Critical interface: Twisting spin glasses at T_c

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We consider two identical copies of a finite dimensional spin glass coupled at their boundaries. This allows to identify the analog for a spin glass of twisted boundary conditions in ferromagnetic system and it leads to a definition of an interface free energy that should scale with a positive power of the system size in the spin-glass phase. In this paper we study within mean-field theory the behavior of this interface at the spin-glass critical temperature T_c. We show that the leading scaling of the interface free energy may be obtained by simple scaling arguments using a cubic field theory of critical spin glasses and neglecting the replica symmetry-breaking dependence.

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I. INTRODUCTION

Sensitivity to boundary conditions (bc's) is a fundamental tool to study the nature of the Gibbs states of extended physical systems. If ergodicity is broken and Gibbs states are not unique, different boundary conditions can select pure phases or induce interfaces in the system. For example, in Ising-type ferromagnets in the ferromagnetic phase, the choice of homogeneous "up" (respectively "down") boundary conditions, where all the spins outside a large region are fixed to point in the up (respectively, down) direction, is enough to select the pure state with positive (respectively, negative) magnetization. Twisted boundary conditions, where along a specific direction + and − conditions are chosen at the opposite boundaries (while neutral boundary condition, e.g., periodic ones, are chosen in the other directions) induce an interface with positive tension: the free-energy cost to impose twisted boundary conditions remains finite in the thermodynamic limit. On the contrary, in the paramagnetic phase the system is insensitive to boundaries and the surface tension is equal to zero. Using boundary conditions one can therefore investigate the stability of the low-temperature phases and the lower critical dimension below which the ferromagnetic phase is unstable. Around the critical temperature, the behavior of the interface tension is described by finite-size scaling, which implies that right at the critical temperature, the interface tension vanishes. The behavior of the interface free energy at the critical temperature is well understood in critical fluctuation theory. Below dimension 4 the interfacial free-energy scales with the system size as $\sigma \sim L^{-(D-1)}$ whereas it scales as $\sigma \sim L^{-3}$ above dimension 4.[1]

In trying to extend this kind of considerations to study the stability of spin-glass phases, one is confronted with the fact that pure phases are in this case glassy random states, strongly correlated with the quenched disorder. Therefore they cannot be selected through boundary conditions uncorrelated with the quenched disorder. The projection unto some pure phase boundary conditions should depend self-consistently on the quenched disorder. A possible way to consider disorder-correlated boundary conditions at low temperature has been introduced in Ref. 2, where, in reference to the Parisi ansatz, two different gauges of the ultrametric matrices were considered at the boundary.

A different procedure is to consider two "clones" of a spin glass[3] with identical disorder coupled at the boundaries. In Ref. 4 a situation was considered where different values of the overlap (a measure of the correlations between configurations) were imposed on the boundaries along one direction. There the stability of a low-temperature phase, with broken replica symmetry (RSB), was investigated against spatial fluctuations of the overlap that would restore replica symmetry; this led to a value of the lower critical dimension, below which the fluctuations destroy RSB, equal to $D_{LCD}=5/2$. In Ref. 7 it was showed that it may be interesting to consider the extreme case where the overlap is 1 at one boundary and −1 (or 1) at the other boundary. These extreme choices simplify the analysis and it can be implemented in numerical simulations in a very simple way.

In the present paper we consider such a situation at the critical temperature itself, where the techniques developed in Ref. 4 do not apply. We then compare two types of boundary conditions. We consider two identical clones of cylindrical spin glasses (i.e., we impose pbc in the transverse directions) and we couple them on the boundaries. In a first type of bc the configurations of the clones are chosen to be identical on both sides of the cylinder. In a second type of bc, the configurations of the two systems are still fixed to be identical on one of the boundaries but they are opposite in the other boundary. We study the problem above the upper critical dimension (which is $D_u=6$ for spin glasses[5]) where we can neglect loop contributions to the interface free energy. We find that the interface free-energy scales as $\sigma \sim L^{-5}$ above dimension 6 and it is natural to conjecture that it scales as $\sigma \sim L^{-(D-1)}$ below that dimension, in analogy with pure systems where it follows from the universality of the free energy in a correlation volume plus finite-size scaling.

The plan of the paper is the following. In Sec. II we discuss in detail the Ising ordered case. This is well understood and it will guide us in the analysis of the more complex spin-glass case. Section III, which constitutes the core of this paper, is devoted to the analysis of the Edwards-Anderson model at T_c. Finally we draw some conclusions.

