# Theory of a supernarrow roton absorption line in the spectrum of a disk-shaped microwave resonator

Vadim M. Loktev* and Maksim D. Tomchenko†

*Bogoliubov Institute for Theoretical Physics, 14-b Metrologichna Street, Kyiv 03680, Ukraine*

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We calculate the probability of the creation of a circular phonon (c-phonon) in He II by a c-photon of the resonator. It is shown that this probability has sharp maxima at frequencies, where the effective group velocity of the c-phonon is equal to zero. For He II, these frequencies correspond to a roton and a maxon. From the probability of the c-roton birth, we calculate the roton linewidth which is found to approximately agree with the experimental one.

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Recently, an unusual effect was discovered.[1,2] In a dielectric disk resonator placed in He II, the azimuth ($l$-) modes were excited. They are of the "whispering-gallery" type and represent a running resonance electromagnetic (EM) waves of the superhigh-frequency (SHF) range in a narrow frequency interval. At the roton frequency $\nu_{rot}=\Delta_{rot}/2\pi\hbar=180.3$ GHz, the supernarrow absorption line with the width $\Delta\nu\simeq 50$ kHz was observed in the spectrum of the resonator. This width is by six orders of magnitude less than that of a roton peak measured in neutron experiments and is comparable with the width of the Mössbauer line. It was assumed in Ref. 2 that the line is related to the Van Hove singularity. In this case, however, one is faced with the problem to satisfy the momentum conservation law since the roton momentum is greater than the momentum of a photon with the same energy by six orders of magnitude. It was supposed in Ref. 2 that the excess momentum of a roton is transferred to helium as a whole.

To clarify this and other points, it is necessary to calculate the probability of the creation of a roton and the widths and the forms of lines for various possible processes, and then to choose a process explaining the experiment. One should take into account that the EM field of a disk-shaped resonator has the circular (c-) symmetry and is concentrated only near the disk according to measurements[1] and the theory.[3] Since the EM field induces the transition, the latter must be characterized by the c-symmetry. Therefore, we assume that the narrow line corresponds to the creation of a c-roton by the EM field of the resonator. Below, the probability of this process and the width of the corresponding absorption line will be found.

To calculate the linewidth, it is necessary to know the probability of the c-photon→c-phonon transition. No problems concerning the conservation laws appear for this process because both c-photon and c-phonon have no momentum, but have angular momenta $L_z=\hbar l$ and $L_z=\hbar l_c$, respectively.[3] Moreover, the condition $l=l_c$ is easily satisfied. The resonator field contains $\sim 10^{12}$ photons with close frequencies. At such occupation number, the photon field can be considered as an external classical perturbation with frequency $\nu$. The width of the azimuth mode, on which the roton line is observed, is about 2.5 MHz, the linewidth $\sim$0.1 MHz, and the frequency $\nu=180.3$ GHz. So, the mode width is very small as compared with the frequency and the latter can be considered constant.

The probability (per unit time) of the creation of a c-phonon in He II due to the action of the EM field is[4]

$$\delta w_{fi}=\frac{2\pi}{\hbar}|F_{fi}|^2\delta(E_f-E_i^{(0)}-\hbar\omega), \tag{1}$$

$$F_{fi}=\int \Psi_f\hat{F}\Psi_i d\Omega^{nuc}d\Omega^{el}, \tag{2}$$

where $d\Omega^{nuc}=d\mathbf{R}_1\ldots d\mathbf{R}_N$ and $d\Omega^{el}=d\mathbf{R}_1^{(1)}d\mathbf{R}_1^{(2)}\ldots d\mathbf{R}_N^{(1)}d\mathbf{R}_N^{(2)}$ are the phase volumes of all nuclei and all electrons, $\Psi_i$ and $\Psi_f$ are the wave functions (WFs) of the initial and final states of helium, respectively, $E_f-E_i^{(0)}=E_c$ is the energy of a c-phonon, and $\omega=2\pi\nu$. The explicit formula for $\hat{F}$ follows from the perturbation operator

