de Haas–van Alphen oscillations for nonrelativistic fermions coupled to an emergent $U(1)$ gauge field

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In this paper we investigate magneto-oscillations in the specific heat and the magnetization of nonrelativistic fermions coupled to a fluctuating $U(1)$ gauge field. This model obtains as an effective model for the underdoped cuprates realizing a so-called "algebraic charge liquid," which is a true non-Fermi-liquid state. Our study is driven by very recent observations of quantum oscillations in the underdoped cuprates. We calculate corrections to the standard Lifshitz-Kosevich expression due to the internal gauge degree of freedom for the oscillation amplitude. We perform this calculation in the dirty limit in a model with $\mathcal{N}$ species of fermions. The $\mathcal{N}\to\infty$ result corresponds to the well-known Fermi-liquid result reproducing the Lifshitz-Kosevich result. We capture the effect of the gauge field on the oscillation amplitude to Gaussian accuracy controlled in the small parameter $1/\mathcal{N}$. Our main finding is the presence of qualitative and quantitative differences compared to standard Fermi-liquid theory.

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I. INTRODUCTION

In this paper we investigate magneto-oscillations in thermodynamic quantities for systems, whose effective low-energy description implies nonrelativistic fermions coupled to an internal $U(1)$ gauge field. The model of two-dimensional nonrelativistic fermions coupled to an internal gauge field has been the subject of intense studies over the last 25 years in a variety of different contexts and systems. It appears as an effective low-energy description of a number of models falling into the class of strongly correlated electronic systems ranging from electrons in the fractional quantum-Hall regime to theories of non-Fermi-liquid phases for underdoped cuprates. Our subsequent studies are carried out with an emphasis on the second scenario but should also be of relevant for formally equivalent problems.

Experimentally, we are motivated by recent observations[1–6] of quantum oscillations in the underdoped cuprate superconductors at high magnetic fields. One of the most surprising features of these experiments was that certain aspects seem to be explainable only if one assumes the presence of fermionic pockets.

So far, these experiments have been consistently interpreted using simple Fermi-liquid models of Fermi pockets. The standard theory of quantum oscillations in Fermi liquids[7] yields oscillatory behavior of thermodynamic and transport quantities as a function of $1/B$, where B is the applied magnetic field, with an amplitude given by the Lifshitz-Kosevich (LK) prefactor.[8]

However, as the precision and range of the observations increase, it would be useful to have theoretical predictions for other candidate metallic ground states of underdoped cuprates. To this end, we will examine the amplitude of quantum oscillations in "algebraic charge liquids" (ACL).[9–11] The charged excitations in these states are described by spinless electrons coupled to an emergent $U(1)$ gauge field. This scenario is reviewed in some detail in Sec. V.

Our main finding is that these states also exhibit oscillations periodic in $1/B$, with a prefactor with small but detectable deviations from the LK theory, stemming from the presence of the internal gauge field. Recently, Ref. 12 has addressed this problem in the clean limit at finite temperatures. However, this approach does not include the oscillatory terms in the gauge field propagator, which are responsible for the main effects we describe below.

Our analysis consists of a computation of the free energy of a system of $\mathcal{N}$ species of fermions coupled to a fluctuating, emergent $U(1)$ gauge field. From the expression of the free energy we obtain all thermodynamic quantities by means of derivatives. Formally, one might think of $\mathcal{N}$ as counting the flavors of electrons. The limit $\mathcal{N}\to\infty$ allows an exact mean-field description of the ACL which coincides with a Fermi-liquid description. Fluctuations around this state are controlled in powers of $1/\mathcal{N}$. We compute the de Haas–van Alphen oscillations in the presence of an applied magnetic field in the dirty limit (specified below) to order $1/\mathcal{N}$ which implies we include the emerging gauge field to Gaussian accuracy.

Our main results for the leading order and $1/\mathcal{N}$ corrections for the specific heat are shown in Eqs. (27), (35), and (43). Most interestingly, we find a qualitative difference in the behavior of the oscillations compared to Fermi-liquid theory[13–15] as a function of T/ω_c, i.e., temperature over the cyclotron frequency. The comparison of Eqs. (27) and (43) is explicitly shown in Figs. 1 and 2. Additionally, for the specific heat, c_V, we find that the gauge field correction to the oscillatory term has a temperature (T) dependence $\sim T\ln(1/T)$ which differs from the $\sim T$ dependence in the LK term and so may potentially be detectable in recent and future experiments.[16] We also calculate corrections to the magneto-oscillations in the magnetization with the main result given in Eq. (45).

As mentioned before, it is also possible that our results have implications for quantum-Hall systems, where similar theories apply to compressible states at even denominator fillings. However, we will not explore this connection further here.

The organization of the paper is as follows. In Sec. II we introduce the generic model for spinless fermionic degrees of freedom subject to a perpendicular magnetic field in the presence of weak scalar disorder coupled to an internal gauge degree of freedom. In Sec. II A we explicitly discuss the Landau level structure and the role of disorder and explain a consistent treatment on a phenomenological level. We conclude this section by deriving the effective theory for the gauge field in the presence of disorder and Landau levels in Sec. II B.

