Persistent currents of noninteracting electrons in one-, two-, and three-dimensional thin rings

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We thoroughly study the persistent current of noninteracting electrons in one-, two-, and three-dimensional thin rings. We find that the results for noninteracting electrons are more relevant for individual mesoscopic rings than hitherto appreciated. The current is averaged over all configurations of the disorder, whose amount is varied from zero up to the diffusive limit, keeping the product of the Fermi wave number and the ring's circumference constant. Results are given as functions of disorder and aspect ratios of the ring. The magnitude of the disorder-averaged current may be larger than the root-mean-square fluctuations of the current from sample to sample even when the mean-free path is smaller, but not too small, than the circumference of the ring. Then a measurement of the persistent current of a typical sample will be dominated by the magnitude of the disorder-averaged current.

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I. INTRODUCTION

One of the consequences of the Aharonov-Bohm (AB) effect[1] is that a finite normal (i.e., nonsuperconducting) mesoscopic ring exhibits a persistent current (PC) when the AB magnetic flux through its opening is nonzero.[2–5] The PC does not decay with time when the dephasing and the thermal lengths are larger than the ring circumference. This results from the fact that the PC reflects an equilibrium state even when the ring has a finite resistance due to defect scattering.[3,6,7] The PC is periodic in the flux Φ with a period given by the magnetic flux quantum $\Phi_0 \equiv 2\pi\hbar c/e$. Measurements of the PC (Refs. 8–12) often stimulated the theoretical studies.[13–24] Today, this fundamental phenomenon of quantum mechanics still challenges both theoreticians and experimentalists of mesoscopic physics.[25–29] Persistent currents are also relevant for the orbital response of semimetals and aromatic molecules,[30] and for the ongoing interest in nanotubes.[31]

At zero disorder, the azimuthal component of the velocity associated with each single-particle eigenstate of the Hamiltonian of noninteracting particles is shifted due to the AB flux $\Phi < \Phi_0/2$, by $\Delta v = 2\pi\hbar\phi/ML$. Here M is the electron mass, L is the circumference of the ring, and $\phi \equiv \Phi/\Phi_0$. One may naively assume that the current density is $-ne\Delta v$, where n is the density of the electrons. In a normal ring, because of level crossing, the occupation of the levels changes with the flux. As a result, once level crossing occurs, the PC density of the normal ring is *much smaller* than $-ne\Delta v$. In a superconducting ring $-ne\Delta v$ gives the value of the PC density at zero temperature and zero disorder. It might be argued that in a perfect superconductor at zero temperature, the above occupation switching is suppressed. Thus, the attractive interaction in a superconductor, which enforces the pairing correlations, strongly enhances the PC compared to the normal-state value. Note that the current of a superconducting ring is an intensive quantity—it does not depend on the size of the system. In the normal state, the current is only a mesoscopic effect—proportional to an inverse power [−1 in the ballistic one-dimensional (1D) case[3]] of the system's length.

The current of noninteracting electrons in two-dimensional (2D) cylinders in the grand-canonical ensemble was studied analytically in the limit of zero disorder and in the diffusive limit.[15–18] In these works the PC was calculated in two geometries: "short" cylinders, $H \ll L$, where H is the height of the cylinder and "long" cylinders, $H \gg L$. Cheung *et al.*[16] studied the case of a three-dimensional (3D) short and thin diffusive cylinder as well. In the zero-disorder limit, the PC was calculated by summing the velocities, with appropriate factors, of all the states that, after the energy shift due to the flux, are below the Fermi energy.[15] In the diffusive limit the PC may be averaged over the configurations of the impurities. It can be calculated as a function of the magnetic flux from the density of states in the diffusive limit.[16,17] Entin-Wohlman and Gefen[18] calculated the impurity-ensemble-averaged current of long cylinders using the linear-response theory in ϕ, which is valid only for $\phi \ll 1/2$.

Our work extends the above research[15–18] in two ways. First, we describe the current for any degree of disorder between the previously studied limits of perfectly clean systems and diffusive systems. Second, we consider 3D thin rings with a finite width W for which $W \ll L$ (in contrast to $W \lesssim a$, where a is the smallest microscopic length of the system).[15,17,18] We also correct, and generalize for any given value of the flux, the expression for the PC as calculated by Entin-Wohlman and Gefen for long 2D cylinders.[18] In the latter, a calculation error[32] gave a result of incorrect sign and magnitude for the prefactor of the dominant (for $L \gg \ell$, where ℓ is the elastic mean-free path) exponential dependence.

The expression[16] for the disorder-averaged PC in the grand-canonical ensemble at zero temperature is given in Sec. II. This expression can be simplified in two regimes, defined in Sec. III, which we name the uncorrelated- and the correlated-channel regimes.[3] In Secs. IV and V we perform the simplifying steps that are allowed in each regime, and then obtain the leading-order expressions of the PC in the zero-disorder and the diffusive limits. The specific conditions for which these two limits hold in both the uncorrelated- and

TABLE I. The results for the PC in the zero-disorder ($\ell/L \to \infty$) and the diffusive ($\ell \ll L$) limits. The conditions defining the uncorrelated- and the correlated-channel regimes are given in the second column for a 2D cylinder. For a 2D annulus the conditions are the same with H replaced by W. For a 3D ring the conditions should be satisfied for both azimuthal directions. In the third column we refer to the appropriate expressions for the PC.

	Conditions associated with the z direction	Results
Uncorrelated: zero disorder	$1 \ll k_F H/\pi < 2L/H$	(24) and (25) (3D rings)
Uncorrelated: diffusive	$\sqrt{L/8\ell} \ll k_F H/\pi < 2L/H$	(26) (3D rings)
Correlated: zero disorder	$H/L > 10m^2 k_F L$	(29) (2D cylinder)
Correlated: diffusive	$H/L > \max\{100\ell/H, \pi/k_F L\}$	(30) (2D cylinder)

the correlated-channel regimes are given in Table I. In Refs. 15–18 the same simplifying assumptions had been used but were referred to as short and long cylinders. We find that these pictorial definitions do not agree with the regimes in which the corresponding results hold. Our results for PC of 2D cylinders in the zero-disorder and the diffusive limits for the uncorrelated-channel regime, and in the zero-disorder limit for the correlated-channel regime, agree with the ones obtained in Refs. 15–17. For 2D cylinders, our result for the PC in the correlated-channel regime in the diffusive limit is new.

The disorder-averaged PC is highly sensitive to the exact value of $k_F L$, as it contains a factor of $\sin(k_F L)$, where k_F is the Fermi wave number. In Sec. VI we discuss the way to compare the measured average PC in an ensemble of rings to the theoretical results depending on the variance of the value of $k_F L$ among the rings. In this section the disorder-averaged PC is also compared with the root-mean-square (rms) fluctuations[16,20] of the PC with respect to the disorder. We find that as long as the system is not too diffusive, the magnitude of the disorder-averaged current may be larger than the current rms fluctuations. As discussed in Sec. VII, our result for the disorder-averaged PC of noninteracting electrons agrees with the PC measured in a 2D clean annulus by Mailly *et al.*,[9] but has a larger magnitude than the one measured by Rabaud *et al.*[10] The results of our study are discussed in Sec. VIII.

