Effect of quasiparticles injection on the ac response of a superconductor

G. Catelani and L. I. Glazman

Departments of Physics and Applied Physics, Yale University, New Haven, Connecticut 06520, USA

K. E. Nagaev

Institute of Radioengineering and Electronics, Mokhovaya 11-7, Moscow 125009, Russia
(Received 8 July 2010; published 1 October 2010)

We calculate the ac linear response of a superconductor in a nonequilibrium electronic state. The nonequilibrium state is produced by injecting quasiparticles into the superconductor from normal leads through asymmetric tunnel contacts. The dissipative part of the response is proportional to the total density of the injected quasiparticles regardless of the imbalance between the numbers of electronlike and holelike excitations. At fixed quasiparticle density, the lower their effective temperature, the stronger is the dissipation.

I. INTRODUCTION

Over the years, measurement of the impedance was used extensively in the studies of spectra of elementary charge carriers in normal metals and superconductors. In the case of normal-metal single crystals, measurements of surface impedance in conditions of cyclotron resonance were instrumental in the reconstruction of the Fermi surface geometry.[1] In superconductors, the temperature dependence of the impedance allowed one to investigate the appearance of the BCS gap in the spectrum of quasiparticles.[2] Recent interest to the ac response of superconductors is driven by its application for studying new materials[3,4] and by the use of well-studied superconductors as material for high-quality resonators. Depending on the problem, it is important to realize the lowest possible dissipation rate of a resonator or its well-defined response to a perturbation. The former goal is central for superconducting qubit physics[5,6] and quantum optics,[7,8] where one is interested in achieving the longest possible coherence times. The latter one is important, e.g., for the use of superconductors in microwave kinetic inductance detectors.[9]

To interpret the measured impedance, one has to compare it with theoretical predictions. However up to now, no microscopic calculations of impedance of superconductors under nonequilibrium conditions were available. The frequency and temperature dependence of the complex conductivity of a superconductor within the BCS theory was first evaluated in the seminal paper of Mattis and Bardeen,[10] where the limit of short electron mean-free path (the "dirty superconductor" limit) was considered. The nonlocal conductivity of a clean superconductor was derived by Abrikosov, Gorkov, and Khalatnikov.[11] Later works of Nam[12] were aimed at bridging the two limiting cases. The theory developed in Refs. [10]–[12] addresses superconductors at thermal equilibrium. If desired, this condition is fairly easy to achieve at not-too-low temperatures. Indeed, even the early measurements have demonstrated the hallmarks of the BCS behavior of the dissipative part of the impedance, including its thermal-activation behavior at subgap frequencies.[13] At the same time, experiments involving impedance measurements in nonequilibrium conditions relied on heuristic extension of Mattis-Bardeen formula, amounting to the replacement of the equilibrium quasiparticle distribution function in the formula by a nonequilibrium one.[9,14]

The goal of this paper is to microscopically evaluate the linear ac conductivity of a superconductor in a concrete setup allowing a controlled perturbation of the quasiparticle distribution function. To this end, we analyze the steady state and linear ac conductivity of a mesoscopic superconductor brought out of equilibrium by electron tunneling through two junctions connecting the superconductor to normal leads (N-I-S-I-N structure—see Fig. 1); the structure is biased by a constant voltage V. The advantage of the mesoscopic setting is in the simplicity of the nonequilibrium quasiparticle distribution: it may be fully controlled by the electron tunneling in the superconductor while for bulk samples it inevitably depends on the relaxation mechanisms.[15–17] Particle-hole imbalance in the bulk may be achieved by electron injection[18] while symmetric particle-hole excitation was widely investigated in connection with the idea of microwave stimulation of superconductivity.[19] In a mesoscopic N-I-S-I-N structure, the imbalance is simply related to the ratio of the conductances of the two tunnel junctions. One may neglect the relaxation of quasiparticles inside the mesoscopic superconductor if the corresponding rate is small compared with the tunneling rates. This sets, of course some limitation on the size of the superconductor and conductances of the junctions.

We find the steady-state electron distribution at arbitrary asymmetry between the conductances of tunnel junctions, and at finite temperature T and voltage V, assuming these
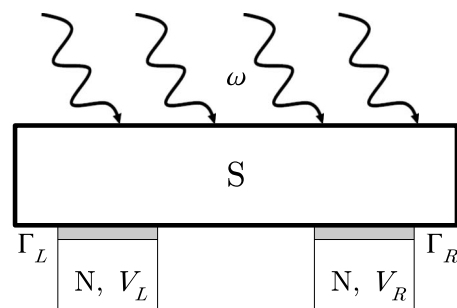

FIG. 1. N-I-S-I-N structure: the superconductor is connected to two normal leads—maintained at different voltages (bias $V=V_R-V_L$)—via tunnel junctions characterized by tunneling rates Γ_R, Γ_L. The superconductor is also subject to a weak electromagnetic field oscillating with frequency ω.

two scales small compared to the quasiparticle energy gap Δ. In the case of unequal conductances of the two tunnel junctions, charge imbalance is created along with a perturbation of the energy distribution of quasiparticles. We evaluate the complex ac conductivity of the superconductor in this nonequilibrium state and show that only the energy mode of the quasiparticle distribution enters in the proper generalization of the Mattis-Bardeen formula. At $eV, k_B T \ll \Delta$ we cast the result for conductivity in terms of T and quasiparticle density n_{qp}. That form extrapolates between the equilibrium result (where n_{qp} is a function of T only) and the nonequilibrium one, where n_{qp} is a function of V and T, with arbitrary ratio $eV/k_B T$.

In the next section, we formulate the problem in terms of matrix Green's functions. The steady state of the electrons in the superconductor formed in the presence of finite bias applied to the N-I-S-I-N structure is found in Sec. III, where we also establish the correspondence between the descriptions in terms of the matrix distribution function $\hat{F}(\varepsilon)$ and the scalar distribution function of quasiparticles f_ξ (the latter was used in the original Mattis-Bardeen theory for the ac conductivity at equilibrium). Using the description in terms of f_ξ, we consider the effect of electron-phonon collisions on the quasiparticle kinetics. It allows us to determine the conditions under which the intrinsic quasiparticle relaxation is negligible compared with the effect of tunneling. The ac conductivity at low temperature and bias (but at arbitrary $eV/k_B T$) is analyzed in Sec. IV. Throughout the paper, we use units $\hbar = k_B = 1$.