II. PURE SYSTEM

We consider an Ising-type system in a cylindrical geometry. The D-dimensional cylinder has a length L and a cross-section area $A \sim L^{D-1}$. The boundary conditions are periodic in the directions transverse to the axis of the cylinder, but on the surfaces at the end of the longitudinal direction the spins may be all up or all down. Below T_c there is an interfacial tension σ defined as the difference of free energy per unit area

$$\sigma = \frac{F_{\uparrow,\downarrow} - F_{\uparrow,\uparrow}}{A}. \quad (1)$$

It is a function of the temperature $t \sim (T - T_c)$ and of the aspect ratio $L/A^{1/(D-1)}$. This tension vanishes near T_c as

$$\sigma \sim (-t)^{\mu}. \quad (2)$$

A scaling law due to Widom[1] relates μ to the correlation length exponent ν

$$\mu = \nu(D-1) \quad (3)$$

and it may be derived in the standard renormalization-group framework[6] (for $D \leq 4$).

Assume that we want to know the behavior of this tension for finite L at T_c. We use finite-size scaling

$$\sigma(t,L) = (-t)^{\mu} f(L/\xi) \quad (4)$$

and, since σ is nonsingular at T_c for L finite, it implies that

$$f(x) \sim_{x\to 0} x^{-\mu/\nu}. \quad (5)$$

Therefore at T_c the interfacial tension vanishes with $1/L$ as

$$\sigma(0,L) \sim L^{-\mu/\nu} = L^{-(D-1)}. \quad (6)$$

In other words at T_c

$$F_{\uparrow,\downarrow} - F_{\uparrow,\uparrow} \sim L^0. \quad (7)$$

In Appendix A we give the details of the verification of this behavior in Eq. (7) in a simple Landau theory (i.e., tree level of a φ^4 theory): i.e., for a free energy at T_c

$$F = A \int_0^L dz \left[\frac{1}{2} \left(\frac{d\varphi}{dz} \right)^2 + \frac{g}{4} \varphi^4 \right] \quad (8)$$

[we have assumed translation invariance in the $(D-1)$ transverse directions]. Up-up bc are defined be the fact that at both boundaries the order parameter is fixed to the value m, up-down conditions correspond to impose the value m on the left boundary and $-m$ on the right one. One finds that independently of the values of m, the surface tension behaves as

$$\frac{F_{\uparrow,\downarrow} - F_{\uparrow,\uparrow}}{A} = \frac{C}{gL^3} \quad (9)$$

with a positive constant C given by

$$C = \frac{5}{192\pi^2} \Gamma^8\left(\frac{1}{4}\right) - \frac{1}{12\pi^{3/2}} \Gamma^6\left(\frac{1}{4}\right) \simeq 44.8. \quad (10)$$

Clearly this mean-field theory should only be compared with Eq. (7) in four dimensions, the result in Eq. (9) is indeed in agreement since $A \sim L^3$.

III. SPIN GLASS

In a spin-glass twisting fixed boundary conditions, uncorrelated with the disordered couplings, do not produce any interface since it can be gauged away in a change in the random couplings. For a given realization of the random disorder, it may increase or decrease the free energy.

In this paper, we consider two identical copies of a cylindrical sample of a spin glass with identical couplings J_{ij} and we study two sets of boundary conditions as follows. (1) In the up-up situation the overlap between the spins of the two copies in the left ($x=0$) plane and in the $x=L$ plane are equal to a positive value l_0. (2) In the up-down situation the overlap is still equal l_0 in the left plane but it is $-l_0$ in the right plane.

In Ref. 7 Contucci *et al.* have recently considered boundary conditions of this kind in the extreme case $l_0=1$. Notice that in this case, the construction leads to effective interactions within the spins in the planes $x=0$ and $x=L$ which are doubled. Choosing the interactions in those planes with the same statistics as for the other planes would lead to effective interactions between the spins in these planes stronger than the other interactions. In order to avoid such a strong difference, the strength of the interaction among spins on the boundary planes was reduced by half in Ref. 7.

Our aim here is to characterize the behavior of the difference $\sigma = \frac{F_{\uparrow,\downarrow} - F_{\uparrow,\uparrow}}{A}$ at T_c as a function of L. Again we shall limit ourselves to mean-field theory. Our purpose is to test whether Widom's scaling law holds in this glassy system by verifying that above the upper critical dimension $D_{UCD}=6$ the interface tension scales as $\sigma \sim L^{-5}$.

