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Fulde-Ferrell-Larkin-Ovchinnikov states and quantum oscillations in mesoscopic superconductors and superfluid ultracold Fermi gases

A. V. Samokhvalov,[1] A. S. Mel'nikov,[1] and A. I. Buzdin[2]

[1]*Institute for Physics of Microstructures, Russian Academy of Sciences, 603950 Nizhny Novgorod, GSP-105, Russia*

[2]*Institut Universitaire de France and Universite Bordeaux, CPMOH, UMR 5798, 33405, Talence, France*

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We have studied the distinctive features of the Fulde-Ferrel-Larkin-Ovchinnikov (FFLO) instability and phase transitions in two-dimensional (2D) mesoscopic superconductors placed in magnetic field of arbitrary orientation and rotating superfluid Fermi gases with imbalanced state populations. Using a generalized version of the phenomenological Ginzburg-Landau theory we have shown that the FFLO states are strongly modified by the effect of the trapping potential confining the condensate. The phenomenon of the inhomogeneous state formation is determined by the interplay of three length scales: (i) length scale of the FFLO instability; (ii) 2D system size; (iii) length scale associated with the orbital effect caused either by the Fermi condensate rotation or magnetic field component applied perpendicular to the superconducting disk. We have studied this interplay and resulting quantum oscillation effects in both superconducting and superfluid finite-size systems with FFLO instability and described the hallmarks of the FFLO phenomenon in a restricted geometry. The finite size of the system is shown to affect strongly the conditions of the observability of switching between the states with different vorticities.

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I. INTRODUCTION

The Zeeman interactions of electron spins with magnetic field is known to be one of the mechanisms destroying the singlet superconducting order (see, e.g., Ref. 1). According to this mechanism a homogeneous superconducting state becomes energetically unfavorable above the Pauli limiting field $H_p=\Delta/\mu_B\sqrt{2}$, where Δ is the gap value and μ_B is the Bohr magneton. However, superconductivity can appear even at the fields exceeding the H_p field provided we consider inhomogeneous states with a spatially modulated Cooper-pair wave function.[2] In this scenario the Cooper pairs consist of electrons with different spin projections and different absolute values of momentum.

There are at least two difficulties in experimental observation of the FFLO instability: (i) first, the strong orbital effect which destroys Cooper pairs above the upper critical field H_{c2} which appears to be much less than H_p in most superconducting compounds; (ii) second, the impurity scattering which is known to prevent the FFLO state formation. Thus, to observe this interesting physical phenomenon we need to find rather clean superconducting materials with very short coherence lengths to increase the critical field corresponding to the orbital effect. Alternatively, we should consider strongly anisotropic quasi-two-dimensional (2D) systems or very thin films in a magnetic field parallel to the superconducting planes. Among the compounds which are usually included in the list of strong candidates for the FFLO states observation one should mention layered organic superconductors[3] and heavy fermion systems like $CeCoIn_5$ (see Ref. 4 and references therein).

During the last decade the attention of both theoreticians and experimentalists have been attracted to a new type of superfluid systems which are considered as promising playground for the study of this intriguing phenomenon, i.e., ultracold Fermi gases in magneto-optical traps.[5] The FFLO-type instability in these systems is caused not by the Zeeman interaction but by the tuning of the population imbalance between two lowest hyperfine states of ^{6}Li atoms. Experimentally this population imbalance is governed by the radio-frequency signal inducing transitions between the hyperfine states. Thus, changing the population imbalance we should get the inhomogeneous FFLO state with a certain intrinsic length scale and this phenomenon is not masked by any kind of the orbital effect. The orbital effect in such neutral atomic condensates is associated not with magnetic field but with system rotation which is known to be an important part of the experimental procedure of detection of superfluidity in the ultracold gases.[6] The FFLO states in an ultracold gas cloud should be, of course, modified by the effect of the trapping potential confining the atomic system. As a result, the physics of this phenomenon will be determined by the interplay of three length scales: (i) length scale of the FFLO instability; (ii) atomic system size; (iii) the length scale associated with the condensate rotation $L_\Omega=\sqrt{\hbar/M\Omega}$, where M is the atomic mass and Ω is the angular velocity. An analogous interplay appears in a thin mesoscopic superconducting disk with FFLO instability caused by the strong magnetic field parallel to the disk plane. The effect of rotation in this case and corresponding length L_Ω should be replaced by the magnetic field component H_z perpendicular to the disk plane and magnetic length $L_H=\sqrt{\hbar c/eH_z}$, respectively. The goal of this paper is to study this interplay in both superconducting and superfluid finite-size systems with FFLO instability and describe the hallmarks of the FFLO phenomenon in a restricted geometry.

In standard superconductors without FFLO instability the finite system size is known to cause the so-called Little-Parks effect, i.e., the oscillatory behavior of the phase transition line on the plane magnetic field-temperature.[7,8] These quantum oscillations originate from the switching between the superconducting states with different vorticities or winding numbers. The quantum oscillations of the critical temperature vs magnetic field (or angular velocity) are known to reveal themselves also in infinite 2D FFLO superconductors and superfluids.[9,10] A manifestation of FFLO state is the special type of the proximity effect at the interface between superconductor and ferromagnet which results in damped oscillatory behavior of the Cooper-pair wave function inside ferromagnet.[11] The proximity induced switching between the superconducting states with different vorticities in multiply connected hybrid superconductor/ferromagnet structures was suggested recently in Ref. 12. Our theoretical work aims to the identification of both the similarities and distinctive features of the quantum oscillations in mesoscopic systems with and without FFLO instability. We focus here on the case of 2D systems when the quantum oscillatory effects appear to be most pronounced. In Sec. II we discuss a modified Ginzburg-Landau (GL) model which takes account of both the FFLO phenomenon and confinement effect. In Sec. III we consider the case of a mesoscopic disk while the Sec. IV is devoted to the rotating Fermi condensates confined in traps. We summarize our results in Sec. V.