In contrast with the Green's function technique used in the main body of this paper, we give in the Appendix an approximation for the PC of a 3D ring in the zero-disorder limit. This approximation is based on the canonical ensemble results for a 1D ring, and on the probabilities that, at a given flux, the number of electrons in a given transverse channel is odd or even.

II. EXPRESSION FOR THE PERSISTENT CURRENT

In this section we obtain an expression[16] for the impurity-ensemble-average zero-temperature PC of noninteracting electrons. We consider spinless electrons in a ring of a mean circumference L, a width W, and a height H. In the absence of disorder, the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2M}\left(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\right)^2. \tag{1}$$

The AB flux, which does not penetrate the ring itself, is given by the magnetic vector potential $\mathbf{A} = \hat{\varphi}\Phi/2\pi r$, where r is the radial coordinate and $\hat{\varphi}$ is a unit vector oriented along the ring. The eigenstates of $\mathcal{H}$, in cylindrical coordinates, are

$$\psi(r,\varphi,z) = e^{in\varphi}\sin\left(\frac{\pi q z}{H}\right) \times [C_1 J_{n+\phi}(kr) + C_2 Y_{n+\phi}(kr)], \tag{2}$$

where $n=0,\pm 1,\pm 2,\ldots$, $q=1,2,\ldots$, and

$$k = \sqrt{2M\epsilon/\hbar^2 - (\pi q/H)^2}. \tag{3}$$

Here J and Y are the Bessel functions of the first and second kind. The boundary conditions $\psi[r=(L/2\pi - W/2)] = \psi[r=(L/2\pi + W/2)] = 0$ set the ratio between the prefactors C_1 and C_2 and the eigenenergies. For $W \ll L$, the eigenenergies are given by[33]

$$\epsilon_{q,s,n} = \frac{\hbar^2\pi^2}{2M}\left[\frac{q^2-1}{H^2} + \frac{s^2-1}{W^2} + \frac{[2(n+\phi)]^2}{L^2}\right] + \frac{1}{L^2}O[(W/L)^2], \tag{4}$$

where s is a positive integer. In this work, all energies are shifted so that the single-particle ground-state energy, for which $q=s=1$, $n=\phi=0$, is zero. We henceforth neglect the term of order $(W/L)^2$ in Eq. (4).

We now introduce disorder, induced by impurities having pointlike potentials. The PC, averaged over a grand-canonical ensemble of disordered systems having the same mean-free path but different impurity configurations, is given by[16]

$$\langle I \rangle = \sum_{q,s,n}\int_{-\infty}^{\infty}\frac{dE}{2\pi i} f(E) \times [G^+([q,s,n],E) - G^-([q,s,n],E)] I_n^{(0)}. \tag{5}$$

Here the Fermi distribution function, $f(E)$, sets the chemical potential as an upper bound on the integration at zero temperature. The current associated with a single-electron wave function is given by

$$I_n^{(0)} = -\frac{2\pi\hbar e}{ML^2}(n+\phi). \tag{6}$$

In Eq. (5), the disorder-averaged retarded and advanced Green's function are denoted by G^+ and G^-, respectively. The expressions for the disorder-averaged Green's function, for $k_F\ell \gg 1$ and within the Born approximation, are[34]

$$\overline{G^{\pm}([q,s,n],E)} = \left[E - \epsilon_{q,s,n} \pm \frac{i\hbar}{2\tau}\right]^{-1}, \tag{7}$$

where τ is the elastic mean-free time. Equation (5) for the disorder-averaged PC is given as a sum over channels (q,s). However, in the corresponding expression for the nonaveraged current, one should use the nonaveraged Green's function and consequently for a specific configuration, the channels are mixed in the expression for the PC.[35]

We note that the (q,s) term in Eq. (5) is given by the averaged PC in a 1D ring[17] with a shifted chemical potential

$$\mu \to \mu(q,s) = \mu - \epsilon(q,s,n=0,\phi=0), \tag{8}$$

namely,

$$\langle I\rangle = \sum_{q,s} \langle I^{1\text{D}}[\mu(q,s)]\rangle. \tag{9}$$

The current of a 1D ring, calculated in Ref. 17, is

$$\langle I^{1\text{D}}\rangle = 2I_0 \sum_{m=1}^{\infty} \frac{\sin(2\pi m\phi)}{\pi m}\cos(mk_F L)e^{-mL/2\ell}. \tag{10}$$

Here $I_0 \equiv ev_F/L$, where v_F is the Fermi velocity.[36] In Eq. (9) each (q,s) term has its Fermi wave number determined by Eq. (8),

$$k_F(q,s) = k_F\sqrt{\mu(q,s)/\mu}. \tag{11}$$

Equation (10) is valid for $\mu \gg \{\hbar/2\tau, \tilde{a}\}$, where $\tilde{a} = 2\pi^2\hbar^2/ML^2$ is the prefactor of $(n+\phi)^2$ in the expression for the eigenenergies, see Eq. (4).

Substituting the 1D result, Eq. (10), in Eq. (9), we obtain that at zero temperature

$$\langle I\rangle = \sum_{m=1}^{\infty} \langle I_m\rangle \sin(2\pi m\phi), \tag{12}$$

where the disorder-averaged harmonics are given by

$$\langle I_m\rangle = \frac{2I_0}{\pi m}\sum_{q=1}^{N_z}\sum_{s=1}^{N_r\sqrt{1-(q^2-1)/N_z^2}} \frac{k_F(q,s)}{k_F}\cos[mk_F(q,s)L]\exp\left[-\frac{mL}{2\ell k_F(q,s)/k_F}\right]. \tag{13}$$

The approximate numbers of the occupied channels corresponding to momenta in the radial and the z directions are

$$N_r = k_F W/\pi, \quad N_z = k_F H/\pi, \tag{14}$$

respectively. In the upper bounds on the summations over q and s, one needs to take the closest integer values for N_r and N_z from below (but not less than 1).

In Eq. (13) we sum over the contributions of the occupied channels, which obey $(s/N_r)^2 + (q/N_z)^2 \leq 1$, so that $\mu(q,s) > 0$. In a diffusive system, one might worry about the contribution to $\langle I_m\rangle$ of channels with high transverse momentum which satisfy

$$\ell[k_F(q,s)/k_F] < 1/k_F(q,s), \tag{15}$$

and are therefore not diffusive. Their contribution is given by an expression similar to Eq. (10), where a term of $\sqrt{4k_F L}$ multiplies the exponent and divides I_0. In Eq. (13) we ignore this extra reduction since only a few channels may satisfy Eq. (15) and their contribution to the PC is anyhow small.

