0$ for any D, a fact which is in turn used in the calculation of the electrical conductivity.

APPENDIX B: COULOMB VERTEX AND THE WARD-TAKAHASHI IDENTITIES IN DIMENSIONAL REGULARIZATION

In this appendix we show in detail that to the leading order in the Coulomb interaction coupling constant e^2 and to $\mathcal{O}(N)$, the Ward-Takahashi identity, questioned to hold in Ref. 18, is satisfied. As a first step, we define the contraction $q^\mu \mathcal{P}_\mu^c \equiv -i\Omega\mathcal{P}_0^c + q_a \mathcal{P}_a^c$. Using Eq. (A18), a straightforward calculation shows that all the terms in the contraction proportional to $i\Omega$ cancel out and the contraction simplifies to

$$q^\mu\mathcal{P}_\mu^c(\mathbf{q},\mathbf{p},i\Omega) = -\frac{\pi e^2}{2}\int_0^1 dx\left\{-\sigma\cdot\mathbf{q}\frac{\Gamma\left[1-\frac{D}{2}\right]}{(4\pi)^{D/2}}\left(1-\frac{\epsilon}{2}\ln\frac{(\mathbf{p}+x\mathbf{q})^2}{16} - \epsilon\tanh^{-1}\sqrt{\frac{\Delta}{(\mathbf{p}+x\mathbf{q})^2+\Delta}}\right) + \frac{1}{4\pi}\sigma\cdot\mathbf{q}\right.$$

$$+\frac{1}{2\pi}\left[-\sigma\cdot\mathbf{q}\sqrt{\frac{\Delta}{(\mathbf{p}+x\mathbf{q})^2+\Delta}} + \sigma\cdot(\mathbf{p}+x\mathbf{q})\frac{(1-2x)(\Omega^2+\mathbf{q}^2)}{(\mathbf{p}+x\mathbf{q})^2}\right.$$

$$\times\left(1-\sqrt{\frac{\Delta}{(\mathbf{p}+x\mathbf{q})^2+\Delta}}\right) - \frac{\Delta}{(\mathbf{p}+x\mathbf{q})^4}\sigma\cdot(\mathbf{p}+x\mathbf{q})\sigma\cdot\mathbf{q}\,\sigma\cdot(\mathbf{p}+x\mathbf{q})$$

$$\left.\left.+\frac{\sigma\cdot(\mathbf{p}+x\mathbf{q})\sigma\cdot\mathbf{q}\,\sigma\cdot(\mathbf{p}+x\mathbf{q})}{2(\mathbf{p}+x\mathbf{q})^2} + \frac{\sigma\cdot(\mathbf{p}+x\mathbf{q})\sigma\cdot\mathbf{q}\,\sigma\cdot(\mathbf{p}+x\mathbf{q})}{(\mathbf{p}+x\mathbf{q})^4}\sqrt{\frac{\Delta^3}{(\mathbf{p}+x\mathbf{q})^2+\Delta}}\right]\right\}. \tag{B1}$$

In order to show the Ward-Takahashi identity, we first note that the self-energy to the first-order in the Coulomb coupling is independent of the frequency. Thus all the terms in the contraction in Eq. (B1) that contain frequency have to vanish if the Ward-Takahashi identity holds. In fact, as we will show in what follows, the contraction can be written in the form

$$q^\mu\mathcal{P}_\mu^c(\mathbf{q},\mathbf{p},i\Omega) = N(\mathbf{p},\mathbf{q}) - \frac{e^2}{4}(\Omega^2+\mathbf{q}^2)W(\mathbf{p},\mathbf{q})$$

$$-\frac{e^2}{4}\sigma\cdot\mathbf{p}\sqrt{(\Omega^2+\mathbf{q}^2)^3}L(\mathbf{p},\mathbf{q},\Omega)$$

$$-\frac{e^2}{4}\sigma\cdot\mathbf{q}\sqrt{\Omega^2+\mathbf{q}^2}M(\mathbf{p},\mathbf{q},\Omega), \tag{B2}$$

where the functions $N(\mathbf{p},\mathbf{q})$, $W(\mathbf{p},\mathbf{q})$, $L(\mathbf{p},\mathbf{q},\Omega)$, and $M(\mathbf{p},\mathbf{q},\Omega)$ are defined in Eqs. (B16), (B3), (B17), and (B18), respectively. This condition is, therefore, satisfied if $W(\mathbf{p},\mathbf{q})=0$, $M(\mathbf{p},\mathbf{q},\Omega)=0$, and $L(\mathbf{p},\mathbf{q},\Omega)=0$. Finally, to

complete the proof of the identity, we will show that $N(\mathbf{p},\mathbf{q})=\Sigma_{\mathbf{p+q}}(i\nu+i\omega)-\Sigma_{\mathbf{p}}(i\nu)$.

Let us first show that term proportional to $\Omega^2+\mathbf{q}^2$ vanishes, i.e., that

$$W(\mathbf{p},\mathbf{q}) \equiv \int_0^1 dx\left[\sigma\cdot(\mathbf{p}+x\mathbf{q})\frac{(1-2x)}{(\mathbf{p}+x\mathbf{q})^2}\right.$$

$$\left.-\frac{x(1-x)}{(\mathbf{p}+x\mathbf{q})^4}\sigma\cdot(\mathbf{p}+x\mathbf{q})\sigma\cdot\mathbf{q}\,\sigma\cdot(\mathbf{p}+x\mathbf{q})\right]$$

$$= 0. \tag{B3}$$

Using the identity

$$\sigma\cdot\mathbf{p}\,\sigma\cdot\mathbf{q}\,\sigma\cdot\mathbf{p} = 2\mathbf{p}\cdot\mathbf{q}\,\sigma\cdot\mathbf{p} - \mathbf{p}^2\sigma\cdot\mathbf{q} \tag{B4}$$

we can rewrite the above integral as

$$W(\mathbf{p},\mathbf{q}) = \sigma \cdot \mathbf{p} \int_0^1 dx \frac{\mathbf{p}^2(1-2x) - (2\mathbf{p} \cdot \mathbf{q} + \mathbf{q}^2)x^2}{(\mathbf{p}+x\mathbf{q})^4} + \frac{\sigma \cdot \mathbf{q}}{\mathbf{q}^2} \int_0^1 dx \left(\frac{x^2[2\mathbf{q}^2\mathbf{p} \cdot \mathbf{q} - \mathbf{p}^2\mathbf{q}^2 + 4(\mathbf{p} \cdot \mathbf{q})^2] + 2xp^2[\mathbf{q}^2 + 2\mathbf{p} \cdot \mathbf{q}] + \mathbf{p}^4}{(\mathbf{p}+x\mathbf{q})^4} - 1 \right)$$

$$\equiv \sigma \cdot \mathbf{p} W_1(\mathbf{p},\mathbf{q}) + \frac{\sigma \cdot \mathbf{q}}{\mathbf{q}^2} W_2(\mathbf{p},\mathbf{q}). \tag{B5}$$

When $\mathbf{p}=\mathbf{q}$, it is easy to show that both W_1 and W_2 vanish, and we thus concentrate on the case $\mathcal{D} \equiv \mathbf{p}^2\mathbf{q}^2 - (\mathbf{p} \cdot \mathbf{q})^2 > 0$. In order to calculate the integrals W_1 and W_2, we use the following identities

$$K_0 \equiv \int_0^1 \frac{dx}{(\mathbf{p}+x\mathbf{q})^4} = \frac{\mathbf{p}^2\mathbf{q}^2 - 2(\mathbf{p} \cdot \mathbf{q})^2 - (\mathbf{p} \cdot \mathbf{q})\mathbf{q}^2}{2\mathbf{p}^2(\mathbf{p}+\mathbf{q})^2\mathcal{D}}$$

$$+ \frac{\mathbf{q}^2}{2\mathcal{D}} \int_0^1 \frac{dx}{(\mathbf{p}+x\mathbf{q})^2}, \tag{B6}$$

$$K_1 \equiv \int_0^1 dx \frac{x}{(\mathbf{p}+x\mathbf{q})^4} = \frac{\mathbf{q}^2 + \mathbf{p} \cdot \mathbf{q}}{2(\mathbf{p}+\mathbf{q})^2\mathcal{D}} - \frac{\mathbf{p} \cdot \mathbf{q}}{2\mathcal{D}} \int_0^1 \frac{dx}{(\mathbf{p}+x\mathbf{q})^2}, \tag{B7}$$

$$K_2 \equiv \int_0^1 dx \frac{x^2}{(\mathbf{p}+x\mathbf{q})^4} = -\frac{\mathbf{p}^2 + \mathbf{p} \cdot \mathbf{q}}{2(\mathbf{p}+\mathbf{q})^2\mathcal{D}} + \frac{\mathbf{p}^2}{2\mathcal{D}} \int_0^1 \frac{dx}{(\mathbf{p}+x\mathbf{q})^2}. \tag{B8}$$

Straightforward calculation yields $W_1(\mathbf{p},\mathbf{q})=0$ and $W_2(\mathbf{p},\mathbf{q})=0$, and thus $W(\mathbf{p},\mathbf{q})=0$. The contraction given by Eq. (B1) then simplifies to

$$q^\mu \mathcal{P}_\mu^c(\mathbf{q},\mathbf{p},i\Omega) = \frac{\pi e^2}{2} \sigma \cdot \mathbf{q} \mathcal{J}_0 - \frac{e^2}{8} \int_0^1 dx \left[\sigma \cdot \mathbf{q} \right.$$

$$+ \frac{\sigma \cdot (\mathbf{p}+x\mathbf{q})\sigma \cdot \mathbf{q}\sigma \cdot (\mathbf{p}+x\mathbf{q})}{(p+x\mathbf{q})^2} + 2 \left[-\sigma \cdot \mathbf{q} \right.$$

$$- \sigma \cdot (\mathbf{p}+x\mathbf{q}) \frac{(1-2x)(\Omega^2+\mathbf{q}^2)}{(\mathbf{p}+x\mathbf{q})^2} + x(1-x)$$

$$\times (\Omega^2+\mathbf{q}^2) \frac{\sigma \cdot (\mathbf{p}+x\mathbf{q})\sigma \cdot \mathbf{q}\sigma \cdot (\mathbf{p}+x\mathbf{q})}{(\mathbf{p}+x\mathbf{q})^4} \right]$$

$$\times \sqrt{\frac{x(1-x)(\Omega^2+\mathbf{q}^2)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}} \right], \tag{B9}$$

where

$$\mathcal{J}_0 \equiv \frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}} \int_0^1 dx \left[1 - \frac{\epsilon}{2}\ln\frac{(\mathbf{p}+x\mathbf{q})^2}{16} \right.$$

$$\left. - \epsilon \tanh^{-1} \sqrt{\frac{x(1-x)(\Omega^2+\mathbf{q}^2)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}} \right]. \tag{B10}$$

After a partial integration in the last term, $\mathcal{J}_0$ becomes

$$\mathcal{J}_0 = \frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}} \int_0^1 dx\{1 + \epsilon \ln 4$$

$$- \epsilon \ln[\sqrt{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}$$

$$+ \sqrt{x(1-x)(\Omega^2+\mathbf{q}^2)}]\}. \tag{B11}$$

Finally, after performing another partial integration in the last term of the previous equation, we obtain

$$\mathcal{J}_0 = \frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}} \int_0^1 dx \left\{ 1 + \epsilon \ln 4 - \frac{\epsilon}{2}\ln[(\mathbf{p}+\mathbf{q})^2] \right.$$

$$+ \epsilon \frac{x(x\mathbf{q}^2 + \mathbf{p} \cdot \mathbf{q})}{(\mathbf{p}+x\mathbf{q})^2} + \frac{\epsilon}{2} \frac{1}{(\mathbf{p}+x\mathbf{q})^2} \left[\frac{1-2x}{1-x}(\mathbf{p}+x\mathbf{q})^2 \right.$$

$$\left. - 2x(x\mathbf{q}^2 + \mathbf{p} \cdot \mathbf{q}) \right] \sqrt{\frac{x(1-x)(\Omega^2+\mathbf{q}^2)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}} \right\}. \tag{B12}$$