II. ELECTRON DYNAMICS IN A SUPERCONDUCTOR SUBJECT TO dc BIAS AND WEAK ac FIELD

We consider a diffusive superconductor connected to two normal leads, left (L) and right (R), via tunnel barriers and exposed to an external, time-dependent electric field. The system properties can be described in terms of disorder-averaged matrix Green's functions for the superconductor, $\check{\mathcal{G}}_S(\mathbf{k}, t, t')$, and for the electrodes, $\check{\mathcal{G}}_i(\mathbf{p}_i, t, t')$, $i = L, R$. Each of the matrices $\check{\mathcal{G}}$ has the form, in Keldysh space[20]

$$\hat{\mathcal{G}} = \begin{pmatrix} \hat{\mathcal{G}}^R & \hat{\mathcal{G}}^K \\ 0 & \hat{\mathcal{G}}^A \end{pmatrix}. \qquad (1)$$

The elements of this matrix are 2×2 matrices in Nambu particle-hole space. The superconductor Green's function obeys the Dyson equation

$$\left[i\hat{\tau}_z \frac{\partial}{\partial t} - \left\{ \frac{[\mathbf{k} - e\mathbf{A}(t)\hat{\tau}_z]^2}{2m} - E_F \right\} + i\hat{\tau}_y \Delta \right] \check{\mathcal{G}}_S(\mathbf{k}, t, t')$$
$$- \int dt'' \frac{1}{2\pi\tau\nu_S} \sum_{\mathbf{k}'} \check{\mathcal{G}}_S(\mathbf{k}', t, t'') \check{\mathcal{G}}_S(\mathbf{k}, t'', t')$$
$$= \hat{1}\delta(t - t') + \int dt'' \sum_{i, \mathbf{p}_i} T_i^2 \check{\mathcal{G}}_i(\mathbf{p}_i, t, t'') \check{\mathcal{G}}_S(\mathbf{k}, t'', t'). \qquad (2)$$

Here $\mathbf{A}(t)$ is the vector potential, which is related to the electric field via $\mathbf{E} = -\partial \mathbf{A}/\partial t$. The coefficient $1/\tau$ is the im-

purity scattering rate and ν_S is the density of states at the Fermi level in the superconductor. The matrix element T_i for tunneling into lead i determines the dimensionless conductance $g_i = 8\pi^2 \nu_S \nu_i T_i^2$ of junction i, where ν_i is the density of states in the lead. The assumption

$$g_L + g_R \ll g_S, \qquad (3)$$

where g_S is the normal-state conductance of the superconductor, justifies the use of the tunneling Hamiltonian from which the last term in Eq. (2) is derived. The same assumption enables us to neglect small spatial variations in the order parameter, which we take to be uniform, real, and time independent. The last two conditions amount to a choice of gauge.

Since we are interested in the linear response to the external field, we first consider the system in the absence of field but with the leads at different potentials. This is the focus of the next section.

III. NONEQUILIBRIUM STEADY STATE

When there is no external field ($\mathbf{A} = 0$), the Green's functions are isotropic in momentum space and depend on the difference $t - t'$. Then the semiclassical Green's functions

$$\check{G}(t - t', \mathbf{n}) = \frac{i}{\pi} \int d\xi_\mathbf{p} \check{\mathcal{G}}(\mathbf{p}, t - t'), \quad \mathbf{n} = \frac{\mathbf{p}}{|\mathbf{p}|}, \qquad (4)$$

where $\xi_\mathbf{p} = p^2/2m - E_F$, depend only on the time difference, and not on the momentum direction $\mathbf{n}$. Therefore, we omit the argument $\mathbf{n}$ for the rest of this section. In the normal leads, the elements of $\check{G}_i$ are (in the frequency domain)

$$\hat{G}_i^R = \hat{\tau}_z, \quad \hat{G}_i^A = -\hat{\tau}_z, \qquad (5)$$

$$\hat{G}_i^K = \hat{G}_i^R \hat{n}_i - \hat{n}_i \hat{G}_i^A \qquad (6)$$

with

$$\hat{n}_i(\varepsilon) = \begin{pmatrix} n(\varepsilon - eV_i) & 0 \\ 0 & n(\varepsilon + eV_i) \end{pmatrix}, \quad n(\varepsilon) = \tanh\left(\frac{\varepsilon}{2T}\right). \qquad (7)$$

The potentials V_i in the leads are measured from the superconductor chemical potential. Using Eq. (5), Eq. (6) can be rewritten as

$$\hat{G}_i^K = 2n_i^0 \hat{\tau}_z + 2n_i^1 \hat{1}, \qquad (8)$$

where

$$n_i^0 = \frac{1}{2}[n(\varepsilon - eV_i) + n(\varepsilon + eV_i)],$$
$$n_i^1 = \frac{1}{2}[n(\varepsilon - eV_i) - n(\varepsilon + eV_i)]. \qquad (9)$$

The superconductor Green's function $\check{G}_s$ is determined by Eq. (2); taking the difference between Eq. (2) and its conjugate, and using Eqs. (4), we find

$$[\varepsilon\hat{\tau}_z + i\Delta\hat{\tau}_y, \check{G}_S] = -i[\Gamma_L\check{G}_L + \Gamma_R\check{G}_R, \check{G}_S], \qquad (10)$$

where $\Gamma_i = \pi\nu_i T_i^2$ are the tunneling rates and ν_i are the densities of states in the leads $i = L, R$. This equation must be supplemented by the normalization condition,[21] which in the frequency domain takes the form

$$\check{G}_S(\varepsilon)\check{G}_S(\varepsilon) = \hat{1}. \qquad (11)$$

Isolating the $R(A)$ component of Eq. (10) gives

$$\hat{H}^{R(A)}\hat{G}_S^{R(A)} - \hat{G}_S^{R(A)}\hat{H}^{R(A)} = 0, \qquad (12)$$

where

$$\hat{H}^{R(A)} = (\varepsilon \pm i\Gamma_L \pm i\Gamma_R)\hat{\tau}_z + i\Delta\hat{\tau}_y. \qquad (13)$$

In view of the $R(A)$ component of the normalization condition in Eq. (11), $(\hat{G}_S^{R(A)})^2 = \hat{1}$, the solution to Eq. (12) can be written as

$$\hat{G}_S^{R(A)} = \hat{H}^{R(A)}/\widetilde{\xi}^{R(A)}, \qquad (14)$$

where

$$\widetilde{\xi}^{R(A)} = \pm[(\varepsilon \pm i\Gamma_L \pm i\Gamma_R)^2 - \Delta^2]^{1/2}. \qquad (15)$$

In the limit $\Gamma_i \to 0$, Eq. (14) reduces to the well-known semiclassical BCS expression.[21] When $\Gamma_i \neq 0$ the excitations in the superconductor have finite lifetime due to tunneling into the normal leads and this causes broadening of the density of states.[22,23] In what follows we assume that the broadening is much smaller than the gap, $\Gamma_L + \Gamma_R \ll \Delta$. Equivalently, we can express this condition as

$$g_L + g_R \ll \Delta/\delta \qquad (16)$$

since the dimensionless conductance, defined in the text before Eq. (3), equals $g_i = 8\pi\Gamma_i/\delta$, where $\delta = 1/(\nu_S V)$ is the level spacing in the superconductor of volume V.

