Tunneling characteristics of a chain of Majorana bound states

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We consider theoretically tunneling characteristic of a junction between a normal metal and a chain of coupled Majorana bound states generated at crossings between topological and nontopological superconducting sections, as a result of, for example, disorder in nanowires. While an isolated Majorana state supports a resonant Andreev process, yielding a zero-bias differential conductance peak of height $2e^2/h$, the situation with more coupled Majorana states is distinctively different with both zeros and $2e^2/h$ peaks in the differential conductance. We derive a general expression for the current between a normal metal and a network of coupled Majorana bound states and describe the differential conductance spectra for a generic set of situations, including regular, disordered, and infinite chains of bound states.

Topological materials are of large current interest, in part because of their potential for topological quantum computing and their interesting non-Abelian quasiparticles.[1] One variant of this is topological superconductors where the low-energy quasiparticles in addition are Majorana fermions.[2,3] Currently, there is an active search for materials that can host such particles, either in certain *p*-wave superconductors,[4–8] or semiconductors with proximity-induced superconductivity and strong spin-orbit coupling.[9–12] Because it takes two Majorana fermions to form a usual fermion that can couple to other degrees of freedom, detection of the *state* of the Majorana fermion system requires nonlocal measurements or interferometry.[13–15] In contrast, a local tunnel current, being independent of the parity of the topological superconductor, does not reveal information about the state of the Majorana fermions. Nevertheless, a tunneling probe could detect the *presence* of a Majorana bound state (MBS) (Refs. [12, 16, and 17]) and the detection of Majorana bound states is the first major challenge in this field. Tunneling contact to an isolated
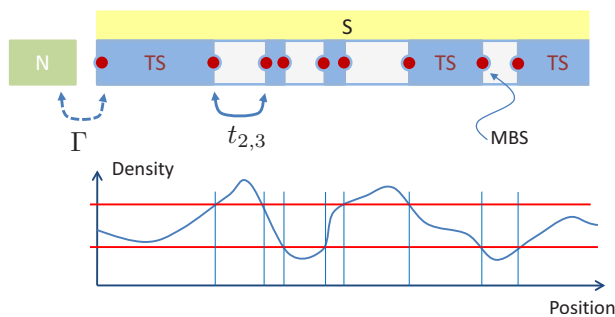

FIG. 1. (Color online) Illustration of a semiconductor with induced superconducting order parameter from an adjacent superconductor (s). Spatial variations in density or superconducting order parameter create crossings between TS and nontopological segment when the density crosses the critical density [vertical (red) lines]. MBS (circles) are located at each crossing point and the distances between them determine the coupling matrix elements t_{ij} of the resulting Majorana network. A tunneling contact (N) probes the network by tunneling into the end Majorana mode, with tunneling density of states Γ.

Majorana state give rise to a resonant Andreev process that gives a zero-bias conductance peak of $2e^2/h$.[16] With two coupled Majorana states cross correlations of the current into each could also show their existence and nonlocal character.[18,19]

However, because of material difficulties, isolated Majorana states might be rare. Rather it is to be expected that density fluctuations will generate a random configuration of topological/nontopological boundaries, at which Majorana states will be located. For strong disorder the distances between these states are sufficiently short for the MBS to overlap and therefore it is important to understand how a network of coupled Majorana fermions maps onto the tunneling characteristic. This problem was recently considered in Ref. [20] in the weak-coupling regime, using a renormalization group to reduce a chain to a sum of single Majorana pairs on a logarithmic energy scale.

In this Rapid Communication, a theory for tunneling between a metallic probe and a collection of coupled Majorana states in the strong-coupling regime is developed, and experimentally relevant situations are addressed. This is done in the limit where the voltage eV, the tunneling broadenings Γ, and the hopping matrix elements between any two MBS all are much smaller than the superconducting energy gap E_g. The regime of large E_g is well suited for characterization and detection of the MBS because in absence of Majorana states the Andreev conductance is on the order of[21] $(e^2/h)(\Gamma/E_g)^2$ and thus much smaller than the resonant Andreev current carried by the Majorana states. Examples for different number and configurations of Majorana states are given and the case of a uniform infinite chain is solved exactly. Finally, disordered Majorana chains are addressed. Disorder is introduced as random nearest-neighbor couplings and it is shown to reduce to a finite chain, truncated by the first weak link (quantified below) in the chain.