III. APPROXIMATIONS FOR THE PC HARMONICS

In this section we identify different regimes in which the expression for the disorder-averaged harmonics, see Eq. (13), can be simplified.

A. Dimensionality of the system

The ring is considered to have a significant thickness along the radial direction when $N_r \gg 1$ [see Eq. (14)] and when the ratio between the exponential in Eq. (13) with a small index s to the following $s+1$ term is much smaller than, say, 10. Thus, for the calculation of $\langle I_m\rangle$ many s values give significant contributions when

$$\frac{k_F W}{\pi} \gg \left\{1 \quad \text{and} \quad \sqrt{\frac{mL}{8\ell}}\right\}. \tag{16}$$

When the "much larger" sign in Eq. (16) is replaced by a "smaller" or "comparable" one, the ring is considered to be of zero dimension along the radial direction, and we use only $s=1$.

Note that condition (16) depends on L/ℓ. This can be understood by the following argument: the phase of the Green's function of a particle that encircles the ring depends on the specific disorder configuration. Averaging the PC over all configurations of disorder results in the exponential decay of $\langle I^{1\text{D}}\rangle$, see Eq. (10).[22] In a multichannel ring, the overall path, and correspondingly the variance of the phase shifts, increase as the transverse momentum increases. This results in the increase in the exponential decay rate in Eq. (13) for increasing channel index. Indeed, as we see in Eq. (16), increasing the disorder may decrease the effective dimensionality of the system. The condition for considering the ring to have a finite height is given by Eq. (16) upon replacing W with H. In this way the system is classified as one of the following: 1D, 2D annulus, 2D hollow cylinder, or a 3D ring. In the 2D annulus case one sums over s taking $q=1$, and in the 2D cylinder case the summation is over q keeping $s=1$.

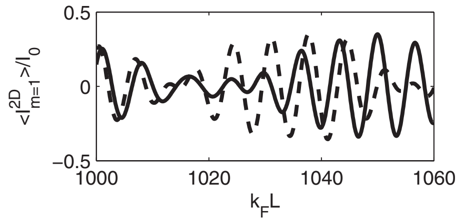

FIG. 1. The disorder-averaged PC depends on $k_F L$, N_z, and N_r in a nontrivial fashion. Here we plot $\langle I^{2D}_{m=1}\rangle$, see Eq. (13), for $L/\ell=5$ and for $N_z=70$ (solid line) and $N_z=100$ (dashed line). The typical magnitude of the disorder-averaged current, given by Eq. (19), for the above two values of N_z is $0.25I_0$ and $0.30I_0$, respectively.

B. Contributions of consecutive channels to $\langle I_m\rangle$

The discrete summation over the channel indices in Eq. (13) makes the expression for $\langle I_m\rangle$ hard to handle analytically. In this section we define two regimes where one can overcome this difficulty. The contributions to the mth harmonic of consecutive transverse channels (s and $s+1$, or q and $q+1$) are uncorrelated when the change in the arguments of the corresponding cosine terms, see Eq. (13), is larger than, say, $\pi/4$. This difference between the arguments of the cosines increases with increasing channel index. Hence, if the lowest two transverse indices obey this condition, then higher indices will fulfill it as well, so that all channels are uncorrelated. Thus, the channels associated with the z direction are uncorrelated when

$$\frac{H}{mL} < \frac{2\pi}{k_F H}. \tag{17}$$

The same rule applies to channels of consecutive s indices upon replacing H with W. The regime defined by Eqs. (16) and (17) will be referred to as the uncorrelated-channel regime.

In the uncorrelated-channel regime the dependence of the PC on the parameters $k_F L$, N_z, and N_r, which appear in the arguments of the cosines in Eq. (13), is nontrivial. This is demonstrated in Fig. 1. We thus turn to calculate the typical magnitude of the disorder averaged harmonics $(\overline{\langle I_m\rangle^2})^{1/2}$. The overline denotes averaging over $k_F L$ within a segment $\delta(k_F L)\ll k_F L$ of a width of $\gtrsim 2\pi$. Note the different notations of averaging over $k_F L$ and averaging over disorder. In the calculation of $(\overline{\langle I_m\rangle^2})^{1/2}$ we use the approximation

$$\overline{\cos[mk_F(q,s)L]\cos[mk_F(q',s)L]} = \frac{\delta_{qq'}}{2} \tag{18}$$

and obtain

$$(\overline{\langle I_m\rangle^2})^{1/2} = \frac{\sqrt{2}}{\pi m} I_0 \times \sqrt{\sum_{q,s}\left[\frac{k_F(q,s)}{k_F}\right]^2 \exp\left[-\frac{mL}{\ell k_F(q,s)/k_F}\right]}. \tag{19}$$

We have confirmed numerically that the standard deviation of $\langle I_m\rangle$ obtained from Eq. (13) gives the same value for $(\overline{\langle I_m\rangle^2})^{1/2}$ as given by Eq. (19). For the calculation of the standard deviation of $\langle I_m\rangle$ we have inserted in Eq. (13) the parameters of the ring used by Mailly *et al.*,[9] see Sec. VII, and considered many values of $k_F L$ in a segment of a width of 10π.

When the first harmonic is in the uncorrelated-channel regime, the harmonics with m up to $m\sim 8k_F^2W^2\ell/\pi^2 L$ are also in that regime, see Eq. (16). In this case, the contribution of higher harmonics is negligible. Therefore, in the approximate expression

$$(\overline{\langle I\rangle^2})^{1/2} = \sqrt{\sum_{m=1}^{\infty}\overline{\langle I_m\rangle^2}\sin^2(2\pi m\phi)}, \tag{20}$$

we can use the expression given in Eq. (19) for $\overline{\langle I_m\rangle^2}$ for all the relevant harmonics.

For a 2D cylinder, the maximal q whose contribution to $\langle I_m\rangle$ is not negligible, see Eq. (13), is

$$q^m_{\max} = \min\left\{N_z\sqrt{\frac{8\ell}{mL}}, N_z\right\}. \tag{21}$$

When Eq. (16) is satisfied and the cosines of sequential indices with $q\le q^m_{\max}$ are correlated, then the sum in Eq. (13) can be replaced by an integral. Since the difference between the arguments associated with sequential channels increases as the index of the channel increases, the condition for the channels to be correlated is

$$mL[k_F(q^m_{\max}-1,1)-k_F(q^m_{\max},1)] < \frac{\pi}{4}. \tag{22}$$

When $q^m_{\max}=N_z$, condition (22) has the form $H/L > 10m^2k_F L$. The correlated-channel regime for a 2D annulus is defined in the same way but the limitation $W\ll L$ of our analysis makes this regime irrelevant for that geometry. We refer to this point in more detail at the end of Sec. V. The expressions for the conditions for the uncorrelated- and the correlated-channel regimes, in the zero-disorder and the diffusive limits are summarized in Table I.