We now consider the Keldysh component of Eq. (10). It may be written in the form

$$\hat{H}^R\hat{G}_S^K - \hat{G}_S^K\hat{H}^A = -i\Gamma_L(\hat{G}_L^K\hat{G}_S^A - \hat{G}_S^R\hat{G}_L^K) - i\Gamma_R(\hat{G}_R^K\hat{G}_S^A - \hat{G}_S^R\hat{G}_R^K). \qquad (17)$$

Its left-hand side can be simplified as follows: first, we use Eq. (14) to rewrite $\hat{H}^{R(A)}$ in terms of $\hat{G}_S^{R(A)}$; then we replace the advanced Green's function with the retarded one employing the orthogonality condition [the Keldysh component of Eq. (11)]

$$\hat{G}_S^R\hat{G}_S^K + \hat{G}_S^K\hat{G}_S^A = 0. $$

Multiplying the resulting equation by $\hat{G}_S^R$ from the left and using again $(\hat{G}_S^R)^2 = \hat{1}$, we arrive at

$$(\widetilde{\xi}^R + \widetilde{\xi}^A)\hat{G}_S^K = i\Gamma_L(\hat{G}_L^K - \hat{G}_S^R\hat{G}_L^K\hat{G}_S^A) + i\Gamma_R(\hat{G}_R^K - \hat{G}_S^R\hat{G}_R^K\hat{G}_S^A). \qquad (18)$$

If $|\varepsilon| < \Delta$, the sum $\widetilde{\xi}^R + \widetilde{\xi}^A$ is of the order of Δ, i.e., much larger than Γ_L and Γ_R. However, if $|\varepsilon| > \Delta$ this quantity is proportional to these small parameters and can be approximated as

$$\widetilde{\xi}^R + \widetilde{\xi}^A \approx i(\Gamma_R + \Gamma_L)\varepsilon\left(\frac{1}{\xi^R} - \frac{1}{\xi^A}\right), \qquad (19)$$

where

$$\xi^{R(A)} = \pm[(\varepsilon \pm i0^+) - \Delta^2]^{1/2}. \qquad (20)$$

Consistently with this approximation, the Green's functions $G_S^{R/A}$ in the right-hand side of Eq. (17) should be taken at zero order in Γ_i's. Using Eqs. (8) and (14), after some algebra we obtain

$$\hat{G}_S^K = \left(\frac{1}{\xi^R} - \frac{1}{\xi^A}\right)\left[n^0(\varepsilon\hat{\tau}_z + i\Delta\hat{\tau}_y) + n^1\frac{\varepsilon^2 - \Delta^2}{\varepsilon}\right], \qquad (21)$$

where $(a = 0, 1)$

$$n^a = \frac{\Gamma_L n_L^a + \Gamma_R n_R^a}{\Gamma_L + \Gamma_R} \qquad (22)$$

with $n_{L,R}^a$ of Eq. (9). We stress that Eqs. (19)–(22) are valid for arbitrary ratio Γ_L/Γ_R, and are not restricted to junctions with conductances smaller than the conductance quantum, the conditions of applicability being given by Eqs. (3) and (16). Equation (21) is the finite-temperature, small-broadening [Eq. (16)] counterpart of the zero-temperature, finite-broadening expression for the Keldysh Green's function of a mesoscopic N-I-S-I-N structure derived in Ref. 24.

The matrix $\hat{G}_S^K$ depends on the voltages V_L, V_R via Eqs. (9) and (22). However, under steady-state conditions these two quantities are not independent and, as we discuss in the next section, the state of the superconductor is fully determined by their difference $V = V_R - V_L$.

A. Current and potentials

The results of the previous sections rest on the steady-state assumption. It means that the total current flowing out of the superconductor must vanish. This requirement, as we show below, defines the division of the applied bias between the two tunnel junctions connecting the superconductor to the normal leads.

The current leaving through the left tunnel contact is given by

$$I_L = \frac{1}{4}e\nu_S\Gamma_L\int d\varepsilon \, \mathrm{Tr}[\hat{\tau}_z(\hat{G}_L^R\hat{G}_S^K + \hat{G}_L^K\hat{G}_S^A - \hat{G}_S^R\hat{G}_L^K - \hat{G}_S^K\hat{G}_L^A)]. \qquad (23)$$

The current I_R into the right contact is found by replacing $L \to R$. Therefore the total outgoing current is

$$I_L + I_R = e\nu_S\int d\varepsilon\left(\frac{1}{\xi^R} - \frac{1}{\xi^A}\right)\frac{\Delta^2}{\varepsilon}(\Gamma_L n_{1L} + \Gamma_R n_{1R}) \qquad (24)$$

and must be zero under steady-state conditions, as discussed above. [We used Eqs. (8), (14), and (21) in derivation of Eq. (24).] In other words, the requirement

$$I_L + I_R = 0 \qquad (25)$$

determines the relationship between the voltages V_L, V_R, that enter as parameters in Eq. (22).

Let us consider explicitly the regime of low temperatures and voltages so that $\max\{eV_L, eV_R, T\} \ll \Delta$. Then the following approximate expression for the functions $n^1_{L(R)}$ are valid at energies $|\varepsilon| > \Delta$:

$$n^1_i(\varepsilon) \simeq -2 \sinh(eV_i/T)\exp(-|\varepsilon|/T). \tag{26}$$

Substituting into Eqs. (24) and (25) we obtain the condition

$$\Gamma_L \sinh(eV_L/T) + \Gamma_R \sinh(eV_R/T) = 0. \tag{27}$$

Introducing the bias $V = V_R - V_L$, we find from this equation that

$$\sinh(eV_L/T) = -\frac{\Gamma_R \sinh(eV/T)}{\sqrt{\Gamma_L^2 + \Gamma_R^2 + 2\Gamma_L\Gamma_R \cosh(eV/T)}}. \tag{28}$$

Thanks to this relation, we can re-express voltages V_L, V_R in terms of the bias V, and the nonequilibrium steady state in the limit $eV, T \ll \Delta$ is fully determined by these quantities.

B. Quasiparticle distribution function

In the preceding sections we have found an explicit expression for the nonequilibrium Keldysh Green's function for the biased superconductor, Eq. (21). In this section we derive the relationship between the Keldysh Green's function and the distribution function f for the Bogoliubov quasiparticles.

