Free-energy distribution functions for the randomly forced directed polymer

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We study the 1+1-dimensional random directed polymer problem, i.e., an elastic string $\phi(x)$ subject to a Gaussian random potential $V(\phi,x)$ and confined within a plane. We mainly concentrate on the short-scale and finite-temperature behavior of this problem described by a short but finite-ranged disorder correlator $U(\phi)$ and introduce two types of approximations amenable to exact solutions. Expanding the disorder potential $V(\phi,x)\approx V_0(x)+f(x)\phi(x)$ at short distances, we study the random-force (or Larkin) problem with $V_0(x)=0$ as well as the shifted random-force problem including the random offset $V_0(x)$; as such, these models remain well defined at all scales. Alternatively, we analyze the harmonic approximation to the correlator $U(\phi)$ in a consistent manner. Using direct averaging as well as the replica technique, we derive the distribution functions $\mathcal{P}_{L,y}(F)$ and $\mathcal{P}_L(F)$ of free energies F of a polymer of length L for both fixed $[\phi(L)=y]$ and free boundary conditions on the displacement field $\phi(x)$ and determine the mean displacement correlators on the distance L. The inconsistencies encountered in the analysis of the harmonic approximation to the correlator are traced back to its nonspectral correlator; we discuss how to implement this approximation in a proper way and present a general criterion for physically admissible disorder correlators $U(\phi)$.

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I. INTRODUCTION

Directed polymers subject to a quenched random potential have been the subject of intense investigations during the past two decades.[1] Diverse physical systems such as domain walls in magnetic films,[2] vortices in superconductors,[3] wetting fronts on planar systems,[4] or Burgers turbulence[5] can be mapped to this model, which exhibits numerous nontrivial features deriving from the interplay between elasticity and disorder. The best understanding, so far, has been reached for the (1+1)-dimensional case, i.e., a string confined to a plane, and it is this geometry we study in the present paper. Specifically, we analyze the situation illustrated in Fig. 1, an elastic string (with elasticity c) of finite length L within an interval $[0,L]$ directed along the x axis. The disorder potential $V(\phi,x)$ drives a finite displacement field $\phi(x)$, which is counteracted by the elastic energy density $c(\partial_x\phi)^2/2$. The problem is conveniently defined through its Hamiltonian

$$H[\phi(x);V]=\int_0^L dx\left\{\frac{c}{2}[\partial_x\phi(x)]^2+V[\phi(x),x]\right\}; \qquad (1)$$

the disorder potential $V(\phi,x)$ is Gaussian distributed with a zero mean $\overline{V(\phi,x)}=0$ and a correlator

$$\overline{V(\phi,x)V(\phi',x')}=\delta(x-x')U(\phi-\phi') \qquad (2)$$

with $U(\phi-\phi')$ the correlation function. Much previous work has been concentrating on the large-scale/high-temperature behavior of this system as described by a δ-correlated disorder potential. In the present work, we are interested in the properties developing at small scales and low temperatures, necessitating the study of short-range correlated disorder potentials. The latter are characterized by their extension ξ and the strength $U_0=U(0)$; these parameters then combine in a curvature (or random-force strength) $u=-U''(0)\approx U(0)/\xi^2$.

Quantities of interest are the scaling behavior

$$\overline{\langle\phi^2\rangle}(L)\equiv\overline{\langle[\phi(L)-\phi(0)]^2\rangle}\propto L^{2\zeta}$$

of the polymer's mean-squared displacement ϕ with length L and the so-called wandering exponent[6] ζ, as well as the distribution function $\mathcal{P}(F)$ of the polymer's free energy F [here, $\langle\cdots\rangle$ and $\overline{(\cdots)}$ denote thermal (temperature T) and disorder (random potential V) averages, respectively]. The polymer's free energy F is defined via its partition function

$$Z[L,y;V]=\int_{\phi(0)=0}^{\phi(L)=y}\mathcal{D}[\phi(x)]\exp(-\beta H[\phi(x);V]), \qquad (3)$$

where $\beta=1/T$ denotes the inverse temperature (we set the Boltzmann constant to unity), from which the free energy

$$F[L,y;V]=-T\ln(Z[L,y;V]) \qquad (4)$$

follows immediately. The free energy F in Eq. (4) is defined for a specific realization of the random potential V and thus defines a random variable; given the above (Gaussian) distributed disorder potential, the task then is to determine the distribution function $\mathcal{P}_{L,y}(F)$. In Eq. (3), we have considered a string starting at $(x,\phi)=(0,0)$ and ending in a fixed position $(x,\phi)=(L,y)$ a distance L away but other cases, e.g., a free boundary condition at $x=L$, see below, may be studied.

When concentrating on large-scale properties and a δ-correlated potential with $U(\phi-\phi')=u\delta(\phi-\phi')$, two types of analytic solutions are known for the (1+1)-dimensional random polymer: (i) mapping the replicated problem to interacting quantum bosons[7] and using the Bethe-Ansatz technique, one can find the spectrum and eigenfunctions of the interacting quantum many-body problem, from which the distribution function $\mathcal{P}_L(F)$ for the free energy F of a polymer of length L and fixed end point at $\phi(L)=0$ can be obtained; we call this the "longitudinal problem." Restricting the solution to the ground-state wave function permits the

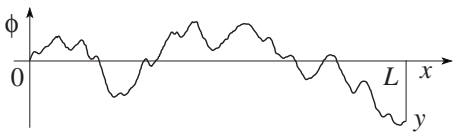

FIG. 1. Thermally averaged trajectory $\langle\phi(x)\rangle_{\rm th}$ of a random directed polymer in a fixed disorder potential $V(\phi,x)$ starting in $(x,\phi)=(0,0)$ and ending in (L,y). The free energy associated with such a configuration is denoted by F. The random choice of the underlying disorder potential $V(\phi,x)$ defines a random process; the free energy then turns into a random variable, whose distribution function $\mathcal{P}_{L,y}(F)$ we seek to calculate.

determination of the far-left tail.[8] First indications that the full distribution function should be of the Tracy-Widom form derived from the work of Prähofer and Spohn[9] on polynuclear growth, a model in the universality class of the Kardar-Parisi-Zhang (KPZ) model to which the random directed polymer problem belongs as well. Recently, both tails of the free-energy distribution function $\mathcal{P}_L(F)$ have been found using instanton techniques,[10] with results consistent with those in Ref. 9. The full distribution function (in the Tracy-Widom form) has been obtained independently by several groups, by Dotsenko[11] and by Calabrese *et al.*[12] using the replica technique including the full spectrum, by Sasamoto and Spohn[13] deriving the exact KPZ solution from the corner growth model, and by Amir *et al.*[14] (ii) An alternative (exact) result has been obtained using a mapping to the Burgers equation via the Cole-Hopf transformation;[15] making use of an invariant distribution, the distribution function $\mathcal{P}_y(F')$ for the free-energy difference F' between two configurations with end points separated by $2y$ has been found[15,16] (the so-called "transverse" problem). Both approaches have been helpful in finding the wandering exponent[15] $\zeta=2/3$ of transverse fluctuations of the polymer. Recently, the interrelation between the longitudinal and the transverse problems has been studied[17] in a calculation of the joint distribution function $\mathcal{P}_{L,y}(\bar{F},F')$ for a polymer of length L involving two configurations of the string ending in points separated by $2y$.

The aforementioned studies describe the behavior of the polymer at large scales and high temperatures (the results for the δ-correlated disorder potential exhibit a singular zero-temperature limit). In order to learn about the short-scale/low-temperature behavior of the system, finite-width correlators $U(\phi-\phi')$ of the random potential have to be studied; this problem has remained unsolved so far. In order to make progress, we linearize the problem, either by an expansion of the original random potential[18] $V(\phi,x)$ for small values of ϕ,

$$V(\phi,x) \approx V_0(x) + f(x)\phi(x), \tag{5}$$

or by an expansion of the correlator $U(\phi-\phi')$,

$$U(\phi-\phi') \simeq U_p(\phi-\phi') = U_0 - \frac{1}{2}u(\phi-\phi')^2. \tag{6}$$

Both approximations lead to quadratic problems which can be solved exactly; their study not only provides solid results for the polymer's short-scale properties but also gives insights on the methodological aspects of the solution (the replica approach). Furthermore, a central question we will address below is which expansion is the more appropriate one.

Studying the destruction of long-range order due to the presence of quenched disorder, i.e., the behavior of the displacement field $\phi(x)$ at large distances x, the random shift $V_0(x)$ can be dropped and one arrives at the *Larkin* or *random-force model* described by the Hamiltonian (1) with

$$V(\phi,x) = f(x)\phi(x), \tag{7}$$

where $f(x)$ denotes a (Gaussian) random-force field with zero mean $\overline{f(x)}=0$ and a correlator

$$\overline{f(x)f(x')} = u\delta(x-x'). \tag{8}$$

Its free-energy distribution function $\mathcal{P}_{y,L}(F)$ has been calculated by Gorokhov and Blatter.[19] However, using this expansion in the analysis of the short-scale properties of the random-potential problem, we are not allowed to ignore the random shift $V_0(x)$. In our study below, we assume the latter to be Gaussian correlated,

$$\overline{V_0(x)V_0(x')} = U_0\delta(x-x'), \tag{9}$$

and uncorrelated with the force, $\overline{V_0(x)f(x')}=0$; we call this approximation the *shifted random-force model*. Both models not only serve as approximations to the random polymer problem on short scales but also describe physical problems where the underlying randomness is properly described by a (shifted) random-force field on all length scales; below we will quote results for these models for arbitrary lengths L.

The alternative short-scale approximation Eq. (6) involving an expansion of the correlator $U(\phi-\phi')$ has been discussed in the literature as well.[1,20] The *harmonic-correlator approximation* relates to the above shifted random-force model via the identification of the force correlator

$$\overline{\partial_\phi V(\phi,x)\partial_{\phi'}V(\phi',x')}\big|_{\phi,\phi'=0} = [-U''(0)]\delta(x-x') = u\delta(x-x')$$

and the shift correlator

$$\overline{V(\phi,x)V(\phi',x')}_{\phi,\phi'=0} = U_0\delta(x-x'),$$

with the additional advantage to preserve the translation invariance of the problem [note, that the shifted random-force model involves only the constant and mixed $(u\phi\phi')$ terms in the correlator, with the quadratic terms $\propto(\phi^2+\phi'^2)$ absent].