IV. UNCORRELATED-CHANNEL REGIME

Consider a 3D ring in the uncorrelated-channel regime, defined by Eqs. (16) and (17). To estimate $(\overline{\langle I_m\rangle^2})^{1/2}$ we replace the sum in Eq. (19) by an integral over $x=\sqrt{(q/N_z)^2+(s/N_r)^2}$, and add the factor $2xN_{\text{tot}}$, where $N_{\text{tot}}=\frac{\pi}{4}N_rN_z$ is the total number of occupied channels

$$(\overline{\langle I^{3D}_m\rangle^2})^{1/2} = \frac{2\sqrt{2}}{\pi m} I_0\sqrt{N_{\text{tot}}} \times \sqrt{\int_0^1 x(1-x^2)\exp\left(-\frac{mL}{\ell\sqrt{1-x^2}}\right)dx}. \tag{23}$$

In Fig. 2 the magnitudes of the first and second harmonics are plotted as a function of L/ℓ using Eq. (23). Here one can see that with increasing disorder, the first harmonic becomes more dominant.

Equation (23) can be further approximated in the zero-disorder and in the diffusive limits. In the first limit

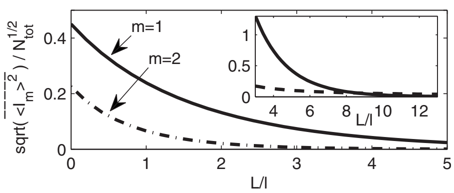

FIG. 2. The PC of a 3D ring in the uncorrelated-channel regime. The typical magnitudes of the first harmonic (solid line) and the second harmonic (dashed-dotted line) are plotted in units of $I_0\sqrt{N_{\rm tot}}$, using Eq. (23). In the inset $(\overline{\langle I^{3D}_{m=1}\rangle^2})^{1/2}/I_0$ (solid line) is obtained by substituting $N_z=N_r=20$ in Eq. (23). For a later discussion, the rms fluctuations of the PC with respect to the disorder, $\delta I/I_0$ (dashed line) are plotted using Eq. (33). Here $(\overline{\langle I_{m=1}\rangle^2})^{1/2}=\delta I$, when $L/\ell=8.5$, in agreement with Eq. (34). The horizontal axis of the inset begins at $L/\ell=3$ since Eq. (33) is valid only in the diffusive limit.

$$[\overline{(I^{3D}_m)^2}]^{1/2}=\frac{1}{\pi m}I_0\sqrt{N_{\rm tot}}. \tag{24}$$

From Eqs. (20) and (24) we obtain[37]

$$[\overline{(I^{3D})^2}]^{1/2}=I_0\sqrt{N_{\rm tot}}\sqrt{|\phi|(1-2|\phi|)}. \tag{25}$$

Note the enhancement of the PC magnitude by the square root of the channel number. Deep enough in the diffusive limit, $L/\ell\geq 10$, the PC is dominated only by its first harmonic. Here, the magnitude of the PC is given by the limit $L/\ell\gg 1$ of Eq. (23),

$$(\overline{\langle I^{3D}\rangle^2})^{1/2}=\frac{2}{\pi}\sqrt{\frac{\ell}{L}}I_0\sqrt{N_{\rm tot}}e^{-L/2\ell}\sin(2\pi\phi). \tag{26}$$

This reproduces the result[38] of Ref. 16.

The PC harmonics of a 2D annulus are given by

$$(\overline{\langle I^{2D}_m\rangle^2})^{1/2}=\frac{\sqrt{2}}{\pi m}I_0\sqrt{N_r}\times\sqrt{\int_0^1(1-x^2)\exp\left(-\frac{mL}{\ell\sqrt{1-x^2}}\right)dx}. \tag{27}$$

Results for a 2D annulus in the uncorrelated-channel regime and the zero-disorder limit are given by Eqs. (24) and (25) with $N_{\rm tot}$ replaced by $4N_r/3$. Here, replacing N_r with N_z gives the expression for the PC in a 2D cylinder obtained[38] by Cheung *et al.*[15] In the diffusive limit, the PC of a 2D annulus or a 2D cylinder in the uncorrelated-channel regime amounts to multiplying the expression in Eq. (26) by the factor $\sqrt{\pi L/8\ell}$ and replacing $N_{\rm tot}$ by N_r or N_z, respectively. The latter yields the results obtained in Refs. 16 and 17. The difference between the powers of L/ℓ between the 2D and the 3D expressions is due to the difference of the densities of states of the transverse channels in these cases.

The similarity between the PC of a 2D annulus and the PC of a 2D cylinder is hardly surprising since these two cases of finite width and of finite height are topologically equivalent for the AB flux, and the eigenenergies are the same as long as $W\ll L$.

V. CORRELATED-CHANNEL REGIME

For a 2D cylinder, the correlated-channel regime is defined by Eq. (16) (with H replacing W) and Eq. (22). In this case we replace the summation over q in Eq. (13) by an integration and obtain

$$\langle I^{2D}_m\rangle=\frac{2}{\pi m}I_0N_z\int_0^1\sqrt{1-x^2}\times\cos(mk_FL\sqrt{1-x^2})\exp\left(-\frac{mL}{2\ell\sqrt{1-x^2}}\right)dx. \tag{28}$$

In the zero-disorder limit, Eq. (28) yields the result[38] of Ref. 15,

$$I^{2D}_m=\sqrt{\frac{2}{\pi m^3}}I_0N_z\frac{1}{\sqrt{k_FL}}\cos(mk_FL-\pi/4). \tag{29}$$

The diffusive limit of the PC of a 2D cylinder in the correlated-channel regime is found here to be given by

$$\langle I^{2D}\rangle=\frac{\sqrt{2}\sin(2\pi\phi)}{\sqrt{\pi k_FL}}I_0N_ze^{-L/2\ell}\cos(k_FL-\pi/4). \tag{30}$$

(The higher harmonics are negligible.) The conditions for the correlated-channel regime in the zero-disorder limit, see Table I, cannot be satisfied for the radial direction together with the restriction $W\ll L$, for most reasonable values of k_FL. The limit of a diffusive annulus, see Table I, is satisfied, for $W\ll L$, only when $L/\ell>130$, but then the disorder-averaged PC is irrelevant.

In Fig. 3 the magnitude of the disorder-averaged PC is plotted using Eq. (30) as a function of L/ℓ in the diffusive regime. The results, Eqs. (29) and (30), are reduced by $1/\sqrt{k_FL}$ compared to the results in the uncorrelated-channel regime in the zero-disorder and the diffusive limits, see Sec. IV. However, these results are enhanced by $\sqrt{N_z}$ and by $\sqrt{N_z}(L/\ell)^{1/4}$, respectively.