We would like to study this problem within a Landau-type free energy close to T_c expanded in powers of an order parameter. Unfortunately, this cannot be done directly if the value l_0 of the order parameter at the boundary is chosen to be large. In particular, we could not directly consider the boundary conditions chosen in Ref. 7. However, the small order-parameter expansion can be saved even in this case, arguing that, analogously to the ferromagnetic case, the result for σ should not depend on the value $|l_0|$ imposed at the boundaries. We will see that this is true within the Landau model. We have also confirmed this in a spherical spin-glass model (see Appendix B) which has a Landau expansion identical to the one of the standard Ising spin glass[8] and for which one can write equations which are valid even for large values of the order parameter.

We start from a short-range standard Edwards-Anderson model.[9] After tracing out over the spins we can rewrite it, in the long-wavelength limit, in terms of a $2n \times 2n$ matrix $Q_{a,b}$ (for the bulk problem we would need only a $n \times n$ matrix Q); when a and b are both smaller than n, Q_{ab} refers to the overlap between the spins in the first copy; when the two

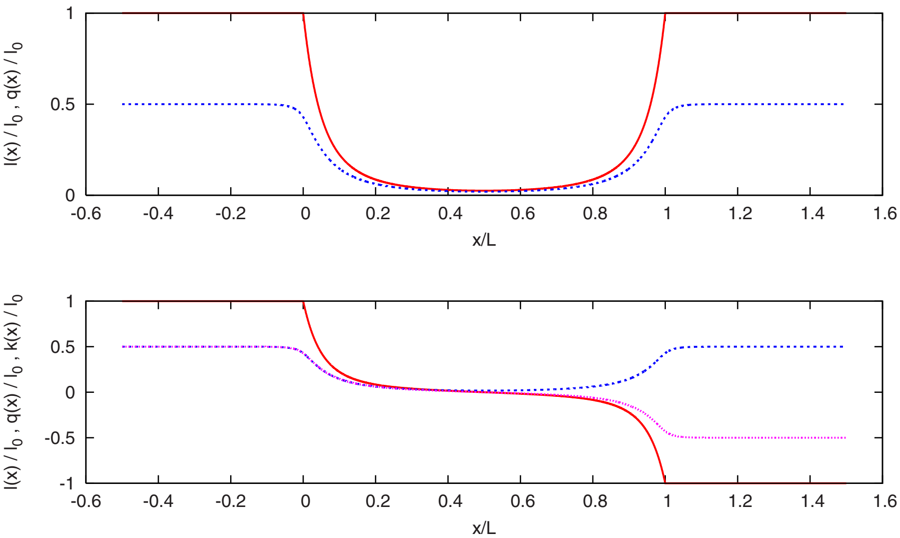

FIG. 1. (Color online) The order parameters in the two solutions for the critical Edwards-Anderson model. Upper panel: the functions $l(x)/l_0$ (solid curve) and $q(x)/l_0 = k(x)/l_0$ (dashed curve) as a function of x/L for $v=1000$ and $r=1/2$ for up-up boundary conditions. Lower panel: the functions $l(x)/l_0$ (solid curve), $q(x)/l_0$ (dashed curve), and $k(x)/l_0$ (dotted curve) for up-down boundary conditions and the same values of the parameters.

indexes are larger than n we are dealing with the overlaps $Q_{\alpha,\beta}$ in the second copy and for one index lower than n and one larger, $Q_{a\alpha}$ is a cross overlap. The critical mean-field free energy is here

$$F = \int d^D x \left[\frac{1}{2} \mathrm{tr}(\nabla Q)^2 + \frac{w}{3} \mathrm{tr}\, Q^3 + \frac{u}{4} \sum_{ab} Q_{ab}^4 \right]. \tag{11}$$

Assuming again translation invariance along the transverse directions $Q(\vec{x}) = Q(z)$, we assume that at T_c the replica symmetry is unbroken. Therefore we assume

$$Q_{ab} = Q_{\alpha\beta} = q(z) \tag{12}$$

and

$$Q_{a\alpha} = k(z)(1 - \delta_{a,\alpha}) + l(z)\delta_{a,\alpha}. \tag{13}$$

Let us note that the free energy Eq. (11) is invariant under $S_n \times S_n$, i.e., independent permutations of the n replicas of system one and of the n replicas of system two. However the last term in Eq. (13) breaks this symmetry down to S_n. Indeed one may imagine the boundary conditions as follows: one adds a coupling $Js^1 s^2$ between the spins of the two systems located in the plane $z=L$. The boundary conditions $s^1 = s^2$ would correspond to J going to $+\infty$ whereas $s^1 = -s^2$ is enforced by the limit $J \to -\infty$. After replicating the coupling becomes $J\Sigma s_a^1 s_a^2$ and thus the boundary conditions lead to a $\delta_{a,\alpha}$ term in Eq. (13).[10]