In Sec. III we present a formalism which allows to extract the oscillatory part of the gauge field contribution to order $1/\mathcal{N}$. To this end we have to calculate oscillatory thermodynamic and transport quantities, which are shown in some detail in Appendix A. In Sec. IV we present a calculation of the specific heat of the system to order $1/\mathcal{N}$. In Sec. IV A we review the derivation of the specific heat and the associated de Haas–van Alphen oscillations of a disordered gas of electrons subject to an external magnetic field. In Sec. IV B we calculate the specific heat of the gauge field. In a first step (Sec. IV B 1) we derive the specific heat of the nonoscillatory part whereas in Sec IV B 2 we calculate the oscillatory correction. Furthermore we calculate the oscillatory contribution of the gauge field to the magnetization. In Sec. V we explain the meaning of our results in the context of a very recent ACL scenario description of the cuprates. Finally, we summarize in Sec. VI.

II. MODEL

We consider a Fermi gas of spinless fermions with quadratic dispersion minimally coupled to an internal $U(1)$ gauge field. The generic Lagrangian for such a system reads

$$\mathcal{L}_f = \bar{f}\left[\partial_\tau - iA_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m} - \mu\right]f, \tag{1}$$

where $\mathbf{A}$ denotes the internal $U(1)$ gauge field and f is the spinless electronic degree of freedom. As already mentioned in the introduction such an effective description occurs in a variety of mean-field descriptions of strongly correlated electronic systems. In Sec.V we will discuss the physical origin of the gauge field in the context of the ACL.

A. Disorder broadened Landau levels in the dirty limit

In the problem at hand we consider the above Lagrangian in the presence of an additional external magnetic field, $\mathbf{B} = \nabla \times \mathbf{a}$, perpendicular to the plane, which implies the fermions live in Landau levels. In the absence of the internal $U(1)$ gauge field $\mathbf{A}$ we can diagonalize the electronic part and cast it as

$$\mathcal{L}_f = \bar{f}\left[\partial_\tau - \omega_c\left(n + \frac{1}{2}\right)\right]f, \tag{2}$$

where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency, n the Landau level index. The degeneracy of the Landau levels is given by

$$\frac{1}{2\pi l_B^2} = \frac{m}{2\pi}\omega_c = \nu_0\omega_c, \tag{3}$$

where $\nu_0 = \frac{m}{2\pi}$ is the density of states of the two-dimensional electrons without magnetic field at the Fermi level, m the band mass, and l_B is the so-called magnetic length.

Additionally, we want to consider the effect of dilute disorder which couples to the electromagnetic charge. In the context of the two-dimensional disordered electron gas in a perpendicular magnetic field this was first considered by Ando[17] in the framework of the self-consistent Born approximation. It was realized, that on a phenomenological level in the oscillatory regime it is sufficient to introduce a finite lifetime for the electronic degrees of freedom. The phenomenological one particle retarded Green's function reads

$$G_f(\omega,n) = \frac{1}{\omega - \omega_c(n+1/2) + i/2\tau}. \tag{4}$$

The major effect of dilute disorder is thus to broaden the Landau levels. The regime of magneto-oscillations we are interested in, the so-called dirty limit, is characterized by $\omega_c\tau \ll 1$. In this regime the density of states has a constant part (just like in a Fermi liquid without magnetic field) with a smooth oscillatory part on top of it, see Eq. (15). In order to derive the effective action for the gauge field we additionally need to deal with disordered two-particle functions, which is the subject of the next section.

B. Effective action for the gauge field in the diffusive limit

To order $1/\mathcal{N}$ we need to derive the photon propagator of the internal $U(1)$ gauge field in the presence of disorder and Landau levels for the thermodynamic potential. Schematically, our derivation goes along the following line: the fermionic action reads

$$\mathcal{S} = \int_0^\beta d\tau d^2x \bar{f}[i\partial_\tau + \mu - \epsilon(-i\nabla - e\mathbf{a} - \mathbf{A}) + ia_\tau]f. \tag{5}$$

In the absence of the magnetic field we assume a parabolic dispersion, i.e., $\epsilon(\mathbf{k}) = \frac{\mathbf{k}^2}{2m}$. Consequently, we integrate out the fermions leading to

$$\mathcal{S}_{\mathrm{eff}} = -\operatorname{tr}\ln[i\partial_\tau + \mu - \epsilon(-i\nabla - e\mathbf{a} - \mathbf{A}) + ia_\tau]. \tag{6}$$

In the above expression the Gaussian integration over the electrons has schematically been performed. Expanding the above expression to second order in $\mathbf{A}$ using the canonical momentum $-i\nabla - e\mathbf{a} = \mathbf{\Pi}$ we obtain the effective Gaussian gauge field propagator whose Kernel is just the polarization operator. In principle, one could derive the effective polarization operator in the basis of the Landau levels.[18] In the presence of disorder this is a tedious calculation since also vertex corrections have to be taken into account eventually requiring a numerical solution of the problem.