II. MODIFIED GINZBURG-LANDAU MODEL FOR 2D FFLO STATES

Hereafter our consideration of the FFLO phase formation will be based on modified GL theory where the appearance of the nonuniform state is caused by a change in the sign of the second-order gradient term in the free-energy expansion. An appropriate GL functional can be derived from the microscopic theory (see Ref. 13). Calculating the superfluid critical temperature one can take the GL free-energy density in the form

$$F = a|\Psi|^2 - \beta|\mathbf{D}\Psi|^2 + \gamma|\mathbf{D}^2\Psi|^2, \tag{1}$$

where Ψ is the superfluid order parameter, $a=\alpha(T-T_{c0})$ and T_{c0} is the critical temperature of the second-order transition into a uniform superconducting or superfluid state, and $\mathbf{D}$ is the gauge-invariant two-dimensional momentum operator. Note that here we omit the terms of the higher order in Ψ which come into play only below the superfluid transition. In the FFLO region the coefficients $\beta,\gamma>0$ and minimum of the free-energy functional does not correspond to uniform state since a spatial variation in the order parameter results in decrease in the system energy. Certainly, the GL functional provides an adequate description of a long-wavelength FFLO modulation only near the Lifshitz tricritical point, however the results of the GL approach can be extrapolated qualitatively to the whole region of the FFLO phase. Note that the above free-energy density includes the terms quadratic in Ψ function and, thus, our model provides an adequate description only for the second-order phase transitions. The order of the transition from the normal phase to the FFLO state strongly depends on the particular system parameters and dimensionality and is controlled by the fourth order in Ψ terms in the free-energy expansion. Explicit calculations[13,14] have demonstrated that in the clean limit in 2D superconducting systems the transition into FFLO state is always of the second order. Of course, for superfluid ultracold Fermi gases the problem of the phase transition order should be treated separately.

A lateral confinement of the condensate can be introduced either using a boundary condition for the order parameter Ψ at the sample edge or adding an external potential well $V(\mathbf{r})$ to the free-energy density (1)

$$F = [a + V(\mathbf{r})]|\Psi|^2 - \beta|\mathbf{D}\Psi|^2 + \gamma|\mathbf{D}^2\Psi|^2, \tag{2}$$

where $\mathbf{r}$ is the in-plane radius vector. Varying an appropriate free-energy functional (2) we find

$$\gamma\mathbf{D}^4\Psi + \beta\mathbf{D}^2\Psi + [a + V(r)]\Psi = 0. \tag{3}$$

We restrict ourselves to the consideration of cylindrically symmetric systems and, thus, assume the confining potential $V(r)$ to depend only on the radius r, where (r,θ,z) are the cylindrical coordinates. The value $k_0=\sqrt{\beta/2\gamma}$ in the above equation plays the role of the inverse characteristic length scale of the FFLO modulation. It is convenient to introduce a dimensionless coordinate $\rho=k_0 r$ and dimensionless shift of the critical temperature $\tau=a/\gamma k_0^4$,

$$T_c = T_{c0} + \frac{\gamma k_0^4}{\alpha}\tau. \tag{4}$$

As a result, one can rewrite Eq. (3) in a dimensionless form

$$\mathbf{D}^4_{\rho,\theta}\Psi + 2\mathbf{D}^2_{\rho,\theta}\Psi + [\tau + v(\rho)]\Psi = 0, \tag{5}$$

where $\mathbf{D}_{\rho,\theta}=\mathbf{D}/k_0$ and $v(\rho)=V/\gamma k_0^4$.

In the following sections we proceed with the calculation of the shift of the critical temperature for FFLO states with different vorticities or winding numbers. We consider two generic examples of restricted FFLO systems: (i) a thin mesoscopic superconducting disk of the radius R placed in an external magnetic field tilted with respect to the disk plane; (ii) rotating 2D superfluid Fermi condensate confined in a harmonic trap.

III. FFLO STATE IN A 2D MESOSCOPIC DISC

A thin superconducting disk of a finite radius R placed in external magnetic field $\mathbf{H}=\mathbf{H}_\parallel+H_z\mathbf{z}_0$ provides a simplest example illustrating the effect of Cooper-pair confinement on the FFLO state. The gauge-invariant 2D momentum operator in the above equations for Ψ takes the form

$$\mathbf{D} = \nabla + \frac{2\pi i}{\phi_0}\mathbf{A},$$

where $\mathbf{A}=(0,A_\theta,0)=(0,H_z r/2,0)$ is the vector potential of the field component $H_z=\mathrm{curl}_z\,\mathbf{A}$, and $\phi_0=\pi\hbar c/|e|$ is the flux quantum. Considering the limit of vanishing disk thickness we neglect here the orbital effect caused by the field component $\mathbf{H}_\parallel$. At the same time the Zeeman interaction energy associated with this parallel field component $\mathbf{H}_\parallel$ is assumed to be crucial and responsible for the FFLO instability. The coefficient $\beta=(H_\parallel,T)$ is a function of temperature T and Zeeman energy $\mu_B H_\parallel$ and vanishes in the tricritical Lifshitz point $[T^*,H^*=H_{c2}(T^*)]$: $\beta(H^*,T^*)=0$. This tricritical point (T^*,H^*) is the meeting point of three transitions lines sepa-