The (shifted) random-force models and the harmonic-correlator approximation produce similar results for various quantities defined at short scales, such as the mean-free energy and the displacement correlator. While all displacement correlators are identical for the two models and the approximation, the random-force model supplies us with the distribution function for the *relaxational* free energy (i.e., the free energy of the distorted string reduced by the energy of the straight string) whereas the shifted random-force model and the harmonic-correlator approximation provide us with the distribution function for the total free energy. Comparing the latter two, we find that the results for the harmonic-correlator approximation give a more consistent description of the original problem Eq. (2) at short distances, as the correlator remains translation invariant, while some terms in the expan-

sion of the random potential are dropped for the shifted random-force problem. On the other hand, the applicability of the harmonic-correlator approximation is limited to short scales, as the free-energy distribution function $\mathcal{P}_L(F)$ suffers from a negative second moment whenever large displacements show up, e.g., at large distances L or at high temperatures T; this is to be expected as the harmonic correlator deviates strongly from the original correlator $U(\phi)$ (and eventually turns negative) at large arguments.

As a technical (but still important) side remark, we point out that expanding the correlator $U(\phi)$ of the random potential or choosing arbitrary forms for the correlator (such as *ad hoc* power-law expressions[1]) is a problematic step, as this action may generate a nonspectral correlator introducing an ill-defined Gaussian measure and thus may lead to an unphysical model at the very start, cf. Sec. IV below for a detailed discussion. This problem manifests itself in the mapping to quantum bosons when using the replica approach. On the other hand, expanding the interaction $-\beta^2 U(\phi)$ between bosons is a perfectly admissible step producing *identical results*; in this case, however, we know that the (perfectly well defined) quantum boson problem *does not* describe the random polymer problem on scales where the approximate quadratic correlator deviates strongly from the original correlator. In the present work, we will discuss the *harmonic-correlator approximation* of the original random polymer problem with the understanding, that the harmonic approximation is done after the mapping to bosons. Furthermore, in Sec. IV, we will present a criterion assuring the consistent definition of a correlator: such correlators have to be spectral. Also, we emphasize the difference in terminology introduced above: while the random force and shifted random-force approximations to the random potential $V(\phi,x)$ define proper models of disordered elastic systems, this is not the case when expanding the potential correlator $U(\phi)$; this is why we refrain from considering a *model* with a harmonically correlated random potential but prefer to talk about a harmonic *approximation* to the correlator.

For a system with quenched disorder, it is usually extremely difficult to find averaged physical quantities, e.g., the mean-free energy $F=-T\overline{\ln Z}$. Replica theory, requiring calculation of the disorder-averaged nth power of the partition function $\overline{Z^n}$ then comes in as a helpful technique. Usually, it is the limit $\lim_{n\to 0}[(\overline{Z^n}-1)/n]=\overline{\ln Z}$, that is to be calculated after analytic continuation of n. It turns out (see below) that the same quantity $\overline{Z^n}$ and its analytic continuation is relevant in the calculation of the free-energy distribution function $\mathcal{P}(F)$, since the latter is nothing but the inverse Laplace transform of the former,[8] hence replica theory seems the technique of choice for the solution of the present problem as well. However, the shifted random-force model defines a quadratic problem that can be analyzed in a straightforward manner, i.e., the partition function $Z[f(x)]$ [involving an integration over the field $\phi(x)$] can be found for any configuration $f(x)$ of the random force and the disorder average of its nth power can be done in the end. This is opposite to the replica approach where the integrations are interchanged, with the first integration over the disorder of the replicated system generating an interacting imaginary-time quantum boson problem, which then is solved in a second step (corresponding to the integration over the field). Below, we will discuss both procedures for the Larkin model and find that they provide similar challenges and identical results.

The disorder (u and U_0) and elastic (c) parameters of the above random polymer problems define convenient and physically relevant length and energy scales: The ratio of U_0 and u defines the transverse length scale ξ where the shifted random-force model approximates well the random polymer problem,

$$\xi=\left(\frac{U_0}{u}\right)^{1/2}. \tag{10}$$

Comparing the elastic energy $E_c=c\xi^2/L=cU_0/uL$ with the disorder energy $E_f=\sqrt{U_0 L}$ accumulated over a distance L, one obtains the corresponding longitudinal scale L_c,

$$L_c=\left(\frac{c^2U_0}{u^2}\right)^{1/3}=\left(\frac{c^2\xi^2}{u}\right)^{1/3}. \tag{11}$$

Finally, the energy scale associated with these length scales is

$$U_c=\left(\frac{cU_0^2}{u}\right)^{1/3}=\frac{c\xi^2}{L_c}. \tag{12}$$

Note that the longitudinal (L_c) and transverse (ξ) scales define the limits of validity where our expansions describe the original random polymer problem. The parameters are not fully appropriate to describe the results of the Larkin model, as the latter is characterized by one disorder parameter (u) only—to allow for proper comparison, below, we will nevertheless express all physical results through ξ, L_c, and U_c. For the Larkin model, these parameters will combine to expressions containing only u and c.

Besides providing various results and insights for the (shifted) random-force models and the harmonic-correlator approximation, as well as a discussion of their use as an approximation to the random-potential problem at short scales, the present study also has its merits from a methodological point of view since this is the only case where the entire analysis (direct and via replica) can be carried through in a complete and consistent manner. Below, we introduce the formalism (Sec. II) and then apply it to the (shifted) random-force models (Sec. III). We then analyze the harmonic-correlator approximation (Sec. IV), analyze its failure due to its nonspectral property, and state the spectral condition to be satisfied by a properly defined random-potential correlator $U(\phi)$; furthermore, we briefly present the results for the displacement correlators which are identical in all three cases. Conclusions are presented in Sec. V.

II. METHODOLOGY

Evaluating the partition function Eq. (3) and the expression (4) for the free-energy F for a given random potential $V(\phi,x)$ defines a sample-dependent random quantity. Its free-energy distribution function $\mathcal{P}_{L,y}(F)$ can be derived from the nth powers of the partition function

$$Z^n(L,y) = \overline{Z[L,y;V]^n} = \overline{\exp(-n\beta F[L,y;V])}. \quad (13)$$

These are equal to the (bilateral) Laplace transform of the free-energy distribution function $\mathcal{P}_{L,y}(F)$ at integer multiples of β,[8]

$$Z^n(L,y) = \int_{-\infty}^{+\infty} dF \mathcal{P}_{L,y}(F) \exp(-n\beta F). \quad (14)$$

Hence, the inversion of this expression through the (inverse) Laplace transformation provides us with the free-energy distribution function $\mathcal{P}_{L,y}(F)$. This requires us to analytically continue the expression $Z^n(L,y)$ for the moments from integer values of n to the complex η plane, $Z^n(L,y) \to Z(\eta;L,y)$ with $\beta n \to \eta$. The free-energy distribution function $\mathcal{P}_{L,y}(F)$ then is given by the inverse Laplace transformation

$$\mathcal{P}_{L,y}(F) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} d\eta Z(\eta;L,y) \exp(\eta F), \quad (15)$$

where the integration goes over the imaginary η axis with $\mathrm{Re}(\eta)$ chosen in such a way as to place all singularities in $Z(\eta;L,y)$ to its right. Furthermore, taking the k-fold derivatives of $Z(\eta;L,y)$ with respect to η provides us with all the moments

$$\overline{\langle F^k \rangle}(L,y) = (-1)^k \left. \frac{\partial^k Z(\eta;L,y)}{\partial \eta^k} \right|_{\eta=0}. \quad (16)$$

The calculation of the moments $Z^n(L,y)$ involves integrations over the displacement field $\phi(x)$ and over the distribution function $P[V(\phi,x)]$ of the disorder potential,

$$Z^n(L,y) = \int \mathcal{D}[V(\phi,x)] P[V(\phi,x)] \prod_{a=1}^{n} \int \mathcal{D}[\phi_a(x)] \times \exp\left(-\beta \sum_{a=1}^{n} H[\phi_a(x)] \right). \quad (17)$$

For the random force or Larkin model, these integrations can be done straightforwardly in the sequence above using the distribution function for the random force

$$P[f(x)] \propto \exp\left(-\int dx f^2(x)/2u \right). \quad (18)$$

This program will be carried through in Sec. III A below. Fixed and free boundary conditions are conveniently imposed by the requirements $\phi(0)=0$, $\phi(L)=y$ and $\phi(0)=0$, $[\partial_x \phi](L)=0$.

On the other hand, for the general situation with a random potential $V(\phi,x)$, the integration in Eq. (17) over the displacement fields $\phi_a(x)$ cannot be done. Interchanging the integrations over $V(\phi,x)$ and $\phi_a(x)$ takes us directly to the replica technique: performing first the integration over the disorder potential $V(\phi,x)$, the remaining integrations over the fields $\phi_a(x)$ have to be done with the replica Hamiltonian $H_n[\{\phi_a\}]$,

$$\Psi[\{y_a\};L] = \left\{ \prod_{a=1}^{n} \int_{\phi_a(0)=0}^{\phi_a(L)=y_a} \mathcal{D}[\phi_a(x)] \right\} \times \exp(-\beta H_n[\{\phi_a(x)\}]), \quad (19)$$

$$H_n[\{\phi_a(x)\}] = \int_0^L dx \left\{ \frac{c}{2} \sum_{a=1}^{n} [\partial_x \phi_a(x)]^2 - \frac{\beta}{2} \sum_{a,b=1}^{n} U[\phi_a(x) - \phi_b(x)] \right\}. \quad (20)$$

Here, we have allowed the individual replicas of the elastic string to end in different locations y_a. The expression Eq. (19) is identical with the imaginary time (x) propagator $\Psi(\{y_a\};x)$ of a many-body problem in a path integral setting. Collapsing the end points $y_a=y$, this propagator coincides with the nth moment [Eq. (17)] of the partition function,

$$\Psi(\{y_a = y\};x = L) = Z_r(n;L,y) = Z^n(L,y), \quad (21)$$

where the last equation holds, provided that the integration over the disorder potential V can be exchanged with the integration over the field ϕ_a. Note that it is this mapping from the polymer statistical mechanics problem to the quantum boson problem which fails when the correlator is nonspectral, e.g., for the (naive version of the) harmonic-correlator approximation.