Our starting point is the definition of $\hat{G}_S^K$ (in the time domain) in terms of creation and annihilation operators

$$\hat{G}_S^K(t,t') = \frac{\hat{\tau}_z}{\pi\nu_S}\sum_m \begin{pmatrix} \langle [c_{m\uparrow}(t), c^\dagger_{m\uparrow}(t')] \rangle & \langle [c_{m\uparrow}(t), c_{m\downarrow}(t')] \rangle \\ \langle [c^\dagger_{m\downarrow}(t), c^\dagger_{m\uparrow}(t')] \rangle & \langle [c^\dagger_{m\downarrow}(t), c_{m\downarrow}(t')] \rangle \end{pmatrix}. \tag{29}$$

The sum over single-particle states $\frac{1}{\nu_S}\sum_m$ with energy ξ_m is equivalent to the integration over their energy $\xi_\mathbf{p}$ in Eq. (4). Next, we perform the Bogoliubov transformation from electron operators $c_{m\uparrow}$, $c_{m\downarrow}$ to quasiparticle operators α_m, β_m

$$c_{m\uparrow} = u_m \alpha_m + v_m \beta^\dagger_m, \quad c_{m\downarrow} = u_m \beta_m - v_m \alpha^\dagger_m \tag{30}$$

with amplitudes u_m, v_m given by

$$|u_m|^2 = 1 - |v_m|^2 = \frac{1}{2}\left(1 + \frac{\xi_m}{\epsilon_m}\right),$$

$$u_m v_m = -\frac{1}{2}\frac{\Delta}{\epsilon_m} \tag{31}$$

and $\epsilon_m = \sqrt{\xi_m^2 + \Delta^2}$ being the quasiparticle energy. Introducing the distribution function (we assume equal population of the two spins)

$$f(\xi_m) = \langle \alpha^\dagger_m \alpha_m \rangle = \langle \beta^\dagger_m \beta_m \rangle \tag{32}$$

using Eq. (31), and taking the Fourier transform with respect to $t - t'$ we obtain

$$\hat{G}_S^K(\varepsilon) = 2\,\text{Re}\frac{1}{\sqrt{\varepsilon^2 - \Delta^2}}\{(\varepsilon\hat{\tau}_z + i\Delta\hat{\tau}_y)[1 - 2f^E(\varepsilon)]$$
$$- 2\sqrt{\varepsilon^2 - \Delta^2}f^Q(\varepsilon)\} \tag{33}$$

with the energy and charge modes of the distribution function $f(\xi_m)$ defined as

$$f^E(\varepsilon) = \frac{1}{2}[f(\sqrt{\varepsilon^2 - \Delta^2}) + f(-\sqrt{\varepsilon^2 - \Delta^2})],$$

$$f^Q(\varepsilon) = \frac{1}{2}[f(\sqrt{\varepsilon^2 - \Delta^2}) - f(-\sqrt{\varepsilon^2 - \Delta^2})]. \tag{34}$$

Hence the Keldysh Green's function can be presented in the form

$$\hat{G}_S^K = \hat{G}_S^R \hat{F} - \hat{F}\hat{G}_S^A, \tag{35}$$

where

$$\hat{F} = \text{sgn}\,\varepsilon[1 - 2f^E(\varepsilon)]\hat{1} - \frac{|\varepsilon|}{\sqrt{\varepsilon^2 - \Delta^2}}\left(\hat{\tau}_z + i\frac{\Delta}{\varepsilon}\hat{\tau}_y\right)2f^Q(\varepsilon). \tag{36}$$

We note that the matrix distribution function $\hat{F}$ is not diagonal. This is in contrast with the usual assumption[25,26] that in nonequilibrium situations this matrix can be expressed as a linear combination of the identity matrix and $\hat{\tau}_z$ only. Nevertheless, there are only two independent degrees of freedom since the off-diagonal $\hat{\tau}_y$ term is proportional to the $\hat{\tau}_z$ term. In fact, Eq. (36) shows that the trace of $\hat{F}$ is directly related to the energy mode f^E while the traceless part possesses an additional density of states factor compared to the charge mode f^Q of the quasiparticle distribution function.

Comparison of Eq. (36) with the solution of the kinetic equation for $\hat{G}_S^K$, Eq. (21), enables us to find the explicit expressions for the distribution function modes

$$f^E(\varepsilon) = \frac{1}{2}[1 - \text{sgn}\,\varepsilon n^0(\varepsilon)],$$

$$f^Q(\varepsilon) = -\frac{1}{2|\varepsilon|}\sqrt{\varepsilon^2 - \Delta^2}n^1(\varepsilon) \tag{37}$$

and the voltages V_L, V_R are related by Eq. (25). At low temperature and bias $T, eV \ll \Delta$, at leading order in $e^{-\Delta/T}$ we obtain, using Eq. (28)

$$f^E(\varepsilon) \approx \Phi(eV/T)e^{-\varepsilon/T} \tag{38}$$

with

$$\Phi(eV/T) = \frac{1}{\Gamma_L + \Gamma_R}\sqrt{\Gamma_L^2 + \Gamma_R^2 + 2\Gamma_L\Gamma_R \cosh(eV/T)} \tag{39}$$

while at this order f^Q vanishes due to Eq. (27).

One may also obtain the distribution function f_ξ by solving the kinetic equation for quasiparticles. The latter method is convenient to assess the effects of relaxation. The relaxation rate of quasiparticles is an intrinsic property of a super-

conductor. On the other hand, the tunneling rate $\Gamma_i = g_i \delta/(8\pi) = g_i/(8\pi \nu_s V)$ scales inversely with the volume of the superconductor at fixed conductance of the junction. Clearly, one may neglect the intrinsic relaxation only if the conductances g_i are sufficiently large and volume V is sufficiently small. The aim of the next section is to quantify these conditions. For intrinsic relaxation, we will concentrate on the dominant at low temperatures mechanism[2,17] of phonon-assisted relaxation.

C. Kinetic equation for quasiparticles

In the kinetic equation approach, the effects of interaction are described by adding to the rate of change in the occupation numbers due to tunneling an appropriate collision integral I. In the case of electron-phonon interaction, one can derive a set of coupled kinetic equations for the two modes of the quasiparticle distribution function, f^E and f^Q

$$\frac{d}{dt}f^E = \Gamma_L f_{0L} + \Gamma_R f_{0R} - \Gamma f^E + I_r^E\{f^E, f^Q, N\} + I_s^E\{f^E, f^Q, N\},$$
(40)

$$\frac{d}{dt}f^Q = -\frac{\sqrt{\varepsilon^2 - \Delta^2}}{\varepsilon}[\Gamma_L f_{1L} + \Gamma_R f_{1R}] - \Gamma f^Q + I_r^Q\{f^E, f^Q, N\} + I_s^Q\{f^E, f^Q, N\},$$
(41)

where $\Gamma = \Gamma_L + \Gamma_R$, $f_{0i} = (1 - n_i^0)/2$, and $f_{1i} = n_i^1/2$. Neglecting the collision integrals, in the steady-state $df/dt = 0$ we recover immediately the result Eq. (37). The subscripts r and s are used to distinguish, in the electron-phonon collision integrals, quasiparticle recombination and scattering processes, respectively, and we take the phonon distribution function N to be the thermal equilibrium one

$$N_\varepsilon = \frac{1}{e^{\varepsilon/T} - 1}.$$
(42)

We characterize the electron-phonon collision rate by the scattering rate, $1/\tau_{ph}$, for normal electrons off phonons at the critical temperature T_c.[27] We consider the limit of low temperature and bias, in which case the electron-phonon interaction has a small effect on the quasiparticle distribution as long as tunneling is the main scattering process. Considering relaxation rate for an arbitrary small perturbation[17] of the quasiparticle distribution function, one might conclude that the corresponding condition for that is $(1/\tau_{ph})(T/\Delta)^{7/2} \ll \Gamma$. (The small factor $(T/\Delta)^{7/2}$ here suppresses the low temperature scattering rate for quasiparticles compared to that of normal electrons at T_c.) However, for the specific case of out-of-equilibrium distribution created by tunneling the condition for the rate $1/\tau_{ph}$ is even softer, see Eq. (50) below.