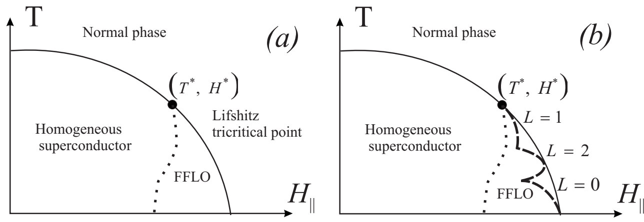

FIG. 1. Schematic phase diagram in the plane $H_\parallel - T$ for an infinite 2D superconducting film (a) and for a 2D disk of a finite radius (b) in a parallel magnetic field $H_\parallel$. Phase transition lines between normal state and different superconducting phases are shown by solid lines. The phase transition line between the FFLO and homogeneous superconducting phases is shown by the dotted line. These phase transition lines cross at the tricritical Lifshitz point (T^*, H^*). The phase transition line between the normal phase and vortex states in panel (b) is shown by the dashed line.

rating the normal, uniform superconducting, and nonuniform FFLO states [see Fig. 1(a)]. A trapping potential is assumed to be absent $[V(r)=0]$ and confinement of the superconducting condensate occurs due to the boundary condition at the disk edge. This Neumann-type boundary condition for a disk in an insulating environment and the gauge $\mathbf{A}=(0,A_\theta,0)$ takes the form

$$\left.\frac{\partial\Psi}{\partial r}\right|_{r=R} = 0.$$

A. FFLO state in a 2D mesoscopic disk in a parallel magnetic field

We start our consideration from the case of zero perpendicular component of external magnetic field: $H_z=0$. The Eq. (5) can be simplified and written as follows:

$$\Delta^2_{\rho,\theta}\Psi + 2\Delta_{\rho,\theta}\Psi + \tau\Psi = 0, \tag{6}$$

where $\Delta_{\rho,\theta}$ is a 2D Laplace operator written in dimensionless coordinates ρ, θ. Equation (6) with the boundary condition

$$\left.\frac{\partial\Psi}{\partial \rho}\right|_{\rho=R_0} = 0 \tag{7}$$

defines a set of eigenfunctions and corresponding eigenvalues τ. Here we introduce the dimensionless disk radius $R_0=k_0R$. The maximum eigenvalue τ gives us a critical temperature of transition into the FFLO phase. The solution can be simplified due to the following obvious observation: the eigenfunctions of the Eq. (6) coincide with eigenfunctions of the Schrödinger-type problem

$$-\Delta_{\rho,\theta}\Psi = q^2\Psi, \tag{8}$$

with the boundary condition (7) at the disk edge. The resulting dimensionless shift of the critical temperature τ depends on the wave number q,

$$\tau(q) = 2q^2 - q^4. \tag{9}$$

The solutions of Eq. (8) characterized by a certain angular momentum L can be expressed via the Bessel function of first kind $J_L(q\rho)$,

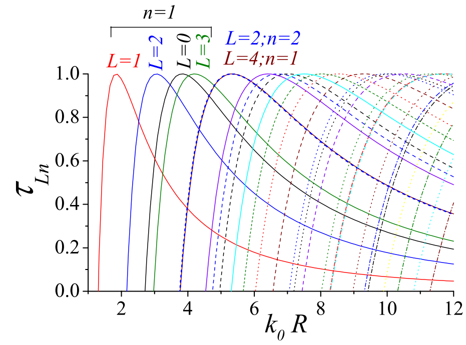

FIG. 2. (Color online) Dependence of the shift of the critical temperature τ vs the dimensionless disk radius k_0R for different values of vorticity L.

$$\Psi = e^{iL\theta}J_L(q\rho). \tag{10}$$

The vorticity parameter L coincides with the angular momentum of the Cooper-pair wave function. The boundary condition (7) gives us a set of zeros z_{Ln} of the derivative of the Bessel function $J_L(z): \partial_z J_L(z_{Ln})=0$. As a consequence, we get a set of eigenvalues $q_{Ln}=z_{Ln}/R_0$. In accordance with Eq. (9) the set of wave numbers q_{Ln} determines a set of critical temperature shifts

$$\tau_{Ln} = 2\left(\frac{z_{Ln}}{R_0}\right)^2 - \left(\frac{z_{Ln}}{R_0}\right)^4, \tag{11}$$

characterizing vortex states with different winding numbers L,

$$\Psi_{Ln} = e^{iL\theta}J_L(q_{Ln}\rho).$$

To get the critical temperature of the superconducting transition into the FFLO state we need to find the maximum of the T_c value, i.e., the maximum of the function