Expansion of the prefactor in ϵ has the form

$$\frac{\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^{D/2}} = \frac{1}{2\pi} \left[\frac{1}{\epsilon} + \frac{1}{2}(-\gamma + \ln 4\pi) \right] + \mathcal{O}(\epsilon), \tag{B13}$$

which, after keeping the terms up to the order ϵ^0 in Eq. (B12), yields

$$\mathcal{J}_0 = \frac{1}{2\pi}\int_0^1 dx\left\{\frac{1}{\epsilon} + \frac{1}{2}[-\gamma + \ln 64\pi] - \frac{1}{2}\ln[(\mathbf{p}+\mathbf{q})^2]\right.$$

$$+ \frac{x(x\mathbf{q}^2 + \mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}+x\mathbf{q})^2} + \frac{1}{2(\mathbf{p}+x\mathbf{q})^2}\left[\frac{1-2x}{1-x}(\mathbf{p}+x\mathbf{q})^2\right.$$

$$\left.\left. - 2x(x\mathbf{q}^2 + \mathbf{p}\cdot\mathbf{q})\right]\sqrt{\frac{x(1-x)(\Omega^2+\mathbf{q}^2)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}}\right\}.$$

$$\tag{B14}$$

Now, after substituting Eq. (B14) into Eq. (B9), the contraction reads

$$q^\mu \mathcal{P}_\mu^c(\mathbf{q},\mathbf{p},i\Omega) = N(\mathbf{p},\mathbf{q}) - \frac{e^2}{4}\sigma\cdot\mathbf{p}\sqrt{(\Omega^2+\mathbf{q}^2)^3}L(\mathbf{p},\mathbf{q},\Omega)$$

$$- \frac{e^2}{4}\sigma\cdot\mathbf{q}\sqrt{\Omega^2+\mathbf{q}^2}M(\mathbf{p},\mathbf{q},\Omega), \tag{B15}$$

which is, in fact, the form in Eq. (B2) of the contraction,

since we have already shown that $W(\mathbf{p},\mathbf{q})=0$. The frequency-independent part in the above equation reads

$$N(\mathbf{p},\mathbf{q}) = \frac{e^2}{8}\int_0^1 dx\left\{\sigma\cdot\mathbf{q}\left[\frac{2}{\epsilon} - \gamma\right.\right.$$

$$+ \ln 64\pi - \ln[(\mathbf{p}+\mathbf{q})^2]$$

$$\left.+ \frac{2x(x\mathbf{q}^2 + \mathbf{p}\cdot\mathbf{q})}{(\mathbf{p}+x\mathbf{q})^2} - 1\right]$$

$$\left. - \frac{\sigma\cdot(\mathbf{p}+x\mathbf{q})\sigma\cdot\mathbf{q}\sigma\cdot(\mathbf{p}+x\mathbf{q})}{(\mathbf{p}+x\mathbf{q})^2}\right\} \tag{B16}$$

while the remaining terms are

$$L(\mathbf{p},\mathbf{q},\Omega) = \int_0^1 dx\frac{x^2(\mathbf{p}+\mathbf{q})^2 - (1-x)^2\mathbf{p}^2}{(\mathbf{p}+x\mathbf{q})^4}\sqrt{\frac{x(1-x)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}} \tag{B17}$$

and

$$M(\mathbf{p},\mathbf{q},\Omega) = \frac{1}{2}\int_0^1 dx\left[\frac{x^2(\mathbf{p}+\mathbf{q})^2 - (1-x)^2\mathbf{p}^2}{(1-x)(\mathbf{p}+x\mathbf{q})^2} - 2\right]\sqrt{\frac{x(1-x)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}}$$

$$+ \int_0^1 dx\left\{\frac{x(\Omega^2+\mathbf{q}^2)[(3x-2)\mathbf{p}^2 + 2x(2x-1)\mathbf{p}\cdot\mathbf{q} + x^3\mathbf{q}^2]}{(\mathbf{p}+x\mathbf{q})^4}\right\}\sqrt{\frac{x(1-x)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+\mathbf{q}^2)}}. \tag{B18}$$

The frequency-independent term in Eq. (B16), after using the identity (B4), and performing the remaining integral, has the form

$$N(\mathbf{p},\mathbf{q}) = \frac{e^2}{8}\left\{\frac{2}{\epsilon}\sigma\cdot\mathbf{q} + (-\gamma + \ln 64\pi)\sigma\cdot\mathbf{q}\right.$$

$$\left. - \sigma\cdot(\mathbf{p}+\mathbf{q})\ln[(\mathbf{p}+\mathbf{q})^2] + \sigma\cdot\mathbf{p}\ln\mathbf{p}^2\right\}$$

$$= \Sigma_{\mathbf{p}+\mathbf{q}}(i\nu + i\Omega) - \Sigma_{\mathbf{p}}(i\nu). \tag{B19}$$

Here, $\Sigma_{\mathbf{p}}(i\nu)$ is the self-energy defined in Eq. (71) in the main text.

APPENDIX C: EVALUATION OF THE FUNCTIONS $L(\mathbf{p},\mathbf{q},\Omega)$ AND $M(\mathbf{p},\mathbf{q},\Omega)$

In this appendix we show that the functions $L(\mathbf{p},\mathbf{q},\Omega)$, Eq. (B17), and $M(\mathbf{p},\mathbf{q},\Omega)$, Eq. (B18), vanish identically, and therefore, the Ward-Takahashi identity is, indeed, satisfied within the dimensional regularization used here. In order to evaluate integrals in Eqs. (B17) and (B18), we introduce new variables

$$p' = \frac{p}{q}e^{-i\phi}, \quad w = \frac{\Omega}{q} \tag{C1}$$

with $p=|\mathbf{p}|$, $q=|\mathbf{q}|$, and $\cos\phi = \mathbf{p}\cdot\mathbf{q}/(pq)$, and thus we can write

$$(\mathbf{p}+x\mathbf{q})^2 = q^2(x+p')(x+p'^*) \tag{C2}$$

with p'^* denoting complex conjugate of p'. The integral in Eq. (B17) can now be rewritten in terms of the new variables as

$$L = \frac{1}{q^2}\sqrt{1+\frac{q^2}{\Omega^2}}\int_0^1 dx$$

$$\times\frac{(1+p')(1+p'^*)x^3(1-x) - x(1-x)^3 p'p'^*}{(x+p')^2(x+p'^*)^2}$$

$$\times\frac{1}{\sqrt{x(x-1)(x-x_+)(x-x_-)}}, \tag{C3}$$

where $x_\pm$ are the roots of the quadratic equation $(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+q^2)=0$

$$x_{\pm} = \frac{(\mathbf{p} + \mathbf{q})^2 - p^2 + \Omega^2 \pm \sqrt{[(\mathbf{p} + \mathbf{q})^2 - p^2 + \Omega^2]^2 + 4\Omega^2 p^2}}{2\Omega^2} \tag{C4}$$

or in terms of the variables p' and w

$$x_{\pm} = \frac{1 + 2\,\mathrm{Re}(p') + w^2 \pm \sqrt{(1 + 2\,\mathrm{Re}(p') + w^2)^2 + 4w^2 p' p'^*}}{2w^2}. \tag{C5}$$

Assuming that $x_+ > 1$, we have the following sequence:

$$x_+ > 1 > 0 > x_-, \tag{C6}$$

which is important for expressing the function $L(\mathbf{p},\mathbf{q},\Omega)$ in terms of the elliptic integrals, as we will see below. We use partial fractions to calculate the integral in Eq. (C3). The first term reads

$$\frac{x^3(1-x)}{(x+p')^2(x+p'^*)^2} = -1 + \left(\frac{ip'^2[p'(1+2p') - (3+4p')p'^*]}{8[\mathrm{Im}(p')]^3(x+p')} \right.$$

$$\left. + \frac{p'^3(1+p')}{4[\mathrm{Im}(p')]^2(x+p')^2} + \mathrm{c.c.} \right)$$

$$\equiv -1 + \left(\frac{A}{x+p'} + \frac{B}{(x+p')^2} + \mathrm{c.c.} \right) \tag{C7}$$

while the second term is

$$\frac{x(1-x)^3}{(x+p')^2(x+p'^*)^2}$$

$$= -1 + \left(\frac{i(1+p')^2[p'^2 - 2p'p'^* - \mathrm{Re}(p')]}{4[\mathrm{Im}(p')]^3(x+p')} \right.$$

$$\left. + \frac{p'(1+p')^3}{4[\mathrm{Im}(p')]^2(x+p')^2} + \mathrm{c.c.} \right)$$

$$\equiv -1 + \left(\frac{A_1}{x+p'} + \frac{B_1}{(x+p')^2} + \mathrm{c.c.} \right). \tag{C8}$$

We therefore reduced the problem of evaluating the integral in Eq. (C3) to the calculation of the following integrals:

$$I_m = \int_0^1 dx \frac{1}{(x+p)^m} \frac{1}{\sqrt{x(x-1)(x-x_+)(x-x_-)}} \tag{C9}$$

with $m = 0,1,2$ and $x_+ > 1 > 0 > x_-$. In terms of the integrals I_m, the integral L reads

$$L = -[1 + 2\,\mathrm{Re}(p')]I_0 + \{[(1+p')(1+p'^*)A - p'p'^*A_1]I_1$$

$$+ [(1+p')(1+p'^*)B - p'p'^*B_1]I_2 + \mathrm{c.c.}\}$$

$$= -[1 + 2\,\mathrm{Re}(p')]I_0 + \left[\frac{-ip'(1+p')(1+2p')}{2\,\mathrm{Im}(p')} I_1 \right.$$

$$\left. + \frac{ip'^2(1+p')^2}{2\,\mathrm{Im}(p')} I_2 + \mathrm{c.c.} \right]. \tag{C10}$$

The integrals I_m with $m = 0,1,2$ and $x_+ > 1 > 0 > x_-$ have the form [Eqs. (255.00), (255.38), (340.01), and (340.02) in Ref. 37]

$$I_0 = gF(k), \tag{C11}$$

$$I_1 = \frac{g}{(1+p')\alpha_1^2}[(\alpha_1^2 - \alpha^2)\Pi(\alpha_1^2, k) + \alpha^2 F(k)], \tag{C12}$$

$$I_2 = \frac{g}{(1+p')^2\alpha_1^4} \left\{ \alpha^4 F(k) + 2\alpha^2(\alpha_1^2 - \alpha^2)\Pi(\alpha_1^2, k) \right.$$

$$+ \frac{(\alpha_1^2 - \alpha^2)^2}{2(\alpha_1^2 - 1)(k^2 - \alpha_1^2)}[\alpha_1^2 E(k) + (k^2 - \alpha_1^2)F(k) + (2\alpha_1^2 k^2$$

$$\left. + 2\alpha_1^2 - \alpha_1^4 - 3k^2)\Pi(\alpha_1^2, k)] \right\}, \tag{C13}$$

where $F(k)$, $E(k)$, and $\Pi(\alpha^2, k)$ are the complete elliptic integrals of the first, second, and third kinds, respectively, defined in terms of the corresponding incomplete integrals as $F(k) \equiv F(\pi/2, k)$, $E(k) \equiv E(\pi/2, k)$, and $\Pi(\alpha^2, k) \equiv \Pi(\pi/2, \alpha^2, k)$ with incomplete elliptic integrals defined as in Ref. 37. Here

$$k^2 = \frac{x_+ - x_-}{x_+(1 - x_-)}, \quad g = \frac{1}{\sqrt{x_+(1 - x_-)}} \tag{C14}$$

and

$$\alpha^2 = \frac{1}{x_+}, \quad \alpha_1^2 = \frac{x_+ + p'}{x_+(1 + p')}. \tag{C15}$$

Therefore, the integral L has the form

$$L = R(p',w)F(k) + P(p',w)E(k) + \mathrm{Im}[G(p',w)\Pi(\alpha_1^2, k)]. \tag{C16}$$

Note that imaginary part of α_1^2 is nonvanishing. In the following, we will show that the coefficient $P(p',w) = 0$. Then, by expressing $\mathrm{Im}[G(p',w)\Pi(p',w)]$ in terms of the functions $F(k)$ and $\Pi(\alpha_2^2, k)$, with α_2^2 defined below purely real, we obtain that the function L vanishes. The coefficient $P(p',w)$ reads

$$P(p',w) = \frac{ip'^2(\alpha_1^2 - \alpha^2)^2}{4\,\mathrm{Im}(p')(\alpha_1^2 - 1)\alpha_1^2(k^2 - \alpha_1^2)} + \mathrm{c.c.}$$

$$= -\frac{ip'x_+(1 - x_+)(1 + p')}{4\,\mathrm{Im}(p')(p' + x_+)(p' + x_-)} + \mathrm{c.c.}$$

$$= \frac{1}{2\,\mathrm{Im}(p')}\mathrm{Im}\left[\frac{p'x_+(1 - x_+)(1 + p')}{(x_+ + p')(x_- + p')} \right] \tag{C17}$$

but

$$\frac{p'x_+(1 - x_+)(1 + p')}{(x_+ + p')(x_- + p)} = -\frac{1}{4w^2(1 + 4w^2)}[1 + 2\,\mathrm{Re}(p') + w^2$$

$$+ \sqrt{(1 + 2\,\mathrm{Re}(p') + w^2)^2 + 4p'p'^*w^2}]$$

$$\times [1 + 2\,\mathrm{Re}(p') - w^2$$

$$+ \sqrt{(1 + 2\,\mathrm{Re}(p') + w^2)^2 + 4p'p'^*w^2}] \tag{C18}$$

is purely real, and thus $P(p',w)$ vanishes identically. The coefficient $R(p',w)$ reads

$$R(p',w) = -g(1 + 2\,\mathrm{Re}(p')) + g\left[\left(\frac{-ip'(1+p')(1+2p')}{2\,\mathrm{Im}(p')(x_+ + p')} + \frac{ip'(1+p')}{4\,\mathrm{Im}(p')(x_+ + p')^2}[2p'(1+p') + x_+(1-x_+)]\right) + \mathrm{c.c.}\right] \tag{C19}$$

while

$$G(p',w) = g\frac{p'(\alpha_1^2 - \alpha^2)}{\mathrm{Im}(p')\alpha_1^2}\left(1 + 2p' + \frac{p'}{\alpha_1^2}\left\{\frac{\alpha^2[3\alpha_1^4 + k^2 - 2\alpha_1^2(1+k^2)] + \alpha_1^2[\alpha_1^4 + 3k^2 - 2\alpha_1^2(1+k^2)]}{2(\alpha_1^2 - 1)(k^2 - \alpha_1^2)}\right\}\right). \tag{C20}$$