Then one has

$$\mathrm{tr}\left(\frac{dQ}{dz}\right)^2 = 2n(n-1)q'^2 + 2n(n-1)k'^2 + 2nl'^2, \tag{14}$$

$$\mathrm{tr}\, Q^3 = 2n(n-1)(n-2)q^3 + 6n(n-1)(n-2)qk^2 + 12n(n-1)qkl, \tag{15}$$

$$\sum Q_{ab}^4 = n(n-1)q^4 + n(n-1)k^4 + nl^4 \tag{16}$$

and thus, in the zero replica limit, the free energy reads

$$\frac{F}{n} = A \int_0^L dz \left[-q'^2 - k'^2 + l'^2 + \frac{w}{3}(4q^3 + 12qk^2 - 12qkl) + \frac{u}{4}(-q^4 - k^4 + l^4) \right]. \tag{17}$$

The quartic term may be dropped at T_c since the functions $q(z)$, $k(z)$, and $l(z)$ remain small for large L. The equations of motion are thus simply

$$-2q'' + 4w(q^2 + k^2 - kl) = 0,$$

$$-2k'' + 4w(2qk - ql) = 0,$$

$$-2l'' + 4wqk = 0. \tag{18}$$

The up-up and up-down boundary conditions are different in the two cases for the functions $l(z)$ and $k(z)$. In the up-up case, if one imposes the same boundary conditions for q and k, $q(z)=k(z)$ is a solution and we are left with only two equations. There is one constant of motion, namely, $4q'^2 - l'^2 + \frac{w}{3}(8q^3 - 3lq)$ but the last integration has to be done numerically. In the up-down, one has to deal with three different functions.

We impose the boundary conditions in the following way: we consider a sample of size $(1+2r)L$ with $-rL \le x \le (1+r)L$ and fix the value of $l(x)$ to preassigned values on the regions $-rL < x < 0$, that we call left boundary, and $L < x < (1+r)L$, that we call right boundary. In the case of up-up, the value $l_0 > 0$ is imposed both at the right and left boundaries, in the up-down case the value $l_0 > 0$ is chosen at the left boundary and $-l_0$ is chosen at the right boundary. The value of $q(x)$ and $k(x)$ on the boundaries are not fixed by the constraints and they are simply determined by extremization of the free energy. Since $r > 0$ and L is large, $q(0)$ and $k(0)$ and $q(L)$ and $k(L)$ should take the same values that that they would take if a uniform overlap profile $l(x) = \pm l_0$ for all x was chosen, namely, $q(0)=k(0)=q(L)=k(L)=l_0/2$ for up-up conditions and $q(0)=k(0)=q(L)=l_0/2$, $k(L)=-l_0/2$ for up-down conditions. In Fig. 1 is depicted the typical behavior of the various functions in space.

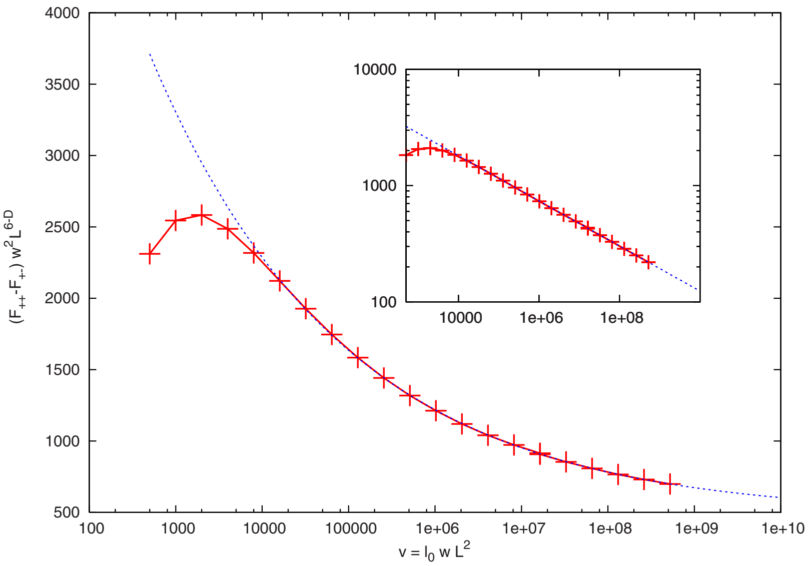

FIG. 2. (Color online) The rescaled free-energy difference, $f=(F_{\uparrow,\downarrow}-F_{\uparrow,\uparrow})w^2L^{6-D}$ as a function of $v=l_o wL^2$ together with a power-law fit of the form $f(v)=a+\frac{b}{v^c}$. Inset: log-log plot of $f(v)-a$ (crosses) together with the same power-law fit (dotted line) as in the main panel. Chi-square fitting for $v>10^4$ gives the parameters $a=480\pm 20$, $b=10790\pm 90$, and $c=0.19\pm 0.01$.