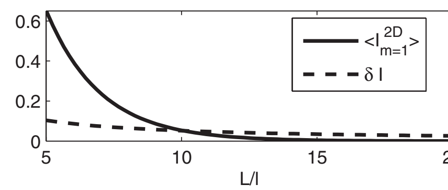

FIG. 3. The disorder-averaged PC of a 2D diffusive cylinder in the correlated-channel regime (solid line) is plotted as a function of ℓ/L. We replace $\sin(k_FL)$ by $1/\sqrt{2}$ in Eq. (30) to obtain the typical magnitude, and use $N_z=10^3$ and $k_FL=5\times10^3$. The rms fluctuations (dashed line), see Eqs. (31) and (33), equals $\langle I^{2D}_{m=1}\rangle$, for the above parameters, at $L/\ell\simeq 10$, in agreement with Eq. (35). Both $\langle I^{2D}_{m=1}\rangle$ and δI are given in units of I_0.

VI. rms FLUCTUATIONS VERSUS $\langle I\rangle$

The disorder-averaged PC is very sensitive to the exact value of k_FL, see, e.g., the cosine factor in Eq. (29). In contrast, the rms fluctuations of the current in respect to the disorder[16,20]

$$\delta I=[\langle I^2\rangle-\langle I\rangle^2]^{1/2} \tag{31}$$

are not sensitive to k_FL. The common practice in PC measurements is to determine the total current, $I_{\rm tot}$, from the measurement of the overall magnetic response of $\tilde{N}$ rings. This current is related to both the disorder-averaged current and to the current rms fluctuations by

$$I_{\rm tot} = \begin{cases} \tilde{N}\langle I\rangle \pm \sqrt{\tilde{N}}\delta I & \delta(k_F L) \ll \pi \\ \pm\sqrt{\tilde{N}}[(\overline{\langle I\rangle^2})^{1/2} \pm \delta I] & \delta(k_F L) > \pi. \end{cases} \quad (32)$$

Here $\delta(k_F L)$ is the variation in $k_F L$ in an ensemble of $\tilde{N}$ rings. Equation (32) hold also for the harmonics (replacing I by I_m). If the ring is in the uncorrelated-channel regime, one may replace $\langle I\rangle$ by $\pm(\overline{\langle I\rangle^2})^{1/2}$ in the top equality of Eq. (32), while if the ring is in the correlated-channel regime, one needs to replace the cosine factor in Eq. (28) for $\langle I_m\rangle$ by $1/\sqrt{2}$ in order to obtain $(\overline{\langle I_m\rangle^2})^{1/2}$ in the bottom equality.

The rms fluctuation due to the disorder of the h/e harmonic of the current for a thin-walled ($L \gg \{W,H\}$) ring in the diffusive limit is given by[16,20]

$$\delta I = \frac{\sqrt{8}}{\pi\sqrt{3}}\frac{\ell}{L} I_0 \sin(2\pi\phi) \quad [\ell \ll L]. \quad (33)$$

This result is independent of the number of channels, i.e., of W and H. These current rms fluctuations do not exist for $\ell/L \gg 1$, see Eq. (31). Thus, the contribution to $I_{\rm tot}$ which is not related to interactions, is expected to be given by Eq. (13) in the zero-disorder limit. Equation (33) for δI is strictly valid in the diffusive regime but is expected to give a correct order of magnitude for systems in which ℓ and L are comparable.

In Figs. 2 and 3, the crossover from the dominance of the disorder-averaged PC to the dominance of δI can be observed. In the uncorrelated-channel regime, the typical magnitude of the disorder-averaged current of a 3D ring is equal to δI at $L/\ell = 5$, 10, and 14 for $N_{\rm tot} = 20$, 10^3, and 10^5, respectively. These values are obtained, for $L/\ell > 1$, by comparing Eq. (26) with Eq. (33),

$$N_{\rm tot} > 0.7\frac{\ell}{L}e^{L/\ell} \Leftrightarrow (\overline{\langle I^{\rm 3D}_{m=1}\rangle^2})^{1/2} > \delta I. \quad (34)$$

The analogous result for a 2D cylinder in the correlated-channel regime is

$$N_z > 0.9\frac{\ell}{L}e^{L/2\ell}\sqrt{k_F L} \Leftrightarrow (\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2} > \delta I. \quad (35)$$

For $k_F L = H/L = 100$, the equality $(\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2} = \delta I$ is satisfied, see Eq. (35), for $N_{\rm tot} = 22$, 135, and 700 at $L/\ell = 5$, 10, and 14, respectively.

VII. DISCUSSION OF EXPERIMENTAL DATA

Since the first harmonic is not expected to be affected by electron-electron interactions,[13,14] we may compare its measurements[9–11,27,28] with calculations of the typical magnitude of $\langle I\rangle$ and δI. Mailly *et al.*[9] studied the PC in an almost ballistic annulus of GaAlAs/GaAs, characterized by $L = 8.5$ μm, $\ell = 11$ μm, $k_F = 1.5\times 10^8$ m^{-1}, $v_F = 2.6 \times 10^5$ m/s, and $W = 0.16$ μm. These parameters, which yield $I_0 = 5$ nA and $N_r = 8$, satisfy conditions (16) and (17) for the uncorrelated-channel regime. We insert these parameters in our result Eq. (27) and in Eq. (33), adding a factor of 2 due to spin degeneracy. This yields $(\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2} = 1.4 I_0$, and $\delta I = 1.3 I_0 \sin(2\pi\phi)$. We see that δI and $(\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2}$ are comparable, and both are in fair agreement with the measured PC of $(0.8 \pm 0.4)I_0$. Using the expression for the PC of a 2D cylinder in the zero-disorder limit obtained in Ref. 15 (replacing H with W) yields a value larger by a factor of ~ 2 compared to our result. When $\ell \sim L$, the ballistic, diffusive and exact expressions should give the same order of magnitude for the PC. Indeed, using the expression for the PC of a diffusive annulus in the uncorrelated-channel regime[16,17] gives a value that is very close to the one obtained from Eq. (23) for the parameters of the annulus measured in Ref. 9.

Rabaud *et al.*[10] measured the PC of an array of 16 ballistic rings of GaAlAs/GaAs. Those rings are in fact squares whose external total edge length is 16 μm and the internal one is 8 μm, yielding $L = 12$ μm. The rings are also characterized by $\ell = 8$ μm, $k_F = 2\times 10^8$ m^{-1}, $W = 0.8$ μm, and $v_F = 3.2\times 10^5$ m/s, implying $I_0 = 4.2$ nA and $N_r = 50$. The measured total PC obtained for disconnected rings, divided by the square root of the number of rings,[39] was $(0.33 \pm 0.07)I_0$. Neither the uncorrelated-channel regime nor the correlated-channel regime can be associated with these rings, since both Eqs. (17) and (22) are not obeyed by the above parameters. Therefore, we use our result Eq. (13), with $q = 1$ and a factor of 2 due to spin degeneracy, and obtain values for $\langle I_m\rangle$ in the regime $(-3I_0, 3I_0)$, whose standard deviation is $(\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2} = 1.1 I_0$. From Eq. (33) we find that $\delta I = 0.7 I_0 \sin(2\pi\phi)$. The discrepancy between the measured value, the above $(\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2}$, and δI may be due to the geometry (squares instead of rings) as well as due to decoherence.[10] The relative large W may also play a role.