However, here we choose a different route using the highly constrained form of density-density-type correlators, which allows us to obtain analytical expressions for the magneto-oscillations of the gauge field. In general, the polarization operator is obtained by expanding to second order in the internal gauge field and performing the functional derivative according to

$$\frac{\delta^2 \mathcal{S}_{\text{eff}}}{\delta \mathbf{A}^i(\mathbf{q},i\nu)\delta \mathbf{A}^j(-\mathbf{q},-i\nu)} = \hat{\Pi}^{ij}(\mathbf{q},i\nu_n), \tag{7}$$

which implies it is just the Kernel of the internal gauge field propagator. Following Halperin *et al.*[19] we keep the following two-component form of the photon propagator

$$\hat{D}^{-1}(\mathbf{q},\omega) = \hat{\Pi}(\mathbf{q},\omega) = \begin{pmatrix} \hat{\Pi}^{00} & \frac{q}{\omega}\hat{\Pi}^{xy} \\ -\frac{q}{\omega}\hat{\Pi}^{xy} & \hat{\Pi}^{yy} \end{pmatrix}. \tag{8}$$

For thermodynamic quantities it suffices to concentrate on the low-energy form of the photon propagator. The overall form of the photon propagator is highly constrained by conservation laws and the respective matrix elements read

$$\hat{\Pi}^{00} = \nu \frac{Dq^2}{Dq^2 - i\omega},$$

$$\hat{\Pi}_{yy} = i\omega\sigma_{yy} + q^2\chi,$$

$$\hat{\Pi}^{xy} = i\omega\sigma_{xy} \tag{9}$$

with ν being the density of states (DOS) at the Fermi level, σ_{yy} and σ_{xy} being the longitudinal and Hall conductivity, respectively, and χ being the diamagnetic susceptibility. The diffusive limit of the system is accounted for by replacing the low-energy form of the density-density correlator $\Pi^{00}=\nu$ by its diffusive counterpart $\Pi^{00}=\nu\frac{Dq^2}{Dq^2-i\omega}$. This, on a formal level, is achieved by including impurity ladders into the vertex function. The dependence upon the magnetic field enters through the field-dependent quantities σ_{xx}, σ_{xy}, ν, and χ. Furthermore, D denotes the diffusion constant and is given by $D=\frac{v_F^2}{d}\tau$, where d is the dimensionality, thus $d=2$ for our purposes. The free energy due to the gauge field is readily calculated using (see Halperin *et al.*[19])

$$\mathcal{F}^A = \int^{\Lambda} \frac{d^2q}{(2\pi)^2}\int \frac{d\omega}{2\pi} n_b(\omega)\arctan\frac{\text{Im det }\hat{D}^{-1}(\mathbf{q},\omega)}{\text{Re det }\hat{D}^{-1}(\mathbf{q},\omega)}. \tag{10}$$

The upper cutoff Λ is set by roughly twice the Fermi momentum, i.e., $\Lambda \approx 2k_F$. This is the upper bound on momentum transfer for the existence of low-energy excitations. Equation (10) is the central expression of this section which will allow to calculate thermodynamic quantities with and without applied magnetic field.

III. OSCILLATORY PART OF THE SPECIFIC HEAT OF THE GAUGE FIELD

In this section we introduce a framework which allows to deduce oscillatory and nonoscillatory contributions of the gauge field to thermodynamic quantities. We decompose the inverse photon propagator [Eq. (8)] into two parts, one containing the nonoscillatory contributions, called $\hat{D}_0$ and another part containing the oscillatory contributions, called $\hat{D}_{\text{osc}}$. The quantities σ_{xx}, σ_{xy}, χ, D, and ν naturally decompose into a nonoscillatory part and an oscillatory part

$$\nu = \nu_0 + \nu^{\text{osc}},$$

$$\sigma_{xx} = \sigma_{xx}^0 + \sigma_{xx}^{\text{osc}},$$

$$\sigma_{xy} = -\omega_c\tau(\sigma_{xx}^0 + \sigma_{xx}^{\text{osc}}),$$

$$\chi = \chi_0 + \chi^{\text{osc}}, \tag{11}$$

where all oscillatory contributions are expressed as a power series in $\exp(-\frac{\pi}{\omega_c\tau})$. This, on a formal level, is facilitated by a Poisson summation duality. As long as $\omega_c\tau \lesssim 1$ it suffices to retain the first moment in this power series to isolate the leading oscillatory contribution. The results for the oscillatory quantities of two-dimensional fermions subject to a magnetic field are given in the subsequent subsection.