The borders of the topological superconductor segments give rise Majorana bound states. These states are zero-energy solutions to the Boguliubov-de Gennes equations for the geometry in question. The general form of a MBS is

$$\gamma_i = \sum_\sigma \int dx [f_{\sigma,i}(x)\Psi_\sigma(x) + f^*_{\sigma,i}(x)\Psi^\dagger_\sigma(x)]. \qquad (1)$$

The Majorana Fermion has the properties that $\gamma_i = \gamma^\dagger_i$ and $\gamma^2_i = 1$. The superconductor Hamiltonian, describing the coupled Majorana state network, is

$$H_S = \frac{i}{2} \sum_{ij} t_{ij} \gamma_i \gamma_j, \qquad (2)$$

where it_{ij} is the matrix element between Majorana bound states i and j. For the examples presented in the plots in this Rapid Communication we assume that t_{ij} includes nearest-neighbor coupling only. This is a good approximation when the correlation length of the disorder is larger than the typical decay length of the Majorana bound states.

The tunnel Hamiltonian between the normal metal and the superconductor is

$$H_T = \sum_{k\sigma} \int dx [t^*_k(x) c^\dagger_{k\sigma} \Psi_\sigma(x) + \text{H.c.}],$$

where $c_{k,\sigma}$ are lead-electron annihilation operators and $\Psi_\sigma(x)$ the superconductor electron field operator. As explained above, for $(eV,\Gamma) \ll E_g$ the Majorana states contribution to the current dominates. Using Nambu representation, $\boldsymbol{\Psi} = (\Psi_\uparrow, \Psi_\downarrow, \Psi^\dagger_\downarrow, \Psi^\dagger_\uparrow)$, the projection of the field operator $\boldsymbol{\Psi}$ onto the manifold of Majorana states is $\boldsymbol{\Psi}(x) \approx \Sigma_i \gamma_i [f_{\uparrow,i}(x), f_{\downarrow,i}(x), f^*_{\downarrow,i}(x), f^*_{\uparrow,i}(x)]$, which then leads to the effective tunnel Hamiltonian describing the coupling between the lead and the Majorana states

$$H_T = \sum_{k\sigma i} (V^*_{k\sigma,i} c^\dagger_{k\sigma} - V_{k\sigma,i} c_{k\sigma}) \gamma_i, \qquad (3)$$

where $V_{k\sigma,i} = \int dx f_{\sigma,i}(x) t_k(x)$. The current operator is given by the rate of change in the number of electrons in the normal lead

$$I = -e\dot{N} = -ie(H_T,N)/\hbar = \frac{2e}{\hbar} \text{Re} \sum_{k\sigma i} [V^*_{k\sigma,i} G^<_{i,k\sigma}(0)], \qquad (4)$$

where the lesser Green's function is defined as $G^<_{i,k\sigma}(t) = i\langle c^\dagger_{k\sigma} \gamma_i(t) \rangle$, which is written as

$$G^<_{i,k\sigma}(t) = \sum_j [G_{ij} V_{k\sigma,j} G^{(0)}_{k\sigma}]^<, \qquad (5)$$

where $G_{ij}(\tau,\tau') = -i\langle T(\gamma_i(\tau)\gamma_j(\tau')\rangle$ is the full Keldysh time-ordered Green's functions for the Majorana operators, and $G^{(0)}_{k\sigma}(\tau,\tau') = -i\langle T[c_{k\sigma}(\tau) c^\dagger_{k\sigma}(\tau')]\rangle_0$ is the unperturbed normal lead Green's function. By choosing the chemical potential of the superconductor as a reference, the general current formula is derived to be[22]

$$I = \frac{e}{h} \int d\omega M(\omega)[f(-\omega+eV) - f(\omega-eV)] \qquad (6)$$

with f being the Fermi-Dirac distribution and

$$M(\omega) = \text{Tr}[\mathbf{G}^R(\omega)\boldsymbol{\Gamma}^*(-\omega)\mathbf{G}^A(\omega)\boldsymbol{\Gamma}(\omega)]. \qquad (7)$$