One may compare our result for $(\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2}$ for the parameters of Ref. 10 with results of previous theoretical studies for these short annuli.[15–17] The latter correspond to $(\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2} = 7.5 I_0$ in the zero-disorder limit and $(\overline{\langle I^{\rm 2D}_{m=1}\rangle^2})^{1/2} = 4.5 I_0$ in the diffusive limit [as given by Eqs. (24) and (26), adapted to 2D and including a factor of 2 due to the spin degree of freedom, see Sec. IV]. Hence, our result is in a smaller disagreement, compared to results of former studies,[15–17] with the measured one. This is due to the fact shown above that the conditions for Eqs. (24) and (26) to be valid are not satisfied by the parameters of the rings measured in Ref. 10.

The first harmonic, measured for the diffusive rings used in the studies of Jariwala *et al.*,[11] and of Bluhm *et al.*,[27] fairly agrees with the theoretical value for δI. Here the rings are deep enough in the diffusive regime and so $(\overline{\langle I\rangle^2})^{1/2} \ll \delta I$. In the very recent work of Bleszynski-Jayich *et al.*,[28] where aluminum rings were used, the high magnetic fields utilized in the experiment cause $\langle I\rangle$ to be negligible, but leave δI unaffected.[29] Indeed, the rms fluctuations, given by Eq. (33), agree with the measured PC.[28]

VIII. DISCUSSION

In this work we have studied the disorder-averaged persistent current of noninteracting electrons. We have extended earlier analytical studies, which considered only the zero-disorder and the diffusive limits,[15–18] and have given an expression, Eq. (13), for a general[35] ratio of L/ℓ, as long as $k_F\ell \gg 1$. We define the uncorrelated- and the correlated-channel regimes in which Eq. (13) can be simplified[38] to expressions (23) and (28), respectively. While previous works[15–19] dealt mostly with 1D rings or 2D cylinders, we have considered here also rings of finite narrow width. In particular, we have obtained an expression for 3D rings. In addition, our expression for the PC in a 2D cylinder in the correlated-channel regime in the diffusive limit is new.

The inset of Figs. 2 and 3 demonstrate that the disorder-averaged PC may be a relevant contribution, compared with the fluctuation δI, for slightly diffusive systems, typically with $L/\ell \lesssim 10$. The relation between the parameters of a ring that satisfy $(\overline{\langle I_m\rangle^2})^{1/2} > \delta I$, is given in Eqs. (34) and (35) for the uncorrelated- and the correlated-channel regimes, respectively. We find that for the parameters of the rings used in Refs. 9 and 10 the disorder-averaged PC is relevant compared to δI.

Interactions, repulsive[13] or attractive,[14] can contribute to an $h/2e$ flux-periodic disorder-averaged PC. However, as long as the sample is not superconducting, the PC remains a mesoscopic effect. We have recently suggested[25,26] that if the effect of pair breaking is taken into account, attractive interactions can explain the $h/2e$ signal measured in ensembles of copper[8] and gold[11] rings. The contribution of interactions to the PC is not sensitive to the exact value of $k_F L$. Therefore, the interaction-induced PC may be compared to measurements using the top equality in Eq. (32), for any value of $\delta(k_F L)$. In contrast, since in reality $\delta(k_F L) > \pi$, the interaction-independent contributions of both δI and $(\overline{\langle I\rangle^2})^{1/2}$ are compared to measurements using the bottom equality in Eq. (32). Thus, as $\tilde{N}$ increases the interaction-dependent contributions to the PC become dominant over the contributions which do not depend on electronic interactions. This explains why measurements on ensembles of 10^5 and 10^7 rings revealed only the $h/2e$ harmonic.[8,12] It seems that the h/e harmonic can be accounted for only by the part of the PC that is independent of interactions, which we study here. However, since the $h/2e$ periodicity of the interaction-dependent part of the PC was obtained from calculations of the disorder-averaged PC,[13,14] further study is needed to assure that the h/e harmonic is not present in the interaction-dependent parts of δI. The special case of a single-channel (pure 1D) interacting system[40] can be solved using bosonization techniques. Qualitative differences exist then between repulsive and attractive interactions. In 1D rings, interactions affect the first harmonic of the sample-specific current.

Each harmonic has a different temperature dependence. Higher harmonics decay faster with temperature since they necessitate multiple paths around the ring.[15,16] For this reason we treated the different harmonics separately, though our calculations are carried out at zero temperature.

We call attention to the appearance of positive powers of the channel number (although the negative power of $k_F L$ in the correlated-channel regime may partially compensate that) in the PC magnitude. This implies that once multichannel ballistic systems would be manufactured, relatively large PCs should appear. Both molecular and clean semiconducting systems come to mind in this connection, and perhaps semimetals, such as Bi (see first reference of Ref. 30). On the other hand, in all regimes, the disorder-averaged PC in the diffusive limit is highly suppressed by a factor of $\exp(-L/2\ell)$. Again, achieving ℓ not too small compared with L, will be helpful.

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APPENDIX: AN ALTERNATIVE STATISTICAL APPROACH FOR THE DESCRIPTION OF THE CURRENT

So far we have used the Green's-function technique for our calculations. In this section we develop an alternative statistical approach to approximate the current in the uncorrelated-channel regime and the zero-disorder limit. The following approach leads to the magnitude of the PC, which is given[15] by Eq. (25), in a more intuitive way. We study here the probabilities that the channels are filled with an odd or an even number of electrons, and use the results for PCs in canonical 1D rings, to obtain the PCs of 2D or 3D rings.

In the regime $-1/2 \le \phi \le 1/2$, the PC of a 1D ring with an odd or with an even number of electrons, see, for example, Ref. 7, is given by

$$I_{\text{odd}} = -2\phi \frac{ev_F}{L}, \tag{A1}$$

$$I_{\text{even}} = [\text{sgn}(\phi) - 2\phi]\frac{ev_F}{L}. \tag{A2}$$

These currents have periodicity of unity in ϕ. Consider a ring of finite width in the grand-canonical ensemble at zero temperature. The contribution of the (q,s) channel to the PC is obtained by replacing v_F in Eqs. (A1) and (A2) by an effective Fermi velocity $v_F(q,s) = Mk_F(q,s)$, see Eqs. (8) and (11). Here, the exact position where the chemical potential crosses the energy levels of each channel determines whether the channel is occupied by an even or an odd number of electrons, see Fig. 4.