Using an expansion controlled in the oscillatory part being small

$$\det \hat{D}^{-1} = \det \hat{D}_0^{-1}\,\det(1 + \hat{D}_0 \cdot \hat{D}_{\text{osc}}^{-1})$$
$$\approx \det \hat{D}_0^{-1}(1 + \text{tr }\hat{D}_0 \cdot \hat{D}_{\text{osc}}^{-1}) \tag{12}$$

we can formulate the contribution of the gauge field to the free energy as

$$\begin{aligned}\mathcal{F}^A &= \int^{\Lambda} \frac{d^2q}{(2\pi)^2}\int \frac{d\omega}{2\pi} n_b(\omega)\arctan\frac{\text{Im det }\hat{D}^{-1}(\mathbf{q},\omega)}{\text{Re det }\hat{D}^{-1}(\mathbf{q},\omega)} \\ &\approx \int^{\Lambda} \frac{d^2q}{(2\pi)^2}\int \frac{d\omega}{2\pi} n_b(\omega)\arctan\frac{\text{Im det }\hat{D}_0^{-1}(\mathbf{q},\omega)}{\text{Re det }\hat{D}_0^{-1}(\mathbf{q},\omega)} \\ &\quad + \int^{\Lambda} \frac{d^2q}{(2\pi)^2}\int \frac{d\omega}{2\pi} n_b(\omega)\text{Im tr }\hat{D}_0 \cdot \hat{D}_{\text{osc}}^{-1} + \mathcal{O}(e^{-2\pi/\omega_c\tau}).\end{aligned} \tag{13}$$

It is important to note that the second term is now proportional to $\exp(-\frac{\pi}{\omega_c\tau})$ and thus small compared to the first term. The first term in the above expression has been analyzed by Halperin *et al.*,[19] and is known to yield a temperature-dependent contribution to the free energy of the type $T\ln T$, which is reviewed later. The additional factor of $\ln(1/T)$ compared to a Fermi liquid is the manifestation of the well-known Altshuler-Aronov[20] correction. Equation (13) constitutes the starting point for all subsequent manipulations.

Oscillatory thermodynamic and transport quantities

We saw in the preceding sections that we can relate the entries of the polarization operator matrix to fundamental thermodynamic and transport properties of the disordered Fermi liquid. In the following calculations we retain the leading order temperature dependence of the density-density and current-current response in μ/T, however we will allow for arbitrary T/ω_c.

The DOS can be calculated in a way analogous to the grand potential (see Appendix A for a detailed presentation). In the regime $\tau\mu \gg 1$, the expression for the DOS at the Fermi level reads

$$\nu(\mu) = \nu_0\left(1 + 2\sum_{l=1}^{\infty}(-1)^l \cos\frac{2\pi l\mu}{\omega_c}e^{-\pi l/\omega_c\tau}\right), \quad (14)$$

whose leading oscillatory behavior is given by

$$\nu(\mu) \approx \nu_0\left(1 - 2\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\right). \quad (15)$$

We can calculate the longitudinal conductivity at finite T/ω_c accordingly,[17] yielding

$$\sigma_{yy} = \sigma_0\frac{1}{1+(\omega_c\tau)^2}\left[1 + 2\sum_{l=1}^{\infty}(-1)^l\cos\frac{2\pi l\mu}{\omega_c}\frac{e^{-\pi l/\omega_c\tau}\lambda_l}{\sinh\lambda_l}\right]$$
$$\approx \sigma_0\frac{1}{1+(\omega_c\tau)^2}\left(1 - 2\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\frac{\lambda_1}{\sinh\lambda_1}\right) \quad (16)$$

with $\sigma_0 = \frac{ne^2\tau}{m}$ and $\lambda_l = \frac{2\pi^2 Tl}{\omega_c}$.

Using the well-known Einstein relation for diffusive systems

$$\sigma = \nu D \quad (17)$$

we can determine the oscillatory part of the diffusion constant D. It turns out that to leading order we have

$$D = D_0\left\{1 + 2\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}[1 - \zeta(T,\omega_c)]\right\}, \quad (18)$$

where we introduced the function

$$\zeta(T,\omega_c) = \frac{2\pi^2 T/\omega_c}{\sinh 2\pi^2 T/\omega_c} \quad (19)$$

for notational convenience. We observe that for $T \ll \omega_c$ $D = D_0$ to all orders in $\exp(-\frac{\pi}{\omega_c\tau})$. Furthermore, it is straightforward to show that

$$\sigma_{xy} = -\omega_c\tau\sigma_{xx} \quad (20)$$

to all orders in $\exp(-\frac{\pi}{\omega_c\tau})$.

The diamagnetic susceptibility can be obtained from the grand potential [Eq. (A14)] by

$$\chi = -\frac{\partial^2\Omega}{\partial B^2}. \quad (21)$$

Using Eq. (A14) in the limit $\omega_c\tau \lesssim 1$, $\frac{\omega_c}{\mu} \ll 1$, and $\mu\tau \gg 1$ we obtain

$$\chi \approx -\frac{1}{24\pi m}\left[1 + 24\frac{\mu^2}{\omega_c^2}\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\zeta(T,\omega_c)\right]$$
$$= \chi_0\left[1 + 24\frac{\mu^2}{\omega_c^2}\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\zeta(T,\omega_c)\right]. \quad (22)$$

In principle, there are further corrections stemming from the oscillations in the chemical potential. These fluctuations are also small in $e^{-\pi/\omega_c\tau}$. Consequently, they lead to higher order corrections which justifies neglecting them.