We expect that, as verified explicitly in the case of the ferromagnet, the interface free energy does not depend on the imposed value l_0. In that case, a scaling argument based on the fact that the dominant interaction terms in the free energy are the cubic ones (namely, the transformation $x\to xL$ $q\to w^{-1}L^{-2}q$ and analogously for k an l), suggests that the interface tension, in absence of accidental cancellations should behave as AL^{-5} for large L, i.e., for $A\sim L^{D-1}$, $\sigma\sim L^{D-6}$. Indeed, one can see by the above rescaling, we can write $F_{\uparrow,\uparrow}=\frac{A}{w^2L^5}g_{\uparrow,\uparrow}(l_0wL^2)$, $F_{\uparrow,\downarrow}=\frac{A}{w^2L^5}g_{\uparrow,\downarrow}(l_0wL^2)$, where $g_{\uparrow,\uparrow}$ and $g_{\uparrow,\downarrow}$ are appropriate scaling functions. This is of course true unless some accidental cancellation occurs. If this is the case, we are indeed authorized to neglect the quartic term. Since this is the term responsible for RSB, we can neglect RSB effects to this leading order.

In order to verify that $g_{\uparrow,\downarrow}-g_{\uparrow,\uparrow}$ remains finite for large L, we have integrated numerically the rescaled Eq. (18), for various values of the parameter $v=l_0wL^2$. This is done by simple a relaxation method, where the initial profiles are iterated until one observes the convergence to a solution of a discretized version of the Eq. (18). In Fig. 2 we have plotted the function $f(v)=g_{\uparrow,\downarrow}(v)-g_{\uparrow,\uparrow}(v)$ together with a power-law fit of the kind $f(v)=a+\frac{b}{v^c}$. While $f(v)$ continues to have an appreciable dependence on v (and thus on L) for very large values of v, the fit indicates that $f_\infty=\lim_{v\to\infty}f(v)$ is different from zero. The attempts to fit $f(v)$ with functions vanishing for large v yield poor results. In addition we have explicitly verified that the inclusion of the quartic terms in the free energy gives a subleading contribution to σ.

IV. CONCLUSIONS

In this paper we have considered the interface free energy obtained by the imposition of different boundary conditions on two identical copies of a spin glass. We have concentrated our study to the behavior of the free-energy difference with the size of the system at T_c. We have found that the behavior of the interface follows the usual pattern of critical phenomena. The free-energy difference above six dimensions is of order L^{D-6} or $\sigma\sim L^{-5}$ as suggested by simple scaling laws. This means that RSB effects may be neglected to leading order at the critical temperature. This is very different from what found at lower temperatures,[4] where the scaling of the interface was found to depend critically on the zero modes associated with RSB. We expect that below dimension 6 the free-energy difference remains finite (or $\sigma\sim L^{D-1}$), a prediction that can be tested in numerical simulations. Indeed since there does not seem to be any RSB at T_c, the usual arguments, namely, (a) the singular part of the free energy scales like a correlation volume, (b) finite-size scaling, should presumably apply.

APPENDIX A: PURE SYSTEMS

The free energy is

$$F=A\int_{-L/2}^{+L/2}dz\left[\frac{1}{2}\varphi'^2+\frac{u}{4}\varphi^4\right]. \tag{A1}$$

1. Up-up boundary conditions

The magnetization m in the planes $z=\pm L/2$ is given and we solve the equation of motion

$$-\varphi''+u\varphi^3=0 \tag{A2}$$

with the bc

$$\varphi(-L/2)=\varphi(+L/2)=m \tag{A3}$$

The mechanical analog is a negative energy bounce off the potential $-\frac{u}{4}\varphi^4$ starting at $\varphi=m$ bouncing at φ_0 and returning to $\varphi=m$. It is given by

$$\int_m^{\varphi(z)}\frac{d\psi}{\sqrt{\psi^4-\varphi_0^4}}=-\sqrt{\frac{u}{2}}(L/2+z). \tag{A4}$$

Then φ_0 is determined by

$$\int_m^{\varphi_0} \frac{d\psi}{\sqrt{\psi^4-\varphi_0^4}} = -\sqrt{\frac{u}{8}}L. \tag{A5}$$