The equivalence to a quantum many-body problem becomes more obvious when going from the path-integral Eq. (19) to an operator formalism; the evaluation of the path-integral Eq. (19) then is equivalent to the solution of the imaginary-time Schrödinger equation

$$-\partial_x \Psi(\{y_a\};x) = \hat{H} \Psi(\{y_a\};x) \quad (22)$$

with the Hamiltonian

$$\hat{H} = -\frac{1}{2\beta c} \sum_{a=1}^{n} \partial_{y_a}^2 - \frac{\beta^2}{2} \sum_{a,b=1}^{n} U(y_a - y_b). \quad (23)$$

The Hamiltonian (23) describes n particles of mass βc interacting via the attractive two-body potential $-\beta^2 U(y)$; the propagation in Eq. (22) starts in the origin at time $x=0$,

$$\Psi(\{y_a\};0) = \Pi_{a=1}^{n} \delta(y_a), \quad (24)$$

and ends at different coordinates $\{y_a\}$ after propagation during the time x. To keep up the formal distinction between the two quantities, we denote the direct physical definition of the moments by $Z^n(L,y)$ (first integration over the field ϕ, raising the result to the power n, and averaging over disorder V) and denote the replica expression (n-fold replication followed by averaging over disorder V and integration over the fields ϕ_a done in the end) by $Z_r(n;L,y)$.

Finally, we note that in the replica technique, free boundary conditions at the end point are more conveniently implemented through an integration over y; the partition function for the polymer with free boundary conditions assumes the form

$$Z[L;V] = N\int_{-\infty}^{+\infty} dy Z[L,y;V] \tag{25}$$

with N a suitable normalization constant. Taking the nth power and averaging over the disorder potential V provides us with the moments $Z^n(L)$. Following the replica procedure, after replication and integration over V, one arrives at the free-boundary replicated partition function through integration over the set $\{y_a\}$ of n different end points,

$$Z_r(n;L) = \left[\prod_{a=1}^{n} N\int_{-\infty}^{+\infty} dy_a\right]\Psi(\{y_a\};x=L). \tag{26}$$

In the next sections, we are going to apply the above general schemes for the calculation of the free-energy distribution functions $\mathcal{P}_{L,y}(F)$ and $\mathcal{P}_L(F)$ for fixed and free boundary conditions, respectively, of the (shifted) random-force model Eqs. (7)–(9) and of the random directed polymer model Eq. (1) with the parabolic approximation for the correlation function, Eq. (6), done after averaging over the disorder. Before doing so, we briefly discuss the results for the free string which determines our normalization N.

Free string

The path integrals Eqs. (3), (17), and (19) over trajectories $\phi(x)$ involve an arbitrary measure of integration. Here, we choose a particular normalization such that the partition function $Z_0(L,y)$ [or wave function $\Psi_0(y,x=L)$] of the free polymer problem with fixed boundary conditions $\phi(0)=0$ and $\phi(L)=y$ assumes the form

$$\begin{aligned} Z_0(L,y) &= \Psi_0(y;L) \\ &= \int_0^y \mathcal{D}[\phi(x)]\exp\left\{-\frac{\beta c}{2}\int_0^L dx[\partial_x\phi(x)]^2\right\} \\ &\equiv \exp\left(-\frac{\beta c}{2L}y^2\right), \end{aligned} \tag{27}$$

the corresponding free energy then is given by

$$F_0(L,y) = \frac{c}{2L}y^2. \tag{28}$$

For the partition function of the free polymer with free boundary conditions we choose the normalization $N=(2\pi L/\beta c)^{1/2}$ and obtain

$$Z_0(L) = \sqrt{\frac{2\pi L}{\beta c}}\int_{-\infty}^{+\infty} dy Z_0(L,y) = 1 \tag{29}$$

and the free energy $F_0(L)=0$. These results will be helpful in the interpretation of the free-energy distribution functions for the random-force model calculated below. With this normalization, all our free energies F are measured with respect to the free thermal energy $F_0^{\mathrm{fs}}=T\ln\sqrt{2\pi LT/c}$ of the free string due to its entropy.

III. RANDOM-FORCE MODEL

We select the simplest case, the Larkin model, for the discussion of the two methodological approaches involving either direct integration over the field ϕ and subsequent disorder average over V or the route following the replica approach. While the first route is preferably done in Fourier space, the replica calculation will be formulated in real space. Also, note that the analysis for the Larkin or random-force model provides the distribution function for the *relaxational* free-energy $F-E_0$ rather then the (total) free energy F of the polymer,

$$\begin{aligned} Z[L,y;V] &= \exp^{-\beta F[L,y;V]} \\ &\approx \int_{\phi(0)=0}^{\phi(L)=y} \mathcal{D}[\phi(x)]\exp(-\beta H[\phi;V_0+f\phi]) \\ &= e^{-\beta E_0}\int_{\phi(0)=0}^{\phi(L)=y} \mathcal{D}[\phi(x)]\exp(-\beta H[\phi;f\phi]) \end{aligned} \tag{30}$$

with $E_0=\int dx V_0(x)$ the disorder energy of a straight string. This latter remark is relevant in the comparison of the random force and the harmonic models.

A further specialty of the Larkin model is the separation between the thermal and the quenched disorder.[21] Indeed, splitting the displacement field $\phi(x)$ into the Hamilonian's minimizer $\phi_q(x)$,

$$c\partial_x^2\phi_q(x) = f(x), \tag{31}$$

and fluctuations $\delta\phi(x)$ around it, $\phi(x)=\phi_q(x)+\delta\phi(x)$, we can decompose the Hamilonian into the free part $H_0[\delta\phi(x)]$ and the energy of the minimizer $H[\phi_q(x)]$,

$$\begin{aligned} H_0[\delta\phi(x)] &= \int_0^L dx\frac{c}{2}[\partial_x\delta\phi(x)]^2, \\ H[\phi_q(x)] &= \int_0^L dx\left\{\frac{c}{2}[\partial_x\phi_q(x)]^2 + f(x)\phi_q(x)\right\}. \end{aligned} \tag{32}$$

In addition, we can account for the boundary condition $\phi(L)=y$ through a simple shift $\phi(x)\to xy/L+\phi(x)$, which adds the terms

$$H_y = \frac{cy^2}{2L} + \frac{y}{L}\int_0^L dx x f(x) \tag{33}$$

to the Hamiltonian. The partition sum then naturally separates into thermal and quenched-disorder averaged factors,[21]

$$\overline{Z(L,y;f)} = Z_0(L,y)\overline{\exp\{-\beta(H[\phi_q(x)]+H_y)\}}, \tag{34}$$

where the factor $Z_0(L,y)$ is the partition function Eq. (27) of the free propagation.

A. Direct integration

The direct integration of the partition function Eq. (3) for the random-force problem $V(\phi,x)=f(x)\phi(x)$ is conveniently done within a Fourier representation. For technical convenience we extend the problem to the interval $[-L,L]$ and define the antisymmetric force and displacement fields $f(-x)\equiv -f(x>0)$ and $\phi(-x)\equiv -\phi(x>0)$. The relevant quan-

tities in Fourier space are the sine transforms

$$g_m = \int_{-L}^{L} dx g(x) \sin(k_m x) \tag{35}$$

with $k_m = m\pi/L$; the backtransformation reads

$$g(x) = \frac{1}{L} \sum_{m=1}^{\infty} g_m(x) \sin(k_m x). \tag{36}$$

The Hamiltonian (on the interval $[0,L]$) assumes the form (we make use of the solution $\phi_{qm} = -f_m/ck_m^2$ of Eq. (31) in Fourier representation)

$$H[\phi_q] + H_y = -\sum_{m=1}^{\infty} \frac{f_m^2}{4cLk_m^2} - \frac{y}{L} \sum_{m=1}^{\infty} \frac{(-1)^m}{k_m} f_m \tag{37}$$

and the partition function reads

$$Z(L,y;f) = \exp\left(-\frac{\beta c y^2}{2L}\right) \times \prod_{m=1}^{\infty} \exp\left[\frac{\beta f_m^2}{4cLk_m^2} + \frac{\beta y}{Lk_m}(-1)^m f_m\right]. \tag{38}$$

The disorder average in the partition function Eq. (34) or (38) has to be taken over the distribution function for the random force f, cf. Eq. (18), or in Fourier space,

$$P(f_m) = \frac{1}{\sqrt{4\pi uL}} \exp(-f_m^2/4uL). \tag{39}$$

Taking the result (38) to the nth power and integrating over the force distribution Eq. (39), we obtain the intermediate result

$$Z^n(L,y) = \exp\left(-\frac{\beta n c y^2}{2L}\right) \times \prod_{m=1}^{\infty} \left[1 - \frac{s}{\pi^2 m^2}\right]^{-1/2} \exp\left(\sum_{m=1}^{\infty} \frac{\beta n c y^2 2s}{2L(\pi^2 m^2 - s)}\right) \tag{40}$$

with

$$s = \frac{\beta n u L^2}{c}. \tag{41}$$

Using the product and partial fraction expansion of circular functions[22]

$$\frac{\sin\sqrt{s}}{\sqrt{s}} = \prod_{m=1}^{\infty} \left[1 - \frac{s}{\pi^2 m^2}\right], \tag{42}$$

$$\frac{\sqrt{s}}{\tan\sqrt{s}} = 1 + \sum_{m=1}^{\infty} \frac{2s}{s - m^2\pi^2}, \tag{43}$$

we obtain the final result

$$Z^n(L,y) \equiv Z(s;\epsilon) = \left(\frac{\sqrt{s}}{\sin\sqrt{s}}\right)^{1/2} \exp\left[-\epsilon \frac{s\sqrt{s}}{\tan\sqrt{s}}\right] \tag{44}$$

with the dimensionless displacement parameter

$$\epsilon = \frac{c^2}{2u} \frac{y^2}{L^3}. \tag{45}$$

With our normalization, the partition sum does not depend on temperature any more [note that in the calculation of the free-energy distribution function, the variable s will be integrated over, cf. Eq. (60)]. The result in Eq. (44) is well defined provided that $0 < s < \pi^2$; the singularity at $s = \pi^2$ will determine the shape of the left tail in the free-energy distribution function, see Eq. (67) below.

As a simple application, we can use the partition function Eq. (38) to find the free energy $\overline{\langle F \rangle} = -T \overline{\ln Z(L,0;f_m)}$ of the string starting and ending in $\phi = 0$. Taking the disorder average over the term $\Sigma_m \beta f_m^2/4cLk_m^2$ in the logarithm of the partition function Eq. (38), we obtain the result

$$\overline{\langle F \rangle} = -\frac{uL^2}{2c\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} = -\frac{U_c}{12}\left(\frac{L}{L_c}\right)^2, \tag{46}$$

where we have used the Riemann zeta function $\zeta(2) = \pi^2/6$ and the definitions Eqs. (11) and (12). Alternatively, we can use the Eqs. (16) and (44) and calculate $\overline{\langle F \rangle}(L,0) = -(uL^2/c)\partial_s Z(s;0)|_{s=0}$. With $Z(s;0) \approx 1 + s/12$ we then easily recover the above result. Note that the result Eq. (46) measures the free energy F with respect to the entropic contribution $F_0^{\mathrm{fs}} = T \ln\sqrt{2\pi LT/c}$ of the free string.