$$\hat{V}=\hat{F}e^{-i\omega t}+\hat{F}^{+}e^{i\omega t}. \tag{3}$$

If the wavelength λ of the EM field is much more than the system size, then the problem is solved in the dipole approximation.[5] In our case, this approximation is not suitable, since the system size exceeds λ by one order of magnitude. In addition, a photon is spent on the excitation of liquid helium as a whole, i.e., on the creation of a c-phonon which is related to the motion of atoms as united objects, rather than on the excitation of electron shells of a single atom or many atoms. Therefore, we will use a general approach, by considering the action of the EM field directly on the charged particles in an atom, i.e., on electrons and the nucleus. Because the atoms interact with one another, the EM field creates the collective excitation, a phonon which is electrically neutral as a whole. For a charge particle in the EM field, we have[4]

$$\hat{V}=-\frac{q}{mc}\left(\mathbf{A}\hat{\mathbf{p}}-\frac{i\hbar}{2}\text{div }\mathbf{A}\right)+\frac{q^2\mathbf{A}^2}{2mc^2}, \tag{4}$$

where $\hat{\mathbf{p}}=-i\hbar\nabla_{\mathbf{r}}$, and $q$ and $\mathbf{r}$ are the charge and the radius vector of a particle, respectively. The term with $\mathbf{A}^2$ induces two-photon transitions which will not be considered here. The field $\mathbf{A}$ should be real and can be presented in the form

 

$$\mathbf{A}(\mathbf{r},t)=\mathbf{A}_0(\mathbf{r})e^{-i\omega t}+\mathbf{A}_0^*(\mathbf{r})e^{i\omega t}+\tilde{\mathbf{A}}(\mathbf{r}). \tag{5}$$

In this case, div $\mathbf{A}$=div $\tilde{\mathbf{A}}(\mathbf{r})$ (see Ref. 3), and we can set[6] $\tilde{\mathbf{A}}(\mathbf{r})=0$, so that div $\mathbf{A}=0$. Thus,

$$\hat{F}=\frac{i\hbar q}{mc}\mathbf{A}_0\nabla_{\mathbf{r}}. \tag{6}$$

Since a $^4$He atom consists of the nucleus with the charge $q_n=-2e$ and two electrons, each possessing the charge $q_e=e$, we have

$$\hat{F}=\sum_j\frac{i\hbar}{c}\left\{\frac{q_n}{m_n}\mathbf{A}_0(\mathbf{R}_j)\frac{\partial}{\partial\mathbf{R}_j}+\frac{q_e}{m_e}\mathbf{A}_0[\mathbf{R}_j^{(1)}]\frac{\partial}{\partial\mathbf{R}_j^{(1)}}+\frac{q_e}{m_e}\mathbf{A}_0[\mathbf{R}_j^{(2)}]\frac{\partial}{\partial\mathbf{R}_j^{(2)}}\right\} \tag{7}$$

for He II. Here, $\mathbf{R}_j$ are coordinates of the nucleus of the $j$th atom, $\mathbf{R}_j^{(1)}$ and $\mathbf{R}_j^{(2)}$ are coordinates of the electrons of this atom, $m_n\approx m_4$ is the $^4$He nucleus mass, and $m_e$ is the electron mass.

To calculate $F_{fi}$, one needs to know the WFs $\Psi_f$ and $\Psi_i$. Let the initial state characterized by $\Psi_i$ be the ground state of He II, and let the final state with $\Psi_f$ be the ground state plus one c-phonon. Then

$$\Psi_i\equiv\Psi_0(\{\mathbf{R}_j,\mathbf{R}_j^{(1,2)}\})=\Psi_0^{nuc}(\{\mathbf{R}_j\})\Psi_0^{el}(\{\mathbf{R}_j^{(1,2)}\}), \tag{8}$$