$$T_c - T_{c0} = \frac{\gamma k_0^4}{\alpha}\max_{Ln}\{\tau_{Ln}\}. \tag{12}$$

In Fig. 2 we plot the dependencies of the dimensionless shift of the critical temperature τ_{Ln} vs the parameter R_0 for different L and n values. These phase transition curves clearly demonstrate the switching between the FFLO states characterized by different winding numbers L. For the fixed value of the disk radius R the parameter R_0 can be tuned by changing the temperature T and/or the in-plane magnetic field $H_\parallel$. We see that for a small disk radius $R \ll 1/k_0 (R_0 \ll 1)$ FFLO instability is suppressed $(\tau<0)$ and only uniform superconducting state appears to be energetically favorable. With the increase in the R_0 value the diameter of the disk becomes comparable with the period of the superconducting order parameter oscillations and, thus, FFLO state in the disk becomes energetically favorable. It is interesting to note that nonuniform FFLO state promotes the vortex states with $L\neq 0$: the mode $L=1$ arises primarily just below T^*. The switching between the FFLO states characterized by different winding numbers L results in an oscillatory behavior of the critical temperature T_c as a function of the external field $H_\parallel$, which can be found as an envelope of a set of curves in Fig. 2. In Fig. 1(b) we show schematically a resulting phase diagram in the plane $H_{\parallel}$-T. The critical temperature appears to be degenerate for FFLO states with opposite vorticity signs and, as a result, the sinusoidally modulated superconducting states below T_c can be formed by superpositions of angular harmonics with L and $-L$ similar to those observed numerically in mesoscopic rings.[15]

B. FFLO state in a mesoscopic disk in the magnetic field of arbitrary orientation: Little-Parks oscillations

Let us now consider the effect of an additional component of the magnetic field H_z, applied perpendicular to the disk plane. We use here the gauge $\mathbf{A}=(0,A_\theta,0)$, where $A_\theta=H_z r/2$, and look for the solution of the Eq. (5) (with $v=0$) characterized by certain angular momentum L

$$\Psi(\rho,\theta)=f_L(\rho)e^{iL\theta}. \tag{13}$$

The function $f_L(\rho)$ satisfies the equation

$$\mathbf{D}_L^2(\mathbf{D}_L^2 f_L)+2\mathbf{D}_L^2 f_L+\tau f_L=0, \tag{14}$$

where the operator $\mathbf{D}_L$ is determined by the expression

$$\mathbf{D}_L^2=\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)-\left(\frac{L}{\rho}+\frac{\rho}{a_H^2}\right)^2. \tag{15}$$

Here $a_H=k_0\sqrt{\phi_0/\pi H_z}$ is the dimensionless magnetic length in the units of k_0^{-1}. The solution in the disk should meet the boundary condition

$$\left.\frac{\partial f_L}{\partial\rho}\right|_{\rho=R_0}=0 \tag{16}$$

at the disk edge. As in the previous section, the eigenvalue τ determines the shift of the critical temperature caused by the FFLO instability. The eigenfunctions $f_L(\rho)$ of problems (14) and (16) coincide with the eigenfunctions of the differential operator $\mathbf{D}_L^2$

$$-\mathbf{D}_L^2 f_L=q^2 f_L, \tag{17}$$

with the same boundary condition (16). The relation between the eigenvalue τ and the eigenvalue of the operator $\mathbf{D}_L^2$ is given by the expression (9).

The solution of the Eq. (17) can be expressed via the confluent hypergeometric function of the first kind (Kummer's function) $F(a,b,z)$,[16]

$$f_L(\phi)=\mathrm{e}^{-\phi/2}\phi^{|L|/2}F(a_L,b_L,\phi), \tag{18}$$

where

$$a_L=\frac{1}{2}\left(|L|+L+1-\frac{q^2a_H^2}{2}\right),\quad b_L=|L|+1,\quad \phi=\rho^2/a_H^2. \tag{19}$$

The boundary condition (16) can be rewritten in terms of the Kummer's functions

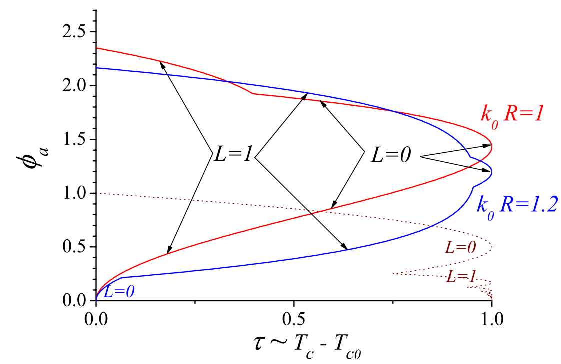

FIG. 3. (Color online) Typical phase diagrams for 2D disks in the plane (τ,ϕ_a) for disk radii $k_0R=1,1.2$. The segments of the $\tau(\phi_a)$ curves corresponding to the different values of the vorticity $L=0,1$ are indicated by the arrows. The dotted line corresponding to an infinite 2D system (see. Ref. 9) is shown for comparison.

$$a_L F(a_L+1,b_L+1,\phi_R)+\frac{b_L}{2}\left(\frac{|L|}{\phi_R}-1\right)F(a_L,b_L,\phi_R)=0, \tag{20}$$

where $\phi_R=\pi R^2 H_z/\phi_0$ is the magnetic flux piercing the disk area in the units of flux quantum. Equations (19) and (20) define an implicit dependence of the eigenvalue q_L on the parameters k_0, R, H_z and the orbital number L. Thus, using Eq. (9) one obtains the dependence of the critical temperature T_L of the state with a vorticity L on the parameters k_0, R, H_z,

$$T_L=T_{c0}+\frac{\gamma k_0^4}{\alpha}\tau_L, \tag{21}$$

where

$$\tau_L=2q_L^2-q_L^4. \tag{22}$$

The critical temperature T_c of superconductivity nucleation is determined by the maximal T_L value

$$T_c=\max_L\{T_L\}. \tag{23}$$

The maximal T_L corresponds to the maximal eigenvalue τ of problems (14) and (16). It has been already shown that for $H_z=0$ the function $\tau(q)$ can be expressed through the zeros of the derivatives of the Bessel functions. These values were taken as the zero approximations to roots of the general boundary condition (20) for $H_z\neq 0$.