The result for the free boundary condition $\partial_x \phi|_{x=L} = 0$ is obtained by using an alternative expansion: first, we symmetrically extend the system from the interval $[0,L]$ to the interval $[0,2L]$ with the definitions $\phi(L+x) \equiv \phi(L-x)$, $f(L+x) \equiv f(L-x)$. Second, we expand the analysis to the interval $[-2L,2L]$ using the same antisymmetric extension as above. As a result, we can expand the displacement and force fields into modes $\sin(q_m L)$ with $q_m = (2m-1)\pi/2L$, $m = 1,\ldots,\infty$ and hence zero slope at $x = L$. Following the same steps as above, we arrive at Eq. (40) with $y = 0$ and the product corresponding to the expansion of the cosine,[22]

$$\cos\sqrt{s} = \prod_{m=1}^{\infty} \left[1 - \frac{4s}{\pi^2(2m-1)^2}\right]. \tag{47}$$

The final result for the partition function with free boundary conditions then reads

$$Z(s) = \frac{1}{\sqrt{\cos\sqrt{s}}}, \tag{48}$$

where the regime of applicability is restricted to the domain $0 < s < \pi^2/4$; again, the singularity at $s = \pi^2/4$ determines the shape of the left tail in the free-energy distribution function, cf. Eq. (71). The alternative procedure of realizing the free boundary condition via integration over the end-point coordinate y, cf. Eq. (25), provides the identical result, although via a much more tedious calculation of determinants.

B. Replica approach

The replica Hamiltonian Eq. (20) for the random-force problem Eq. (7) reads

$$H_n[\{\phi_a\}] = \frac{1}{2}\int_0^L dx \left\{ c\sum_{a=1}^{n} [\partial_x \phi_a(x)]^2 - \beta u \sum_{a,b=1}^{n} \phi_a(x)\phi_b(x) \right\} \tag{49}$$

$$= -\frac{1}{2}\int_0^L dx \sum_{a,b=1}^{n} \phi_a(x) U_{ab} \phi_b(x) \tag{50}$$

with the matrix

$$U_{ab} = c\,\delta_{ab}\partial_x^2 + \beta u. \tag{51}$$

Accounting for the random shift $V_0(x)$, cf. Eq. (9), adds an additional term $-n^2\beta U_0 L/2$ to the Hamiltonian (49).

The matrix U_{ab} can be easily diagonalized and we find one $(n-1)$-fold degenerate eigenvalue $\lambda_1 = c\partial_x^2$ pertinent to the free string with the $(n-1)$ orthonormal eigenvectors ξ_i^a obeying the constraint $\Sigma_{a=1}^n \xi_i^a = 0$, $i = 1, \ldots, n-1$. The nth eigenvalue $\lambda_2 = c\partial_x^2 + \beta n u$ is nondegenerate and appertains to an inverted harmonic potential problem; the associated eigenvector is $\xi_n^a = 1/\sqrt{n}$, $a = 1, \ldots, n$. The coefficients ξ_i^a of the $(n \times n)$ transformation matrix ξ_i^a satisfy the conditions $\Sigma_{a=1}^n \xi_i^a \xi_j^a = \delta_{ij}$ (completeness) and $\Sigma_{i=1}^n \xi_i^a \xi_i^b = \delta_{ab}$ (orthonormality). In terms of the new fields $\varphi_i(x)$ and boundary conditions $q_i = \varphi_i(L)$,

$$\varphi_i(x) = \sum_{a=1}^{n} \xi_i^a \phi_a(x), \quad q_i = \sum_{a=1}^{n} \xi_i^a y_a, \tag{52}$$

the wave function (or propagator) Eq. (19) takes the form

$$\Psi[(q_i);L] = \prod_{i=1}^{n-1} \left\{ \int_0^{q_i} \mathcal{D}[\varphi_i(x)] \right\} \int_0^{q_n} \mathcal{D}[\varphi_n(x)] \times \exp\left\{ -\frac{\beta c}{2}\int_0^L dx \sum_{i=1}^{n-1} [\partial_x \varphi_i(x)]^2 \right\} \times \exp\left[-\frac{\beta c}{2}\int_0^L dx \{[\partial_x \varphi_n(x)]^2 - \varphi_n^2(x)/\lambda^2\} \right], \tag{53}$$

where we have introduced the length parameter λ, cf. Eq. (41),

$$\lambda^2 = \frac{c}{\beta n u} = \frac{L^2}{s}. \tag{54}$$

The $(n-1)$ free propagators are given by $\Psi_0(q_i;L)$, $i = 1, \ldots, n-1$, cf. Eq. (27). The propagator $\Psi_{\rm ih}(q_n;L)$ for the inverted harmonic potential problem is obtained by solving the imaginary-time Schrödinger equation

$$\partial_x \Psi_{\rm ih}(q;x) = \frac{1}{2}\left[\frac{1}{\beta c}\partial_q^2 + \frac{\beta c}{\lambda^2} q^2 \right] \Psi_{\rm ih}(q;x) \tag{55}$$

with the initial condition $\Psi_{\rm ih}(q;x=0) = \delta(q)$. With the Gaussian Ansatz $\Psi_{\rm ih}(q;x) = \chi(x)\exp[-a(x)q^2/2]$ and proper accounting of the initial condition, we find the solution (cf. Ref. 23)

$$\Psi_{\rm ih}(q_n;L) = \left(\frac{\sqrt{s}}{\sin\sqrt{s}} \right)^{1/2} \exp\left(-\frac{\beta c}{2L}\frac{\sqrt{s}}{\tan\sqrt{s}} q_n^2 \right). \tag{56}$$

Inserting the free [Eq. (27)] and harmonic [Eq. (56)] factors into the full propagator Eq. (53) and transforming back to original variables, $\Sigma_{i=1}^{n-1} q_i^2 = \Sigma_{i=1}^n q_i^2 - q_n^2 = \Sigma_{a=1}^n y_a^2 - (1/n)(\Sigma_{a=1}^n y_a)^2$, we obtain the result

$$\Psi(\{y_a\};L) = \left[\prod_{a=1}^{n} \Psi_0(y_a;L) \right] \frac{\Psi_{\rm ih}\left(\sum_b y_b/\sqrt{n};L \right)}{\Psi_0\left(\sum_b y_b/\sqrt{n};L \right)}. \tag{57}$$

Choosing the appropriate boundary conditions $y_a = y$, $a = 1, \ldots, n$, we obtain the replica partition function Eqs. (19) and (21) identical to the previous result Eq. (44), $Z_r(s;\epsilon) = Z(s;\epsilon)$, $0 < s < \pi^2$.

The result for the (replica) partition function has been derived for positive integer n. Since $Z(s;\epsilon)$ depends on n only via the parameter s, the expression (44) can be analytically continued to the complex half plane restricted by the condition $\mathrm{Re}[s] < \pi^2$.

For the free boundary condition, we obtain the replica partition function via integration of Eq. (57) over all end points $\{y_a\}$, cf. Eq. (26); the integration is conveniently done in the variables q_a and we make use of the normalization Eq. (29) to obtain the result identical to Eq. (48), $Z_r(s) = 1/\sqrt{\cos\sqrt{s}}$, $0 < s < \pi^2/4$. Furthermore, the analytic continuation to real *negative* values of the parameter n provides the expression

$$Z(s) = \frac{1}{\sqrt{\cosh\sqrt{|s|}}} \quad (s < 0), \tag{58}$$

alternatively, this result is obtained via the solution of the Schrödinger equation Eq. (55) for negative n involving a summation over the discrete spectrum of the parabolic potential, see Appendix A.

C. Distribution function: Fixed boundary condition

We now turn to the calculation of the free-energy distribution function $\mathcal{P}_{L,y}(F)$ from the partition function $Z^n(L,y)$. Following the procedure described in Sec. II, specifically Eqs. (14) and (15), the Laplace transform and its inverse assume the form

$$Z(s;\epsilon) = \int_{-\infty}^{+\infty} df p_\epsilon(f) \exp(-sf), \tag{59}$$

$$p_\epsilon(f) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds Z(s;\epsilon)\exp(sf), \tag{60}$$

where

$$f(F,L) = \frac{F}{F_f(L)}, \quad F_f(L) = \frac{u}{c}L^2 = U_c\left(\frac{L}{L_c}\right)^2, \tag{61}$$

$$\epsilon(y,L) = \frac{c^2}{2u}\frac{y^2}{L^3} \tag{62}$$

are the rescaled free energy of the system and the rescaled displacement parameter; the original free-energy distribution function $\mathcal{P}_{L,y}(F)$ then derives from the rescaled expression $p_\epsilon(f)$ through the relation

$$\mathcal{P}_{L,y}(F) = \frac{1}{F_f(L)} p_{\epsilon(y,L)}[f(F,L)]. \tag{63}$$

Note, that the parameter ξ drops out in the combination U_c/L_c^2, as has to be the case for the random-force model where the disorder is characterized by only one parameter, its strength u. Or in other words, using the results below for a random-force (rather then a random potential) problem, these are valid for all length scales. For the relaxational free-energy distribution function of the system with fixed boundary condition, we obtain the expression

$$p_\epsilon(f) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds \left(\frac{\sqrt{s}}{\sin\sqrt{s}}\right)^{1/2} \times \exp\left[-\epsilon\frac{s\sqrt{s}}{\tan\sqrt{s}} + fs\right], \tag{64}$$

which simplifies drastically for the special case of fixed boundary conditions with $\phi(0)=\phi(L)=0$,

$$p_0(f) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds \left(\frac{\sqrt{s}}{\sin\sqrt{s}}\right)^{1/2} \exp(fs). \tag{65}$$

The above result already expresses an important property of the distribution function $p_0(f)$: for $f>0$ the expression under the integral is analytic and quickly goes to zero at $s\to-\infty$, hence the contour of integration in the complex plane can be safely shifted to $-\infty$. This implies that the function $p_0(f)$ must be equal to zero for $f>0$ and the relaxational free energy of the directed polymer with zero boundary conditions is bounded from above, $F<0$. This constraint then is easily understood, as the presence of a random force can only *reduce* the relaxational free energy of the directed polymer.