For L large the order parameter φ_0 at the center of the sample is small, much smaller than the given finite value m on the boundaries, and it is thus given asymptotically by

$$L\varphi_0 = \sqrt{\frac{8}{u}}\int_1^{\infty} \frac{dt}{\sqrt{t^4-1}} + O(1/L). \tag{A6}$$

The free energy is then given by

$$\begin{aligned}\frac{F_{\uparrow\uparrow}}{A} &= \frac{u}{2}\int_{-L/2}^{+L/2} dz(\varphi^4-\varphi_0^4) + \frac{u}{4}L\varphi_0^4\\ &= -\sqrt{2u}\int_m^{\varphi_0} d\psi\sqrt{\psi^4-\varphi_0^4} + \frac{u}{4}L\varphi_0^4\\ &= \frac{1}{3}\sqrt{2u}m^3 - \sqrt{2u}\frac{\varphi_0^3}{3} + \frac{u}{4}L\varphi_0^4 + \sqrt{2u}\varphi_0^3\int_1^{\infty}\frac{dt}{\sqrt{t^4-1}+t^2}.\end{aligned} \tag{A7}$$

The corrections to the first leading term are of order $1/L^3$. We finally note that

$$\int_1^{\infty} \frac{dt}{\sqrt{t^4-1}} = \frac{\Gamma^2(1/4)}{\sqrt{32\pi}} \tag{A8}$$

and

$$\int_1^{\infty} \frac{dt}{\sqrt{t^4-1}+t^2} = \frac{\Gamma^2(1/4)}{\sqrt{72\pi}} - \frac{1}{3}. \tag{A9}$$

2. Up-down boundary conditions

We are still dealing with the mechanical analog of a motion in the inverted potential $-u\varphi^4/4$ but now with a positive-energy solution going from $\varphi(L/2)=m$ to $\varphi(-L/2)=-m$. The solution is given by

$$\int_m^{\varphi} \frac{d\psi}{\sqrt{\psi^4+\varphi_1^4}} = -\sqrt{\frac{u}{2}}\left(\frac{L}{2}-z\right) \tag{A10}$$

and φ_1 is determined by

$$\int_{-m}^{+m} \frac{d\psi}{\sqrt{\psi^4+\varphi_1^4}} = \sqrt{\frac{u}{2}}L. \tag{A11}$$

Again φ_1 is of order $1/L$ and given asymptotically by

$$L\varphi_1 = \sqrt{\frac{2}{u}}\int_{-\infty}^{+\infty} \frac{dt}{\sqrt{t^4+1}}. \tag{A12}$$

Then the free energy

$$\frac{F_{\uparrow,\downarrow}}{A} = -\frac{u}{4}L\varphi_1^4 + \sqrt{2u}\varphi_1^3\int_0^{+m/\varphi_1} dt\sqrt{t^4+1}. \tag{A13}$$

The last asymptotic estimate

$$\int_0^{+m/\varphi_1} dt\sqrt{t^4+1} = \frac{1}{3}\left(\frac{m}{\varphi_1}\right)^3 + \int_0^{\infty}\frac{dt}{\sqrt{t^4+1}+t^2} \tag{A14}$$

and the values of the integrals

$$\int_{-\infty}^{+\infty} \frac{dt}{\sqrt{t^4+1}} = \frac{\Gamma^2(1/4)}{\sqrt{16\pi}}, \tag{A15}$$

$$\int_0^{\infty} \frac{dt}{\sqrt{t^4+1}+t^2} = \frac{\Gamma^2(1/4)}{\sqrt{36\pi}} \tag{A16}$$

complete the calculation.

We may now compute $F_{\uparrow,\downarrow}-F_{\uparrow,\uparrow}$; the constant m^3 term cancels and we are left with a difference proportional to A/L^3 as expected

$$\frac{F_{\uparrow,\downarrow}-F_{\uparrow,\uparrow}}{A} = \frac{1}{uL^3}\left[\frac{5}{3\times 2^{10}\pi^2}\Gamma^8(1/4) - \frac{1}{3\times 4^3\pi\sqrt{2\pi}}\Gamma^6(1/4)\right]. \tag{A17}$$

APPENDIX B: A SPHERICAL SPIN-GLASS MODEL

As observed in the main text, the boundary conditions defined in Ref. 7, imply high overlaps on the boundaries, and they cannot be analyzed in the context of Landau small order-parameter expansions of the free energy. We define here a model with the following properties. (a) The replica free energy of the model admits a simple closed form in terms of the overlap matrix $Q_{a,b}(x)$, and it can be explicitly continued to the zero replica limit $n\to 0$, when the Parisi ansatz is assumed. (b) The Landau expansion close to T_c coincides, up to the fourth-order term, with that of the Edwards-Anderson model.