Here the retarded Majorana Green's function is

$$\mathbf{G}^R_\omega = 2[\omega - 2i\mathbf{t} + i(\boldsymbol{\Gamma}_\omega + \boldsymbol{\Gamma}^*_{-\omega}) - (\boldsymbol{\Lambda}_\omega - \boldsymbol{\Lambda}^*_{-\omega})]^{-1}, \qquad (8)$$

where $\mathbf{t}$ is an antisymmetric matrix while the Hermitian matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$ are

$$\Gamma_{ij}(\omega) = 2\pi \sum_{k\sigma} V_{k\sigma,i} V^*_{k\sigma,j} \delta(\omega - \varepsilon_{k\sigma}), \qquad (9)$$

$$\Lambda_{ij}(\omega) = \mathcal{P} \int \frac{d\omega'}{2\pi} \frac{\Gamma_{ij}(\omega')}{\omega - \omega'}. \qquad (10)$$

If the coupling matrix respects particle-hole symmetry, $\boldsymbol{\Gamma}(\omega) = \boldsymbol{\Gamma}^*(-\omega)$, the current is antisymmetric $I(V) = -I(-V)$.[22]

The expression in Eq. (6) is a general finite temperature expression for the current into a Majorana state network in terms of matrices describing the coupling to the normal lead and the Majorana network. The general formula is straightforwardly extended to the case with more normal-metal contact connected to the network.[23]

The remainder of this Rapid Communication deals with the case where only a single Majorana bound state is coupled to the lead, i.e., $\Gamma_{ij} \equiv \delta_{i,1}\delta_{j,1}\Gamma$. Moreover, assuming an energy independent Γ (so-called wide-band limit), which leads to $\Lambda_{ij} = 0$, the differential conductance reduces to

$$\frac{dI}{dV} = \frac{2e^2}{h} \int d\omega \Gamma \, \text{Im}[G^R_{11}(eV)]\left[\frac{df(\omega-eV)}{d\omega}\right] \qquad (11)$$

with

$$\mathbf{G}^R(\omega) = 2[\omega - 2i\mathbf{t} + i2\boldsymbol{\Gamma}]^{-1}. \qquad (12)$$

For a single isolated Majorana state with tunnel broadening coupling the Green's function is: $G^R_{ii} = 2/(\omega + i2\Gamma)$ and the zero-temperature differential conductance is easily obtained as

$$\frac{dI}{dV} = \frac{2e^2}{h} \frac{4\Gamma^2}{(eV)^2 + 4\Gamma^2}, \qquad (13)$$

which confirms that the resonant Andreev tunneling with zero-bias conductance $G = 2e^2/h$.[16] With two Majoranas coupled by tunneling t and only one of them coupled to the lead the differential conductance is

$$\frac{dI}{dV} = \frac{2e^2}{h} \frac{(2eV\Gamma)^2}{((eV)^2 - 4t^2)^2 + (2eV\Gamma)^2}, \qquad (14)$$

which has a dip at zero voltage and peaks at $eV = \pm 2t$, where the conductance again reaches $2e^2/h$. In fact, a very general statement holds for tunneling into the end of a chain, namely, that *with an odd number of coupled Majorana states the zero-bias conductance is $2e^2/h$, and with an even number the zero-bias conductance is zero*. This can be shown by the inversion in Eq. (12) setting $\omega = 0$ and for an arbitrary chain matrix t_{ij}. Moreover, *for a cluster with n MBS the differential conductance versus bias voltage has $n-1$ zeros and n voltages where $dI/dV = 2e^2/h$*. If the normal metal electrode overlaps with more than one MBS these conclusions change, as shown in Fig. 2, where the conductance for some examples is plotted.

With many coupled MBS in the chain the conductance oscillates between $2e^2/h$ and 0, as seen in Fig. 2. As the
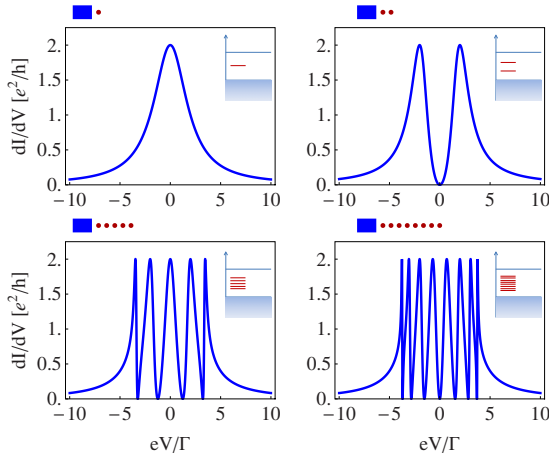
FIG. 2. (Color online) The conductance for tunneling into the end of a chain with one, two, five, and eight Majorana states coupled by $t=\Gamma$, illustrated with a small icon over each plot. The n coupled Majoranas give rise to a n states inside the gap, illustrated by the insets. The number of zeros and peaks with conductance $2e^2/h$ is also determined by n, see text.

number of sites in the chain is increased, the period of the oscillations decreases. If the period is smaller than temperature the conductance will average to a value between the two extremes. The same occurs for an infinite homogeneous chain, which is considered next.