Evaluating the correction to the charge mode f^Q due to the weak electron-phonon relaxation at low temperatures, we find that such correction to Eq. (37) vanishes at leading order in $e^{-\Delta/T}$ due to the relation Eq. (27). We now consider the energy mode. The collision integrals $I_{r,s}^E$ contain terms quadratic in f^Q, which we can neglect since f^Q vanishes at leading order, as discussed above. The nonvanishing terms are[15]

$$I_r^E = \frac{1}{\tau_{ph}\Delta^3}\int_\Delta^\infty d\varepsilon' \frac{\varepsilon'}{\sqrt{\varepsilon'^2 - \Delta^2}}(\varepsilon + \varepsilon')^2\left(1 + \frac{\Delta^2}{\varepsilon\varepsilon'}\right)$$
$$\times[(1 - f_\varepsilon^E)(1 - f_{\varepsilon'}^E)N_{\varepsilon+\varepsilon'} - f_\varepsilon^E f_{\varepsilon'}^E (N_{\varepsilon+\varepsilon'} + 1)]$$
(43)

for recombination processes and

$$I_s^E = \frac{1}{\tau_{ph}\Delta^3}\left\{\int_\Delta^\varepsilon d\varepsilon' \frac{\varepsilon'}{\sqrt{\varepsilon'^2 - \Delta^2}}(\varepsilon - \varepsilon')^2\left(1 - \frac{\Delta^2}{\varepsilon\varepsilon'}\right)\right.$$
$$\times[(1 - f_\varepsilon^E)f_{\varepsilon'}^E N_{\varepsilon-\varepsilon'} - f_\varepsilon^E(1 - f_{\varepsilon'}^E)(N_{\varepsilon-\varepsilon'} + 1)]$$
$$+ \int_\varepsilon^\infty d\varepsilon' \frac{\varepsilon'}{\sqrt{\varepsilon'^2 - \Delta^2}}(\varepsilon - \varepsilon')^2\left(1 - \frac{\Delta^2}{\varepsilon\varepsilon'}\right)$$
$$\left.\times[(1 - f_\varepsilon^E)f_{\varepsilon'}^E(N_{\varepsilon'-\varepsilon} + 1) - f_\varepsilon^E(1 - f_{\varepsilon'}^E)N_{\varepsilon'-\varepsilon}]\right\}$$
(44)

for scattering processes.

We solve the kinetic Eq. (40) by iterations, writing f^E in the form

$$f^E \simeq f^{E(0)} + \frac{1}{\tau_{ph}\Gamma}f^{E(1)}$$
(45)

with $f^{E(0)}$ given in Eq. (38). At leading order in $e^{-\Delta/T}$, only the scattering collision integral I_s^E is present. It is satisfied by any Boltzmann distribution function, such as the one in Eq. (38). Indeed, let us consider the terms in square brackets in the second line of Eq. (44); in the low temperature and bias regime they are approximately

$$\Phi e^{-\varepsilon'/T}N_{\varepsilon-\varepsilon'} - \Phi e^{-\varepsilon/T}(N_{\varepsilon-\varepsilon'} + 1) = 0$$
(46)

in agreement with the detailed balance in the absence of recombination. Similarly, the terms in square brackets in the last line of Eq. (44) add up to zero. Therefore, at order $e^{-\Delta/T}$ there are no corrections to the distribution function.

To calculate the correction due to recombination, we note that $\varepsilon + \varepsilon' > 2\Delta$, so that in the last term in square brackets in Eq. (43) we can neglect $N_{\varepsilon+\varepsilon'}$ in comparison to unity. The square brackets then becomes approximately

$$e^{-(\varepsilon+\varepsilon')/T}(1 - \Phi^2),$$
(47)

i.e., they are of order $e^{-2\Delta/T}$. For the correction we find

$$f^{E(1)} = e^{-\varepsilon/T}(1 - \Phi^2)\frac{1}{\Delta^3}\int_\Delta^\infty d\varepsilon' \frac{\varepsilon'}{\sqrt{\varepsilon'^2 - \Delta^2}}$$
$$\times (\varepsilon + \varepsilon')^2\left(1 + \frac{\Delta^2}{\varepsilon\varepsilon'}\right)e^{-\varepsilon'/T}.$$
(48)

After the substitution $\varepsilon' = \Delta + Tx$, the integral can be evaluated to give, at leading order

$$f^{E(1)} \simeq e^{-\varepsilon/T}(1 - \Phi^2)e^{-\Delta/T}\sqrt{\frac{\pi T}{2\Delta}}\left(\frac{\varepsilon}{\Delta}\right)^2\left(1 + \frac{\Delta}{\varepsilon}\right)^3.$$
(49)

[Note that at zero bias $\Phi(0) = 1$ and the correction vanishes.] From this equation we can estimate the ratio between the leading term and the correction in Eq. (45) and find that the iterative solution is applicable if

$$\frac{1}{\tau_{ph}\Gamma} \ll \sqrt{\frac{\Delta}{T}}e^{\Delta/T}. \qquad (50)$$

Under this condition, the electron-phonon relaxation leads only to a small modification of the distribution function given by Eqs. (45) and (49). Albeit the correction Eq. (49) at low energies ($\varepsilon \approx \Delta$) scales with temperature predominantly as $e^{-2\Delta/T}$, it may exceed the corresponding term of expansion of Eq. (37) in $e^{-\Delta/T}$. That happens at large values of $1/\tau_{ph}\Gamma$, which still may satisfy Eq. (50). Note that according to Eq. (50) the lower the temperature, the weaker is the constraint on Γ.

This concludes our analysis of the nonequilibrium steady state of a superconductor for the specific problem of a finite-bias tunnel injection of quasiparticles. In the next section we consider the linear response of an out-of-equilibrium superconductor to an external ac field.

IV. ac CONDUCTIVITY OF N-I-S-I-N STRUCTURE

In this section we study the linear response of the superconductor to an external ac electric field oscillating at frequency ω. Within linear response, the Green's function $\check{G}_S$ is the sum of the zero-approximation isotropic part $\check{G}_0$ and a small correction $\check{G}_1$, linear in $\mathbf{A}$, which is anisotropic in momentum space

$$\check{G}_S(\mathbf{n}) = \check{G}_0 + \check{G}_1(\mathbf{n}), \quad \mathbf{n} = \mathbf{p}/|\mathbf{p}|. \qquad (51)$$

The correction $\check{G}_1(\mathbf{n})$ determines the current density via

$$\mathbf{j}_\omega = -\frac{1}{4}e\nu_S \int_{-\infty}^{\infty} d\varepsilon \, \mathrm{Tr}\langle \mathbf{v}\hat{\tau}_z \hat{G}_1^K \rangle, \qquad (52)$$

where $\mathbf{v} = v_F \mathbf{n}$, v_F is the Fermi velocity, and angular brackets denote averaging over the momentum direction. We emphasize that the calculation of the ac response that we present here is not restricted to the particular nonequilibrium state considered in the previous section; rather, it is valid for a generic matrix distribution function $\hat{F}$, which determines the Keldysh part of the zero-approximation Green's function $\check{G}_0$ via Eq. (35).