Consider a system where on each site, labeled by its longitudinal and transverse coordinates x and $\mathbf{y}$, respectively, there are K spins $\sigma^r_{x,\mathbf{y}}$, $r=1,\ldots,K$, subject to the spherical constraint $\Sigma_r(\sigma^r_{x,\mathbf{y}})^2=K$. We write the Hamiltonian of the model as a sum of terms that couple spins on the same plane and terms that couple spins in adjacent planes.

$$H = \sum_{x=0}^{L}\mathcal{H}_x^{ort} + \sum_{x=0}^{L-1}\mathcal{H}_x^{par}, \tag{B1}$$

where the Hamiltonians $\mathcal{H}_x^{ort}$ and $\mathcal{H}_x^{par}$ are Gaussian random variables with variances

$$\begin{aligned}\langle\mathcal{H}_x^{ort}[\sigma]\mathcal{H}_x^{ort}[\tau]\rangle &= \sum_{\mathbf{y}} f_x\left(\frac{1}{2^{D-1}}\sum_{\mathbf{z}\in\mathbf{V_y}} q^{x,x}_{\mathbf{y},\mathbf{z}}\right),\\ \langle\mathcal{H}_x^{par}[\sigma]\mathcal{H}_x^{par}[\tau]\rangle &= \sum_{\mathbf{y}} f(q^{x,x+1}_{\mathbf{y},\mathbf{y}}),\end{aligned} \tag{B2}$$

where we denoted by $\sigma^x_{\mathbf{y}}$ the value of the spin on site $\mathbf{y}$ of the xth plane, $q^{x,w}_{\mathbf{y},\mathbf{z}}=\frac{1}{K}\Sigma_{r=1}^K\sigma^r_{x,\mathbf{y}}\tau^r_{w,\mathbf{z}}$, the overlaps between spin configurations σ and τ on different sites. The different functions of the overlap are chose to be $f(q)=\frac{1}{2}(q^2+yq^4)$,

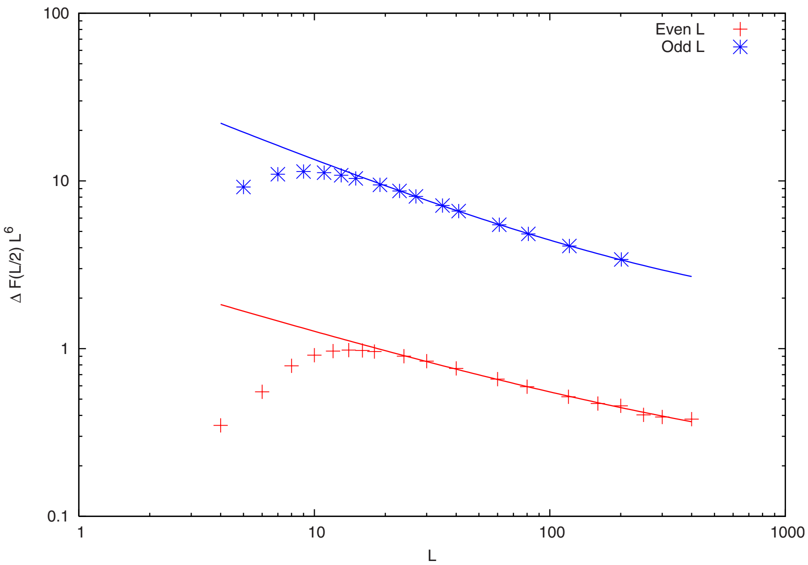

FIG. 3. (Color online) The rescaled difference of free-energy density in the center of the box for the two kinds of boundary conditions $\sigma(L/2)L^6$ as a function of the system size L. Notice that even and odd values of L give rise to different curves. The points are plotted together with a power-law fit of the form $f(L)=a+\frac{b}{L^c}$ (solid lines). Chi-square fitting for $L>20$ gives the parameters $a=0.15\pm0.04$, $b=3.1\pm0.3$, and $c=0.44\pm0.04$ for L even and $a=1.3\pm0.1$, $b=46.9\pm1.7$, and $c=0.58\pm0.04$ for L odd.