The evaluation of the inverse Laplace transform Eq. (64) is discussed in Appendix B and provides the free-energy distribution function $p_\epsilon(f)$ as illustrated in Fig. 2 (note that all temperature dependence has vanished since we measure our free energy with respect to the entropic contribution $f_0^{\rm fs} = F_0^{\rm fs}/F_f = (cT/uL^2)\ln\sqrt{2\pi LT/c}$ of the free string). The various asymptotic forms of the distribution function $p_\epsilon(f)$ in the limits $f\to\pm\infty$ and $f\to-0$ are derived via a saddle-point integration and read,

$$p_\epsilon(f\to-\infty) \sim \exp(-\pi^2|f|), \tag{66}$$

$$p_\epsilon(f\to+\infty) \sim \exp\left(-\frac{4}{27\epsilon^2}[f]^3\right), \tag{67}$$

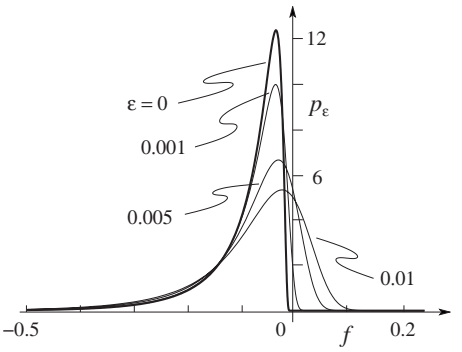

FIG. 2. Relaxational free-energy distribution function $p_\epsilon(f)$ for the randomly forced directed polymer for several values of the dimensionless displacement parameter $\epsilon=(y/\xi)^2(L_c/L)^3/2$: $\epsilon=0, 0.001, 0.005, 0.01$.

$$p_{\epsilon=0}(f\to-0) \sim \exp\left(-\frac{1}{32|f|}\right). \tag{68}$$

Note that the shape of the left tail is determined by the singularity of $Z(s;\epsilon)$ at $s=\pi^2$, cf. Eq. (44). The above results agree with those obtained before in Ref. 19.

D. Distribution function: Free boundary conditions

The result (48) provides us with all the moments of the relaxational free-energy distribution function, of which the first one, the average free energy, is given by

$$\overline{\langle F\rangle} = -U_c(L/L_c)^2[\partial_s Z(s)]_{s=0} = -\frac{U_c}{4}\left(\frac{L}{L_c}\right)^2. \tag{69}$$

In order to obtain the full distribution function, we perform the inverse Laplace transform $(\mathcal{P}_L(F)=p[f=F/F_f(L)]/F_f(L))$

$$p(f) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} ds \frac{1}{\sqrt{\cos\sqrt{s}}}\exp(fs). \tag{70}$$

Given the scaling form in $f=F/F_f(L)$, the result is valid at all scales. Again, for $f>0$ the integrand is analytic and rapidly approaches zero as $s\to-\infty$ and hence the function $p(f)$ must vanish identically for $f>0$. The functional form for $f<0$ is found as before, see Appendix B. The relaxational free-energy distribution function $\mathcal{P}(f)$ assumes a universal form with no parameters; it vanishes identically for $f>0$ and its overall form is shown in Fig. 3.

Note that the free energy of the "trivial" configuration $\phi(x)\equiv 0$ is equal to zero and any deviation due to the action of the random force can only reduce the energy, providing a simple explanation for the cutoff at positive energies. The asymptotic behavior in the limits $f\to-\infty$ and $f\to-0$ can be easily estimated by a saddle-point calculation,

$$p(f\to-\infty) \sim \exp\left(-\frac{\pi^2}{4}|f|\right), \tag{71}$$

$$p(f\to-0) \sim \exp\left(-\frac{1}{32|f|}\right). \tag{72}$$

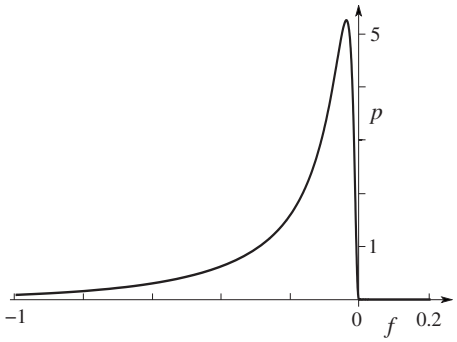

FIG. 3. Relaxational free-energy distribution function $p(f)$ of the randomly forced directed polymer with free boundary conditions.

E. Shifted random-force model

In order to find the distribution function for the total free energy (rather than its relaxational part), we have to account for the random shift $V_0(x)$, cf. Eq. (9). Here, we concentrate on the situation with free boundary conditions. The multiplication of Eq. (48) with the Gaussian $\exp(\beta^2 n^2 U_0 L/2) = \exp[s^2(L_c/L)^3/2]$ and subsequent Laplace transform of

$$Z_r(s) = \frac{1}{\sqrt{\cos\sqrt{s}}} e^{(s^2/2)(L_c^3/L^3)} \tag{73}$$

generates the (rescaled) free-energy distribution function $p^t(f)$ shown in Fig. 4.

The (total) free-energy distribution function $\mathcal{P}^t$ derives from a convolution of the distribution function $\mathcal{P}^f(F)$ of the relaxational free energy, cf. Eq. (70) and Fig. 3, and the factor $\mathcal{P}^V$ originating from the random shift $V_0(x)$,

$$\mathcal{P}^t(F) = \int_{-\infty}^{\infty} dF' \mathcal{P}^f(F') \mathcal{P}^V(F - F'). \tag{74}$$

The contribution $\mathcal{P}^V$ from the random shift assumes the simple Gaussian form

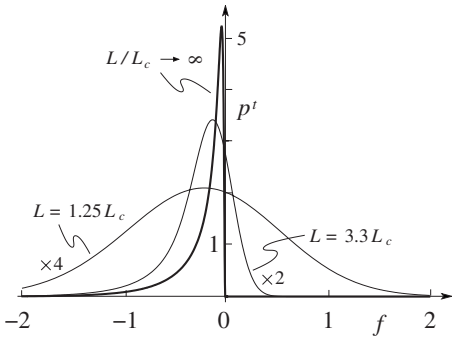

FIG. 4. Free-energy distribution function $p^t(f)$ of the randomly forced directed polymer with free boundary conditions including the random shift $V_0(x)$. For $L \gg L_c$, the relaxational part of the free energy dominates the distribution; in the regime $L < L_c$, where the shifted random-force model provides an approximation to the random polymer problem, the free-energy distribution function is dominated by the Gaussian part originating from the shift $V_0(x)$.

$$\mathcal{P}^V(F) = \frac{1}{\sqrt{2\pi} F_V(L)} \exp\{-[F/F_V(L)]^2/2\} \tag{75}$$

with the scaling parameter

$$F_V(L) = U_c \sqrt{L/L_c}. \tag{76}$$

For $L \gg L_c$, the result coincides with that for the relaxational energy, cf. Eq. (70) and Fig. 3, and scales as $F/F_f(L)$. In the limit of short lengths $L \lesssim L_c$, where the model can be used as an approximation of the random-potential problem, the distribution function is dominated by the Gaussian due to the random shift $V_0(x)$. More specifically, we find the width of the distribution's body through the calculation of the second cumulant: expanding $Z_r(s)$ for small values of s and using Eq. (16), we obtain

$$Z_r(s) \simeq 1 + \frac{s}{4} + \frac{s^2}{2}\left[\frac{7}{48} + \left(\frac{L_c}{L}\right)^3\right], \tag{77}$$

$$\overline{\langle F \rangle} = -\frac{U_c}{4}\left(\frac{L}{L_c}\right)^2, \tag{78}$$

$$\overline{F^2} - \bar{F}^2 = U_c^2 \frac{L}{L_c}\left[1 + \frac{1}{12}\left(\frac{L}{L_c}\right)^3\right]. \tag{79}$$

The leading term $\propto U_c/L_c^2 = u/c$ in first moment Eq. (78) derives from the random-force part of the disorder. This is different from the second cumulant in Eq. (79), where the first term $\propto U_c^2/L_c = U_0$ derives from the random shift $V_0(x)$ and dominates over the contribution from the random force [second term $\propto (u/c)^2$] at short lengths $L < L_c$, hence the width of $\mathcal{P}^t$ is given by F_V [see below for a discussion of the corrections $\propto (L/L_c)^3$].

Besides the first two moments/cumulants, we can easily determine the scaling of higher moments. Starting from the convolution Eq. (74), we note the different scaling of the arguments in the two distribution functions, $\propto F/L^2$ for the relaxational part $\mathcal{P}^f$ and $\propto F/\sqrt{L}$ for the random shift part $\mathcal{P}^V(F)$. Hence, for small distances L, the function $\mathcal{P}^f(F')$ peaks narrowly near zero, while $\mathcal{P}^V(F-F')$ retains a broader shape; expanding $\mathcal{P}^V(F-F')$ around F and integrating over F', we obtain the following expansion for the total distribution function,

$$\mathcal{P}^t(F) \approx \mathcal{P}^V(F) - \mathcal{P}^{V'}(F)\bar{F}^f + \frac{1}{2}\mathcal{P}^{V''}(F)\overline{F^2}^f + \cdots, \tag{80}$$

where $\mathcal{P}^{V'}(F)$ is the derivative of $\mathcal{P}^V$ with respect to the argument F and $\overline{(\cdots)}^f$ denotes averaging over the random-force part $\mathcal{P}^f$. Using the scaling $\bar{F}^f \propto uL^2$ and $\overline{F^2}^f \propto u^2 L^4$ for the moments of the relaxational free energy, we can calculate the dependence of the moments $\overline{F^k}$ on the length L. Thereby, we exploit the fact that the leading term $\mathcal{P}^V(F)$ in the expansion (80) is symmetric in F, cf. Eq. (75), and hence determines the even moments, while the next term is antisymmetric and generates the odd moments; finally, the third term provides the corrections to the even moments. The combination of the scaling of $\mathcal{P}^V(F)$ [deriving from the random shift $V_0(x)$] and of the first two moments $\bar{F}^f$ and $\overline{F^2}^f$ [deriving

from the random force $f(x)$] then generates the following nontrivial scaling of the moments with different powers in L for the even and odd moments,

$$\overline{F^{2k}} \propto L^k + \mathcal{O}(L^{k+3}), \tag{81}$$

$$\overline{F^{2k+1}} \propto L^{k+2}, \tag{82}$$

where $\mathcal{O}(L^{k+3})$ denotes a correction term $\propto L^{k+3}$. In particular, $\bar{F} \propto uL^2$, $\overline{F^2} \propto U_0 L + \mathcal{O}(u^2 L^4)$, $\overline{F^3} \propto uU_0 L^3$, $\overline{F^4} \propto U_0^2 L^2 + \mathcal{O}(u^2 U_0 L^5)$, $\overline{F^5} \propto uU_0^2 L^4$.