The two terms in the sum in Eq. (51) obey the orthogonality condition

$$\check{G}_+ \check{G}_1(\varepsilon, \omega) + \check{G}_1(\varepsilon, \omega)\check{G}_- = 0, \qquad (53)$$

where

$$\check{G}_\pm = \check{G}_0(\varepsilon \pm \omega/2). \qquad (54)$$

An equation for $\check{G}_1$ can obtained as before by considering the difference between Eq. (2) and its conjugate and using Eqs. (4). Assuming $\mathbf{A}(t) = \mathbf{A}_\omega e^{-i\omega t}$ we find, at zeroth order in the tunneling rates Γ_i

$$-i\hat{H}_+ \check{G}_1 + i\check{G}_1 \hat{H}_- + \frac{1}{2\tau}\check{G}_+ \check{G}_1 - \frac{1}{2\tau}\check{G}_1 \check{G}_-$$

$$= ie\mathbf{v}\mathbf{A}_\omega(\hat{\tau}_z \check{G}_- - \check{G}_+ \hat{\tau}_z), \qquad (55)$$

where

$$\hat{H}_\pm = (\varepsilon \pm \omega/2)\hat{\tau}_z + i\Delta \hat{\tau}_y. \qquad (56)$$

The zero order in Γ_i approximation is valid when $\Gamma_L + \Gamma_R \ll \min\{1/\tau, \omega, \Delta\}$. In particular, the condition $\Gamma_L + \Gamma_R \ll 1/\tau$ implies that the tunneling has a negligible influence on the quasiparticle states; in fact, the only role of the leads is to generate a nonequilibrium quasiparticle population, and the calculation of the ac response does not depend on the specific way in which a nonequilibrium population is established (as long as it does not substantially alter the quasiparticle states).

Equation (55) can be solved for a generic relation between $1/\tau$ and Δ—see Appendix. Here we consider the dirty limit $\tau\Delta \ll 1$. To calculate the current, we only need the following expression for the Keldysh component:

$$\hat{G}_1^K = ie\mathbf{v}\mathbf{A}_\omega \tau [\hat{G}_+^R \hat{\tau}_z (\hat{G}_-^R - \hat{G}_-^A)\hat{F}_- + \hat{F}_+ (\hat{G}_+^R - \hat{G}_+^A)\hat{\tau}_z \hat{G}_-^A]$$
$$(57)$$

with $\hat{F}_\pm = \hat{F}(\varepsilon \pm \omega/2)$. The retarded and advanced Green's functions in this equation are given by Eqs. (13) and (14). Substituting Eq. (57) into Eq. (52), we arrive at

$$\mathbf{j}_\omega = \sigma(\omega)\mathbf{E}_\omega, \qquad (58)$$

where $\sigma(\omega)$ is the complex conductivity. It is convenient to isolate its equilibrium zero-temperature kinetic part

$$\sigma_0(\omega) = -\frac{\pi\sigma_N \Delta}{i\omega}, \qquad (59)$$

which represents the purely inductive response of the superconducting condensate to the external field in the absence of quasiparticles and where $\sigma_N = 2e^2 \nu_S D$ is the normal-state conductivity of the superconductor with $D = v_F^2 \tau/3$ the diffusion constant. Using the relationship Eq. (36) between matrix and quasiparticle distribution functions, assuming $0 < \omega < 2\Delta$, and performing simple rearrangements, we find

$$\sigma(\omega) = \sigma_0(\omega) + \frac{2\sigma_N}{\omega}\left[\int_\Delta^\infty d\varepsilon \frac{\varepsilon(\varepsilon + \omega) + \Delta^2}{\sqrt{(\varepsilon^2 - \Delta^2)[(\varepsilon + \omega)^2 - \Delta^2]}}f_\varepsilon^E \right.$$

$$-i\int_\Delta^{\Delta+\omega} d\varepsilon \frac{\varepsilon(\varepsilon - \omega) + \Delta^2}{\sqrt{(\varepsilon^2 - \Delta^2)[\Delta^2 - (\varepsilon - \omega)^2]}}f_\varepsilon^E$$

$$\left. -\int_{\Delta+\omega}^\infty d\varepsilon \frac{\varepsilon(\varepsilon - \omega) + \Delta^2}{\sqrt{(\varepsilon^2 - \Delta^2)[(\varepsilon - \omega)^2 - \Delta^2]}}f_\varepsilon^E \right]. \qquad (60)$$

Equation (60) may be viewed as a generalization of the Mattis-Bardeen formula to an arbitrary quasiparticle distribution function f_ξ. Note that Eq. (60) contains only the symmetric in ξ part of the quasiparticle distribution function.

We now use Eq. (60) to find the ac conductivity for the biased N-I-S-I-N structure considered in Sec. III, concentrating on the limit of low temperatures and voltages, $\max\{eV_L, eV_R, T\} \ll \Delta$. In this limit, f_ε^E is exponentially small and approximately given by Eq. (38). If in addition to the assumptions of low temperature and voltages made above, the frequency is also much smaller than the gap, $\omega \ll \Delta$, we find at leading order in Δ

$$\sigma(\omega) = \frac{\sigma_N \Delta}{i\omega}\{-\pi + 2e^{-\Delta/T}\Phi(eV/T)$$
$$\times [\pi \exp(-\omega/2T)I_0(\omega/2T)$$
$$+ 2i \sinh(\omega/2T)K_0(\omega/2T)]\}, \qquad (61)$$

where I_0 and K_0 denote the modified Bessel functions of zeroth order.