IV. FREE ENERGY, SPECIFIC HEAT, AND MAGNETIZATION

In the spirit of the large-$\mathcal{N}$ treatment we can expand the free energy $\mathcal{F}$ to order $1/\mathcal{N}$, which yields the following result:

$$\mathcal{F} = \mathcal{N}\mathcal{F}^{0f} + \mathcal{F}^A, \quad (23)$$

where $\mathcal{F}^{0f}$ is the free energy of the noninteracting fermionic system and $\mathcal{F}^A$ denotes the free energy associated with fluctuations of the emergent gauge field.

The specific heat can be obtained by differentiation, i.e., $c_V = -T\frac{\partial^2\mathcal{F}}{\partial T^2}$. This implies that the specific heat decomposes into two parts

$$c_V = \mathcal{N}c_V^f + c_V^A, \quad (24)$$

which are analyzed independently in Secs. IV A and IV B. The same kind of relation holds for the magnetization, which is calculated in Sec. IV C.

A. Specific heat of the electrons

The free energy is related to the grand potential via Legendre transform

$$\mathcal{F}^{0f} = \mu N + \Omega. \quad (25)$$

The dependence of the chemical potential upon the magnetic field is subdominant, thus

$$c_V = -T\frac{\partial^2\mathcal{F}^{0f}}{\partial T^2} \approx -T\frac{\partial^2\Omega}{\partial T^2} \quad (26)$$

with the oscillatory contribution obtained from Eq. (A14)

$$c_V^{\rm osc} = \alpha(T)\frac{\nu_0}{2\pi^2}T\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau} = \alpha(T)\frac{mT}{4\pi^3}\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}, \quad (27)$$

where

$$\alpha(T) = \omega_c^2 \frac{\partial^2}{\partial T^2}\zeta(T,\omega_c) = \frac{4\pi^6 T}{\omega_c \sinh^3\left(\frac{2\pi^2 T}{\omega_c}\right)}\left[3 + \cosh\left(\frac{4\pi^2 T}{\omega_c}\right)\right] - \frac{4\pi^4 \sinh\left(\frac{4\pi^2 T}{\omega_c}\right)}{\sinh^3\left(\frac{2\pi^2 T}{\omega_c}\right)}. \quad (28)$$

We compare this function to the oscillatory contribution from the gauge field in Sec. IV B 2.

In the limit $\frac{T}{\omega_c} \ll 1$ this reduces to the well-known expression

$$c_V = \frac{\pi^2 \nu_0}{3} T\left(1 - 2\cos\frac{2\pi\mu}{\omega_c} e^{-\pi/\omega_c \tau}\right) \quad (29)$$

per species.

B. Specific heat of the gauge field

We proceed to calculate the nonoscillatory and oscillatory contributions of the gauge field to the specific heat, separately.

1. Nonoscillatory contribution

We start with a calculation of the nonoscillatory part of the specific heat of the gauge field. In order to do so we analyze

$$\int^{\Lambda} \frac{d^2 q}{(2\pi)^2} \int \frac{d\omega}{2\pi} n_b(\omega) \arctan \frac{\operatorname{Im} \det \hat{D}_0^{-1}(\mathbf{q},\omega)}{\operatorname{Re} \det \hat{D}_0^{-1}(\mathbf{q},\omega)}. \quad (30)$$

We can rewrite the contribution to the specific heat as

$$c_V^A = \frac{T}{16\pi^2} \int_0^{\Lambda/\sqrt{T}} dk \int_{-\infty}^{\infty} dx \times \frac{k\left[2x - x^2 \coth\left(\frac{x}{2}\right)\right]}{\sinh(x/2)^2} \arctan \frac{cxk^2}{ak^4 + bx^2} \quad (31)$$

with

$$a = D\chi_0[1 + 48\pi^2(\omega_c\tau)^2(\sigma_{yy}^0)^2],$$

$$b = -\sigma_{yy}^0(1 + \omega_c^2\tau^2),$$

$$c = D\sigma_{yy}^0 + \chi_0 \approx D\sigma_{yy}^0 \quad (32)$$

for $\mu\tau \gg 1$. In order to analyze the asymptotic behavior of the above x integral, we start noting that the x integration is cutoff by the factor $\frac{1}{\sinh^2 \frac{x}{2}}$ on the order of $x=10$. If we consider the factor $\frac{1}{ak^4+bx^2}$, we know that for $k \gg \sqrt{10}(\frac{b}{a})^{1/4}$ it becomes $\frac{1}{ak^4}$ whereas for $k \ll 4$ it becomes $\frac{1}{x^2}$. Only in the former case will the integral contribute a logarithmic dependence upon temperature hence

$$c_V^A \approx -\frac{cT}{6a} \int_{\sqrt{10}\left(\frac{b}{a}\right)^{1/4}}^{\Lambda/\sqrt{T}} \frac{dk}{k} = -\frac{cT}{12a} \ln \frac{\Lambda^2 a^{1/2}}{10 b^{1/2} T}. \quad (33)$$

We furthermore introduce the short form

$$\kappa = 1 + 48(\mu\tau)^2(\omega_c\tau)^2, \quad (34)$$

which is a number of order 1. This leaves us with

$$c_V^A = 8\frac{\mu\tau m}{\kappa} T \ln\left(\frac{8\mu}{5T}\sqrt{\frac{\kappa}{3}}\right) \quad (35)$$

as the nonoscillatory contribution.