In an ensemble of rings with similar but not identical parameters, the energy levels of a given channel are shifted (among the rings) due to fluctuations in H and W, see Eq. (4). Also, the variation in these levels with ϕ is changing due to fluctuations in L. Therefore, the exact position of μ relative to the energy levels of a given channel is distributed randomly in the ensemble. When the levels with $E \le E_{q,s,-n}$ in Fig. 4 are occupied the channel consists of an even number of electrons, and when the levels with $E \le E_{q,s,n}$ are oc-

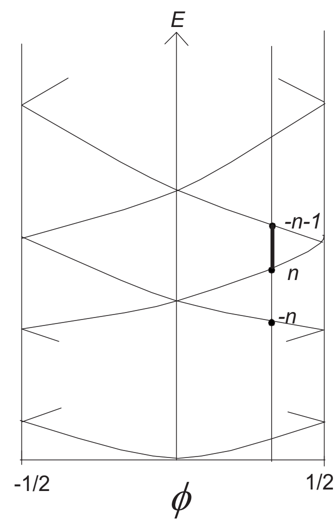

FIG. 4. The energy levels of a single channel are plotted as a function of the flux. The consecutive energy levels for a given positive flux and longitudinal indices $-n$, n, and $-n-1$, are marked by full circles. The bottom level corresponds to $n=0$. The random choice of μ in the interval $[E_{q,s,-n}(\phi),E_{q,s,-n-1}(\phi)]$ yields an odd number of occupied levels when $\mu > E_{q,s,n}(\phi)$ and an even number of occupied levels when $\mu < E_{q,s,n}(\phi)$. The former regime is marked by the bold line in the figure. Here, without loss of generality, we take $n>0$.

cupied the channel consists of an odd number. Taking $E_{q,s,n}(\phi)\simeq \mu$ the probability that a channel consists of an odd number of electrons is determined by

$$P_{\rm odd}=\frac{E_{q,s,-n-1}(\phi)-E_{q,s,n}(\phi)}{E_{q,s,-n-1}(\phi)-E_{q,s,-n}(\phi)}. \tag{A3}$$

We assumed here $\phi>0$ and $n>0$. The difference appearing in the nominator is shown in Fig. 4 as a vertical line. Inserting the eigenenergies, Eq. (4), in Eq. (A3) (considering $n \gg 1$), yields

$$P_{\rm odd}=1-2|\phi|, \quad P_{\rm even}=2|\phi|. \tag{A4}$$

These probabilities are independent of the channel index.

We calculate the average current in an ensemble of similar rings using the currents and the probabilities given in Eqs. (A1), (A2), and (A4), and find

$$\overline{I(q,s)}=P_{\rm odd}I_{\rm odd}(q,s)+P_{\rm even}I_{\rm even}(q,s)=0,$$

$$\bar{I}=\sum_{q,s}\overline{I(q,s)}=0. \tag{A5}$$

For $|\phi|\ll 1$, the probability to have an odd number of electrons in a channel is much larger than the probability to have an even number, see Eq. (A4). However, since $|I_{\rm even}|\gg|I_{\rm odd}|$, see Eqs. (A1) and (A2), the average PC is zero. This suggests very large fluctuations of the current at small flux. The typical magnitude of $I(q,s)$ is given by

$$[\overline{I^2(q,s)}]^{1/2}=\sqrt{P_{\rm odd}I^2_{\rm odd}(q,s)+P_{\rm even}I^2_{\rm even}(q,s)}$$
$$=\sqrt{2|\phi|(1-2|\phi|)}\frac{ev_F(q,s)}{L}. \tag{A6}$$

We add the assumption that the contributions of different channels to the PC are uncorrelated, which, together with Eq. (A5), yields

$$\overline{I(q,s)I(q',s')}=\delta_{qq'}\delta_{ss'}\overline{I^2(q,s)}. \tag{A7}$$

Using Eqs. (A6) and (A7) we obtain the standard deviation of the current

$$(\overline{I^2})^{1/2}=\left[\sum_{q,s}\overline{I^2(q,s)}\right]^{1/2}=\sqrt{2|\phi|(1-2|\phi|)}\frac{ev_F}{L}C_D. \tag{A8}$$

Here

$$C_D=\left[\sum_{q,s}\frac{v_F^2(q,s)}{v_F^2}\right]^{1/2}=\begin{cases}1 & \text{1D}\\ \sqrt{2N_z/3} & \text{2D}\\ \sqrt{N_{\rm tot}/2} & \text{3D},\end{cases} \tag{A9}$$

depends on the dimensionality of the ring. The nonanalytic $\sqrt{\phi}$ behavior at $\phi\ll 1$ at zero temperature is due to the paramagnetic contributions, since $P_{\rm even}\propto\phi$, while $I_{\rm even}\propto\pm$const at $\phi\to 0$. Thus, the slope of Eq. (A8) at $\phi=0$ diverges.[41]

Equation (A8) reproduces Eq. (25) obtained for the uncorrelated-channel regime in the zero-disorder limit for 3D rings. For one- and two-dimensional rings, Eq. (A8) reproduces the results of Refs. 15 and 19. The reason for this equivalence is that Eq. (18), which yields Eq. (25), is equivalent to Eq. (A7).

For a finite ensemble of $\tilde{N}$ clean rings, whose typical number of channels is $N_{\rm tot}$, the probability that all channels in all rings will be occupied by an odd number of electrons is given for small ϕ by

$$(P_{\rm odd})^{\tilde{N}N_{\rm tot}}\underset{\phi\tilde{N}N_{\rm tot}\ll 1}{\longrightarrow}1-2\phi\tilde{N}N_{\rm tot}. \tag{A10}$$

This probability becomes arbitrarily close to unity for $\phi\tilde{N}N_{\rm tot}\ll 1$. Therefore, such a measurement will produce the diamagnetic linear response of a clean superconductor (see Sec. I). By increasing the flux in a given finite ensemble (or by increasing $\tilde{N}N_{\rm tot}$), even channels will appear one by one, each giving a large paramagnetic contribution, eventually causing the zero average and anomalously large fluctuations of the current.

Note that an ensemble of 1D rings, with equal probability for an odd and for an even number of electrons in a ring, should exhibit a very large paramagnetic response, see Eqs. (A1) and (A2).