In Figs. 3 and 4 we show typical phase diagrams on the plane $(\tau,\phi_a=1/a_H^2)$ for different disk radii. Here ϕ_a is a dimensionless magnetic field component along the z axis. The phase boundary exhibits Little-Parks oscillations, caused by transitions between the states with different angular momenta L. The decrease in the disk radius results in the decrease in the observable number of transitions between different vortex states. For rather small disk radii (Fig. 3) one can clearly observe the regime of the magnetic field induced superconductivity. It should be noted that the switching between the vortex states in the disk can occur with large jumps in vorticity $\Delta L>1$ [see Fig. 4(b)]. Similar jumps in

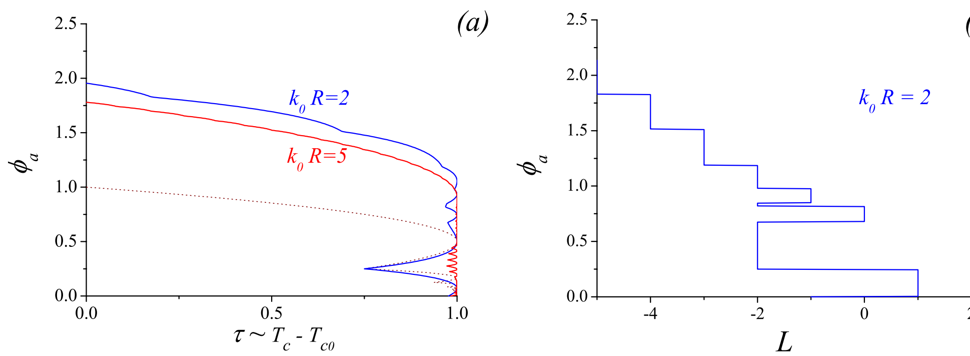

FIG. 4. (Color online) (a) Typical phase diagrams for 2D disks in the plane (τ,ϕ_a) for disk radii $k_0R=2,5$. The dotted line corresponding to an infinite 2D system (see Ref. 9) is shown for comparison. (b) Jumps in vorticity L vs the dimensionless magnetic field ϕ_a for $k_0R=2$.

vorticity are known to occur in mesoscopic rings[17] and hybrid FS structures.[18]

C. Vortex solution in a disk beyond the range of FFLO instability: Critical field of the vortex entry

Unconventional behavior of the vortex states in thin disks placed in a strong parallel magnetic fields reveals, of course, not only in the peculiarities of the oscillatory behavior of the superconducting phase transition line. To illustrate the effect of Zeeman interaction energy on the basic properties of vortices in finite-size samples we consider here the critical field of the first vortex entry into a homogeneous superconducting state close (but beyond) the range of FFLO instability. In order to find this critical field we need to calculate the energy difference between the states with and without vortex. Neglecting the contribution of the vortex core we can assume the order-parameter absolute value to be homogeneous $(\Psi\simeq e^{-i\theta})$ and consider only the gradient part of the free-energy functional

$$\frac{F_g}{F_0}=\int d^2r(\xi_1^2|\mathbf{D}\Psi|^2+\xi_2^4|\mathbf{D}^2\Psi|^2), \tag{24}$$

where F_0 is a constant normalization factor. Approaching the tricritical point one can change the balance between two gradient terms in the above expression: for $H_\parallel\to H^*$ we obtain $\xi_2\gg\xi_1$ and, thus, the fourth-order gradient term becomes a dominant one.

We consider a vortex placed in the center of a disk of finite radius R and take the gauge $A_\theta=H_zr/2$. The energy difference between the states with and without such vortex takes the form

$$\frac{\delta F_g}{F_0}=2\pi\int_{\xi_m}^{R}rdr\left[\xi_1^2\left(\frac{1}{r}-\frac{2\pi}{\phi_0}A_\theta\right)^2-\xi_1^2\left(\frac{2\pi}{\phi_0}A_\theta\right)^2+\xi_2^4\left(\frac{1}{r}-\frac{2\pi}{\phi_0}A_\theta\right)^4-\xi_2^4\left(\frac{2\pi}{\phi_0}A_\theta\right)^4\right],$$

where $\xi_m=\max[\xi_1,\xi_2]$. Integrating over r we find

$$\frac{\delta F_g}{F_0}=2\pi\xi_1^2\left[\ln\frac{R}{\xi_m}-\phi_R+\frac{\xi_2^4}{2\xi_m^2\xi_1^2}+\frac{\xi_2^4}{R^2\xi_1^2}\left(3\phi_R^2-\phi_R^3-4\phi_R\ln\frac{R}{\xi_m}\right)\right],$$

where $\phi_R=\pi R^2H_z/\phi_0$. The condition $\delta F=0$ gives us the field of first vortex entry

$$\ln\tilde{R}-\phi_R+\frac{\alpha}{2}+\frac{\alpha}{\tilde{R}^2}(3\phi_R^2-4\phi_R\ln\tilde{R}-\phi_R^3)=0.$$