2. Oscillatory contribution of the gauge field

In this section we analyze the oscillatory contribution of the gauge field. Following the prescription given in Eq. (12) we decompose the gauge-field propagator according to

$$\hat{D}^{-1} = \begin{pmatrix} \nu \frac{Dq^2}{Dq^2 - i\omega} & iq\sigma_{xy} \\ -iq\sigma_{xy} & i\omega\sigma_{yy} + \chi q^2 \end{pmatrix} = \begin{pmatrix} \nu_0 \frac{D_0 q^2}{D_0 q^2 - i\omega} & iq\sigma_{xy}^0 \\ -iq\sigma_{xy}^0 & i\omega\sigma_{yy}^0 + \chi_0 q^2 \end{pmatrix} + \begin{bmatrix} \left(\nu \frac{Dq^2}{Dq^2 - i\omega}\right)_{\mathrm{osc}} & iq\sigma_{xy}^{\mathrm{osc}} \\ -iq\sigma_{xy}^{\mathrm{osc}} & i\omega\sigma_{yy}^{\mathrm{osc}} + \chi^{\mathrm{osc}} q^2 \end{bmatrix} = \hat{D}_0^{-1} + \hat{D}_{\mathrm{osc}}^{-1}. \quad (36)$$

Using the oscillatory expressions derived in Sec. III we identify

$$\hat{D}_{\rm osc}^{-1} \approx -2\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\hat{D}_0^{-1} + \cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\times\begin{bmatrix} -2\frac{\nu_0 D_0^2 q^4}{(D_0q^2-i\omega)^2}[1-\zeta(T,\omega_c)] & 0 \\ 0 & 24\frac{\mu^2}{\omega_c^2}\chi_0 q^2\zeta(T,\omega_c)\end{bmatrix}, \quad (37)$$

where $\frac{\omega_c}{\mu}\ll 1$ was used. Calculating Im tr $\hat{D}_0\cdot\hat{D}_{\rm osc}^{-1}$ we realize that the first term on the right-hand side of Eq. (37) does not contribute an imaginary part, thus

$$\text{Im tr } \hat{D}_0\cdot\hat{D}_{\rm osc}^{-1} = -24\frac{\mu^2}{\omega_c^2}\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau} \times\Bigg\{\zeta(T,\omega_c)\frac{Cq^2\omega}{Aq^4+B\omega^2}+\frac{\omega_c^2}{12\mu^2} \times[\zeta(T,\omega_c)-1]\frac{\tilde{C}q^2\omega}{Aq^4+B\omega^2}\frac{\tilde{a}q^4+\tilde{b}\omega^2}{D^2q^4+\omega^2}\Bigg\}. \quad (38)$$

Looking at Eq. (38) it is interesting to note that there are two contributions. The first contribution survives in the limit $T/\omega_c\to 0$, whereas the second part goes to zero. Additionally, the second term is down by a factor $\frac{\omega_c^2}{\mu^2}$ and is thus parametrically small compared to the first term. Consequently, we will discard the second contribution for our analysis. The remaining constants read

$$A = D_0[(\sigma_{xy}^0)^2 - \nu_0\chi_0]^2,$$

$$B = [(\sigma_{xy}^0)^2 + (\sigma_{yy}^0)^2]^2,$$

$$C = D_0\nu_0\chi_0[(\sigma_{xy}^0)^2 + (\sigma_{yy}^0)^2]. \quad (39)$$

The constants of the disordered electronic system involved in the above expression assume the well-known values

$$\sigma_{yy}^0 = \frac{1}{1+(\omega_c\tau)^2}\sigma_0,$$

$$\sigma_{xy}^0 = -\frac{\omega_c\tau}{1+(\omega_c\tau)^2}\sigma_0,$$

$$\nu_0 = \frac{m}{2\pi}, \quad \chi_0 = -\frac{1}{24\pi m},$$

$$\frac{\mu^2}{\omega_c^2} = \left(\frac{2\pi}{\omega_c\tau}\right)^2\sigma_0^2, \quad D_0 = \frac{2\pi\sigma_0}{m}. \quad (40)$$