Finally, we can estimate the tails of $\mathcal{P}^t$ from the convolution Eq. (74) using the asymptotic behavior of $\mathcal{P}^f$ and $\mathcal{P}^V$ and find a left tail $\mathcal{P}^t(F<-F_{\text{tail}}) \propto \exp(F/F_f)$ and a Gaussian tail on the right, $\mathcal{P}^t(F>F_{\text{tail}}) \propto \exp[-(F/F_V)^2/2]$, where $F_{\text{tail}} = F_V[1+(L_c/L)^{3/2}]$. On short scales $L<L_c$, we have $F_{\text{tail}} \approx F_V(L_c/L)^{3/2} > F_V$ and the random-force behavior appears only quite beyond the body.

F. Joint distribution function

We add a note on the joint free-energy distribution function $\mathcal{P}_{L,y}(\bar{F},F')$, where $\bar{F}=(F_+ + F_-)/2$ and $F'=(F_+ - F_-)/2$ denote the mean-free energy and the free-energy difference for two polymer trajectories starting at the origin $\phi=0$ at $x=0$ and ending in the symmetric points $\phi=\pm y$ at $x=L$, $F_\pm \equiv F(L,\pm y;V)$. Opposite to the δ-correlated potential, cf. Ref. 17, the present case of the random-force model is less revealing and we keep the discussion short.

Starting with the original (random) Hamiltonian $H[\phi(x);V]$ with the random-force potential Eq. (7), we account for the boundary condition $\phi(L)=y$ through the shift $\phi(x) \to (y/L)x + \phi(x)$ [the $T=0$ solution for the string ending in $\phi(L)=y$ derives from the solution ending in $\phi(L)=0$ by adding the shift $(y/L)x$] and obtain the Hamiltonian described by Eqs. (33) and (37). The relaxational free energy of the system with the boundary condition $\phi(L)=y$ separates into the terms

$$F[L,y;f] = \frac{cy^2}{2L} + \frac{y}{L}\int_0^L dx x f(x) + F[L,0;f]. \tag{83}$$

The first term is the trivial part of the elastic energy, the second is a random constant, and finally, the third is the (random) relaxational free energy of the polymer with zero boundary conditions; its randomness is *correlated* with the randomness in the second term. Then, for the free energies $\bar{F}$ and F' introduced above, we find that

$$F'[L,y;f] = \frac{y}{L}\int_0^L dx x f(x), \tag{84}$$

$$\bar{F}[L,y;f] = \frac{cy^2}{2L} + F[L,0;f], \tag{85}$$

and hence F' and $\bar{F}$ carry the information on the second and third terms in Eq. (83), respectively. Although the joint distribution function for the random and *correlated* quantities $\bar{F}$ and F' must be nontrivial, we can conclude that the *separate* statistics of $\bar{F}$ and F' must be simple: according to Eq. (84), the distribution for F' is Gaussian with zero mean and width $\overline{(F')^2} = y^2 uL/3$ while the distribution for $\bar{F}$ must coincide with that for the free energy with zero boundary conditions $\mathcal{P}_{L,y=0}(F)$, cf. Eq. (65), shifted by the trivial elastic term $cy^2/2L$. Also note, that a change in the final coordinate y modifies the polymer's trajectory over the entire length L and hence the joint distribution function is not expected to factorize, in contrast to the results found for the short-range correlated random polymer problem.[17] The detailed replica calculation, which represents a straightforward extension of the above analysis, produces results in full agreement with these simple arguments.

IV. HARMONIC-CORRELATOR APPROXIMATION

We consider the random directed polymer described by the Hamiltonian Eq. (1) and approximate the interaction $-\beta^2 U$ in the replica Hamiltonian Eq. (20) by the harmonic expression (6) to arrive at,

$$H_n[\{\phi_a\}] = \int_0^L dx \left\{ \frac{c}{2}\sum_{a=1}^n [\partial_x \phi_a(x)]^2 + \frac{\beta u}{4}\sum_{a,b=1}^n [\phi_a(x) - \phi_b(x)]^2 \right\} - \frac{n^2}{2}\beta U_0 L$$
$$= -\frac{1}{2}\int_0^L dx \sum_{a,b=1}^n \phi_a(x)\tilde{U}_{ab}\phi_b(x) - \frac{n^2}{2}\beta U_0 L \tag{86}$$

with the matrix $\tilde{U}_{ab} = (c\partial_x^2 - \beta n u)\delta_{ab} + \beta u$ (note that the parabolic approximation of the correlator should be implemented *after* the integration over the disorder potential). Diagonalization produces the $(n-1)$-fold degenerate eigenvalue $\lambda_1 = c\partial_x^2 - \beta n u$ of the harmonic oscillator problem with the $(n-1)$ orthonormal eigenvectors ξ_i^a constrained by the condition $\Sigma_{a=1}^n \xi_i^a = 0$, $i=1,\ldots,n-1$, and one nondegenerate eigenvalue $\lambda_2 = c\partial_x^2$ of the free problem with the eigenvector $\xi_n^a = 1/\sqrt{n}$, $a=1,\ldots,n$. The propagator for the harmonic-correlator approximation then assumes the form [cf. Eq. (57)]

$$\Psi(\{y_a\};L) = \left[\prod_{a=1}^n \Psi_{\text{h}}(y_a;L)\right] \frac{\Psi_0(\sum_b y_b/\sqrt{n};L)}{\Psi_{\text{h}}(\sum_b y_b/\sqrt{n};L)}, \tag{87}$$

where Ψ_{h} derives from Ψ_{ih} by the substitution $\lambda \to i\lambda$. For simplicity, we only consider the model with free boundary conditions and find the shifted random-force result Eq. (73) replaced by the expression $[s=L^2/\lambda^2 = n(U_c/T)^2(L/L_c)^2$; see also Appendix A]

$$\tilde{Z}_r(s) = \left[\frac{1}{\sqrt{\cosh\sqrt{s}}}\right]^{(n-1)} \exp[s^2(L_c/L)^3/2]$$
$$= [\cosh\sqrt{s}]^{1/2}[\cosh\sqrt{s}]^{-(s/2)(T/U_c)(L_c^2/L^2)} e^{-(s^2/2)(L_c^3/L^3)}. \tag{88}$$

Although the inverse Laplace transform can be performed,

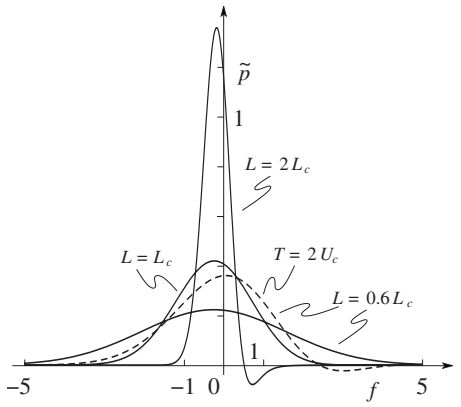

FIG. 5. Free-energy distribution function $\tilde{p}(f)$ for the directed polymer with free boundary conditions using the harmonic-correlator approximation. Solid curves refer to $T=0$ while the dashed curve attains to $T=2U_c$.

the resulting (total) free-energy probability distribution $\tilde{p}(f)$ develops a negative right tail at zero and low temperatures, see Fig. 5; at large temperatures $T \gg U_c$ the right tail exhibits pronounced oscillations. These unphysical results are due to the departure of the approximate harmonic interaction $U_p(\phi)$ from the true interaction $U(\phi)$, becoming relevant at large scales $L > L_c$, $\phi > \xi$, $F > U_c$, and the large fluctuations of the string at high temperatures $T \gg U_c$. Note that the inverse Laplace transform cannot be performed at all in case the random shift $V_0(x)$ is ignored.

The breakdown of the harmonic-correlator approximation is conveniently observed in the second moment: expanding $\tilde{Z}_r(s)$ for small values of s we find to second order

$$\tilde{Z}_r(s) \simeq 1 + \frac{s}{4} + \frac{s^2}{2}\left[\left(\frac{L_c}{L}\right)^3\left(1 - \frac{T}{2U_c}\frac{L}{L_c}\right) - \frac{1}{48}\right] \quad (89)$$

and using Eq. (16), we find the average free energy $\bar{F} = -U_c(L/L_c)^2/4$ [cf. Eq. (78)] and the second cumulant reads

$$\overline{F^2} - \bar{F}^2 = U_c^2 \frac{L}{L_c}\left[1 - \frac{T}{2U_c}\frac{L}{L_c} - \frac{1}{12}\left(\frac{L}{L_c}\right)^3\right]. \quad (90)$$

Comparing with the result Eq. (79) for the shifted random force, we note the additional dependence on temperature and the sign change in the correction term $\propto (L/L_c)^3$. This decrease in width is in accord with the behavior of the free-energy fluctuations in the random directed polymer problem (2), which scales as $\delta F \propto L^{2\zeta-1}$ at large distances; with a wandering exponent[6] $\zeta = 2/3$, we have $\delta F \propto L^{1/3}$. The negative correction $\propto (L/L_c)^3$ to the linear growth in δF^2 observed in Eq. (90) then is consistent with the sublinear growth $\delta F^2 \propto L^{2/3}$ of the exact solution.

As before, we can analyze the higher moments of the distribution function and compare the results for the harmonic-correlator approximation at $T=0$ with those obtained for the shifted random-force model. Making use of the product form of $\tilde{Z}_r$ and expanding each factor in s, we find identical leading terms for all even and odd moments [corresponding to equal expressions for the first two terms in Eq. (80)], while the corrections to the even moments [described by the third term in Eq. (80)] are different. Furthermore, we note that the even moments $\overline{F^{2k}} \propto U_c^{2k}(L/L_c)^k[1 - \mathcal{O}(L^3/L_c^3)] \propto U_0^k L^k$ are large with a small negative correction, while the odd moments $\overline{F^{2k+1}} \propto -U_c^{2k+1}(L/L_c)^{k+2}[1 + \mathcal{O}(L^3/L_c^3)] \propto u U_0^k L^{k+2}$ are small, their ratios being

$$(\overline{F^{2k+1}})^2/\overline{F^{2(2k+1)}} \propto (L/L_c)^3.$$

To leading order, the free-energy distribution function for the random-potential model at small scales then is a trivial Gaussian generated by $V_0(x)$, with a small negative shift and a small reduction in width due to the random-force term in the potential.

At $T=0$, the second cumulant turns negative for $L > 2\sqrt{3/2}L_c$ and the result in Eq. (90) makes no longer any sense, hence the harmonic approximation to a random-potential problem cannot be used on scales larger then ξ (along the transverse direction) or L_c (along the longitudinal direction); at finite temperatures the regime of validity is further reduced.