Here we introduce the dimensionless parameters: $\tilde{R}=R/\xi_m\gg 1$ and $\alpha=\xi_2^4/(\xi_m\xi_1)^2$. It is natural to consider now two limiting cases. Far from the range of FFLO instability we can put $\xi_1\gg\xi_2(\alpha\ll 1)$ and find a standard logarithmic expression $\phi_R\simeq\ln(R/\xi_1)$. The field of the first vortex entry can be written as follows:

$$H_z^{(c)}\propto\frac{1}{R^2}\ln\frac{R}{\xi_1},\quad \xi_1\gg\xi_2.$$

Close to the range of FFLO instability we need to consider an opposite limit $\xi_1\ll\xi_2(\alpha\gg 1)$ and obtain $\phi_R\simeq(R/\xi_2)^{2/3}$. The scaling behavior of the field of the first vortex entry changes dramatically

$$H_z^{(c)}\propto\left(\frac{1}{R}\right)^{4/3},\quad \xi_2\gg\xi_1.$$

Both coherence lengths (ξ_1 and ξ_2) diverge as one approaches the tricritical point. Considering the above asymptotical expressions for $H_z^{(c)}$ one can see that for $H_\parallel$ well below H^* the field $H_z^{(c)}$ diverges as a function of variable $H^*-H_\parallel$ while close to H^* the critical field $H_z^{(c)}$ tends to zero. Thus, the dependence of the critical field $H_z^{(c)}$ vs $H^*-H_\parallel$ should reveal a peak in the vicinity of the Lifshitz tricritical point.

IV. FFLO STATE IN A SUPERFLUID CONDENSATE CONFINED IN A TRAP

As a second example of the effect of the condensate confinement on the FFLO states we consider a superfluid Fermi gas trapped by a harmonic potential

$$V(r) = \frac{1}{2}M\omega^2 r^2. \quad (25)$$

Here ω is a trapping frequency and M is the atomic mass. Similarly to the previous section we start from the free-energy density [Eq. (2)] written in notations which are adequate for a rotating superfluid gas. In this case the two-dimensional momentum operator $\mathbf{D}$ can be expressed through the angular velocity vector $\mathbf{\Omega}=\Omega\mathbf{z}_0$ directed along the z axis

$$\mathbf{D} = \nabla - \frac{2iM}{\hbar}[\Omega,\mathbf{r}],$$

and the coefficient β of the term $\beta|\nabla\Psi|^2$ in the expression (2) depends on the population imbalance $\delta\mu$. The rotation of superfluid gases plays a similar role as the orbital effect in superconductors. Varying the free-energy functional and introducing a dimensionless radial coordinate $\rho=k_0 r$ we find

$$\mathbf{D}^4_{\rho,\theta}\Psi + 2\mathbf{D}^2_{\rho,\theta}\Psi + (\tau + v_0\rho^2)\Psi = 0, \quad (26)$$

where $\mathbf{D}_{\rho,\theta}=\mathbf{D}/k_0$ and the parameter $v_0=M\omega^2/2\gamma k_0^6$ characterizes the trapping potential.

A. FFLO state in a parabolic trapping potential

In the absence of rotation ($\Omega=0$) the Eq. (26) can be simplified

$$\Delta^2_{\rho,\theta}\Psi + 2\Delta_{\rho,\theta}\Psi + (\tau + v_0\rho^2)\Psi = 0, \quad (27)$$

where $\Delta_{\rho,\theta}$ is a 2D Laplace operator written in ρ,θ coordinates. Introducing a 2D Fourier transform

$$\Psi = \int d^2\mathbf{q}e^{i\mathbf{qr}'}\psi(\mathbf{q}) \quad (28)$$

one can write the Eq. (27) in the momentum representation as the Schrödinger-type equation with the potential $U(q)=q^4-2q^2$,

$$-v_0\Delta_{\mathbf{q}}\psi + U(q)\psi = -\tau\psi. \quad (29)$$

One can see that the solution of Eq. (27) with minimal energy $-\tau$ should correspond to the zero angular momentum: $L=0$. Indeed, the momentum-dependent contribution to energy is positive and proportional to L^2. For rather small v_0 values the lowest-energy level $-\tau$ in this Schrödinger-type equation is close to the value -1 and the wave function is localized near the potential minimum. As a result, one can introduce the coordinate $s=q-1$ and expand the potential near the minimum $U\simeq -1+4s^2$ to consider an approximate oscillator-type solution. Indeed, for $|s|\ll 1$ we obtain

$$-v_0\frac{\partial^2}{\partial s^2}\psi + 4s^2\psi = (1-\tau)\psi. \quad (30)$$