where $\Psi_0^{el}$ is the WF of all electrons, and $\Psi_0^{nuc}$ is the WF of all nuclei of helium atoms. The modern microscopic models describe the properties of He II well enough. According to them, a $^4$He atom is a very elastic object. Therefore, we can consider that the electron shell of each atom follows the nucleus without deformation caused by the interaction with neighboring atoms. In this case, $\Psi_0^{nuc}(\mathbf{R}_j)$ coincides with $\Psi_0(\mathbf{R}_j)$ written as a function of the coordinates of atoms, and the electron part looks as

$$\Psi_0^{el}=\prod_{j=1}^{N}\psi_j(\mathbf{R}_j^{(1)},\mathbf{R}_j^{(2)})\approx\prod_{j=1}^{N}\tilde{\psi}_{1s}(\mathbf{r}_j^{(1)})\tilde{\psi}_{1s}(\mathbf{r}_j^{(2)}), \tag{9}$$

where $\mathbf{r}_j^{(1)}=\mathbf{R}_j^{(1)}-\mathbf{R}_j$, $\mathbf{r}_j^{(2)}=\mathbf{R}_j^{(2)}-\mathbf{R}_j$. For the WF of the ground state of a $^4$He atom, we use the simple approximation with $\tilde{\psi}_{1s}(\mathbf{r})=\pi^{-1/2}a^{-3/2}e^{-r/a}$, $a=0.313$ Å.

The WF of the state with a single c-phonon is[3]

$$\Psi_f=\Psi_0\Psi_c(l_c,k_z,k_\rho), \tag{10}$$

$$\Psi_c\approx\sum_{j=1}^{N}\frac{c_{l_c,k_z,k_\rho}}{\sqrt{N}}e^{il_c\varphi_j+ik_zZ_j}J_{l_c}(k_\rho\rho_j), \tag{11}$$

where $\rho_j$, $\varphi_j$, and $Z_j$ are the coordinates of nuclei, $J_l(x)$ is the Bessel function. The probability of the creation of a c-phonon with "momentum" $k$ is

$$w_{fi}=\sum_{l_c,n_z,n_\rho}\delta w_{fi}=\frac{2\pi}{\hbar}\sum_{l_c,n_z,n_\rho}|F_{fi}|^2\delta(E_c-\hbar\omega). \tag{12}$$

In Eq. (12), the quantities $k_z$ and $k_\rho$ satisfy the relation $k_\rho^2+k_z^2=k^2$ and the quantization conditions[3]

$$k_z=\frac{2\pi n_z}{H-h_d},\quad n_z=\pm1,\ldots;k_\rho=\frac{\pi n_\rho}{R_\infty-R_d},\quad n_\rho\geqslant l, \tag{13}$$

where $h_d$=0.1 cm and $R_d$=0.95 cm are, respectively, the height and the radius of the disk resonator, and $H\approx4.2$ cm and $R_\infty\approx2.1$ cm are the same for the chamber with helium[7] (somewhat more exact relations are presented in Ref. 3).

Using a solution[3] for the EM field of a resonator and performing some awkward calculations,[6] we obtain

$$\begin{aligned}w_{fi}&\approx w_{fi}(k_z\to0)+w_{fi}(k_\rho\to0)\\&\approx\frac{2\pi^2(1-h_d/H)n\hbar(eA_m)^2|S(k)-f|}{c^2k^2m_e^2\,\partial E/\partial k}\\&\quad\times\left\{\frac{8.6h_d}{\pi R_d}[(l-1)J_{l-1}(Q_1R_d)]^2+\frac{2a(k,l)}{1-f}f\right\}.\end{aligned} \tag{14}$$

Here, $c$ is the light velocity in vacuum, $A_m$ is the amplitude of the EM field, $f=R_d/R_\infty$, $Q_1R_d\approx72.1$, and $S(k)$ is the structure factor.