Imaginary part of the product $G(p',w)\Pi(\alpha_1^2,k)$ can be expressed in terms of the complete elliptic function of the first kind and the third kind, as given by Eq. (419.00) in Ref. 37

$$\mathrm{Im}[G(p',w)\Pi(\alpha_1^2,k)] = \frac{g}{\mathrm{Im}(p')}\frac{1}{m_2 r^2(s_2 t_1 - s_1 t_2)}\{[a_1(s_1 r^2 - k^2 s_2 m_2) + b_1(k^2 t_2 m_2 - t_1 r^2)]F(k) + n_2 m_2 r^2(a_1 s_1 - b_1 t_1)\Pi(\alpha_2^2,k)\}, \tag{C21}$$

where

$$G = a_1 + ib_1, \quad \alpha_1^2 = -\gamma_1 - i\gamma_2, \quad r^2 = \gamma_1^2 + \gamma_2^2,$$

$$m_2 = -\frac{2\gamma_1 + k^2}{\gamma_1}, \quad \alpha_2^2 = \frac{k^2 m_2^2}{r^2},$$

$$s_1 = 1 - \frac{k^2}{r^2}, \quad n_2 = \frac{m_2[\alpha_2^4 - (2 + m_2)\alpha_2^2 + (1 - 2\alpha_2^2)k^2] - r^2}{m_2[\gamma_2^2 + (\gamma_1 + \alpha_2^2)^2]},$$

$$t_1 = \frac{2(k^2 + 2\gamma_1 + \gamma_1^2)}{\gamma_2(2\gamma_1 + k^2)},$$

$$t_2 = \frac{m_2^2 + (\gamma_1 + 2 - \alpha_2^2)m_2 + n_2 m_2(\gamma_1 + \alpha_2^2)}{m_2\gamma_2},$$

$$s_2 = 1 - n_2 - \frac{1}{m_2} \tag{C22}$$

since $r^2 + 2\gamma_1 + k^2 = 0$ in our case, with k^2 given by Eq. (C14), and a_1, b_1, γ_1, and γ_2 real. Thus, the integral L acquires the form

$$L = [R(p',w) + R_1(p',w)]F(k) + S_1(p',w)\Pi(\alpha_2^2,k), \tag{C23}$$

where

$$R_1(p',w) = \frac{g}{\mathrm{Im}(p')}\frac{a_1(s_1 r^2 - k^2 s_2 m_2) + b_1(k^2 t_2 m_2 - t_1 r^2)}{m_2 r^2(s_2 t_1 - s_1 t_2)} \tag{C24}$$

and

$$S_1(p',w) = \frac{g}{\mathrm{Im}(p')}\frac{n_2(a_1 s_1 - b_1 t_1)}{s_2 t_1 - s_1 t_2} \tag{C25}$$

while $R(p',w)$ is given by Eq. (C19).

Straightforward calculation shows that $a_1 s_1 - b_1 t_1 = 0$ and $s_2 t_1 - s_1 t_2 \neq 0$, and thus the coefficient $S_1(p',w)$ vanishes identically. Furthermore, using that $a_1 s_1 - b_1 t_1 = 0$, the form of the coefficient $R_1(p',w)$, given by Eq. (C24), can be simplified to

$$R_1(p',w) = -\frac{g}{\mathrm{Im}(p')}\frac{a_1 k^2}{r^2 t_1}. \tag{C26}$$

Finally, the previous equation together with Eqs. (C19) and (C22), since w^2 is purely real, yields $R(p',w) + R_1(p',w) = 0$, and, therefore, the integral $L(\mathbf{p},\mathbf{q},\Omega)$ given by Eq. (B17) vanishes. When the two vectors are (anti)collinear, i.e., when $\mathrm{Im}(p') = 0$, the continuity implies $L(\mathbf{p},\mathbf{q},\Omega) = 0$. When $x_+ < 1$, the analogous calculation shows that $L(\mathbf{p},\mathbf{q},\Omega) = 0$, as well.

Let us now turn to the integral $M(\mathbf{p},\mathbf{q},\Omega)$, and consider the case $\mathrm{Im}(p') \neq 0$, and $x_+ > 1$. In terms of partial fractions, this integral reads

$$M = \frac{1+w^2}{w}\int_0^1 dx\left\{\left[\frac{p'^3(1+p')^2}{2i\,\mathrm{Im}(p')(x+p')^2} + \frac{ip'^2(1+p')(1+2p')}{2\mathrm{Im}(p')(x+p')} + \mathrm{c.c.}\right] + p'^{*2} + 2(1+p')\mathrm{Re}(p') + x - x^2\right\}$$

$$\times \frac{1}{\sqrt{x(x-1)(x-x_+)(x-x_-)}} + \frac{1}{2w}\int_0^1 dx\left\{\left[\frac{p'^2(1+p')}{x+p'}\right.\right.$$

$$\left.\left. + \mathrm{c.c.}\right] - 2\,\mathrm{Re}[p'(1+p')] + [2\,\mathrm{Re}(p') - 1]x + 2x^2\right\}$$

$$\times \frac{1}{\sqrt{x(x-1)(x-x_+)(x-x_-)}}. \tag{C27}$$

It follows from Eq. (255.17) in Ref. 37 that

$$J_1 \equiv \int_0^1 \frac{x\,dx}{\sqrt{x(x-1)(x-x_+)(x-x_-)}} = \frac{g}{\alpha^2}[(\alpha^2-1)\Pi(\alpha^2,k)$$
$$+ F(k)], \tag{C28}$$

$$J_2 \equiv \int_0^1 \frac{x^2\,dx}{\sqrt{x(x-1)(x-x_+)(x-x_-)}} = \frac{g}{\alpha^4}\Bigg\{ F(k) + 2(\alpha^2$$

$$-1)\Pi(\alpha^2,k) + \frac{\alpha^2-1}{2(k^2-\alpha^2)}[\alpha^2 E(k) + (k^2-\alpha^2)F(k)$$

$$+ (2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2)\Pi(\alpha^2,k)]\Bigg\}, \tag{C29}$$

which together with Eqs. (C11)–(C13) allows us to express the function M in terms of the elliptic integrals as

$$M = \frac{g}{w}[A(p',w)\Pi(\alpha_1^2,k) + A(p',w)^*\Pi(\alpha_1^2,k)^* + B(p',w)F(k)$$
$$+ X(p',w)\Pi(\alpha^2,k) + Y(p',w)E(k)] \tag{C30}$$

with the coefficients in the above equation of the form

$$A(p',w) = \frac{p'^2}{2}\left(1 - \frac{\alpha^2}{\alpha_1^2}\right)\Bigg\{ 1 + \frac{i(1+2p')(1+w^2)}{\mathrm{Im}(p')}$$

$$- \frac{ip'(1+w^2)}{\alpha_1^2 \mathrm{Im}(p')}\Bigg[2\alpha^2$$

$$+ \frac{(\alpha_1^2-\alpha^2)(2\alpha_1^2 k^2 + 2\alpha_1^2 - \alpha_1^4 - 3k^2)}{2(\alpha_1^2-1)(k^2-\alpha_1^2)}\Bigg]\Bigg\}, \tag{C31}$$

$$B(p',w) = \frac{1+w^2}{\mathrm{Im}(p')}\mathrm{Im}\Bigg\{ \frac{p'^3}{\alpha_1^4}\Bigg[\alpha^4 + \frac{(\alpha_1^2-\alpha^2)^2}{2(\alpha_1^2-1)}\Bigg]\Bigg\}$$

$$+ \mathrm{Re}\Bigg\{ \frac{\alpha^2 p'^2}{\alpha_1^2}\Bigg[1 + \frac{i(1+2p')(1+w^2)}{\mathrm{Im}(p')}\Bigg]\Bigg\} + (1+w^2)$$

$$\times [p'^{*2} + 2(1+p')\mathrm{Re}(p')] - \mathrm{Re}[p'(1+p')]$$

$$+ \frac{1}{\alpha^2}\Bigg[\frac{1}{2} + w^2 + \mathrm{Re}(p')\Bigg] \tag{C32}$$

$$- \frac{w^2(1+\alpha^2)}{2\alpha^4}, \tag{C33}$$

$$X(p',w) = \left(1 - \frac{1}{\alpha^2}\right)\Bigg[\frac{1}{2} + \mathrm{Re}(p') + w^2$$

$$- \frac{w^2(2\alpha^2 k^2 - \alpha^4 + k^2 - 2\alpha^2)}{2\alpha^2(k^2-\alpha^2)}\Bigg],$$

$$Y(p',w) = \frac{1+w^2}{2\,\mathrm{Im}(p')}\mathrm{Im}\Bigg[\frac{p'^3(\alpha_1^2-\alpha^2)^2}{\alpha_1^2(\alpha_1^2-1)(k^2-\alpha_1^2)}\Bigg]$$

$$- \frac{w^2(\alpha^2-1)}{2\alpha^2(k^2-\alpha^2)}. \tag{C34}$$

Here, k, α, and α_1 are defined by Eqs. (C14) and (C15).

Since w^2 is purely real, the functions $A(p',w)$, $B(p',w)$, $X(p',w)$, and $Y(p',w)$ vanish, and thus the integral $M(\mathbf{p},\mathbf{q},\Omega)$ given by Eq. (B18) is identically equal to zero. This result also implies that, because of the continuity, in the case of (anti) collinear vectors $\mathbf{p}$ and $\mathbf{q}$, i.e., when $\mathrm{Im}(p')=0$, the function $M(\mathbf{p},\mathbf{q},\Omega)$ also vanishes. When the root x_+ in Eq. (C4) is less than one, the analogous calculation shows that the function $M=0$, as well.

APPENDIX D: VIOLATION OF THE WARD-TAKAHASHI IDENTITY WITHIN HARD-CUT-OFF REGULARIZATION

In this appendix we show that Ward-Takahashi identity does not hold within the hard cut-off regularization. In order to show that, we will follow the same steps as in the previous two appendices. Let us first calculate the self-energy, given by Eq. (62)

$$\Sigma_{\mathbf{p}}(i\omega) = \int \frac{d\omega'}{2\pi}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{2\pi e^2}{|\mathbf{p}-\mathbf{k}|}\frac{i\omega' + \sigma\cdot\mathbf{k}}{\omega'^2 + k^2}$$

$$= \pi e^2 \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{\sigma\cdot\mathbf{k}}{p|\mathbf{p}-\mathbf{k}|}, \tag{D1}$$

which after using the Feynman parametrization in Eq. (63), shifting the momentum variable, and performing angular integration becomes

$$\Sigma_{\mathbf{p}}(i\omega) = \frac{e^2}{2\pi}\sigma\cdot\mathbf{p}\int_0^1 dy\sqrt{\frac{y}{1-y}}\int_0^\Lambda dk\frac{k}{k^2 + y(1-y)p^2}, \tag{D2}$$

where Λ is the momentum cutoff regulating the ultraviolet divergence of the integral. Remaining integrations then yield

$$\Sigma_{\mathbf{p}}(i\omega) = \frac{e^2}{4}\sigma\cdot\mathbf{p}(\ln\Lambda - \ln p + 2\ln 2). \tag{D3}$$

Therefore, the divergent part of the momentum integral appears as the logarithm of the cutoff which corresponds to $1/\epsilon$ pole in the dimensional regularization, see Eq. (71). Note that the divergent parts appear with precisely the same coefficients but the finite parts are different within the two regularizations.