Although the results for the shifted random force remain valid at any length L, we emphasize that the harmonic correlator provides a better approximation for the behavior of the short-range correlated random polymer: Both results agree in lowest order, providing the same first moment $\bar{F}$ due to the random force $f(x)$ and the same leading term in the second cumulant $\overline{F^2} - \bar{F}^2$ generated by the random shift $V_0(x)$. The correction $\propto (L/L_c)^3$ in the second cumulant is due to the random force $f(x)$ and contributes with the opposite sign in the shifted random-force model as compared to the harmonic-correlator approximation. While the shifted random-force result in Eq. (79) is correct (at all scales) when dealing with a true random-force model, the correction $\propto (L/L_c)^3$ carries the wrong sign when used as an approximation to the random-potential model and it is the result of the harmonic-correlator approximation Eq. (90) which should be trusted. Indeed, the harmonic correlator preserves the translation invariance of the problem whereas some quadratic terms are dropped from the shifted random-force model. Expanding the potential to second order,

$$V(\phi, x) = V_0(x) + f(x)\phi(x) - \frac{1}{2}g(x)\phi^2(x), \quad (91)$$

we identify the terms in $\overline{V(\phi,x)V(\phi',x')}$ with the harmonic expansion, Eq. (6), to obtain the correlators Eqs. (8) and (9), $\overline{V_0(x)g(x')} = u\delta(x-x')$, and vanishing mixed terms $\overline{V_0(x)f(x')} = \overline{f(x)g(x')} = 0$. A scaling estimate of the second moment $\overline{F^2}$ using Eq. (91) then provides a leading term $\propto L$ from V_0 and subleading terms $\propto L^4$ from f and from g. The contribution from the random force provides the positive contribution $U_c(L/L_c)^4/12$ in the cumulant Eq. (79) while the mixed terms $\overline{V_0(x)g(x')}$ contribute with a negative weight, generating the negative correction $\propto L^4$ in Eq. (90). Note that higher order terms do not change this result but contribute to the next order term $\propto L^7$.

Given that the harmonic correlator provides the better approximation to the random polymer problem at short scales,

one may wonder why we end up with unphysical results (negative distribution function, negative second moment) at larger scales. Also, different types of correlators, e.g., power-law type, have been studied in the past, cf. Refs. 1 and 20, and one would like to know, what properties of a disorder correlator guarantee consistent results; this question is addressed in the following section.

A. Correlators with nonpositive spectrum

It is important to identify problematic correlators right from the beginning; indeed, the proper definition of the disorder potential is subject to important constraints[24] regarding its shape $U(\phi)$ and failure to respect these constraints may lead to unphysical results. Consider a random potential $V(\phi)$ and its Fourier representation $V(q)=\int d\phi V(\phi)\exp(-iq\phi)$; then the Gaussian distribution function of the random function $V(q)$ has the form

$$\mathcal{P}[V(q)] = P_0 \exp\left(-\int \frac{dq}{2\pi}\frac{|V(q)|^2}{2G(q)}\right). \tag{92}$$

The width $G(q)$ has to be positive and relates to the correlation function $U(\phi)$ via

$$U(\phi) = \int \frac{dq}{2\pi} G(q)\exp(iq\phi). \tag{93}$$

Expanding both sides in powers of ϕ,

$$U(0) + \sum_{k=1}^{\infty} \frac{U^{(2k)}(0)}{(2k)!}\phi^{2k} = \int \frac{dq}{2\pi} G(q) + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!}\left[\int \frac{dq}{2\pi} G(q) q^{2k}\right]\phi^{2k}, \tag{94}$$

and comparing coefficients, we find that the $2k$th derivative of $U(\phi)$ in the origin relates to the integral $\int dqG(q)q^{2k}$, which is a positive quantity. Hence, we have to be careful in our choice of the correlator $U(\phi)$. For example, truncating the expansion of $U(\phi)$ beyond some k^*, such that $U^{(2k)}(0)=0$ for $k \geq k^*$, we impose the condition

$$\int dqG(q)q^{2k} = 0 \quad \text{for} \quad k \geq k^*, \tag{95}$$

which cannot be satisfied for a positively defined $G(q)$.

Obviously then, choosing a parabolic correlator $U_p(\phi)$ as in Eq. (6) is in severe conflict with the constraint Eq. (95). The averaging over the disorder potential $V(\phi,x)$ is undefined for those modes (in Fourier space) where $G(q)$ is negative. Hence, going over from the disordered directed polymer (a statistical mechanics problem) to the quantum boson problem is an ill-defined step and the results cannot be trusted any longer. On the other hand, performing the integration over the random potential $V(\phi,x)$ with a well defined, i.e., spectral, correlator $U(\phi)$ and expanding the resulting interaction $-\beta^2 U(\phi)$ between bosons is perfectly admissible and leads to an *identical result*; in this case, however, we know that the quantum boson problem *does not* describe the random polymer problem on scales where the approximate quadratic correlator deviates strongly from the original correlator. Nevertheless, in the end we have to appreciate, that the harmonic-correlator approximation (6), although breaking down at lengths beyond L_c, does produce more accurate approximate results for the short-range correlated random-potential problem (2) than the shifted random-force model (5), although the latter remains formally valid at all length scales L. The (shifted) random-force model then should be used whenever the disorder landscape is given by a force field as defined by Eqs. (5), (8), and (9) but not as an approximation to a random-potential problem.

B. Displacement correlator

Another quantity of interest in the random polymer problem is the displacement correlator $\overline{\langle\phi^2\rangle}(L) \equiv \overline{\langle[\phi(L)-\phi(0)]^2\rangle}$, with $\langle\cdots\rangle$ and $\overline{(\cdots)}$ denoting thermal and disorder averages, respectively. Choosing free boundary conditions with $\phi(0)=0$ and an arbitrary position $\phi(L)=y$ for the end point, the averages $\overline{\langle y^2\rangle}$ and $\overline{\langle y\rangle^2}$ are easily calculated within replica theory.[20] Defining

$$\overline{\langle y_a y_b\rangle} = \left[\prod_{c=1}^{n}\int dy_c\right] y_a y_b \Psi(\{y_c\};L), \tag{96}$$

we obtain the two types of averages

$$\overline{\langle y^2\rangle} = \lim_{n\to 0}\overline{\langle y_a y_b\rangle}|_{a=b},$$

$$\overline{\langle y\rangle^2} = \lim_{n\to 0}\overline{\langle y_a y_b\rangle}|_{a\neq b}. \tag{97}$$

The Hamiltonians for the shifted random-force model and the harmonic-correlator approximation differ only by the term $(\beta un/2)\Sigma_{a=1}^{n}\phi_a^2$, which vanishes in the limit $n\to 0$, hence both schemes produce identical results for the displacement correlators in Eq. (97). We then concentrate on the random-force case and calculate the expression

$$\overline{\langle y_a y_b\rangle} = C\left[\prod_{c=1}^{n}\int dy_c\right] y_a y_b \exp\left[-\frac{1}{2}\sum_{c,d} K_{cd} y_c y_d\right] \tag{98}$$

with $K_{cd}=A\delta_{cd}+B$ and

$$A = \frac{\beta c}{L}, \quad B = \frac{\beta c}{nL}\left[\frac{\sqrt{s}}{\tan\sqrt{s}} - 1\right],$$

$$C = \left(\frac{\beta c}{2\pi L}\right)^{n/2}\left(\frac{\sqrt{s}}{\sin\sqrt{s}}\right)^{1/2}.$$

In the calculation of $\overline{\langle y_a y_b\rangle}|_{a\neq b}$, we combine all diagonal terms into a sum $(D/2)\Sigma_c y_c^2$ with $D=A+B$, leaving the nondiagonal in the form $(B/2)\Sigma_{c\neq d} y_c y_d$; the nondiagonal average then follows from the derivative

$$\overline{\langle y_a y_b\rangle}|_{a\neq b} = -\frac{2}{n(n-1)}\frac{\partial}{\partial B}\left[\prod_{c=1}^{n}\int dy_c\right]\Psi(\{y_c\};L)|_D$$

while the diagonal average is given by the derivative

$$\overline{\langle y_a^2\rangle} = \frac{2}{n}\frac{\partial}{\partial A}\left[\prod_{c=1}^{n}\int dy_c\right]\Psi(\{y_c\};L)|_B. \tag{99}$$

The final results assume the form

$$\overline{\langle y^2\rangle} = \lim_{n\to 0}\left(\frac{1}{A}-\frac{B}{A^2}\right) = \xi^2\frac{T}{U_c}\frac{L}{L_c}+\frac{\xi^2}{3}\left(\frac{L}{L_c}\right)^3,$$

$$\overline{\langle y\rangle^2} = -\lim_{n\to 0}\frac{B}{(D-B)^2} = -\lim_{n\to 0}\frac{B}{A^2} = \frac{\xi^2}{3}\left(\frac{L}{L_c}\right)^3. \tag{100}$$

The relation $\overline{\langle y^2\rangle}-\overline{\langle y\rangle^2}=\langle y^2\rangle|_{V=0}=TL/c$ (here, $\langle y^2\rangle|_{V=0}$ denotes the thermal average in the absence of any disorder, $V=0$), is a constraint holding true for any disorder potential uncorrelated in x, cf. Refs. 25 and 26.

V. SUMMARY AND CONCLUSIONS

The (shifted) randomly forced polymer model and the random disordered polymer described through a harmonic-correlator approximation define quadratic problems and hence admit exact solutions. For the random-force models, different approaches can be taken, either a direct integration of the path integrals within a Fourier representation or using the (real-space) replica technique; in retrospect, the preferred method is a matter of taste. We have determined the free-energy distribution functions $\mathcal{P}_{L,y}(F)$ and $\mathcal{P}_L(F)$ for fixed and free boundary conditions. This calculation necessitates the determination of all powers $Z_r(n;L,y)=\overline{Z^n[L,y;V]}$ (rather than the usual $n\to 0$ limit) and subsequent inverse Laplace transformation of the analytically continued replica partition function $Z_r(n\in\mathbb{C};L,y)$. The displacement correlators $\overline{\langle y^2\rangle}$ and $\overline{\langle y\rangle^2}$ have been found as well. The simplicity of the quadratic models allows to carry through the entire program and thus serves to study not only the physical properties of the problem but its methodological aspects as well.

Regarding the shape of the distribution functions for the random-force model, a number of interesting features has been obtained: for the free boundary, the probability to find a positive free energy F vanishes exactly, with an essential singularity appearing in $\mathcal{P}_L(F)\propto\exp(-uL^2/32c|F|)$ as F approaches zero from the left, cf. Eq. (72). For fixed boundary conditions, a similar result has been found for $\mathcal{P}_{L,y=0}(F)$, see also Ref. 19. Furthermore, the left and right tails provide a consistent scaling $F\propto L^2$ and $y\propto L^{3/2}$, $\mathcal{P}_{L,y}(F\to-\infty)\propto\exp(-\pi^2c|F|/uL^2)$ and $\mathcal{P}_{L,y}(F\to\infty)\propto\exp[-(16/27)|F|^3/ucy^4]$, cf. Eqs. (66) and (67).