With an infinite chain of Majorana states with nearest-neighbor couplings, t_{ij}, the Green's function for the first MBS is $G_{11}^R = 2g_{11}$, where

$$g_{11} = \frac{1}{(g_{11}^0)^{-1} - 4|t_{12}|^2 \widetilde{g}_{22}}, \qquad (15)$$

and where $\widetilde{g}_{22}$ is the Green's function for the network starting with site 2, decoupled from site 1, and where $(g_{11}^0)^{-1} = \omega + 2i\Gamma$. An illustrative example is a homogeneous chain, i.e., with all couplings identical $t_{ij} = t$. Then the Dyson equation for $\widetilde{g}_{22}$ is

$$\widetilde{g}_{22} = \frac{1}{\omega + i\eta} + \frac{4t^2}{\omega + i\eta} \widetilde{g}_{33} \widetilde{g}_{22}. \qquad (16)$$

Since all connections are equal $\widetilde{g}_{22} = \widetilde{g}_{33}$, and hence $(4t^2/\omega)\widetilde{g}_{22}^2 - \widetilde{g}_{22} + 1/\omega = 0$, which gives can be solved for $\widetilde{g}_{22}$. Choosing the correct branch cuts[24] and setting this into the Green's function (15), the differential conductance is derived to be

$$\frac{dI}{dV} = \frac{2e^2}{h} \begin{cases} \dfrac{4\Gamma(4\Gamma + \sqrt{(4t)^2 - (eV)^2})}{(eV)^2 + (4\Gamma + \sqrt{(4t)^2 - (eV)^2})^2}, & |eV| < 4|t|, \\[4mm] \dfrac{(4\Gamma)^2}{(|eV| + \sqrt{(eV)^2 - (4t)^2})^2 + (4\Gamma)^2}, & |eV| > 4|t|. \end{cases}$$

$$(17)$$

This is an interesting expression with a line shape that strongly depends on the ratio t/Γ, which is shown in Fig. 3. Furthermore, for $eV=0$ it reduces to
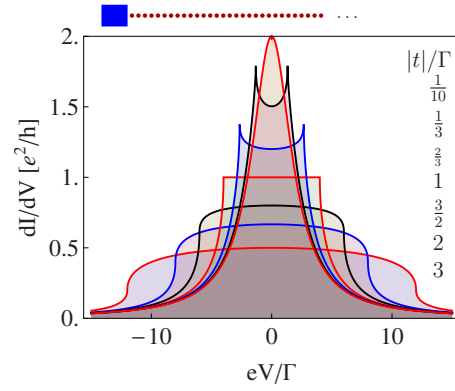

FIG. 3. (Color online) The conductance for tunneling into the end of an infinite Majorana chain with identical tunneling couplings. The parameters $|t|/\Gamma$ ranges from 5 to 1/8, from outside in.

$$\left. \frac{dI}{dV} \right|_{V=0} = \frac{2e^2}{h} \frac{2\Gamma}{|t| + 2\Gamma}, \qquad (18)$$

which shows that for small tunnel broadening compared to the bandwidth of the chain the zero-bias conductance is $2e^2/h$, which was expected because it corresponds to tunneling into an effectively isolated Majorana state.

As the last situation, which might also be the most experimentally relevant, we now discuss different realizations of long disordered chains, sampled, for example, by scanning the average density. The tunneling coupling between two neighboring MBS depends both on the distance between them and the deviation from the critical value for the topological/nontopological transition, with exponential dependence on both, as was shown by Shivamoggi et al.[20] for a specific example. The distribution of tunneling couplings is thus a complicated convolution of amplitude fluctuations and the level-crossings statistics,[25] in this case with two crossings. The resulting distribution of tunneling couplings is an interesting problem in itself and several regimes can be identified depending on the ratio of the amplitude to the topological/nontopological threshold and the ratio of the correlation length to the decay length of the Majorana bound states. For example, for large correlation length and for the disorder amplitude either close to or much larger than the S-TS threshold, the Majorana bound states would tend to come in pairs. In all other regimes, there is no correlation between the MBS couplings. Here, we focus on the generic behavior expected for a given configurations of the tunneling couplings in the chain, also taking the temperature broadening into account.