We now consider the Coulomb vertex function defined in Eq. (70) which after introducing the Feynman parameters may be written in the form given by Eq. (A10) with the integral performed in $D=2$. After following the same steps as in Appendix A, and performing straightforward integrations, we obtain

$$\int \frac{d^2\ell}{(2\pi)^2}\frac{1}{|\mathbf{p}+x\mathbf{q}-\ell|}\frac{1}{\sqrt{\ell^2 + x(1-x)(\Omega^2+q^2)}}$$

$$= \frac{1}{2\pi}\Bigg[\ln\Lambda - \frac{1}{2}\ln\frac{(\mathbf{p}+x\mathbf{q})^2}{16}$$

$$- \tanh^{-1}\sqrt{\frac{x(1-x)(\Omega^2+q^2)}{(\mathbf{p}+x\mathbf{q})^2 + x(1-x)(\Omega^2+q^2)}}\Bigg]. \tag{D4}$$

Analogously, we have

$$\int \frac{d^2\ell}{(2\pi)^2} \frac{(\sigma \cdot \ell - xS)\sigma_\mu[\sigma \cdot \ell + (1-x)S]}{|\mathbf{p} + x\mathbf{q} - \ell|(\ell^2 + \Delta)^{3/2}}$$

$$= \frac{\delta_{\mu 0}}{2\pi}\left\{ \ln \Lambda - \frac{1}{2}\ln \frac{(\mathbf{p} + x\mathbf{q})^2}{16} \right.$$

$$- \tanh^{-1}\sqrt{\frac{x(1-x)(\Omega^2 + q^2)}{(\mathbf{p} + x\mathbf{q})^2 + x(1-x)(\Omega^2 + q^2)}}$$

$$\left. - \frac{\sqrt{\Delta[(\mathbf{p} + x\mathbf{q})^2 + \Delta]} - \Delta}{(\mathbf{p} + x\mathbf{q})^2} \right\} + \frac{1}{2\pi^2}\{ -x(1-x)S\sigma_\mu S\mathcal{I}_0[(\mathbf{p}$$

$$+ x\mathbf{q})^2, \Delta] + [-xS\sigma_\mu\sigma \cdot (\mathbf{p} + x\mathbf{q}) + (1-x)(\mathbf{p}$$

$$+ x\mathbf{q}) \cdot \sigma\sigma_\mu S]\mathcal{I}_1[(\mathbf{p} + x\mathbf{q})^2, \Delta] + \sigma \cdot (\mathbf{p} + x\mathbf{q})\sigma_\mu\sigma \cdot (\mathbf{p}$$

$$+ x\mathbf{q})\mathcal{I}_2[(\mathbf{p} + x\mathbf{q})^2, \Delta]\} \tag{D5}$$

with $\mathcal{I}_0$, $\mathcal{I}_1$, and $\mathcal{I}_2$ defined in Eqs. (A15)–(A17). Note that the term $\sim \delta_{\mu a}\sigma_a$ is not present in the above equation since the Pauli matrices here are treated strictly in $D = 2$, and thus $\sigma_a\sigma_\mu\sigma_a = 2\delta_{\mu 0}$. Taking the contraction $q^\mu\mathcal{P}_\mu^c$, and following the steps in Appendix B, its frequency-independent part reads

$$\tilde{N}(\mathbf{p}, \mathbf{q}) = \frac{e^2}{4}\int_0^1 dx\left\{ \sigma \cdot \mathbf{q}\left[\ln \Lambda + \ln 4 - \ln[|\mathbf{p} + \mathbf{q}|] \right. \right.$$

$$\left. + \frac{x(x\mathbf{q}^2 + \mathbf{p} \cdot \mathbf{q})}{(\mathbf{p} + x\mathbf{q})^2} \right] - \frac{\sigma \cdot (\mathbf{p} + x\mathbf{q})\sigma \cdot \mathbf{q}\sigma \cdot (\mathbf{p} + x\mathbf{q})}{2(\mathbf{p} + x\mathbf{q})^2} \right\}, \tag{D6}$$

which after performing straightforward integrals yields

$$\tilde{N}(\mathbf{p}, \mathbf{q}) = \Sigma_{\mathbf{p}+\mathbf{q}}(i\nu + i\Omega) - \Sigma_{\mathbf{p}}(i\nu) + \frac{e^2}{8}\sigma \cdot \mathbf{q} \tag{D7}$$

and therefore the Ward-Takahashi identity is violated within the hard-cut-off regularization scheme. The other frequency-dependent terms, actually, vanish, since they have the same form as within the dimensional regularization but even if this were not the case they could not cancel purely momentum-dependent term on the right-hand side of Eq. (D7) that spoils the Ward-Takahashi identity. In fact, using exactly the same procedure, one can show that within dimensional regularization with Pauli matrices treated in strictly $D = 2$ the Ward-Takahashi identity is also violated precisely because of the last term on the right-hand side in the frequency-independent part of the contraction $q^\mu\mathcal{P}_\mu^c$.

APPENDIX E: KUBO FORMULA AND THE ac CONDUCTIVITY WITHIN DIMENSIONAL REGULARIZATION WITH PAULI MATRICES IN $D = 2 - \epsilon$

In this appendix we perform explicit calculation of the Coulomb correction to the conductivity within the dimensional regularization, in which both the momentum integrals and the Pauli matrices are treated in $D = 2 - \epsilon$, which, as we demonstrated, is consistent with the $U(1)$ gauge symmetry of the theory, and show that this regularization yields $\mathcal{C} = (11 - 3\pi)/6$ in Eq. (1). Let us first calculate contribution coming

from the self-energy part. Using Eq. (65) and the identity

$$\sigma_x\sigma \cdot \mathbf{k}\sigma_x = 2\sigma_x k_x - \sigma \cdot \mathbf{k} \tag{E1}$$

the self-energy part of $\delta\Pi_{\mu\nu}(i\Omega, 0)$ for $\mu = \nu = x$, $\delta\Pi_{xx}^{(a)}(i\Omega, 0)$, can be written as

$$\delta\Pi_{xx}^{(a)}(i\Omega, 0) = 2Ne^2\frac{\Gamma\left(1 - \dfrac{D}{2}\right)\Gamma\left(\dfrac{D+1}{2}\right)\Gamma\left(\dfrac{D-1}{2}\right)}{(4\pi)^{D/2}\Gamma(D)}$$

$$\times \int \frac{d\omega}{2\pi}\int \frac{d^D\mathbf{k}}{(2\pi)^D}\frac{k^{D-2}}{(\omega^2 + k^2)^2[(\omega + \Omega)^2 + k^2]}$$

$$\times \mathrm{Tr}\{(i\omega + \sigma \cdot \mathbf{k})[i(\omega + \Omega) + 2\sigma_x k_x - \sigma \cdot \mathbf{k}](i\omega$$

$$+ \sigma \cdot \mathbf{k})\sigma \cdot \mathbf{k}\}. \tag{E2}$$

Performing the trace and the frequency integral, we obtain

$$\delta\Pi_{xx}^{(a)}(i\Omega, 0) = -4Ne^2\frac{\left(1 - \dfrac{1}{D}\right)\Gamma\left(1 - \dfrac{D}{2}\right)\Gamma\left(\dfrac{D+1}{2}\right)}{(4\pi)^{D/2}\Gamma(D)}$$

$$\times \Gamma\left(\frac{D-1}{2}\right)\int \frac{d^D\mathbf{k}}{(2\pi)^D}\frac{k^{D-1}(4k^2 - \Omega^2)}{(\Omega^2 + 4k^2)^2}. \tag{E3}$$

After subtracting the zero-frequency part of the Coulomb correction to the polarization tensor, and using Eq. (41), the self-energy part of the Coulomb interaction correction to the conductivity reads

$$\sigma_a = -32\sigma_0\Omega e^2\frac{\left(1 - \dfrac{1}{D}\right)\Gamma\left(1 - \dfrac{D}{2}\right)\Gamma\left(\dfrac{D+1}{2}\right)\Gamma\left(\dfrac{D-1}{2}\right)}{(4\pi)^D\Gamma(D)\Gamma\left(\dfrac{D}{2}\right)}$$

$$\times \int_0^\infty dk k^{2D-4}\frac{\Omega^2 + 12k^2}{(\Omega^2 + 4k^2)^2}, \tag{E4}$$

where σ_0 is the Gaussian conductivity of the Dirac fermions given by Eq. (51). The remaining integral then yields

$$\sigma_a = -\sigma_0 e^2\Omega^{2D-4}\frac{\left(1 - \dfrac{1}{D}\right)\Gamma\left(1 - \dfrac{D}{2}\right)\Gamma\left(\dfrac{D+1}{2}\right)\Gamma\left(\dfrac{D-1}{2}\right)}{(4\pi)^D\Gamma(D)\Gamma\left(\dfrac{D}{2}\right)}$$

$$\times \frac{2^{8-2D}(D-1)\pi}{\cos(D\pi)}. \tag{E5}$$

Taking $D = 2 - \epsilon$ and expanding up to the order ϵ^0, we obtain the self-energy part of the Coulomb contribution to the conductivity

$$\sigma_a = \frac{1}{2}\sigma_0 e^2\left[-\frac{1}{\epsilon} + \frac{3}{2} + \gamma - \ln(64\pi) + \mathcal{O}(\epsilon) \right]. \tag{E6}$$

Let us now concentrate on the vertex part of the Coulomb correction to the conductivity. Taking the trace in $\delta\Pi_{xx}^{(b)}$ given by Eq. (78), we have

$$\delta\Pi_{xx}^{(b)}(i\Omega,0) = 2N\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}$$

$$\times\frac{1}{[(\omega+\Omega)^2+k^2][(\omega'+\Omega)^2+p^2]}$$

$$\times\frac{1}{(\omega^2+k^2)(\omega'^2+p^2)}\{\omega(\omega+\Omega)\omega'(\omega'+\Omega)$$

$$-[\omega\omega'+(\omega+\Omega)(\omega'+\Omega)]\mathbf{k}\cdot\mathbf{p}+4\mathbf{k}\cdot\mathbf{p}k_xp_x$$

$$-2k_x^2p^2-2p_x^2k^2+p^2k^2-[\omega(\omega'+\Omega)+\omega'(\omega$$

$$+\Omega)](2k_xp_x-\mathbf{k}\cdot\mathbf{p})-\omega(\omega+\Omega)(2p_x^2-p^2)$$

$$-\omega'(\omega'+\Omega)(2k_x^2-k^2)\}. \tag{E7}$$

Integration over the frequencies then yields

$$\delta\Pi_{xx}^{(b)}(i\Omega,0) = 2N\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}$$

$$\times\frac{1}{kp(\Omega^2+4k^2)(\Omega^2+4p^2)}[\Omega^2(k_xp_x-\mathbf{k}\cdot\mathbf{p})$$

$$+4(\mathbf{k}\cdot\mathbf{p}k_xp_x+k^2p^2-p_x^2k^2-p^2k_x^2)]. \tag{E8}$$

After subtracting the zero-frequency part of $\delta\Pi_{xx}(i\Omega,0)$, we obtain the vertex part of the Coulomb correction to the conductivity

$$\sigma_b = 8\sigma_0\Omega\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}\frac{1}{k^3p^3(\Omega^2+4k^2)}$$

$$\times\frac{1}{\Omega^2+4p^2}\{(\mathbf{k}\cdot\mathbf{p}k_xp_x+k^2p^2-p_x^2k^2-p^2k_x^2)$$

$$\times[\Omega^2+4(k^2+p^2)]+4k^2p^2(\mathbf{k}\cdot\mathbf{p}-k_xp_x)\}$$

$$= \sigma_{b1}+\sigma_{b2}, \tag{E9}$$

where

$$\sigma_{b1} = 8\sigma_0\Omega\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}\frac{\Omega^2+4(k^2+p^2)}{k^3p^3(\Omega^2+4k^2)}$$

$$\times\frac{1}{\Omega^2+4p^2}\{(\mathbf{k}\cdot\mathbf{p}k_xp_x+k^2p^2-p_x^2k^2-p^2k_x^2)\}$$

$$\tag{E10}$$

and

$$\sigma_{b2} = 32\sigma_0\Omega\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}$$

$$\times\frac{\mathbf{k}\cdot\mathbf{p}-k_xp_x}{kp(\Omega^2+4k^2)(\Omega^2+4p^2)}. \tag{E11}$$

The contribution σ_{b1}, by adding and subtracting Ω^2 in the first term in the numerator, may be rewritten as

$$\sigma_{b1} = \sigma_{b1}^{(1)}+\sigma_{b1}^{(2)}+\sigma_{b1}^{(3)}, \tag{E12}$$

where

$$\sigma_{b1}^{(1)} = 2\sigma_0\Omega\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}\frac{\mathbf{k}\cdot\mathbf{p}k_xp_x+k^2p^2-p_x^2k^2-p^2k_x^2}{k^3p^3\left[k^2+\left(\frac{\Omega}{2}\right)^2\right]\left[p^2+\left(\frac{\Omega}{2}\right)^2\right]}\left[k^2+\left(\frac{\Omega}{2}\right)^2+p^2+\left(\frac{\Omega}{2}\right)^2\right]$$

$$= 4\sigma_0\Omega\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}\frac{\mathbf{k}\cdot\mathbf{p}k_xp_x+k^2p^2-p_x^2k^2-p^2k_x^2}{k^3p^3\left[k^2+\left(\frac{\Omega}{2}\right)^2\right]}, \tag{E13}$$

$$\sigma_{b1}^{(2)} = -\frac{1}{2}\sigma_0\Omega^3\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}\frac{1}{kp\left[k^2+\left(\frac{\Omega}{2}\right)^2\right]\left[p^2+\left(\frac{\Omega}{2}\right)^2\right]} \tag{E14}$$

and

$$\sigma_{b1}^{(3)} = -\frac{1}{2}\sigma_0\Omega^3\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}V_{\mathbf{k}-\mathbf{p}}\frac{\mathbf{k}\cdot\mathbf{p}k_xp_x-2p^2k_x^2}{k^3p^3\left[k^2+\left(\frac{\Omega}{2}\right)^2\right]\left[p^2+\left(\frac{\Omega}{2}\right)^2\right]}. \tag{E15}$$

The advantage of this decomposition of the integral σ_b is that its diverging part is now isolated, and it is contained in the integral $\sigma_{b1}^{(1)}$, whereas all the other integrals are finite in $D=2$.