We note that the main contribution to sum (12) is given by the regions $(k_z\to0)$ and $(k_\rho\to0)$. So, the quantity $w_{fi}$ is determined by the creation of c-phonons with the smallest $k_z$ and with the smallest $k_\rho$ (almost planar c-phonons). For the roton line ($l\approx66$, $k=k_{rot}=1.93$ Å$^{-1}$) under conditions of the experiments,[1,2] we have[6] $a(k,l)\approx0.2$ (for $T\lesssim1.4$ K). The function $a(k,l)$ is almost independent of $k$. However, if $k$ are such that $S(k)\to f$, then $a(k,l)$ increases by several orders of magnitude, approximately as $|S(k)-f|^{-1}$ (but, for the probability $w_{fi}$, such a growth is cancelled by the factor $|S(k)-f|$ in Eq. (14) in front of the large parentheses). Using these numbers, we obtain that the probability of the creation of c-rotons with large $k_\rho$ is ~5 times more than that for the almost planar ones (with small $k_\rho$). Thus, a resonator creates c-rotons mainly of two "extreme" types: almost completely circular and almost planar.

The spectrum of the SHF emission of a disk resonator contains the very narrow absorption line at the roton frequency. Let the linewidth be the distance between the points located on both sides from the line center, for which the intensity of a signal is about 0.8 of the background one. Then the linewidth[1,2] is about 50 kHz at $T$=1.8 K and decreases, at lower $T$, to the resolving power of a spectroscope ($\simeq$30 kHz). At $T\lesssim1.6$ K, the width approaches a constant $\simeq$30 kHz. The minimum experimental temperature $T$=1.4 K but formula (14) is applicable at $T$=0. However, the width does not depend on $T$ at $T\lesssim1.6$ K, so we may assume that the width at $T$=0 is the same as that at $T$=1.4 K.

To estimate the probability $\tilde{w}_{fi}=w_{fi}/N_{phot}$ of the c-photon$\to$c-roton process, we consider that in a vicinity of the roton minimum $\partial E(k)/\partial k=\hbar^2|k-k_{rot}|/m_{rot}$(where $m_{rot}\approx0.165m_4$), the number of c-photons in a resonator is equal to[6] $N_{phot}\approx5.2\times10^{11}$ (for the roton mode), and the experimental width $\Delta\nu$=30 kHz. Then we obtain

$$\tilde{w}_{fi} \approx \frac{4\pi^2}{9.2}\frac{(1-h_d/H)ne^2\omega m_{rot}|S(k)-f|}{m_e^2k^2c^2h_d|k-k_{rot}|}\left(\frac{0.95\ \text{cm}}{R_d}\right)^2 \times\left\{\frac{4.3h_d}{\pi R_d}[(l-1)J_{l-1}(Q_1R_d)]^2+\frac{a(k,l)f}{1-f}\right\} \approx 3.36\times10^{-7}\omega_{rot}. \quad (15)$$

The c-photon→c-roton process weakens the flow of photons propagating from the resonator. In the experiment,[7] the pumping signal of the power $w^0_{pump}\simeq 10^{-3}$ W within the frequency band $\Delta\nu_{pump}\simeq 50$ kHz was used. In view of losses, the resonator received $w_{pump}\simeq 10^{-4}-10^{-5}$ W. The resonator amplifies the pumping signal but the energy losses increase simultaneously, until a stationary equilibrium state is established. In this state, the losses in the resonator due to the emission are compensated by the pumping,

$$w_{pump}=N_{phot}\hbar\omega/\tau_{ren}, \quad (16)$$

where $\tau_{ren}$ is the mean emission time of a c-photon by the resonator (it is also the period of renewal of EM modes).