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[1] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
[2] I. O. Kulik, JETP Lett. **11**, 275 (1970).
[3] L. Gunther and Y. Imry, Solid State Commun. **7**, 1391 (1969).
[4] E. N. Bogachek and G. A. Gogadze, Sov. Phys. JETP **36**, 973 (1973).
[5] N. B. Brandt, D. V. Gitsu, A. A. Nikolaeva, and Ya. G. Ponomarev, Sov. Phys. JETP **45**, 1226 (1977).
[6] M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. **96A**, 365 (1983).
[7] Y. Imry, *Introduction to Mesoscopic Physics*, 2nd ed. (Oxford University Press, Oxford, 2002).
[8] L. P. Lévy, G. Dolan, J. Dunsmuir, and H. Bouchiat, Phys. Rev. Lett. **64**, 2074 (1990).
[9] D. Mailly, C. Chapelier, and A. Benoit, Phys. Rev. Lett. **70**, 2020 (1993).
[10] W. Rabaud, L. Saminadayar, D. Mailly, K. Hasselbach, A. Benoit, and B. Etienne, Phys. Rev. Lett. **86**, 3124 (2001).
[11] E. M. Q. Jariwala, P. Mohanty, M. B. Ketchen, and R. A. Webb, Phys. Rev. Lett. **86**, 1594 (2001).
[12] R. Deblock, Y. Noat, B. Reulet, H. Bouchiat, and D. Mailly, Phys. Rev. B **65**, 075301 (2002).
[13] V. Ambegaokar and U. Eckern, Phys. Rev. Lett. **65**, 381 (1990).
[14] V. Ambegaokar and U. Eckern, Europhys. Lett. **13**, 733 (1990).
[15] H. F. Cheung, Y. Gefen, and E. K. Riedel, IBM J. Res. Dev. **32**, 359 (1988).
[16] H. F. Cheung, E. K. Riedel, and Y. Gefen, Phys. Rev. Lett. **62**, 587 (1989).
[17] E. K. Riedel, H. F. Cheung, and Y. Gefen, Phys. Scr. **T25**, 357 (1989).
[18] O. Entin-Wohlman and Y. Gefen, Europhys. Lett. **8**, 477 (1989).
[19] H. F. Cheung, Y. Gefen, E. K. Riedel, and W. H. Shih, Phys. Rev. B **37**, 6050 (1988).
[20] E. K. Riedel and F. von Oppen, Phys. Rev. B **47**, 15449 (1993).
[21] L. Wendler, V. M. Fomin, and A. A. Krokhin, Phys. Rev. B **50**, 4642 (1994).
[22] N. Argaman, Y. Imry, and U. Smilansky, Phys. Rev. B **47**, 4440 (1993).
[23] B. L. Altshuler, Y. Gefen, and Y. Imry, Phys. Rev. Lett. **66**, 88 (1991).
[24] G. Montambaux, H. Bouchiat, D. Sigeti, and R. Friesner, Phys. Rev. B **42**, 7647 (1990).
[25] H. Bary-Soroker, O. Entin-Wohlman, and Y. Imry, Phys. Rev. Lett. **101**, 057001 (2008).
[26] H. Bary-Soroker, O. Entin-Wohlman, and Y. Imry, Phys. Rev. B **80**, 024509 (2009).
[27] H. Bluhm, N. C. Koshnick, J. A. Bert, M. E. Huber, and K. A. Moler, Phys. Rev. Lett. **102**, 136802 (2009).
[28] A. C. Bleszynski-Jayich, W. E. Shanks, B. Peaudecerf, E. Ginossar, F. von Oppen, L. Glazman, and J. G. E. Harris, Science **326**, 272 (2009).
[29] E. Ginossar, L. I. Glazman, T. Ojanen, F. von Oppen, W. E. Shanks, A. C. Bleszynski-Jayich, and J. G. E. Harris, Phys. Rev. B **81**, 155448 (2010).
[30] P. Ehrenfest, Physica (The Hague) **5**, 388 (1925); Z. Phys. **58**, 719 (1929); C. V. Raman and K. S. Krishnan, Proc. R. Soc. London **A113**, 511 (1927); L. Pauling, J. Chem. Phys. **4**, 673 (1936); K. Lonsdale, Proc. R. Soc. London **A159**, 149 (1937); F. London, J. Phys. Radium **8**, 397 (1937); for an early reference on mesoscopic orbital magnetism in normal metals, see, F. Hund, Ann. Phys. **32**, 102 (1938).
[31] F. Kuemmeth, S. Ilani, D. C. Ralph, and P. L. McEuen, Nature (London) **452**, 448 (2008).
[32] The error is in the expansion of Eq. (14) of Ref. 18.
[33] M. Abramovich and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972), see Eqs. 9.5.(27)–(29) therein.
[34] S. Doniach and E. H. Sondheimer, *Green's Functions for Solid State Physicists*, 2nd ed. (Imperial College Press, London, 1998). Besides $k_F\ell \gg 1$, the Born approximation requires $\nu(0) \ll n_i\tau/\hbar$, where $\nu(0)$ is the density of states and n_i is the concentration of the impurities in the system. The latter condition is equivalent to $\sigma \ll k_F^{-2}$, where σ is the impurity scattering cross section.
[35] We use Eq. (7) for a finite system although it is strictly valid only in the thermodynamic limit. Thus the case $L \ll \ell$ is not rigorously covered by this formulation. It may be hoped though that this is a reasonable approximation for a system with periodic boundary conditions.
[36] Equation (10) is derived for the grand-canonical ensemble. However, a proper choice of the chemical potential (which reflects on the value of k_FL) will give the sawtooth shape of the PC in a ring with a fixed number of electrons at zero disorder, see Ref. 19, and references therein.
[37] The nonanalytic behavior at small flux of Eq. (25) [which follows from an effective $1/\phi$ cutoff of the summation over m in Eq. (20) in the zero-disorder limit] exists only at the $T \to 0$ limit. At any temperature smaller than the single channel level spacing, Δ_1, there will be a small linear portion for $\phi \ll T/\Delta_1$, with a slope proportional to Δ_1/T.
[38] The definitions of long and short cylinders, used in Refs. 15–19 do not agree with the regimes for which the results hold.
[39] Due to a difference in the definition of the parameter I_{tot} in Ref. 10 and in this paper, we multiplied the value of I_{tot} given in Ref. 10 by the square root of the number of rings used in that experiment.
[40] D. Schmeltzer, Phys. Rev. B **47**, 7591 (1993); A. O. Gogolin and N. V. Prokofev, *ibid.* **50**, 4921 (1994); D. Schmeltzer and R. Berkovitz, Phys. Lett. A **253**, 341 (1999); M. Kamal, Z. H. Musslimani, and A. Auerbach, J. Phys. I **5**, 1487 (1995).
[41] It is nontrivial to produce simple formulae for P_{odd} and P_{even} at finite temperatures using the statistical approximation. The singularities at $\phi=0$ and $\phi=\pm 1/2$ of, respectively, I_{even} and I_{odd} are rounded at finite temperatures. We expect that $P_{even}(\phi=0)$ and $P_{odd}(\phi=\pm 1/2)$ will have a finite contribution, which will keep $\overline{I}=0$ and eliminate (as in Ref. 37) the square-root singularities in $[\overline{I^2}(q,s)]^{1/2}$ for small ϕ and $\phi=\pm 1/2$.