We first consider the term $\sigma_{b1}^{(1)}$. Using the Feynman parametrization

$$\frac{1}{A^\alpha B^\beta C^\gamma} = \frac{\Gamma[\alpha + \beta + \gamma]}{\Gamma[\alpha]\Gamma[\beta]\Gamma[\gamma]}\int_0^1 dx \int_0^{1-x} dy$$
$$\times \frac{(1-x-y)^{\alpha-1}x^{\beta-1}y^{\gamma-1}}{\{(1-x-y)A + xB + yC\}^{\alpha+\beta+\gamma}} \quad (E16)$$

we write

$$\frac{1}{k^3|\mathbf{k}-\mathbf{p}|\left[\left(\frac{\Omega}{2}\right)^2 + k^2\right]}$$
$$= \frac{4}{\pi}\int_0^1 dx \int_0^{1-x} dy \frac{(1-x-y)^{1/2}x^{-1/2}}{\left[(\mathbf{k}-x\mathbf{p})^2 + x(1-x)p^2 + y\left(\frac{\Omega}{2}\right)^2\right]^3}. \quad (E17)$$

Shifting the momentum $\mathbf{k}-x\mathbf{p} \to \mathbf{k}$, and retaining terms even in $\mathbf{k}$ as these are the only nonvanishing ones due to the rotational invariance of the integrand, we obtain

$$\sigma_{b1}^{(1)} = 32\sigma_0\Omega e^2 \int_0^1 dx \int_0^{1-x} dy(1-x-y)^{-1/2}x^{1/2}$$
$$\times \int \frac{d^D\mathbf{p}}{(2\pi)^D}\frac{p^2-p_x^2}{p^3}\int \frac{d^D\mathbf{k}}{(2\pi)^D}\left[\left(1-\frac{1}{D}\right)k^2 + x^2p^2\right]$$
$$\times \frac{1}{\left[k^2 + x(1-x)p^2 + y\left(\frac{\Omega}{2}\right)^2\right]^3}. \quad (E18)$$

After performing the integral over $\mathbf{k}$, we have

$$\sigma_{b1}^{(1)} = \frac{32\sigma_0\Omega e^2}{(4\pi)^D\Gamma\left(\frac{D}{2}\right)}\left(1-\frac{1}{D}\right)\int_0^1 dx \int_0^{1-x}$$
$$\times dy x^{-1/2}(1-x-y)^{1/2}$$
$$\times \int_0^\infty dp \frac{p^{D-2}}{\left[x(1-x)p^2 + y\left(\frac{\Omega}{2}\right)^2\right]^{2-D/2}}$$
$$\times \left\{\frac{1}{2}(D-1)\Gamma\left(2-\frac{D}{2}\right) + \frac{x^2p^2\Gamma\left(3-\frac{D}{2}\right)}{x(1-x)p^2 + y\left(\frac{\Omega}{2}\right)^2}\right\}. \quad (E19)$$

Integration over p then yields

$$\sigma_{b1}^{(1)} = \sigma_0 e^2\Omega^{2D-4}\frac{2^{9-2D}\Gamma\left(\frac{5}{2}-D\right)\Gamma\left(\frac{D+1}{2}\right)\left(1-\frac{1}{D}\right)}{(4\pi)^D\Gamma\left(\frac{D}{2}\right)}$$
$$\times \int_0^1 dx x^{-D/2}(1-x)^{-(D+1)/2}\int_0^{1-x} dy y^{D-(5/2)}$$
$$\times (1-x-y)^{1/2}. \quad (E20)$$

Using Eq. (64) and the identity

$$\int_0^{1-x} dy(1-x-y)^{1/2}y^{D-5/2} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(D-\frac{3}{2}\right)}{\Gamma(D)}(1-x)^{D-1} \quad (E21)$$

after integration over y and x, $\sigma_{b1}^{(1)}$ acquires the form

$$\sigma_{b1}^{(1)} = \sigma_0 e^2\Omega^{2D-4}\frac{4^{4-D}\left(1-\frac{1}{D}\right)\Gamma\left(\frac{5}{2}-D\right)\Gamma\left(1-\frac{D}{2}\right)}{(4\pi)^D\Gamma\left(\frac{D}{2}\right)\Gamma(D)}$$
$$\times \Gamma\left(D-\frac{3}{2}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D-1}{2}\right). \quad (E22)$$

Finally, expanding the previous result in the parameter ϵ, we obtain

$$\sigma_{b1}^{(1)} = \frac{1}{2}\sigma_0 e^2\left[\frac{1}{\epsilon} - \frac{1}{2}(1 + 2\gamma - 12\ln 2 - 2\ln \pi) + \mathcal{O}(\epsilon)\right]. \quad (E23)$$

Therefore, poles coming from the self-energy and the vertex parts cancel out, as it should be, since the theory of Coulomb interacting Dirac fermions is renormalizable, at least to the second order in the Coulomb coupling.[36]

Let us turn to the remaining contributions which are all finite in $D=2$. We first consider the term $\sigma_{b1}^{(2)}$ in Eq. (E14). Using the Feynman parametrization in Eq. (E16), we have

$$\frac{1}{p|\mathbf{k}-\mathbf{p}|\left[\left(\frac{\Omega}{2}\right)^2 + p^2\right]}$$
$$= \frac{1}{\pi}\int_0^1 dx \int_0^{1-x} dy \frac{(1-x-y)^{-1/2}x^{-1/2}}{\left[(\mathbf{p}-x\mathbf{k})^2 + x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2\right]^2}. \quad (E24)$$

After shifting the momentum $\mathbf{p}-x\mathbf{k} \to \mathbf{p}$, and integrating over $\mathbf{p}$, the term $\sigma_{b1}^{(2)}$ in Eq. (E14) becomes

$$\sigma_{b1}^{(2)} = -\frac{1}{4\pi}\sigma_0 e^2 \Omega^3 \int_0^1 dx \int_0^{1-x} dy (1-x-y)^{-1/2}$$

$$\times \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{x^{-1/2}}{k\left[x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2\right]\left[k^2 + \left(\frac{\Omega}{2}\right)^2\right]}.$$

$$(E25)$$

Integration over the remaining momentum variable then yields

$$\sigma_{b1}^{(2)} = -\frac{1}{2\pi}\sigma_0 e^2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{\sqrt{x(1-x-y)}}$$

$$\times \frac{1}{y + \sqrt{xy(1-x)}}.$$

$$(E26)$$

After integrating out the Feynman parameter y, the term $\sigma_{b1}^{(2)}$ is

$$\sigma_{b1}^{(2)} = \frac{i}{\pi}\sigma_0 e^2 \int_0^1 dx \frac{\sec^{-1}\sqrt{x}}{\sqrt{x(1-x)}} = -\frac{\pi}{2}\sigma_0 e^2. \quad (E27)$$

We now evaluate the term $\sigma_{b1}^{(3)}$ in Eq. (E15). Using Eq. (E17), after shifting the momentum variable, and retaining only terms even in $\mathbf{p}$, we have

$$\sigma_{b1}^{(3)} = -4\sigma_0 \Omega^3 e^2 \int_0^1 dx \int_0^{1-x} dy (1-x-y)^{1/2} x^{-1/2}$$

$$\times \int \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{k_x^2}{k^3\left[k^2 + \left(\frac{\Omega}{2}\right)^2\right]}$$

$$\times \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{\left(\frac{1}{D} - 2\right)p^2 - x^2 k^2}{\left[p^2 + x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2\right]^3}. \quad (E28)$$

After integrating over $\mathbf{p}$ and setting $D=2$, we obtain

$$\sigma_{b1}^{(3)} = \sigma_0 \Omega^3 e^2 \frac{1}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy (1-x-y)^{1/2} x^{-1/2}$$

$$\times \int_0^\infty \frac{dk}{\left[k^2 + \left(\frac{\Omega}{2}\right)^2\right]\left[x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2\right]}$$

$$\times \left[\frac{3}{2} + \frac{x^2 k^2}{x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2}\right]. \quad (E29)$$

Integration over k then gives

$$\sigma_{b1}^{(3)} = -\frac{1}{4\pi}\sigma_0 e^2 \int_0^1 dx \int_0^{1-x} dy (1-x-y)^{1/2}$$

$$\times (1-x)^{-1/2} \frac{3(x-1)\sqrt{\dfrac{y}{x(1-x)}} + 2x - 3}{\sqrt{y}[\sqrt{y} + \sqrt{x(1-x)}]^2},$$

$$(E30)$$

which, after integrating over the remaining variables, yields

$$\sigma_{b1}^{(3)} = \frac{1}{12}(4 + 3\pi)\sigma_0 e^2. \quad (E31)$$

Let us now calculate the term σ_{b2} given by Eq. (E11). Using Eq. (E24), shifting the momentum $\mathbf{p} - x\mathbf{k} \to \mathbf{p}$, and retaining only terms even in $\mathbf{p}$, we have

$$\sigma_{b2} = 4\sigma_0 e^2 \Omega \int_0^1 dx \int_0^{1-x} dy (1-x-y)^{-1/2} x^{-1/2}$$

$$\times \int \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{x(k^2 - k_x^2)}{k\left[k^2 + \left(\frac{\Omega}{2}\right)^2\right]}$$

$$\times \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{1}{\left[p^2 + x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2\right]^2}. \quad (E32)$$

After the integration over $\mathbf{p}$ and setting $D=2$, we obtain

$$\sigma_{b2} = \frac{\sigma_0 e^2 \Omega}{(2\pi)^2} \int_0^1 dx \int_0^{1-x} dy (1-x-y)^{-1/2} x^{1/2}$$

$$\times \int_0^\infty dk \frac{k^2}{\left[k^2 + \left(\frac{\Omega}{2}\right)^2\right]\left[x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2\right]}. \quad (E33)$$

Integration over k then gives

$$\sigma_{b2} = \frac{\sigma_0 e^2}{4\pi} \int_0^1 dx \int_0^{1-x} dy \frac{[(1-x)(1-x-y)]^{-(1/2)}}{\sqrt{y} + \sqrt{x(1-x)}}$$

$$= \sigma_0 e^2 \frac{1}{4\pi} \int_0^1 dx \frac{1}{\sqrt{1-x}}\left[\pi + 2i\sqrt{\frac{x}{1-x}}\sec^{-1}\sqrt{x}\right],$$

$$(E34)$$

where $\sec^{-1} x$ is the inverse function of $\sec x \equiv 1/\cos x$. Finally, integration over x yields

$$\sigma_{b2} = \frac{4 - \pi}{4}\sigma_0 e^2. \quad (E35)$$

Therefore, using Eqs. (E6), (E23), (E27), (E31), and (E35) we obtain the first-order correction to the ac conductivity due to the Coulomb interaction

$$\delta\sigma^{(c)} = \sigma_a + \sigma_{b1}^{(1)} + \sigma_{b1}^{(2)} + \sigma_{b1}^{(3)} + \sigma_{b2} = \frac{11 - 3\pi}{6}\sigma_0 e^2,$$
(E36)

which corresponds to the value

$$\mathcal{C} = \frac{11 - 3\pi}{6}$$
(E37)

in Eq. (1).

APPENDIX F: LONGITUDINAL CONDUCTIVITY USING DENSITY-DENSITY CORRELATOR

In order to obtain the longitudinal conductivity, we expand $\delta\Pi_{00}^{(c)}(i\Omega, \mathbf{q})$ in Eq. (83), which is the time component ($\mu = \nu = 0$) of the Coulomb correction to the polarization tensor in Eq. (59), to the order $\mathbf{q}^2$. Let us first consider the self-energy part in Eq. (84) which may be written as

$$\delta\Pi_{00}^{(a)}(i\Omega, \mathbf{q}) = \delta\Pi_{00}^{(a1)}(i\Omega, \mathbf{q}) + \delta\Pi_{00}^{(a1)}(-i\Omega, -\mathbf{q}), \quad (F1)$$

where

$$\delta\Pi_{00}^{(a1)}(i\Omega, \mathbf{q}) = N \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d\omega}{2\pi} \int \frac{d^2\mathbf{p}}{(2\pi)^2} \int \frac{d\omega'}{2\pi} V_{\mathbf{k}-\mathbf{p}}$$
$$\times \text{Tr}[G_{\mathbf{k}}(i\omega)G_{\mathbf{p}}(i\omega')G_{\mathbf{k}}(i\omega)G_{\mathbf{k}+\mathbf{q}}(i\omega + i\Omega)].$$
(F2)

Using Eq. (65) to integrate over the momentum $\mathbf{p}$, and taking the trace over Pauli matrices, we obtain

$$\delta\Pi_{00}^{(a1)}(i\Omega, \mathbf{q}) = 2Ne^2 \frac{\Gamma\left(1 - \frac{D}{2}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{D/2}\Gamma(D)}$$
$$\times \int \frac{d\omega}{2\pi} \int$$
$$\times \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{k^{D-2}}{(\omega^2 + k^2)^2[(\omega + \Omega)^2 + (\mathbf{k} + \mathbf{q})^2]}$$
$$\times [(-\omega^2 + k^2)\mathbf{k} \cdot (\mathbf{k} + \mathbf{q}) - 2\omega(\omega + \Omega)k^2].$$
(F3)