The effect of the finite bias on the ac response of the superconductor described by Eq. (61) is contained in full in the function Φ, Eq. (39). In fact, as we now show, this function accounts for the nonequilibrium density of quasiparticles. The quasiparticle density is given by [cf. Eq. (32)]

$$n_{qp} = 2\nu_S \int d\xi f(\xi) = 4\nu_S \int_\Delta^\infty d\varepsilon \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}} f_\varepsilon^E \qquad (62)$$

with f^E defined in Eq. (34) and the factor 2 accounting for spin. At low temperature and voltages, we can approximate f^E as [see Eq. (38)]

$$f_\varepsilon^E \simeq \Phi(eV/T)e^{-\varepsilon/T} \qquad (63)$$

and evaluating the integral at leading order in Δ we arrive at

$$n_{qp} = 4\nu_S \Phi(eV/T) \sqrt{\frac{\pi \Delta T}{2}} e^{-\Delta/T}. \qquad (64)$$

Using this equation, Eq. (61) can be written as

$$\sigma(\omega) = \sigma'(\omega) + i\sigma''(\omega),$$

$$\sigma'(\omega) = \sigma_N \frac{2\Delta}{\omega}\sinh\frac{\omega}{2T}K_0\left(\frac{\omega}{2T}\right)\sqrt{\frac{\Delta}{2\pi T}}\frac{n_{qp}}{\nu_S \Delta},$$

$$\sigma''(\omega) = \sigma_N \frac{\pi \Delta}{\omega}\left[1 - e^{-\omega/2T}I_0\left(\frac{\omega}{2T}\right)\sqrt{\frac{\Delta}{2\pi T}}\frac{n_{qp}}{\nu_S \Delta}\right]. \quad (65)$$

At low temperature, even a small bias, $V < \Delta$, may cause an exponential increase in the quasiparticle density n_{qp}, see Eq. (64). That, in turn, enhances exponentially the dissipative response $\sigma'(\omega)$. We note that the above result for the ac conductivity in terms of the quasiparticle density preserves its form even if we include the correction to the distribution function due to the electron-phonon interaction [assuming the condition Eq. (50) is satisfied]. Of course, the quasiparticle density Eq. (64) should be corrected to account for Eq. (49).

Equation (65) agrees with the results of Ref. 9, even though in that work a different mechanism was considered to be the source of the nonequilibrium distribution—a flux of photons which break Cooper pairs. In Ref. 9, the ac response of a nonequilibrium superconductor was calculated by replacing the Fermi distribution function in the expressions of Mattis and Bardeen[10] by a phenomenological one with an effective chemical potential μ^*. In the resulting formulas, μ^* was eliminated in favor of the density of quasiparticles by means of Eq. (62). The pair breaking does not create imbalance and the resulting quasiparticle distribution function contains only the energy mode. Thus our Eq. (60) supports the validity of the approach employed in Ref. 9.

V. SUMMARY AND DISCUSSION

In this work we have derived an extension of the Mattis-Bardeen formula for the ac conductivity of a dirty superconductor to include a nonequilibrium occupation of the quasiparticle states. Equation (60) shows that the charge imbalance does not affect the linear response to external radiation.

As an explicit example, we considered properties of a superconductor connected via tunnel junctions to two normal leads. Application of a constant bias to the leads creates a steady nonequilibrium state in the superconductor, see Sec. III. We have evaluated bulk ac conductivity $\sigma(\omega)$ of such N-I-S-I-N setup for low frequency $\omega \ll \Delta$ of a weak external field and low temperature ($T \ll \Delta$) while keeping fixed the dc bias V. The role of the finite bias is to create a nonequilibrium quasiparticle density n_{qp}. Remarkably, it is possible to express the conductivity $\sigma(\omega)$ as a function of temperature and n_{qp} only, see Eq. (65). The dependence of $\sigma(\omega)$ on bias V enters only through the V dependence of n_{qp}. That simplification works even if electron-phonon relaxation is taken into account [if the condition Eq. (50) is satisfied]. At zero bias ($V=0$) the distribution n_{qp} becomes the equilibrium distribution function, and Eq. (65) reduces to the conventional Mattis-Bardeen formula.

As a further extension one may include the electron-electron interaction. If sufficiently strong, it would lead to the replacement of temperature T in the quasiparticle distribution function with an effective temperature T_{eff}. Interestingly, for a fixed nonequilibrium quasiparticle density, an increase in temperature (or effective temperature) leads to a lower dissipation.

APPENDIX: SOLUTION TO EQ. (55)

In this Appendix we give some details on how to solve Eq. (55). We begin with the equation for $\hat{G}_1^R$

$$-i\hat{H}_+\hat{G}_1^R + i\hat{G}_1^R\hat{H}_- + \frac{1}{2\tau}\hat{G}_+^R\hat{G}_1^R - \frac{1}{2\tau}\hat{G}_1^R\hat{G}_-^R$$
$$= -ie\mathbf{v}\mathbf{A}_\omega(\hat{G}_+^R\hat{\tau}_z - \hat{\tau}_z\hat{G}_-^R). \qquad (A1)$$

Expressing $\hat{H}_\pm$ in Eq. (A1) in terms of $\hat{G}_\pm^R$ by means of Eq. (14) and making use of the retarded component of the orthogonality condition in Eq. (53)

$$\hat{G}_+^R\hat{G}_1^R + \hat{G}_1^R\hat{G}_-^R = 0 \qquad (A2)$$

we bring Eq. (A1) to the form

$$(-i\xi_+^R - i\xi_-^R + 1/\tau)\hat{G}_+^R\hat{G}_1^R = -ie\mathbf{v}\mathbf{A}_\omega(\hat{G}_+^R\hat{\tau}_z - \hat{\tau}_z\hat{G}_-^R) \quad (A3)$$

with the obvious notation [cf. Eq. (20)]

$$\xi_{\pm}^R = [(\varepsilon \pm \omega/2 + i0^+)^2 - \Delta^2]^{1/2}. \quad (A4)$$

Multiplying both sides of Eq. (A3) by $\hat{G}_+^R$ and making use of the normalization condition $(G_0^R)^2 = 1$, we arrive at

$$\hat{G}_1^R = \frac{e\mathbf{v}\mathbf{A}_\omega}{\xi_+^R + \xi_-^R + i/\tau}(\hat{\tau}_z - \hat{G}_+^R\hat{\tau}_z\hat{G}_-^R). \quad (A5)$$

Consider now the equation for $\hat{G}_1^K$

$$-i\hat{H}_+\hat{G}_1^K + i\hat{G}_1^K\hat{H}_- + \frac{1}{2\tau}(\hat{G}_+^R\hat{G}_1^K - \hat{G}_1^K\hat{G}_-^A)$$

$$= -ie\mathbf{v}\mathbf{A}_\omega(\hat{G}_+^K\hat{\tau}_z - \hat{\tau}_z\hat{G}_-^K) - \frac{1}{2\tau}(\hat{G}_+^K\hat{G}_1^A - \hat{G}_1^R\hat{G}_-^K) \quad (A6)$$

and the corresponding orthogonality condition

$$\hat{G}_+^R\hat{G}_1^K + \hat{G}_+^K\hat{G}_1^A + \hat{G}_1^R\hat{G}_-^K + \hat{G}_1^K\hat{G}_-^A = 0. \quad (A7)$$