When the frequency of the EM field approaches the roton one, then, according to Eq. (15), the probability of the emission of c-rotons by the resonator becomes large. For the frequency interval $\Delta\nu_0=\Delta\nu_{pump}/100\simeq 0.5$ kHz (which is much less than the line width and the pumping bandwidth but contains a macroscopic number of levels), the condition of equilibrium takes the form

$$0.01w_{pump}=0.01N_{phot}\hbar\omega/\tau_{ren}+N^0_{rot}\Delta_{rot}/\tau_{em}, \quad (17)$$

where $\tau_{em}=1/\tilde{w}_{fi}$ is a duration of the emission of a c-roton by a c-photon of the resonator, and $N^0_{rot}$ is the number of c-rotons emitted for a time interval $\tau_{em}$ by c-photons from the band $\Delta\nu_0$. In this case, the losses of the resonator are separated into the channels of emission of c-photons and c-rotons. Respectively, the flow of emitted c-photons decreases, which is manifested in the resonator spectrum as the absorption. As was mentioned above, the signal is equal to 80% of the background one on the edge of lines. Hence, 20% of the energy losses of the resonator are transformed into the energy of c-rotons whereas 80% are related to c-photons. Thus, each c-photon of the resonator is emitted as a c-photon for the time $\tau_{ren}$ with a probability of 0.8 or creates a c-roton with a probability of 0.2. So, $\tau_{em}=4\tau_{ren}$, and the condition for the line edge is

$$\tilde{w}_{fi}\equiv 1/\tau_{em}=1/(4\tau_{ren}). \quad (18)$$

Relation (16) yields $\tau_{ren}\simeq 6.25(10^{-7}-10^{-6})$ s, which corresponds to the experimental value $\tau_{ren}\simeq 10^{-6}$ s. Using the last value of $\tau_{ren}$ and Eq. (18), we obtain $\tilde{w}_{fi}\simeq 2.2\times10^{-7}\omega_{rot}$ for the line edge, which is lower only by a factor of 1.5 than the theoretical value of $\tilde{w}_{fi}$ [Eq. (15)].

The theoretical and experimental lines are shown in Fig. 1. At $\nu>\nu_{rot}$, the amplitude of the former is given by the formula $a\approx(1+\sqrt{4.4\ \text{kHz}(\nu-\nu_{rot})^{-1}})^{-1}$. In this case, the error of the theoretical line width is about one order of magnitude; it related to the approximate character of the solutions for the WF of a c-phonon and for the EM field of the resonator. As is seen

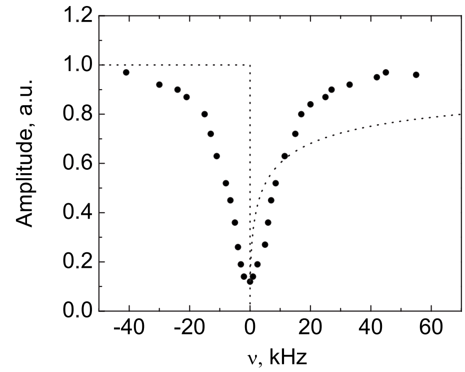


FIG. 1. Intensity of the signal received by the antenna vs the frequency near a narrow roton absorption line, $T=1.63$ K. ●●●—experiment (Ref. 1), dotted line—theory (by the present work). The frequency is equal to that indicated in the figure plus 175.7 GHz (this number is the roton energy at $T=1.63$ K).

from Fig. 1, the theoretical line corresponds approximately to the experimental one by width but does not by shape. The density of states $\varrho(E)=\int\delta(E(k)-\hbar\omega)d\mathbf{k}=4\pi k^2\,\partial k/\partial E$ is high in the region above the energy $\Delta_{rot}$ of a roton $[\varrho(E)=4\pi k^2m_{rot}(\hbar^2|k-k_{rot}|)^{-1}\rightarrow\infty$ at $k\rightarrow k_{rot}]$. But, below $\Delta_{rot}$, no roton states are present, and the density of states decreases sharply to a value corresponding to the linear phonon curve: $\varrho(E)=4\pi k^2/(u_1\hbar)$. Therefore, for the c-photon→c-roton transition, the theoretical line sharply falls on the side of lower frequencies whereas the experimental line is almost symmetric. This difference means that the process is more complicated and involves at least two particles, rather than one particle (c-photon→c-roton), as was accepted above. In particular, one more c-phonon can be created, or a part of the angular momentum of a c-photon can be absorbed by the disk. We hope to clarify the mechanism in the future investigations.