Note that above expression contains a part divergent in $D = 2$ that arises from the self-energy in Eq. (65) and when multiplied with the remaining terms of the order ϵ gives a finite result, i.e., the final result does not have a pole in ϵ. Therefore, in order not to overlook this subtle cancellation, we have to perform the integrations in D-dimensions first, and only at the end of the calculation to take $D = 2 - \epsilon$, with $\epsilon \to 0$. Expanding the $\mathbf{q}$-dependent term in the denominator to the quadratic order in $\mathbf{q}$, and keeping only the terms quadratic in $\mathbf{q}$ in the expression for $\delta\Pi_{00}^{(a1)}(i\Omega, \mathbf{q})$, we find

$$\delta\Pi_{00}^{(a1)}(i\Omega, \mathbf{q}) = 2Ne^2 \frac{\Gamma\left(1 - \frac{D}{2}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{D/2}\Gamma(D)}$$
$$\times \int \frac{d\omega}{2\pi} \int \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{k^{D-2}}{(\omega^2 + k^2)^2[(\omega + \Omega)^2 + k^2]^2}$$
$$\times [(-\omega^2 + k^2)\mathbf{k} \cdot (\mathbf{k} + \mathbf{q}) - 2\omega(\omega + \Omega)k^2]$$
$$\times \left(-2\mathbf{k} \cdot \mathbf{q} - q^2 + \frac{4(\mathbf{k} \cdot \mathbf{q})^2}{(\omega + \Omega)^2 + k^2}\right)$$
$$= \delta\Pi_{00}^{(a11)}(i\Omega, \mathbf{q}) + \delta\Pi_{00}^{(a12)}(i\Omega, \mathbf{q}),$$
(F4)

where

$$\delta\Pi_{00}^{(a11)}(i\Omega, \mathbf{q}) = -4Ne^2 \frac{\Gamma\left(1 - \frac{D}{2}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{D/2}\Gamma(D)}$$
$$\times \int \frac{d\omega}{2\pi} \int$$
$$\times \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{(-\omega^2 + k^2)(\mathbf{k} \cdot \mathbf{q})^2}{k^{2-D}(\omega^2 + k^2)^2[(\omega + \Omega)^2 + k^2]^2}$$
(F5)

and

$$\delta\Pi_{00}^{(a12)}(i\Omega, \mathbf{q}) = 2Ne^2 \frac{\Gamma\left(1 - \frac{D}{2}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{D/2}\Gamma(D)}$$
$$\times \int \frac{d\omega}{2\pi} \int \frac{d^D\mathbf{k}}{(2\pi)^D}$$
$$\times \frac{(-\omega^2 + k^2)k^2 - 2\omega k^2(\omega + \Omega)}{k^{2-D}(\omega^2 + k^2)^2[(\omega + \Omega)^2 + k^2]^2}$$
$$\times \left(-q^2 + \frac{4(\mathbf{k} \cdot \mathbf{q})^2}{(\omega + \Omega)^2 + k^2}\right).$$
(F6)

We first consider the term $\delta\Pi_{00}^{(a11)}$ in Eq. (F5). After integrating over ω and using the rotational symmetry of the integrand, we have

$$\delta\Pi_{00}^{(a11)}(i\Omega, \mathbf{q}) = -Ne^2 \frac{\Gamma\left(1 - \frac{D}{2}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{D/2}\Gamma(D)}$$
$$\times \frac{q^2}{D} \int \frac{d^D\mathbf{k}}{(2\pi)^D} k^{D-3} \frac{32k^4 - 12k^2\Omega^2 - \Omega^4}{(\Omega^2 + k^2)^3}.$$
(F7)

After integrating over $\mathbf{k}$, and expanding the result in ϵ, we obtain to the order ϵ^0

$$\delta\Pi_{00}^{(a11)}(i\Omega,\mathbf{q}) = \frac{3}{64}Ne^2\frac{q^2}{\omega} = \frac{3}{4}\sigma_0 e^2\frac{q^2}{|\Omega|}. \qquad (F8)$$

The term $\delta\Pi_{00}^{(a12)}(i\Omega,\mathbf{q})$ given by Eq. (F6), after integration over the frequency and using the rotational symmetry of the integrand, acquires the form

$$
\delta\Pi_{00}^{(a12)}(i\Omega,\mathbf{q}) = -Ne^2 \frac{\Gamma\left(1-\frac{D}{2}\right)\Gamma\left(\frac{D+1}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{D/2}\Gamma(D)}
$$
$$
\times \frac{q^2}{2D}\int\frac{d^D\mathbf{k}}{(2\pi)^D}\frac{k^{D-3}}{(\Omega^2+4k^2)^3}
$$
$$
\times[16(D-5)k^4 + 24k^2\Omega^2 + (3-D)\Omega^4]. \qquad (F9)
$$

Integration over $\mathbf{k}$ in the last expression, and expansion in ϵ then yield

$$\delta\Pi_{00}^{(a12)}(i\Omega,\mathbf{q}) = -\frac{1}{32}Ne^2\frac{q^2}{|\Omega|} = -\frac{1}{2}\sigma_0 e^2\frac{q^2}{|\Omega|}. \qquad (F10)$$

Therefore, using Eq. (F4), we have

$$\delta\Pi_{00}^{(a1)}(i\Omega,\mathbf{q}) = \frac{1}{64}Ne^2\frac{q^2}{|\Omega|} = \frac{1}{4}\sigma_0 e^2\frac{q^2}{|\Omega|}, \qquad (F11)$$

which together with Eq. (F1) gives for the self-energy part

$$\delta\Pi_{00}^{(a)}(i\Omega,\mathbf{q}) = \frac{1}{2}\sigma_0 e^2\frac{q^2}{|\Omega|}. \qquad (F12)$$

Let us now concentrate on the vertex part of the density-density correlator given by Eq. (85). In order to calculate this contribution, we need the following integral over the frequency:

$$
\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}G_{\mathbf{p}}(i\omega')G_{\mathbf{p}-\mathbf{q}}(i\omega'-i\Omega) = \frac{1}{2}\frac{1}{(|\mathbf{p}-\mathbf{q}|+p)^2+\Omega^2}
$$
$$
\times\left[-|\mathbf{p}-\mathbf{q}|-p+i\Omega\sigma\cdot\frac{2\mathbf{p}-\mathbf{q}}{|\mathbf{p}-\mathbf{q}|}+(-i\Omega\sigma\cdot\mathbf{p}+p^2-\mathbf{p}\cdot\mathbf{q}\right.
$$
$$
\left.-i\sigma_3\mathbf{p}\times\mathbf{q})\frac{|\mathbf{p}-\mathbf{q}|+p}{|\mathbf{p}-\mathbf{q}|p}\right]. \qquad (F13)
$$

Since the frequency integrals in Eq. (85) are decoupled, we use the above equation to perform them separately, and then take the trace using the identity

$$\text{Tr}[(a+\mathbf{b}\cdot\sigma)(c+\mathbf{d}\cdot\sigma)] = 2ac + 2\mathbf{b}\cdot\mathbf{d}. \qquad (F14)$$

Integrations over ω and ω' in the vertex part, given by Eq. (85), yield the coefficients

$$a = -(|\mathbf{p}-\mathbf{q}|+p)\left[1-\frac{\mathbf{p}\cdot(\mathbf{p}-\mathbf{q})}{p|\mathbf{p}-\mathbf{q}|}\right],$$

$$c = -(|\mathbf{k}-\mathbf{q}|+k)\left[1-\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})}{k|\mathbf{k}-\mathbf{q}|}\right],$$

$$
\mathbf{b} = \left[i\Omega\left(\frac{\mathbf{p}-\mathbf{q}}{|\mathbf{p}-\mathbf{q}|}-\frac{\mathbf{p}}{p}\right), \quad -i(\mathbf{p}\times\mathbf{q})_z\frac{|\mathbf{p}-\mathbf{q}|+p}{|\mathbf{p}-\mathbf{q}|p}\right],
$$
$$
\mathbf{d} = \left[i\Omega\left(\frac{\mathbf{k}-\mathbf{q}}{|\mathbf{k}-\mathbf{q}|}-\frac{\mathbf{k}}{k}\right), \quad i(\mathbf{k}\times\mathbf{q})_z\frac{|\mathbf{k}-\mathbf{q}|+k}{|\mathbf{k}-\mathbf{q}|k}\right].
$$
$$(F15)$$

in Eq. (F14), where $(\mathbf{p}\times\mathbf{q})_z = p_xq_y - p_yq_x$. Notice that $\mathbf{b}$ and $\mathbf{d}$ are three-dimensional vectors. Expanding the expressions in Eq. (F14) to the order q^2, we obtain

$$a = -\frac{q^2p^2-(\mathbf{q}\cdot\mathbf{p})^2}{p^3},$$

$$c = -\frac{q^2k^2-(\mathbf{q}\cdot\mathbf{k})^2}{k^3},$$

$$\mathbf{b} = \left\{i\Omega\frac{\mathbf{p}(\mathbf{p}\cdot\mathbf{q})-\mathbf{q}p^2}{p^3}, \quad -2i\frac{(\mathbf{p}\times\mathbf{q})_z}{p}\right\},$$

$$\mathbf{d} = \left\{i\Omega\frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{q})-\mathbf{q}k^2}{k^3}, \quad 2i\frac{(\mathbf{k}\times\mathbf{q})_z}{k}\right\} \qquad (F16)$$

thus $ac = \mathcal{O}(q^4)$ and therefore does not contribute to the conductivity whereas

$$\mathbf{b}\cdot\mathbf{d} = -\Omega^2\frac{[\mathbf{p}(\mathbf{p}\cdot\mathbf{q})-\mathbf{q}p^2]\cdot[\mathbf{k}(\mathbf{k}\cdot\mathbf{q})-\mathbf{q}k^2]}{p^3k^3}$$
$$+4\frac{(\mathbf{p}\times\mathbf{q})_z(\mathbf{k}\times\mathbf{q})_z}{pk}. \qquad (F17)$$

Setting $D=2$ in the momentum integrals, since there are no divergent subintegrals in the vertex part as it was the case in the self-energy term, the vertex part of the density-density correlator to the order q^2 has the form

$$\delta\Pi_{00}^{(b)}(i\Omega,\mathbf{q}) = \delta\Pi_{00}^{(b1)}(i\Omega,\mathbf{q}) + \delta\Pi_{00}^{(b2)}(i\Omega,\mathbf{q}),$$

where

$$
\delta\Pi_{00}^{(b1)}(i\Omega,\mathbf{q}) = -N\frac{\Omega^2}{2}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\int\frac{d^2\mathbf{p}}{(2\pi)^2}V_{\mathbf{k}-\mathbf{p}}
$$
$$
\times\frac{[\mathbf{p}(\mathbf{p}\cdot\mathbf{q})-\mathbf{q}p^2]\cdot[\mathbf{k}(\mathbf{k}\cdot\mathbf{q})-\mathbf{q}k^2]}{p^3k^3[(2p)^2+\Omega^2][(2k)^2+\Omega^2]}
$$
$$
\equiv N\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{[\mathbf{k}(\mathbf{k}\cdot\mathbf{q})-\mathbf{q}k^2]\cdot\mathbf{I}_1(\mathbf{k},\mathbf{q},\Omega)}{k^3[(2k)^2+\Omega^2]}
$$
$$(F18)$$

and

$$\delta\Pi_{00}^{(b2)}(i\Omega,\mathbf{q}) = 2N\int\frac{d^2\mathbf{k}}{(2\pi)^2}\int\frac{d^2\mathbf{p}}{(2\pi)^2}V_{\mathbf{k}-\mathbf{p}}$$

$$\times\frac{\mathbf{p}\cdot\mathbf{k}q^2 - \mathbf{p}\cdot\mathbf{q}\mathbf{q}\cdot\mathbf{k}}{pk[(2k)^2+\Omega^2][(2p)^2+\Omega^2]}$$

$$\equiv N\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{I_2(\mathbf{k},\mathbf{q},\Omega)}{k[(2k)^2+\Omega^2]}. \qquad \text{(F19)}$$

Here, we defined

$$\mathbf{I}_1(\mathbf{k},\mathbf{q},\Omega) \equiv -\frac{\Omega^2}{8}\int\frac{d^2\mathbf{p}}{(2\pi)^2}V_{\mathbf{k}-\mathbf{p}}\frac{\mathbf{p}(\mathbf{p}\cdot\mathbf{q})-\mathbf{q}p^2}{p^3\left[p^2+\left(\frac{\Omega}{2}\right)^2\right]} \qquad \text{(F20)}$$

and

$$I_2(\mathbf{k},\mathbf{q},\Omega) \equiv \frac{1}{2}\int\frac{d^2\mathbf{p}}{(2\pi)^2}V_{\mathbf{k}-\mathbf{p}}\frac{\mathbf{p}\cdot\mathbf{k}q^2 - \mathbf{p}\cdot\mathbf{q}\mathbf{q}\cdot\mathbf{k}}{p\left[p^2+\left(\frac{\Omega}{2}\right)^2\right]}. \qquad \text{(F21)}$$

We consider the term $\delta\Pi_{00}^{(b1)}(i\Omega,\mathbf{q})$, given by Eq. (F18), and, as a first step, calculate the integral $\mathbf{I}_1(\mathbf{k})$. Using the Feynman parametrization in Eq. (E16), this integral can be written as

$$\mathbf{I}_1(\mathbf{k},\mathbf{q},\Omega) = -e^2\Omega^2\int_0^1 dx\int_0^{1-x}dy\frac{\sqrt{1-x-y}}{\sqrt{x}}$$

$$\times\int\frac{d^2\mathbf{p}}{(2\pi)^2}\frac{\mathbf{p}(\mathbf{p}\cdot\mathbf{q})-\mathbf{q}p^2}{\left[(\mathbf{p}-x\mathbf{k})^2+x(1-x)\mathbf{k}^2+\frac{y}{4}\Omega^2\right]^3}, \qquad \text{(F22)}$$

which after shifting the momentum variable $\mathbf{p}-x\mathbf{k}\to\mathbf{p}$, and integrating over $\mathbf{p}$, acquires the form

$$\mathbf{I}_1(\mathbf{k},\mathbf{q},\Omega) = -e^2\frac{\Omega^2}{8\pi}\int_0^1 dx\int_0^{1-x}dy\frac{\sqrt{1-x-y}}{\sqrt{x}}$$

$$\times\left\{\frac{-\frac{1}{2}\mathbf{q}}{x(1-x)\mathbf{k}^2+\frac{y}{4}\Omega^2}\right.$$

$$\left.+\frac{x^2\mathbf{k}(\mathbf{k}\cdot\mathbf{q})-\mathbf{q}x^2\mathbf{k}^2}{\left[x(1-x)\mathbf{k}^2+\frac{y}{4}\Omega^2\right]^2}\right\}. \qquad \text{(F23)}$$