It is convenient to separate $\hat{G}_1^K$ into a "regular" part, similar in form to Eq. (35), and an "anomalous" one as follows:

$$\hat{G}_1^K = \hat{G}_1^r + \hat{G}_1^a, \quad \hat{G}_1^r = \hat{G}_1^R\hat{F}_- - \hat{F}_+\hat{G}_1^A, \quad (A8)$$

where $\hat{F}_{\pm} = \hat{F}(\varepsilon \pm \omega/2)$. A substitution of this definition into the orthogonality condition in Eq. (A7) yields the much simpler equation

$$\hat{G}_+^R\hat{G}_1^a + \hat{G}_1^a\hat{G}_-^A = 0. \quad (A9)$$

Substituting Eq. (A8) into Eq. (A6) and using Eq. (A1) we find the equation for $\hat{G}_1^a$

$$-i\hat{H}_+\hat{G}_1^a + i\hat{G}_1^a\hat{H}_- + \frac{1}{2\tau}(\hat{G}_+^R\hat{G}_1^a - \hat{G}_1^a\hat{G}_-^A)$$

$$= -ie\mathbf{v}\mathbf{A}_\omega[\hat{G}_+^R(\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_-) - (\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_-)\hat{G}_-^A]. \quad (A10)$$

The solution to this equation can be found by following the procedure similar to that used above to solve for $\hat{G}_1^R$. First, we express $\hat{H}_+$ and $\hat{H}_-$ in terms of $\hat{G}_+^R$ and $\hat{G}_-^A$, respectively,

$$\left(-i\xi_+^R + \frac{1}{2\tau}\right)\hat{G}_+^R\hat{G}_1^a - \left(-i\xi_-^A + \frac{1}{2\tau}\right)\hat{G}_1^a\hat{G}_-^A$$

$$= -ie\mathbf{v}\mathbf{A}_\omega[\hat{G}_+^R(\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_-) - (\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_-)\hat{G}_-^A]. \quad (A11)$$

Then, using the orthogonality condition Eq. (A9), we eliminate $\hat{G}_-^A$ in favor of $\hat{G}_+^R$

$$(-i\xi_+^R - i\xi_-^A + 1/\tau)\hat{G}_+^R\hat{G}_1^a$$

$$= -ie\mathbf{v}\mathbf{A}_\omega[\hat{G}_+^R(\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_-) - (\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_-)\hat{G}_-^A]. \quad (A12)$$

The solution of this equation is obtained by multiplying both sides by $\hat{G}_+^R$ and employing the normalization condition $(\hat{G}_0^R)^2 = 1$

$$\hat{G}_1^a = \frac{e\mathbf{v}\mathbf{A}_\omega}{\xi_+^R + \xi_-^A + i/\tau}[\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_- - \hat{G}_+^R(\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_-)\hat{G}_-^A]. \quad (A13)$$

Equations (A5) and (A13) are valid for $\Gamma_i \ll \Delta, 1/\tau$ and arbitrary relation between Δ and $1/\tau$. In the dirty limit $\Delta\tau \ll 1$, Eqs. (A5) and (A13) can be, respectively, approximated as

$$\hat{G}_1^R = -ie\mathbf{v}\mathbf{A}_\omega\tau(\hat{\tau}_z - \hat{G}_+^R\hat{\tau}_z\hat{G}_-^R), \quad (A14)$$

$$\hat{G}_1^a = -ie\mathbf{v}\mathbf{A}_\omega\tau[\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_- - \hat{G}_+^R(\hat{F}_+\hat{\tau}_z - \hat{\tau}_z\hat{F}_-)\hat{G}_-^A]. \quad (A15)$$

Substituting these equations into Eq. (A8) we find Eq. (57).

---

[1] J. M. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, 1979).

[2] M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1996).

[3] D. N. Basov and T. Timusk, Rev. Mod. Phys. **77**, 721 (2005).

[4] O. V. Dolgov, A. A. Golubov, and D. Parker, New J. Phys. **11**, 075012 (2009).

[5] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **69**, 062320 (2004).

[6] A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nature (London) **431**, 162 (2004).

[7] S. Gleyzes, S. Kuhr, C. Guerlin, J. Bernu, S. Deleglise, U. B. Hoff, M. Brune, J.-M. Raimond, and S. Haroche, Nature (London) **446**, 297 (2007).

[8] S. Kuhr, S. Gleyzes, C. Guerlin, J. Bernu, U. B. Hoff, S. Deleglise, S. Osnaghi, M. Brune, J.-M. Raimond, S. Haroche, E. Jacques, P. Bosland, and B. Visentin, Appl. Phys. Lett. **90**, 164101 (2007).

[9] J. Gao, J. Zmuidzinas, A. Vayonakis, P. Day, B. Mazin, and H. Leduc, J. Low Temp. Phys. **151**, 557 (2008).

[10] D. C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958).

[11] A. A. Abrikosov, L. P. Gorkov, and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **35**, 265 (1958) [Sov. Phys. JETP **8**, 182 (1959)].

[12] S. B. Nam, Phys. Rev. **156**, 470 (1967); **156**, 487 (1967).

[13] See, e.g., J. P. Turneaure and I. Weissman, J. Appl. Phys. **39**, 4417 (1968), and references therein.

[14] R. Barends, J. J. A. Baselmans, S. J. C. Yates, J. R. Gao, J. N.

Hovenier, and T. M. Klapwijk, Phys. Rev. Lett. **100**, 257002 (2008).

[15] V. M. Galitskii, V. F. Elesin, and Yu. V. Kopaev, in *Nonequilibrium Superconductivity*, edited by D. N. Langenberg and A. I. Larkin (North-Holland, Amsterdam, 1986), Chap. 9.

[16] V. F. Elesin and Yu. V. Kopaev, Sov. Phys. Usp. **24**, 116 (1981).

[17] S. B. Kaplan, C. C. Chi, D. N. Langenberg, J. J. Chang, S. Jafarey, and D. J. Scalapino, Phys. Rev. B **14**, 4854 (1976).

[18] M. Tinkham, Phys. Rev. B **6**, 1747 (1972).

[19] See, e.g., G. M. Eliashberg and B. I. Ivlev, in *Nonequilibrium Superconductivity*, edited by D. N. Langenberg and A. I. Larkin (North-Holland, Amsterdam, 1986), Chap. 6 and references therein.

[20] For a review of the Keldysh approach see, e.g., Ref. 21.

[21] J. Rammer and H. Smith, Rev. Mod. Phys. **58**, 323 (1986).

[22] W. L. McMillan, Phys. Rev. **175**, 537 (1968).

[23] A. V. Zaitsev, A. F. Volkov, S. W. D. Bailey, and C. J. Lambert, Phys. Rev. B **60**, 3559 (1999).

[24] I. Snyman and Yu. V. Nazarov, Phys. Rev. B **79**, 014510 (2009).

[25] A. I. Larkin and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **68**, 1915 (1975) [Sov. Phys. JETP **41**, 960 (1976)].

[26] N. Kopnin, *Theory of Nonequilibrium Superconductivity* (Clarendon, Oxford, 2001).

[27] More precisely, in Eqs. (43) and (44) we use for $1/\tau_{ph}$ the definition of Ref. 15, which differs from the standard one (Ref. 17) by the numerical factor $(T_c/\Delta_0)^3$. Since we are interested in the low-temperature, low-bias regime, we neglect the difference between the gap Δ and its zero-temperature, equilibrium value Δ_0.