We proceed to calculate the temperature dependent part of the free energy (see Appendix B). This part reads

$$\mathcal{F}_{\rm osc}^A = -\frac{\mu^2}{\omega_c^2}\frac{C}{A}\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\zeta(T,\omega_c)T^2\ln\left(\sqrt{\frac{A}{B}}\frac{\Lambda^2}{T}\right) - \xi\frac{\mu^2}{\omega_c^2}\frac{C}{A}\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\zeta(T,\omega_c)T^2 \quad (41)$$

with $\xi\approx 0.352822$ and constitutes the main result since it allows to obtain all oscillatory contributions due to the gauge field via differentiation. Consequently, thermodynamic quantities obtained via differentiation of Eq. (41) will carry the labels A and osc. We introduce the dimensionless functions

$$\beta(T) = -\frac{\partial^2}{\partial T^2}T^2\zeta(T,\omega_c),$$

$$\gamma(T) = -\zeta(T,\omega_c) + \frac{1}{T}\frac{\partial}{\partial T}T^2\zeta(T,\omega_c) + \xi\beta(T), \quad (42)$$

which allow to express the specific heat as

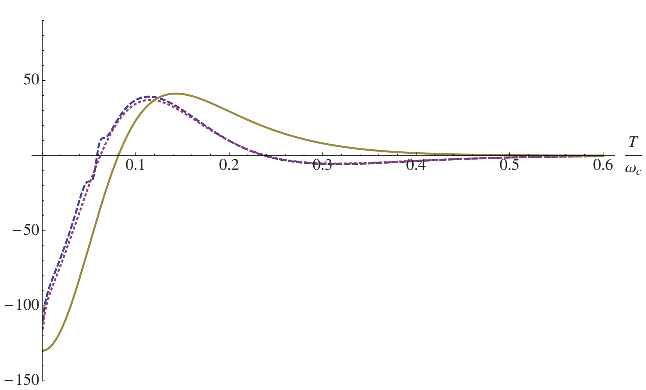

FIG. 1. (Color online) Function $\alpha(T)$ (yellow/solid), $5\delta(T)$ (blue/dashed), and $5\beta(T)\ln(100/T)$ (red/dotted) where T is measured in units of ω_c and $\frac{4\kappa}{3\tau}=100$ in units of ω_c.

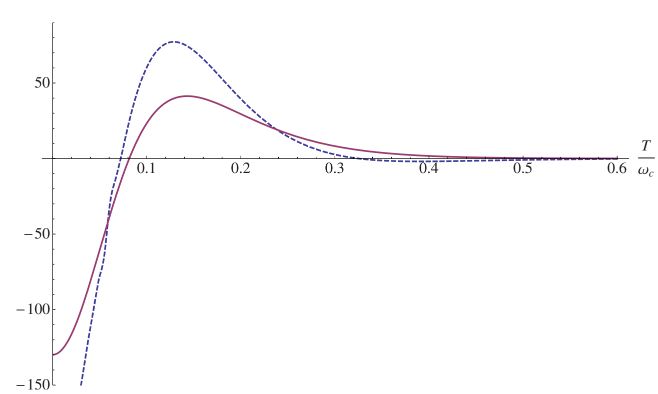

FIG. 2. (Color online) ACL result for the oscillation amplitude $5\delta(T)+\alpha(T)$ (blue/dashed) compared to the Fermi-liquid result $\alpha(T)$ (red/thick) where T is measured in units of ω_c and $\frac{4\kappa}{3\tau}=100$ in units of ω_c.

$$c_V^{A\mathrm{osc}} = \cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\frac{24(\mu\tau)^3}{(\omega_c\tau)^2\kappa^2}mT\ln\left(\sqrt{\frac{A}{B}}\frac{\Lambda^2}{T}\right)\beta(T) + \cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}\frac{24(\mu\tau)^3}{(\omega_c\tau)^2\kappa^2}mT\gamma(T) \quad (43)$$

with κ as defined in Eq. (34). It is important to note here that for $T \ll \omega_c$ both $\beta(T)$ and $\gamma(T)$ approach constant values. Thus, at low temperatures the first term with the logarithmic temperature dependence will dominate the second term. If we compare standard Fermi-liquid oscillations [Eq. (27)] with the $1/\mathcal{N}$ corrections we realize that their ratio behaves as

$$\mathcal{N}\frac{c_V^{A\mathrm{osc}}(T)}{c_V^{\mathrm{osc}}(T)} = \frac{96\pi^3(\mu\tau)^3}{(\omega_c\tau)^2\kappa^2}\frac{\beta(T)\ln\left(\sqrt{\frac{A}{B}}\frac{\Lambda^2}{T}\right)+\gamma(T)}{\alpha(T)} = \frac{96\pi^3(\mu\tau)^3}{(\omega_c\tau)^2\kappa^2}\frac{\delta(T)}{\alpha(T)}. \quad (44)$$

In order to get a meaningful comparison at different temperatures we plot the dimensionless quantities $\alpha(T)$, $\beta(T)\ln(100/T)$, and $\delta(T)$ in Fig. 1, where $\delta(T)$ denotes the oscillation amplitude of the oscillations associated with the gauge field, and $\alpha(T)$ is the standard Fermi liquid result. We refrain from plotting $\gamma(T)$, since it is, as mentioned before, subdominant with respect to the logarithmic part. For a reasonable qualitative comparison of the functional forms of the two contributions we take the dimensionless oscillation amplitude prefactor in Eq. (44) to be on the order of 5.