Using the previous result for the integral $\mathbf{I}_1(\mathbf{k},\mathbf{q},\Omega)$, after integration over $\mathbf{k}$, we obtain

$$\delta\Pi_{00}^{(b1)}(i\Omega,\mathbf{q}) = -\frac{Ne^2\mathbf{q}^2}{64\pi|\Omega|}\int_0^1 dx\int_0^{1-x}dy\frac{\sqrt{1-x-y}}{\sqrt{xy}}$$

$$\times\left[\frac{1}{\sqrt{y}+\sqrt{(1-x)x}}\right.$$

$$\left.+\frac{x^{3/2}}{\sqrt{1-x}[\sqrt{(1-x)x}+\sqrt{y}]^2}\right]$$

$$= -N\frac{e^2q^2}{64\pi|\Omega|}\left[\frac{4\pi}{3}+\pi\left(\pi-\frac{8}{3}\right)\right]$$

$$= N\frac{e^2q^2}{|\Omega|}\frac{1}{16}\left(\frac{1}{3}-\frac{\pi}{4}\right) = \frac{e^2q^2}{|\Omega|}\sigma_0\left(\frac{1}{3}-\frac{\pi}{4}\right). \qquad \text{(F24)}$$

We now evaluate the term $\delta\Pi_{00}^{(b2)}(i\Omega,\mathbf{q})$ in Eq. (F19). First, we compute the integral $I_2(\mathbf{k},\mathbf{q},\Omega)$ in Eq. (F21) using the Feynman parametrization in Eq. (E16)

$$I_2(\mathbf{k},\mathbf{q},\Omega) = e^2\int_0^1 dx\int_0^{1-x}dy\frac{1}{\sqrt{x}\sqrt{1-x-y}}$$

$$\times\int\frac{d^2\mathbf{p}}{(2\pi)^2}\frac{\mathbf{p}\cdot\mathbf{k}q^2 - \mathbf{p}\cdot\mathbf{q}\mathbf{q}\cdot\mathbf{k}}{\left((\mathbf{p}-x\mathbf{k})^2+x(1-x)\mathbf{k}^2+\frac{y}{4}\Omega^2\right)^2}$$

$$= \frac{e^2q^2}{8\pi}\int_0^1 dx\int_0^{1-x}dy\frac{\sqrt{x}}{\sqrt{1-x-y}}$$

$$\times\frac{k^2}{\left[x(1-x)k^2+\frac{y}{4}\Omega^2\right]}. \qquad \text{(F25)}$$

We use this result to calculate integral over $\mathbf{k}$ in Eq. (F19). After performing straightforward integrations, we find

$$\delta\Pi_{00}^{(b2)}(i\Omega,\mathbf{q}) = \frac{Ne^2q^2}{64\pi|\Omega|}\int_0^1 dx\int_0^{1-x}dy\frac{\sqrt{x}}{\sqrt{1-x-y}}$$

$$\times\frac{1}{x(1-x)+\sqrt{x(1-x)}y}$$

$$= N\frac{e^2q^2}{64|\Omega|}(4-\pi) = \frac{e^2q^2}{|\Omega|}\frac{4-\pi}{4}. \qquad \text{(F26)}$$

Using Eqs. (F24) and (F26), the vertex part of $\delta\Pi_{00}^{(c)}$ reads

$$\delta\Pi_{00}^{(b)}(i\Omega,\mathbf{q}) = \frac{e^2q^2}{|\Omega|}\frac{8-3\pi}{6}, \qquad \text{(F27)}$$

which together with the self-energy part in Eq. (F11) yields up to the order q^2

$$\delta\Pi_{00}^{(c)}(i\Omega,\mathbf{q}) = \frac{e^2q^2}{|\Omega|}\frac{11-3\pi}{6}. \qquad \text{(F28)}$$

Finally, using Eq. (40), we obtain the Coulomb interaction correction to the longitudinal conductivity

$$\sigma_\parallel^{(c)} = \frac{11 - 3\pi}{6}\sigma_0 e^2 \qquad (F29)$$

in agreement with the result in Eq. (81) obtained from the current-current correlator, which is expected, since the dimensional regularization explicitly preserves the $U(1)$ gauge symmetry of the theory of the Coulomb interacting Dirac fermions.

APPENDIX G: KUBO FORMULA AND THE ac CONDUCTIVITY WITHIN DIMENSIONAL REGULARIZATION WITH PAULI MATRICES IN $D=2$

In this appendix we present the calculation of the Coulomb correction to the ac conductivity using dimensional regularization, but treating the Pauli matrices in $D=2$ spatial dimensions strictly. As we commented earlier, this leads to the violation of the Ward identity, and thus it is incompatible with the $U(1)$ gauge symmetry of the theory, but yields the number obtained in Ref. 9. We use Eq. (41) with the Coulomb correction to the polarization tensor given by Eq. (76). Since the system of Dirac fermions interacting only via the long-range Coulomb interaction is isotropic, translationally and time-reversal invariant, the trace over spatial indices of the polarization tensor at zero momentum and a finite frequency is $\Pi_{aa}(i\Omega,0) = D\Pi_B(i\Omega,0)$, which we use to calculate the Coulomb correction to the conductivity.

Let us first consider the self-energy part obtained from Eq. (77). Using that in $D=2$, $\sigma_a\sigma_\mu\sigma_a = 2\delta_{0\mu}$, taking the trace

over the Pauli matrices, integrating over the frequencies, and subtracting the zero-frequency part, the self-energy contribution reads

$$\tilde{\sigma}_a = -\frac{8}{D}\sigma_0\Omega \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{2\pi e^2}{|\mathbf{k}-\mathbf{p}|}$$
$$\times \frac{\mathbf{k}\cdot\mathbf{p}}{kp} \frac{\Omega^2 + 12k^2}{k^2(\Omega^2 + 4k^2)^2}. \qquad (G1)$$

Performing the momentum integrals, we have

$$\tilde{\sigma}_a = -\sigma_0 e^2 \Omega^{2D-4} \frac{2^{8-4D}\Gamma\left(\dfrac{D+1}{2}\right)\left(1 - \dfrac{1}{D}\right)}{\pi^{D-1}\Gamma(D/2)\Gamma(D)\cos(\pi D)}$$
$$\times \Gamma\left(1 - \frac{D}{2}\right)\Gamma\left(\frac{D-1}{2}\right) = \frac{1}{D-1}\sigma_a \qquad (G2)$$

with σ_a given by Eq. (E5). This result, after expanding in ϵ, reads

$$\tilde{\sigma}_a = \frac{1}{2}\sigma_0 e^2 \left[\frac{1}{\epsilon} + \frac{1}{2} + \gamma - \ln 64\pi + \mathcal{O}(\epsilon)\right]. \qquad (G3)$$

The vertex part of the Coulomb correction to the conductivity is obtained from Eq. (78). The trace over spatial indices of $\delta\Pi_{\mu\nu}^{(b)}$ in Eq. (78) at the momentum $\mathbf{q}=0$, using the standard anticommutation and trace relations for the Pauli matrices in $D=2$, assumes the form

$$\delta\Pi_{aa}^{(b)}(i\Omega,0) = -4N \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^D\mathbf{p}}{(2\pi)^D} V_{\mathbf{k}-\mathbf{p}} \frac{[(\omega+\Omega)(\nu+\Omega) - 2\mathbf{k}\cdot\mathbf{p}](-\omega\nu + \mathbf{k}\cdot\mathbf{p}) - \omega\nu\mathbf{k}\cdot\mathbf{p} + k^2p^2}{(\omega^2+k^2)[(\omega+\Omega)^2+k^2](\nu^2+p^2)[(\nu+\Omega)^2+p^2]}. \qquad (G4)$$

Performing the integrals over the frequencies ω and ν in the above equation, we have

$$\delta\Pi_{aa}^{(b)}(i\Omega,0) = 2N \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int \frac{d^D\mathbf{p}}{(2\pi)^D} V_{\mathbf{k}-\mathbf{p}} \frac{\mathbf{k}\cdot\mathbf{p}(4\mathbf{k}\cdot\mathbf{p} - \Omega^2)}{kp(\Omega^2+4k^2)(\Omega^2+4p^2)}. \qquad (G5)$$

After subtracting the zero-frequency part of $\delta\Pi_{aa}(i\Omega,0)$, we obtain the vertex part of the Coulomb correction to the conductivity

$$\tilde{\sigma}_b = \frac{8}{D}\sigma_0\Omega \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{2\pi e^2}{|\mathbf{k}-\mathbf{p}|} \frac{\mathbf{k}\cdot\mathbf{p}[\mathbf{k}\cdot\mathbf{p}(\Omega^2+4k^2+4p^2) + 4k^2p^2]}{k^3p^3(\Omega^2+4k^2)(\Omega^2+4p^2)} = \tilde{\sigma}_{b1} + \tilde{\sigma}_{b2} + \tilde{\sigma}_{b3}, \qquad (G6)$$

where

$$\tilde{\sigma}_{b1} = \frac{4}{D}\sigma_0\Omega \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{2\pi e^2}{|\mathbf{k}-\mathbf{p}|} \frac{\Omega^2 + 4(k^2+p^2)}{kp(\Omega^2+4k^2)(\Omega^2+4p^2)} \qquad (G7)$$

diverges in $D=2$, as we will show in the following, and the remaining integrals

$$\tilde{\sigma}_{b2} = \frac{32}{D}\sigma_0\Omega \int \frac{d^D\mathbf{k}}{(2\pi)^D} \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{2\pi e^2}{|\mathbf{k}-\mathbf{p}|} \frac{\mathbf{k}\cdot\mathbf{p}}{kp(\Omega^2+4k^2)(\Omega^2+4p^2)} \qquad (G8)$$

and

$$\tilde{\sigma}_{b3} = \frac{4}{D}\sigma_0\Omega\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{2\pi e^2}{|\mathbf{k}-\mathbf{p}|}$$
$$\times\frac{[2(\mathbf{k}\cdot\mathbf{p})^2-k^2p^2][\Omega^2+8k^2]}{k^3p^3(\Omega^2+4k^2)(\Omega^2+4p^2)} \quad \text{(G9)}$$

are finite in two dimensions. The above decomposition is obtained from Eq. (G6) by adding and subtracting Ω^2 in the term in the numerator multiplying $(\mathbf{k}\cdot\mathbf{p})^2$. We now further decompose the integral $\tilde{\sigma}_{b1}$ in order to isolate its diverging part

$$\tilde{\sigma}_{b1} = \tilde{\sigma}_{b1}^{(1)} + \tilde{\sigma}_{b1}^{(2)}. \quad \text{(G10)}$$

Here, the term

$$\tilde{\sigma}_{b1}^{(1)} = \frac{8}{D}\sigma_0\Omega\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{2\pi e^2}{|\mathbf{k}-\mathbf{p}|}\frac{1}{kp(\Omega^2+4k^2)} \quad \text{(G11)}$$

has the pole in Laurent expansion in ϵ, while the remaining one

$$\tilde{\sigma}_{b1}^{(2)} = -\frac{4}{D}\sigma_0\Omega^3\int\frac{d^D\mathbf{k}}{(2\pi)^D}\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{2\pi e^2}{|\mathbf{k}-\mathbf{p}|}$$
$$\times\frac{1}{kp(\Omega^2+4k^2)(\Omega^2+4p^2)} \quad \text{(G12)}$$

is finite in $D=2$.

We first consider the term divergent in $D=2$, namely, $\tilde{\sigma}_{b1}^{(1)}$. Integration over $\mathbf{p}$ in Eq. (G11), after performing the standard steps, yields

$$\int\frac{d^D\mathbf{p}}{(2\pi)^D}\frac{1}{p|\mathbf{k}-\mathbf{p}|} = \frac{k^{D-2}}{\pi}\frac{\Gamma\left(1-\frac{D}{2}\right)\left[\Gamma\left(\frac{D-1}{2}\right)\right]^2}{(4\pi)^{D/2}\Gamma(D-1)} \quad \text{(G13)}$$

while after integrating over $\mathbf{k}$, we find

$$\tilde{\sigma}_{b1}^{(1)} = \sigma_0 e^2\Omega^{2D-4}$$
$$\times\frac{\Gamma\left(1-\frac{D}{2}\right)\left[\Gamma\left(\frac{D-1}{2}\right)\right]^2\Gamma\left(D-\frac{3}{2}\right)\Gamma\left(\frac{5}{2}-D\right)}{2^{4D-7}\pi^D D\Gamma(D-1)\Gamma\left(\frac{D}{2}\right)}. \quad \text{(G14)}$$

The previous expression, after expanding in ϵ, reads

$$\tilde{\sigma}_{b1}^{(1)} = \frac{1}{2}\sigma_0 e^2\left[\frac{1}{\epsilon}+\frac{1}{2}-\gamma+\ln(64\pi)+O(\epsilon)\right]. \quad \text{(G15)}$$

Thus the poles in $\tilde{\sigma}_a$, given by Eq. (G3), and $\tilde{\sigma}_{b1}^{(1)}$ cancel out, and

$$\tilde{\sigma}_{b1}^{(1)} + \tilde{\sigma}_a = \frac{1}{2}\sigma_0 e^2 + \mathcal{O}(\epsilon). \quad \text{(G16)}$$

The remaining integrals are finite in $D=2$, and can be calculated as follows. Consider the term $\tilde{\sigma}_{b2}$ given by Eq. (G8).