$f_x(q)=f(q)$ for $x\neq 0,L$ and $f(0)=\frac{1}{4}f(q)$ for $x=0,L$. Notice that with this choice, it is possible to express Hamiltonian (B1) in terms of two-body and four-body Gaussian couplings between the spins. We can now introduce two copies of the system with up-up and up-down boundary conditions. The up-up conditions consist in considering two copies $\sigma^r_{x,\mathbf{y}}$ and $\tau^r_{x,\mathbf{y}}$ constrained to be identical for $x=0,L$: $\sigma^r_{0,\mathbf{y}}=\tau^r_{0,\mathbf{y}}$ and $\sigma^r_{L,\mathbf{y}}=\tau^r_{L,\mathbf{y}}$. The up-down conditions consist of two copies $\sigma^r_{x,\mathbf{y}}$ and $\tau^r_{x,\mathbf{y}}$ constrained to be identical for $x=0$ but with opposite values for $x=L$: $\sigma^r_{0,\mathbf{y}}=\tau^r_{0,\mathbf{y}}$ and $\sigma^r_{L,\mathbf{y}}=-\tau^r_{L,\mathbf{y}}$. As for the reduced model, the replica treatment of the problem involves the introduction of two local $n\times n$ overlap matrices, which under the replica symmetric ansatz and the assumption of independence of the overlap profiles of the transverse spatial coordinate, can be parametrized in terms of the functions $q(x)$, $l(x)$, and $k(x)$, $x=0,\ldots,L$ of the main text. The resulting $F_{\uparrow,\uparrow}$ free energies and $F_{\uparrow,\downarrow}$ as a function of these parameters can be decomposed as

$$F_{\uparrow,\uparrow}[\{q,l,k\}]=F^{bulk}[\{q,l,k\}]+F_0^{boundary}+F_{\uparrow,\uparrow}^{boundary}, \tag{B3}$$

$$F_{\uparrow,\downarrow}[\{q,l,k\}]=F^{bulk}[\{q,l,k\}]+F_0^{boundary}+F_{\uparrow,\downarrow}^{boundary}, \tag{B4}$$

where

$$\begin{aligned}-\beta F^{bulk}=&\,\beta^2\sum_{x=1}^{L-1}\{f(1)+f[l(x)]-f[q(x)]-f[k(x)]\}+\beta^2\sum_{x=1}^{L-2}f(1)+f\{[l(x)+l(x+1)]/2\}\\&-f\{[q(x)+q(x+1)]/2\}-f\{[k(x)+k(x+1)]/2\}\\&+\frac{1}{2}\sum_{x=1}^{L-1}\log[1+l(x)-q(x)-k(x)]+\frac{q(x)+k(x)}{1+l(x)-q(x)-k(x)}\\&+\log[1-l(x)-q(x)+k(x)]+\frac{q(x)-k(x)}{1-l(x)-q(x)+k(x)},\end{aligned} \tag{B5}$$

$$\begin{aligned}-\beta F_0^{boundary}=&\,\frac{1}{2}\beta^2\{2f(1)-f[q(0)]-f[q(L)]\}+\beta^2(f(1)+f\{[1+l(1)]/2\}-f\{[q(1)+q(0)]/2\}-f\{[q(0)+k(1)]/2\}\\&+\frac{1}{2}\left\{\log[1-q(0)]+\frac{q(0)}{1-q(0)}+\log[1-q(L)]+\frac{q(L)}{1-q(L)}\right\},\end{aligned} \tag{B6}$$

$$-\beta F_{\uparrow,\uparrow}^{boundary}=+\beta^2(f(1)+f\{[1+l(L-1)]/2\}-f\{[q(L-1)+q(L)]/2\}-f\{[q(L)+k(L-1)]/2\}, \tag{B7}$$

$$-\beta F_{\uparrow,\downarrow}^{boundary}=+\beta^2(f(1)+f\{[1+l(L-1)]/2\}-f\{[q(L-1)+q(L)]/2\}-f\{[q(L)-k(L-1)]/2\}. \tag{B8}$$

Given the complexity of the expression, we have manipulated them through the MATHEMATICA software in order to obtain the equations of motion and integrated the resulting equations numerically by the relaxation method. As a proxy of the free-energy difference $F_{\uparrow,\downarrow}-F_{\uparrow,\uparrow}$, in Fig. 3 we plot $\sigma(L/2)L^6$, the difference in free-energy density in the center of the box multiplied by L^6, which according to the argument given in the main text should tend to a constant for large L. The numerical error on $\sigma(L/2)$, which is an extremely small quantity, limited the range of L that we could investigate to $L\leq 400$. The integration gives result compatible with the analysis of the reduced model, confirming the independence on the detailed boundary conditions imposed. As in the case of the reduced model, the curves can be fitted by the form $f(L)=a+b/L^c$, though in this case a logarithmic fit of the form $f(L)=a/\log(L)^c$ give a fit of comparable quality.

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