Note that the number of photons approaching the receiver is diminished due to the birth of c-rotons by c-photons of the resonator and by c-photons emitted by the resonator. However, the line is formed due to the first process.[6] Morover, the signal of the antenna is caused by photons emitted by the resonator, rather than by its stationary EM field.

As was mentioned above, probability (15) of the c-photon→c-roton process grows strongly as $\partial E(k)/\partial k\rightarrow 0$, i.e., near the extremum points of the dispersion curve $E(k)$. This peculiarity explains why the narrow line is observed just at the roton frequency and predicts one more line at the frequency of a maxon $\nu_{\max}\approx 290$ GHz with width of the same order of magnitude.

The peculiarity at $\partial E(k)/\partial k\rightarrow 0$ is well known in solid-state physics as the Van Hove singularity. At $\partial E(k)/\partial k\rightarrow 0$, the states of c-rotons falling in a narrow energy interval are strongly concentrated. Respectively, the transition probability in this energy interval sharply increases. In crystals, the narrow lines were registered a lot of times in light absorption[8] and neutron-scattering[9] experiments. However, in crystals the linewidths are larger by several orders of magnitude than those of SHF lines in helium and corresponds to the observed in helium[2] "pedestal" (base). Thus, the narrow SHF line in helium is related to the Van Hove singularity, like the lines of crystals but its widening is caused by a different mechanism.

In the neutron experiments with liquid helium, the analogous very narrow peaks must be observed on the scattering curve $S(k=\text{const},\omega)$ at the frequencies of the roton and maxon extrema. However, the high errors of neutron measurements ($\delta\omega \approx 0.1$ K) do not allow one to register these peaks.

In work,[2] the explanation of the appearance of a narrow roton line is based on the assumption that the main process is the plane (p-) photon → p-roton transition plus the transfer of a momentum to helium as a whole. Such a consideration is equivalent to our one, if the EM field is assumed classical and is expanded in plane waves outside the resonator. However, the real EM field is not classical and is a collection of quanta. A roton is created as a result of the disappearance of one of these quanta *as a whole*. Due to the significantly different dielectric permittivities of the disk and helium, a quantum of the EM field cannot be represented[3] as a superposition of planar photons and is characterized by the circular symmetry. Therefore, the approach based on c-photons is more correct, in our opinion.

In work,[6] the model is presented in more details. There, we also consider the possibility for a p-roton to have a dipole moment, the quantization of the amplitude and the Stark effect for the roton line.

With regard for the zero boundary conditions for the WF of He II, it becomes obvious that the real energy spectrum of He II is not a Landau continuous curve but it is a collection of separate disconnected points very densely lying on this curve. According to Eq. (13), the observed roton line[1,2] consists of $\sim 10^5$ individual roton lines. If the experiment would be executed with a film of helium $\sim 100$ Å in thickness, the distances between lines increases by six orders of magnitude, and they would become resolvable. In this case, instead of a single roton line, we would measure many lines in a wide range of the frequencies. Thus, it would be possible to observe the line spectrum of a fluid consisting of a huge number of discrete lines, like the spectrum of an atom. It is only necessary that the intensities of lines be sufficiently high. However, if the resonator disk is only covered by a helium film, no lines will be observed. Indeed, by Eq. (14), the intensities of lines will be of the same order of magnitude as those for a thick layer of helium.[1,2] But the line registered for the thick layer consists of $\sim 10^5$ individual lines which are too weak to be observed separately. To resolve them, one needs to increase their intensity by 4–5 orders of magnitudes. This is a task for future studies.

In conclusion, it is seen from the above-presented analysis that the experiment[1,2] has revealed the existence of particular excitations in He II: circular rotons which are azimuth sound waves. In the present work, we have approximately calculated the probability of the creation of a circular roton by the EM field of the resonator and, on its basis, have evaluated the width of the absorption line at the roton frequency. The theoretical line is close to the experimental one by width but differs by shape.

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*vloktev@bitp.kiev.ua
†mtomchenko@bitp.kiev.ua

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