Using Feynman parametrization in Eq. (E16), we write

$$\frac{1}{p|\mathbf{k}-\mathbf{p}|\left[\left(\frac{\Omega}{2}\right)^2+p^2\right]} = \frac{1}{\pi}\int_0^1 dx\int_0^{1-x}dy$$
$$\times\frac{(1-x-y)^{-1/2}x^{-1/2}}{\left[(\mathbf{p}-x\mathbf{k})^2+x(1-x)k^2+y\left(\frac{\Omega}{2}\right)^2\right]^2}. \quad \text{(G17)}$$

As usual, we now shift the momentum $\mathbf{p}-x\mathbf{k}\to\mathbf{p}$, and integrate over $\mathbf{p}$, to obtain

$$\tilde{\sigma}_{b2} = \frac{1}{2\pi}\sigma_0 e^2\Omega\int_0^1 dx\int_0^{1-x}dy(1-x-y)^{-1/2}x^{1/2}$$
$$\times\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{k}{\left[x(1-x)k^2+y\left(\frac{\Omega}{2}\right)^2\right]\left[k^2+\left(\frac{\Omega}{2}\right)^2\right]}. \quad \text{(G18)}$$

The integral over $\mathbf{k}$ in the previous equation then yields

$$\tilde{\sigma}_{b2} = \frac{1}{4\pi}\sigma_0 e^2\int_0^1 dx\int_0^{1-x}dy$$
$$\times\frac{1}{\sqrt{(1-x)(1-x-y)}(\sqrt{x(1-x)}+\sqrt{y})} \quad \text{(G19)}$$

while the integration over the variable y gives

$$\tilde{\sigma}_{b2} = \frac{1}{4\pi}\sigma_0 e^2\int_0^1 dx\frac{1}{\sqrt{1-x}}\left[\pi+2i\sqrt{\frac{x}{1-x}}\sec^{-1}\sqrt{x}\right]. \quad \text{(G20)}$$

Finally, after performing the remaining integral over x, we obtain

$$\tilde{\sigma}_{b2} = \frac{4-\pi}{4}\sigma_0 e^2. \quad \text{(G21)}$$

Similarly, using Eq. (G17) after shifting $\mathbf{p}-x\mathbf{k}\to\mathbf{p}$, and integrating over $\mathbf{p}$, the term $\tilde{\sigma}_{b1}^{(2)}$ in Eq. (G12) becomes

$$\tilde{\sigma}_{b1}^{(2)} = -\frac{1}{16\pi}\sigma_0 e^2\Omega^3\int_0^1 dx\int_0^{1-x}dy(1-x-y)^{-1/2}x^{-1/2}$$
$$\times\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{k\left[x(1-x)k^2+y\left(\frac{\Omega}{2}\right)^2\right]\left[k^2+\left(\frac{\Omega}{2}\right)^2\right]}. \quad \text{(G22)}$$

Integration over the remaining momentum variable yields

$$\tilde{\sigma}_{b1}^{(2)} = -\frac{1}{8\pi}\sigma_0 e^2\int_0^1 dx\int_0^{1-x}dy\frac{1}{\sqrt{x(1-x-y)}[y+\sqrt{xy(1-x)}]}. \quad \text{(G23)}$$

After integrating out the Feynman parameter y, the term $\tilde{\sigma}_{b1}^{(2)}$ is

$$\tilde{\sigma}_{b1}^{(2)} = \frac{i}{4\pi}\sigma_0 e^2 \int_0^1 dx \frac{\sec^{-1}\sqrt{x}}{\sqrt{x(1-x)}} = -\frac{\pi}{8}\sigma_0 e^2. \quad (G24)$$

Let us now calculate the term σ_{b3} in Eq. (G9). Using Feynman parametrization in Eq. (E16), we can write

$$\frac{1}{p^3|\mathbf{k}-\mathbf{p}|\left[\left(\frac{\Omega}{2}\right)^2 + p^2\right]}$$

$$= \frac{4}{\pi}\int_0^1 dx \int_0^{1-x} dy \frac{(1-x-y)^{1/2}x^{-1/2}}{\left[(\mathbf{p}-x\mathbf{k})^2 + x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2\right]^3}. \quad (G25)$$

After shifting $\mathbf{p} - x\mathbf{k} \rightarrow \mathbf{p}$, and integrating over $\mathbf{p}$, we have

$$\tilde{\sigma}_{b3} = \frac{1}{8\pi}\sigma_0 e^2 \Omega \int_0^1 dx \int_0^{1-x} dy(1-x-y)^{1/2}x^{3/2}$$

$$\times \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{k(\Omega^2+8k^2)}{\left[x(1-x)k^2 + y\left(\frac{\Omega}{2}\right)^2\right]^2 \left[k^2 + \left(\frac{\Omega}{2}\right)^2\right]} \quad (G26)$$

while integration over $\mathbf{k}$ then gives

$$\tilde{\sigma}_{b3} = \frac{1}{8\pi}\sigma_0 e^2 \int_0^1 dx \int_0^{1-x} dy$$

$$\times \frac{\sqrt{1-x-y}[x(1-x) + 4\sqrt{xy(1-x)} + 2y]}{(1-x)^{3/2}[y + \sqrt{xy(1-x)}][\sqrt{y} + \sqrt{x(1-x)}]}, \quad (G27)$$

which after calculating the integral over the variable y becomes

$$\tilde{\sigma}_{b3} = \frac{1}{8\pi}\sigma_0 e^2 \int_0^1 dx(1-x)^{-3/2}[2(x-1)\sqrt{x} + \pi(1-x^2)$$

$$+ 2i\sqrt{x^3(1-x)} \sec^{-1}\sqrt{x}]$$

$$= \frac{14 - 3\pi}{24}\sigma_0 e^2. \quad (G28)$$

Therefore, we calculated all the terms needed to obtain the Coulomb correction to the conductivity within this regularization scheme. Using Eqs. (G16), (G21), (G24), and (G28), we obtain the final result as found in Ref. 9

$$\tilde{\sigma}^{(c)} = \tilde{\sigma}_a + \tilde{\sigma}_b = \tilde{\sigma}_a + \tilde{\sigma}_{b1}^{(1)} + \tilde{\sigma}_{b1}^{(2)} + \tilde{\sigma}_{b2} + \tilde{\sigma}_{b3} = \frac{25 - 6\pi}{12}\sigma_0 e^2, \quad (G29)$$

which thus yields $\mathcal{C} = (25 - 6\pi)/12$, different than one in Eq. (82) obtained using dimensional regularization with Pauli matrices in $D = 2 - \epsilon$ spatial dimensions.

APPENDIX H: DIRECT EVALUATION OF THE ac CONDUCTIVITY FROM THE KUBO FORMULA IN TWO DIMENSIONS

In this section we show that the value of the coefficient $\mathcal{C}$ may also be unambiguously computed directly in two spatial dimensions, provided extra care is taken in evaluations of the integral that defines it. The result is then in agreement with the general dimensional-regularization scheme used in the rest of the paper.

As suggested by Mishchenko,[10] the expression for the first-order correction to conductivity may also be conveniently written as a sum of three terms

$$\sigma' = \sigma'_a + \sigma'_b + I, \quad (H1)$$

where

$$\sigma'_a = e^2\omega \int \frac{d^2\mathbf{p}d^2\mathbf{k}}{(2\pi)^4}V_{\mathbf{p}-\mathbf{k}}\cos\theta_{\mathbf{pk}}\frac{\omega^2 - 4p^2}{p^2(\omega^2 + 4p^2)^2}, \quad (H2)$$

$$\sigma'_b = e^2\omega \int \frac{d^2\mathbf{p}d^2\mathbf{k}}{(2\pi)^4}V_{\mathbf{p}-\mathbf{k}}\cos\theta_{\mathbf{pk}}\frac{4 - (\omega^2/pk)\cos\theta}{(\omega^2 + 4k^2)(\omega^2 + 4p^2)}, \quad (H3)$$

$$I = -2e^2\omega \int \frac{d^2\mathbf{p}d^2\mathbf{k}}{(2\pi)^4}V_{\mathbf{p}-\mathbf{k}}\cos\theta_{\mathbf{pk}}\frac{k - p\cos\theta}{p^2k(\omega^2 + 4p^2)}. \quad (H4)$$

Here we have set the fermi velocity to unity for simplicity and taken ω to be the Matsubara frequency.

Now we show that all three terms are UV convergent and when summed yield the same result as the dimensional regularization. First[10]

$$\sigma_0 + \sigma'_a = 2e^2\omega \int \frac{d^2\mathbf{p}}{(2\pi)^2}\frac{v_p}{p(1 + 4v_p^2p^2)} + O(V^2), \quad (H5)$$

where

$$v_p = v_F\left(1 + \frac{e^2}{4}\ln\frac{\Lambda}{p}\right) \quad (H6)$$

is the renormalized velocity. A simple change in variables then gives

$$\frac{\sigma'_a}{\sigma_0} = \beta_v(e^2), \quad (H7)$$

where

$$\beta_v(e^2) = -\frac{dv_p}{v_p d\ln(p)} = \frac{e^2}{4} + O(e^4) \quad (H8)$$

is the beta-function for the velocity. The coefficients in the series expansion of $\beta_v(e^2)$ are universal numbers and therefore

$$\frac{\sigma'_a}{\sigma_0} = \frac{e^2}{4} \quad (H9)$$

in agreement with Ref. 10. The result also agrees with the brute force numerical integration, in which the integral over

the angle is computed first and exactly, to be followed by the UV-convergent integrals over the momenta, which are then computed up to a large cutoff.

The second integral is completely convergent and was solved by Mishchenko[10] with the result

$$\frac{\sigma_b'}{\sigma_0} = \left(\frac{4}{3} - \frac{\pi}{2}\right)e^2. \tag{H10}$$

We have also reproduced this value numerically within a tenth of a percent.

Finally, the third integral may be written as

$$\frac{I}{\sigma_0} = -\frac{2e^2}{\pi^2}\lim_{\omega \to 0}\int_0^{\Lambda_1/\omega}\frac{dp}{p(1+4p^2)}\int_0^{\Lambda_2/\omega}dk\frac{\partial}{\partial k}\int_0^{2\pi}d\theta\cos\theta(p^2$$
$$+ k^2 - 2pk\cos\theta)^{1/2}, \tag{H11}$$

where we have carefully retained finite upper cutoffs on the momentum integrals. One finds

$$\frac{I}{\sigma_0} = -\frac{2e^2}{\pi^2}\lim_{\omega \to 0}\int_0^{\Lambda_1/\omega}\frac{dp}{p(1+4p^2)}F\left(p, \frac{\Lambda_2}{\omega}\right), \tag{H12}$$

where

$$F(x,y) = \frac{2|x-y|}{3xy}\left\{(x+y)^2K\left[-\frac{4xy}{(x-y)^2}\right]\right.$$
$$\left. - (x^2+y^2)E\left[-\frac{4xy}{(x-y)^2}\right]\right\} \tag{H13}$$

and $K(z)$ and $E(z)$ are the elliptic functions. The single remaining numerical integration quickly converges to a value

quite independent of the ratio Λ_1/Λ_2 and we obtain

$$\frac{I}{\sigma_0} = 0.2498e^2 \to \frac{e^2}{4}, \tag{H14}$$

where the last equality is the conjectured exact result. All put together gives

$$\frac{\sigma'}{\sigma_0} = \left[\frac{1}{4} + \left(\frac{4}{3} - \frac{\pi}{2}\right) + \frac{1}{4}\right]e^2 = \left(\frac{11}{6} - \frac{\pi}{2}\right)e^2 \tag{H15}$$

in agreement with the procedure of dimensional regularization.

It is instructive to see how in this calculation the value $\mathcal{C} = (25 - 6\pi)/12$ would arise. If we follow the usually safe practice and take the UV cutoffs to infinity before all the integrals have been performed, in the present case the result turns out to depend on the order of integration. In this way performing exactly the integral over k first we find

$$\frac{I}{\sigma_0} = -\frac{2e^2}{\pi^2}\int_0^\infty \frac{dp}{p(1+4p^2)}(-\pi p) = \frac{e^2}{2}, \tag{H16}$$

which ultimately yields $\mathcal{C} = (25 - 6\pi)/12$. Computing first numerically the integral over p and then the remaining integral over k, on the other hand, leads to $e^2/4$ instead. Of course, by the very definition of the integral, the correct way is to take any limits of the integration bounds only after all the integrals have been already performed. It is pleasing to see that this then leads to the same result that is obtained by the general dimensional regularization, which is constructed so to preserve the crucial symmetries of the theory.

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