The lowest-energy level of this harmonic oscillator and corresponding wave function take the form

$$\tau = 1 - 2\sqrt{v_0}, \quad \psi = e^{-s^2/\sqrt{v_0}}.$$

The expression $\tau=1-2\sqrt{v_0}$ gives us the critical temperature of the FFLO state. One can see that the FFLO instability appears only for rather small trapping frequencies: $v_0<1/4$. To find the eigenfunction in the $\mathbf{r}$ space we should consider the inverse Fourier transform

$$\Psi \simeq \int_{-\infty}^{+\infty} ds e^{-s^2/\sqrt{v_0}} J_0[(1+s)\rho],$$

where J_0 is a Bessel function of the zeroth order. Considering the asymptotical expression for the Bessel function at $\rho\gg 1$ we find

$$\Psi \simeq \frac{1}{\sqrt{\rho}}\int_{-\infty}^{+\infty} ds e^{-s^2/\sqrt{v_0}}\cos[(1+s)\rho - \pi/4]$$

$$= \frac{1}{\sqrt{\rho}}\mathrm{Re}\int_{-\infty}^{+\infty} ds e^{-s^2/\sqrt{v_0}} e^{i(1+s)\rho - i\pi/4}$$

$$= \sqrt{\frac{\pi}{\rho}}\cos(\rho - \pi/4)e^{-\rho^2\sqrt{v_0}/4}.$$

Thus, the wave function strongly decays with increase in the trapping frequency and the number of observable oscillations is on the order of $2v_0^{-1/4}=2k_0(\beta/M\omega^2)^{-1/4}$.

B. FFLO states in a rotating superfluid gas in a parabolic trapping potential

As a next step we study the effect of rotation ($\Omega\neq 0$) on the superfluid states of the Fermi gas trapped in the parabolic potential well [Eq. (25)]. We look for the solution of Eq. (26) characterized by the conserving angular momentum L

$$\Psi_L(\rho,\theta) = f_L(\rho)e^{iL\theta}, \quad (31)$$

where f_L satisfies the equation

$$\mathbf{D}^2_L(\mathbf{D}^2_L f_L) + 2\mathbf{D}^2_L f_L + (\tau + v_0\rho^2)f_L = 0, \quad (32)$$

$$\mathbf{D}^2_L = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) - \left(\frac{L}{\rho} + \phi_a\rho\right)^2, \quad (33)$$

and $\phi_a=2M\Omega/\hbar k_0^2$ is the dimensionless rotation frequency. Let us consider the following expansion for the order parameter:

$$f_L(\rho) = \sum_{n=0}^{\infty} c_n u_{nL}(\rho), \quad (34)$$

where u_{nL} are the eigenfunctions of the operator $-\mathbf{D}^2_L$ corresponding to the eigenvalues

$$q^2_{nL} = 2\phi_a(2n + L + |L| + 1),$$

and the coefficients c_n satisfy the equation

$$(2q^2_{nL} - q^4_{nL})c_n - \sum_m v^L_{nm}c_m = \tau c_n. \quad (35)$$

The matrix elements

$$v^L_{nm} = v_0\int_0^{\infty}\rho d\rho(u_{mL}\rho^2 u_{nL})$$

are nonzero if $m=n$ or $m=n\pm 1$,

$$v_{nn}^{L}=\frac{v_0}{\phi_a}(2n+|L|+1),\quad v_{n(n+1)}^{L}=-\frac{v_0}{\phi_a}\sqrt{(n+1)(n+|L|+1)},$$

$$v_{(n-1)n}^{L}=-\frac{v_0}{\phi_a}\sqrt{n(n+|L|)}. \qquad (36)$$

The set of normalized eigenfunctions $u_{nL}(\rho)$ can be written as follows:

$$u_{nL}(\rho)=\sqrt{2\phi_a\frac{(n+|L|)!}{n!(|L|)!)^2}}e^{-\phi_a\rho^2/2}(\phi_a\rho^2)^{|L|/2}\times F(-n,|L|+1,\phi_a\rho^2),$$

$$\int_0^\infty \rho d\rho(u_{nL}u_{mL})=\delta_{nm}. \qquad (37)$$

The maximal eigenvalue τ of the above problem determines the shift in the critical temperature of the FFLO transition. Within the first-order perturbation theory in v_0 one can get the following expression for the temperature shift τ_{nL} vs the dimensionless rotation frequency ϕ_a:

$$\tau_{nL}=\tau_{nL}^{(0)}-v_{nn}^{L}. \qquad (38)$$

Thus, perturbation theory provides us a simple estimate for the FFLO transition temperature

$$\tau=\max_{L\geq 0}[(4\phi_a-v_0/\phi_a)(2L+1)+v_0L/\phi_a-4\phi_a^2(2L+1)^2].$$

In Fig. 5 we show the results of the numerical calculation of the dependencies $\tau(\phi_a)$ for different trapping frequencies. These phase diagrams appear to be in good qualitative agreement with the above estimate for not too small ϕ_a values. For rather large trapping frequencies one can clearly observe the regime of the rotation-induced superfluid transition. It is important to note that the increase in the trapping frequency v_0 causes a decrease in the critical temperature of vortex states with higher vorticities and, thus, is responsible for suppression of quantum oscillations of the superfluid phase transition line.

V. CONCLUSIONS

To sum up, we have studied the effect of confinement of superconducting and superfluid condensates on the phenomenon of FFLO instability. We have found the following hallmarks of the FFLO phenomenon in a restricted geometry: (i) both the finite system size and parabolic trapping potential are responsible for suppression of the quantum oscillations of the superfluid critical temperature; (ii) the spatial oscillations of the superfluid order parameter in the FFLO regime are suppressed by the increase in the trapping frequency; (iii) change in the Zeeman interaction energy in the mesoscopic superconducting system can induce phase transitions between different inhomogeneous FFLO states; (iv) switching between the vortex states in confined geometry can be accompanied by giant jumps in vorticities; (v) rotation-induced superfluid transition in a Fermi gas cloud for rather large trapping frequency; (vi) superconducting transition induced by the perpendicular magnetic field component in a mesoscopic superconducting disk; (vii) unusual scaling in the dependence of the field of the vortex entry vs system size in the vicinity of FFLO instability. We believe that these theoretical predictions can be used for experimental identification of the FFLO phases in both mesoscopic superconductors and superfluid Fermi gases. Note in conclusion, that we consider our results to be useful for analysis of experimental data both in systems with second- and first-order phase transitions: in the latter case the phase diagrams studied above can be observed under the supercooling conditions.