When interested in the short-distance behavior of the random directed polymer Eq. (2), two types of approximations offer a drastic simplification of the problem: these are the expansion of the random potential $V(\phi,x)$ according to Eq. (5) (generating the shifted random-force problem) or the expansion of the correlator Eq. (6) (leading to the harmonic-correlator approximation). While both approximations generate the same results for the even and odd moments to leading order, the next to leading-order terms turn out different. In this situation, the results of the harmonic-correlator approximation have to be trusted, as it consistently accounts for the relevant terms preserving the translation invariance of the problem. Collecting all results, we find that the free-energy distribution function for the random-potential model at ($T=0$ and) small scales is a trivial Gaussian of width $U_c\sqrt{L/L_c}$ generated by $V_0(x)$, with a small negative shift $-U_c(L/2L_c)^2$ and a small reduction $-(U_c/24)(L/L_c)^{7/2}$ in width due to the random-force term in the potential.

Finally, we mention a few useful insights regarding the replica technique which derive from our analysis above. The replica technique provides a link between two seemingly unrelated problems, the classical statistical theory of disordered polymers and the quantum many-body theory of attractive bosons. Several stumbling blocks can be eliminated by properly appreciating the subtleties in this mapping. As is well known, after the mapping from polymers to bosons the disorder correlator assumes the role of the interaction potential. While many shapes for the interaction potential may produce meaningful results for the quantum boson problem, only a restricted set of them (those describing a correlator with positive spectrum) relate to a meaningful random polymer problem. Hence, the original choice of physical correlators and any modification thereof during the calculation should be done with great care; in particular, a simple power-law form[1] might not work. For example, there is nothing wrong in studying quantum bosons with a simple harmonic interaction $U(\phi)=-U_0+u\phi^2/2$ and the results obtained for the quantum propagator are perfectly acceptable for any constant shift U_0. However, interpreting the result for the propagator in terms of a replica partition function and transforming back (via the inverse Laplace transformation) to random polymers, the resulting distribution function becomes unphysical when setting $U_0=0$; dropping a shift U_0 in the potential for the bosons is a trivial shift in energy while ignoring the same shift in the correlator produces unphysical results for the polymer problem after Laplace transformation.

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APPENDIX A: NEGATIVE REPLICA NUMBER

We determine the replica partition function $Z_r(s)$, Eq. (58), for the polymer with free boundary conditions via direct solution of the Schrödinger equation, Eq. (55), for negative n. We confirm, that the result analytically continued from positive n agrees with the one obtained for negative n. The wave function $\Psi(q,x)$ satisfies the Schrödinger equation [cf. Eq. (55)]

$$\partial_x\Psi(q;x) = \frac{1}{2}\left[\frac{1}{\beta c}\partial_q^2-\frac{\beta c}{\lambda^2}q^2\right]\Psi(q;x) \tag{A1}$$

with $\lambda^2=c/\beta|n|u$. We are seeking the solution

$$\Psi(q;x)=\sum_{k=0}^{\infty}A_k\exp(-E_kx)\Psi_k(q), \tag{A2}$$

satisfying the initial condition $\Psi(q;x=0)=\delta(q)$; the energies and corresponding orthonormal eigenfunctions E_k and $\Psi_k(q)$ satisfy the stationary equation,

$$E_k\Psi_k(q)=-\frac{1}{2}\left[\frac{1}{\beta c}\partial_q^2+\frac{\beta c}{\lambda^2}q^2\right]\Psi_k(q). \tag{A3}$$

The coefficients A_k in Eq. (A2) derive from the initial condition

$$A_k=\int_{-\infty}^{+\infty}dq\Psi_k^*(q)\Psi(q;x=0)=\Psi_k^*(0).$$

The spectrum of the harmonic problem is given by $E_k=(k+1/2)/\lambda$ and the corresponding eigenfunctions are (see, e.g., Ref. 27)

$$\Psi_k(q)=\left(\frac{\beta c/\lambda}{\sqrt{\pi}2^k k!}\right)^{1/2}\exp[-(\beta c/2\lambda)q^2]H_k[\sqrt{\beta c/\lambda}q],$$

where $H_k(x)$ are the Hermite polynomials $H_k(x)=(-1)^k\exp(x^2)\partial_x^k[\exp(-x^2)]$. Substituting A_k and Ψ_k into Eq. (A2) and taking into account that $H_{2l+1}(0)=0$, we obtain the wave function

$$\Psi(q;x)=\sqrt{\frac{\beta c}{\pi\lambda}}\sum_{l=0}^{\infty}\frac{1}{2^{2l}(2l)!}\exp(-E_{2l}x)\times\exp[-(\beta c/\lambda)q^2]H_{2l}[\sqrt{\beta c/\lambda}q]H_{2l}(0). \tag{A4}$$

With the spectrum $E_k(\lambda)$ and the normalization $H_{2l}(0)=(-1)^l(2l)!/2^l l!$, we obtain the replica partition function for free boundary conditions [cf. Eq. (48)]

$$\begin{aligned}Z_r(n;L)&\equiv Z(s)\\&=\int_{-\infty}^{+\infty}dq\Psi(q;x=L)\\&=\frac{\exp(-\sqrt{s}/2)}{\sqrt{\pi}}\sum_{l=0}^{\infty}\frac{(-1)^l}{2^l l!}\exp(-2\sqrt{s}l)C_{2l},\end{aligned} \tag{A5}$$

where $s=L^2/\lambda^2$ and

$$C_k=\int_{-\infty}^{+\infty}dx\exp(-x^2/2)H_k(x). \tag{A6}$$

Using the recurrence relation $H_{k+1}(x)=2xH_k(x)-2kH_{k-1}(x)$, we find that $C_{k+2}=2(k+1)C_k$ and with $C_0=\sqrt{2\pi}$, we obtain the coefficients $C_k=\sqrt{2\pi}(2l)!/l!$. Substitution into Eq. (A5) provides the replica partition function in the form

$$Z(s)=\sqrt{2}\exp(-\sqrt{s}/2)R[\eta(s)] \tag{A7}$$

with the function $R(\eta)$ defined by the series

$$R(\eta)=\sum_{l=0}^{\infty}\frac{(2l)!}{(l!)^2}\eta^l \tag{A8}$$

and we have introduced the shorthand $\eta(s)=-\exp(-2\sqrt{s})/4$.

In order to find the explicit form of the function $R(\eta)$, we implement the shift $l\to l+1$ in the sum (A8) and obtain,

$$\begin{aligned}R(\eta)&=1+\sum_{l=1}^{\infty}\frac{(2l)!}{(l!)^2}\eta^l\\&=1+\sum_{l=0}^{\infty}\frac{(2l+2)!}{(l!)^2}\eta^{l+1}\\&=1+4\eta\sum_{l=0}^{\infty}\frac{(2l)!}{(l!)^2}\eta^l-2\sum_{l=0}^{\infty}\frac{(2l)!}{(l+1)(l!)^2}\eta^{l+1}\\&=1+4\eta R(\eta)-2S(\eta),\end{aligned} \tag{A9}$$

$$S(\eta)=\sum_{l=0}^{\infty}\frac{(2l)!}{(l+1)(l!)^2}\eta^{l+1}. \tag{A10}$$

With R the derivative of S, $R(\eta)=\partial_\eta S(\eta)$, we obtain the differential equation $\partial_\eta S(\eta)=1+4\eta\partial_\eta S(\eta)-2S(\eta)$ and the initial condition $S(0)=0$ determines the solution $S(\eta)=(1-\sqrt{1-4\eta})/2$, from which $R(\eta)=1/\sqrt{1-4\eta}$ follows via integration. Substitution into Eq. (A7) produces the final result $Z(s)=1/\sqrt{\cosh\sqrt{s}}$, in agreement with Eq. (58).

APPENDIX B: INVERSE LAPLACE TRANSFORMATIONS

The inverse Laplace transforms Eqs. (64) and (70) are reduced to the following expressions: Using the transformation $s=\rho\exp(\pm i\pi/2)$ in Eq. (64), we analytically continue the expression for the distribution function $p_\epsilon(f)$ to the imaginary axis,

$$p_\epsilon(f)=\frac{1}{\pi}\mathrm{Re}\int_0^{\infty}d\rho\left(\frac{\sqrt{\rho}\exp(i\pi/4)}{\sin[(1+i)\sqrt{\rho/2}]}\right)^{1/2}\times\exp\left[\epsilon\frac{(1-i)\rho\sqrt{\rho/2}}{\tan[(1+i)\sqrt{\rho/2}]}+if\rho\right]. \tag{B1}$$

A change in the integration variable $\rho=2t^2$ provides, after some algebra, the final expression

$$p_\epsilon(f)=\frac{2^{5/2}}{\pi}\int_0^{\infty}dtt^{3/2}\exp[-\epsilon\omega_-(t)]\times\frac{\cos[\gamma(t)/2+2t^2f+\pi/8-\epsilon\omega_+(t)]}{\sqrt{\Phi(t)}}. \tag{B2}$$

The functions $\Phi(t)$, $\omega_\pm(t)$, and $\gamma(t)$ are defined as,

$$\Phi(t)=\sqrt{[\sin(t)\cosh(t)]^2+[\cos(t)\sinh(t)]^2},$$

$$\omega_\pm(t)=t^3[\sinh(2t)\pm\sin(2t)]/\Phi^2(t),$$

$$\sin(\gamma(t))=-\cos(t)\sinh(t)/\Phi(t),$$

$$\cos(\gamma(t))=\sin(t)\cosh(t)/\Phi(t).$$

Similarly, substituting $s=2t^2\exp(\pm i\pi/2)$ in Eq. (70), one obtains

$$p(f) = \frac{4}{\pi}\int_0^\infty dtt\frac{\cos[\zeta(t)/2 + 2ft^2]}{\sqrt{\Psi(t)}} \tag{B3}$$

with the functions $\Psi(t)$ and $\zeta(t)$ defined by

$$\Psi(t) = \sqrt{[\cos(t)\cosh(t)]^2 + [\sin(t)\sinh(t)]^2},$$

$$\sin(\zeta(t)) = \sin(t)\sinh(t)/\Psi(t),$$

$$\cos(\zeta(t)) = \cos(t)\cosh(t)/\Psi(t).$$

The remaining integrals in Eqs. (B2) and (B3) have to be done numerically.

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