As we learned for the infinite chain, different behaviors occur depending on the ratio of Γ to the tunneling couplings t_{ij}. For the random chain the ratio of Γ to the spread of tunneling couplings turns out to be crucial. Clearly, if the spread in tunneling couplings is much smaller than Γ, the average conductance resembles that of the homogenous infinite chain, which we have verified by numerical simulation.

In contrast, with large fluctuations in the tunneling couplings, *the infinite chain will be effectively truncated into a finite chain, where one of the tunneling coupling happens to*

be much smaller than k_BT. To see this, invert the matrix in Eq. (12) and pull out the dependence on the weak link and write is as

$$G^R(11,\omega) = \frac{2D_{2,\infty}}{D_{1,\infty} - 4t_{n,n+1}^2 D_{1,n}D_{n+1,\infty} + i\Gamma D_{2,\infty}}, \quad (19)$$

where $t_{\text{weak}} = t_{n,n+1}$ is the weak link, and where $D_{i,j}$ is the determinant of the matrix $(\omega - 2\mathbf{t})$ for the isolated chain between sites i and j, but with $t_{n,n+1} = 0$. The weak link has two effects, (1) it gives small shifts of the existing resonances and (2) it creates new resonances. The new resonances, however, have widths that scale with the square of the weak coupling $\Gamma_{\text{weak}} \propto t_{\text{weak}}^2 / \langle t \rangle \Gamma$, where $\langle t \rangle$ denotes typical couplings in the first part of the chain. Therefore the new resonances introduced by the chain after the weak link is not resolved if $k_B t \gtrsim \Gamma_{\text{weak}}$. An example of this is shown in Fig. 4, where the conductance of a truncated chain is compared with that of a full chain.

In summary, the differential conductance for a junction between a normal metal and topological superconductor hosting a network of Majorana bound states has been studied. Different configurations of the interacting network of bound states give rise to distinct tunneling spectra. Long chains with fluctuating tunneling couplings is truncated into a finite chain once a coupling becomes smaller than a certain critical value, determined by temperature.
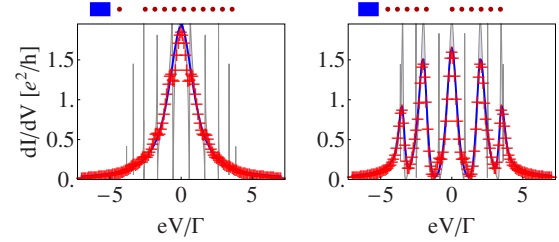

FIG. 4. (Color online) Both panels show the conductance for a chain with ten sites and one weak link with tunnel coupling $t_{\text{weak}} = 0.25\Gamma$ while the other links have $t = \Gamma$. The weak links are the 1–2 and 5–6 connections for the left/right panel, respectively. The thick (blue) curve is the conductance for $kT = 0.1\Gamma$, the thin gray curve for $kT = 0$ while the crosses show the conductance for the chain truncate at the weak link also with $kT = 0.1\Gamma$. It is clearly seen that the truncated approximation works well, because the chain after the truncations leads to structure barely resolvable, because $k_B T$ is not much smaller than the width (≈ 0.125) of the additional resonances. It should be noted that the situation with only a single deviating link is not the generic situation but it is chosen here to illustrate the point, that the first weak link determines the effective length of the chain.

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[23] by generalizing the k-sum in Eq. (9) to run over the different contacts as well and inserting a matrix that specifies the measurement contact in the trace in Eq. (7).

[24] For $|\omega| < 4t$ this is $\tilde{g}_{22} = (\omega - i\sqrt{(4t)^2 - \omega^2})/8t^2$ whereas for $|\omega| > 4t$ the result is $\tilde{g}_{22} = [\omega - \text{sgn}(\omega)\sqrt{\omega^2 - (4t)^2}]/8t^2$.

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