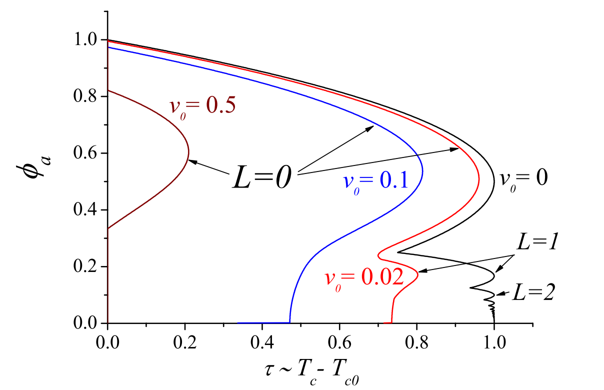

FIG. 5. (Color online) Typical phase diagrams in the plane (τ,ϕ_a) of a rotating Fermi condensate for different trapping frequencies (v_0=0;0.02;0.1;0.5). The segments of the $\tau(\phi_a)$ curves corresponding to the different values of the vorticity L=0,1,2 are indicated by the arrows.

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[1] D. Saint-James, G. Sarma, and E. J. Thomas, *Type-II Superconductivity* (Pergamon Press, New York, 1969).

[2] P. Fulde and R. A. Ferrell, Phys. Rev. **135**, A550 (1964); A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **47**, 1136 (1964) [Sov. Phys. JETP **20**, 762 (1965)].

[3] S. Uji, T. Terashima, M. Nishimura, Y. Takahide, T. Konoike, K. Enomoto, H. Cui, H. Kobayashi, A. Kobayashi, H. Tanaka, M. Tokumoto, E. S. Choi, T. Tokumoto, D. Graf, and J. S. Brooks,

Phys. Rev. Lett. **97**, 157001 (2006).

[4] Y. Matsuda and H. Shimahara, J. Phys. Soc. Jpn. **76**, 051005 (2007).

[5] M. W. Zwierlein, A. Schirotzek, C. H. Schunck, and W. Ketterle, Science **311**, 492 (2006).

[6] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, Nature (London) **435**, 1047 (2005).

[7] W. A. Little and R. D. Parks, Phys. Rev. Lett. **9**, 9 (1962); R. D. Parks and W. A. Little, Phys. Rev. **133**, A97 (1964).

[8] H. J. Fink and A. G. Presson, Phys. Rev. **151**, 219 (1966); V. V. Moshchalkov, L. Gielen, C. Strunk, R. Jonckheere, X. Qiu, C. Van Haesendonck, and Y. Bruynseraede, Nature (London) **373**, 319 (1995); A. K. Geim, I. V. Grigorieva, S. V. Dubonos, J. G. S. Lok, J. C. Maan, A. E. Filippov, and F. M. Peeters, *ibid.* **390**, 259 (1997); V. A. Schweigert and F. M. Peeters, Phys. Rev. B **57**, 13817 (1998); H. T. Jadallah, J. Rubinstein, and P. Sternberg, Phys. Rev. Lett. **82**, 2935 (1999); L. F. Chibotaru, A. Ceulemans, V. Bruyndoncx, and V. V. Moshchalkov, Nature (London) **408**, 833 (2000).

[9] A. I. Buzdin and M. L. Kulic, J. Low Temp. Phys. **54**, 203 (1984).

[10] M. L. Kulić, A. Sedrakian, and D. H. Rischke, Phys. Rev. A **80**, 043610 (2009).

[11] A. I. Buzdin, Rev. Mod. Phys. **77**, 935 (2005).

[12] A. V. Samokhvalov, A. S. Mel'nikov, and A. I. Buzdin, Phys. Rev. B **76**, 184519 (2007); A. V. Samokhvalov, A. S. Mel'nikov, J.-P. Ader, and A. I. Buzdin, *ibid.* **79**, 174502 (2009).

[13] A. I. Buzdin and H. Kachkachi, Phys. Lett. A **225**, 341 (1997).

[14] H. Burkhardt and D. Rainer, Ann. Phys. **506**, 181 (1994).

[15] F. Ye, Y. Chen, Z. D. Wang, and F. C. Zhang, J. Phys.: Condens. Matter **21**, 355701 (2009).

[16] *Handbook of Mathematical Functions*, Natl. Bur. Stand. Appl. Math. Series No. 55, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, DC, 1965).

[17] A. A. Zyuzin and A. Yu. Zyuzin, Pis'ma Zh. Eksp. Teor. Fiz. **88**, 147 (2008) [JETP Lett. **88**, 136 (2008)]; Phys. Rev. B **79**, 174514 (2009).

[18] A. Yu. Aladyshkin, D. A. Ryzhov, A. V. Samokhvalov, D. A. Savinov, A. S. Mel'nikov, and V. V. Moshchalkov, Phys. Rev. B **75**, 184519 (2007).