It is also useful to compare the sum of the oscillatory contribution in the ACL to standard Fermi-liquid behavior. This plot is shown in Fig. 2 where the same numerical values as for Fig. 1 were used. From the figures we see that in an experiment the most clear indications for an ACL ground state would be a lot more structure on the curve compared to the Fermi liquid as well as a logarithmically diverging amplitude as function of temperature T once the linear in T behavior is divided out.

C. Oscillatory magnetization

A quantity which is also often measured in magneto-oscillations measurements is the magnetization. For this reason we also make predictions for experimental signatures of the ACL ground state. Again, we calculate the oscillatory contribution due to the gauge field from Eq. (41). From elementary thermodynamic relations we obtain

$$M_{\mathrm{osc}}^A = -\frac{\partial F_{\mathrm{osc}}^A}{\partial B} = \frac{C}{A\nu_0}\mu\frac{(\mu\tau)^2}{(\omega_c\tau)^2}\zeta(T,\omega_c)\sin\left(\frac{2\pi\mu}{\omega_c}\right) \times \frac{T^2}{\omega_c^2}\left[\ln\left(\sqrt{\frac{A}{B}}\frac{\Lambda^2}{T}\right)+\xi\right]e^{-\pi/\omega_c\tau}. \quad (45)$$

The temperature behavior of this result constitutes a stark deviation from the standard result in a Fermi liquid due to Champel *et al.*[21] reading

$$M = \frac{2\nu_0\omega_c\mu}{\pi B}\zeta(T,\omega_c)\sin\left(\frac{2\pi\mu}{\omega_c}\right)e^{-\pi/\omega_c\tau}. \quad (46)$$

Again, as in the case of the specific heat, the oscillation period is unaffected, but the amplitude is modified and strongly temperature dependent (note the $T^2\ln 1/T$). Again, such a deviation from Fermi-liquid predictions could be expected to show if the underlying ground state is of the ACL type.

V. APPLICATION TO THE UNDERDOPED CUPRATE SUPERCONDUCTORS

Recent experiments performed in the underdoped regime of the cuprate superconductors show great promise to shed light onto some aspects of these still mysterious materials. In our discussion we focus on quantum oscillation measurements in the underdoped region of $YBa_2Cu_3O_{7-\delta}$.[1–6] In this context, LeBoeuf *et al.*[6] observed signatures indicating the presence of pockets of carriers of charge $-e$ (in contrast to holelike charge carriers).

Here, we will discuss these experiments using a specific theoretical model[9–11,22,23] for the interplay between spin-density wave (SDW) and superconductivity in the underdoped regime. We investigate the magneto-oscillations for magnetic fields greater than H_{c2}, i.e., in the absence of superconductivity. In contrast to earlier works[10] which investigated Shubnikov-de Haas oscillations, we emphasis the role of de Haas–van Alphen oscillations, i.e., oscillation in thermodynamic quantities.

The theory has two nonsuperconducting metallic phases. One has long-range SDW order and so is a conventional Fermi liquid at low-enough temperatures: here the magneto-oscillations will be given by the LK theory. The other metallic phase has no SDW order but retains aspects of the Fermi pocket structure of the SDW-ordered phase: this is the algebraic charge liquid. The ACL has an emergent gapless $U(1)$ photon which will lead to corrections to the LK theory, as we have discussed above. The photon acquires a Higgs mass, Δ_{AF}, across the transition from the ACL to the SDW phase and so its effects are quenched in the SDW phase. We want to stress here that the gauge field is an important component in the description of states with the mentioned electron/hole pocket structure. *Ad hoc* models usually violate the Luttinger Fermi area law. However, it was argued that metallic states with non-Luttinger-Fermi surfaces must have topological order, which means there is an additional collective excitation which is faithfully described by the emergent gauge field.[24,25] The above described scenario is explained in great detail in Refs. 10 and 22.

The specific model has the Lagrangian

$$\mathcal{L} = \mathcal{L}_z + \mathcal{L}_g + \mathcal{L}_f \quad (47)$$

with the respective constituents explained below.

The first term describes the magnetic degrees of freedom. Conventionally, the slow magnetic degrees of freedom are expressed in the framework of $O(3)$ nonlinear sigma model. Here, however, we map the SDW order parameter $\vec{\phi}$ to bosonic degrees of freedom z, which carry spin $S=\frac{1}{2}$, via

$$\vec{\phi} = \sum_{\alpha,\beta} z_\alpha^\star \vec{\sigma}_{\alpha\beta} z_\beta \tag{48}$$

and the effective Lagrangian assumes the form of the so-called CP^1 model

$$\mathcal{L}_z = \frac{1}{t}\left[\sum_{\alpha=1}^{N}\left(|(\partial_\tau - iA_\tau)z_\alpha|^2 + v^2|(\nabla - i\mathbf{A})z_\alpha|^2\right) + i\rho\sum_{\alpha=1}^{\mathcal{N}}\left(|z_\alpha|^2 - \mathcal{N}\right)\right], \tag{49}$$

where A_τ and $\mathbf{A}$ denote an internal $U(1)$ gauge field emerging from the redundant parametrization of the SDW order parameter shown in Eq. (48). Furthermore, t denotes the stiffness, v the spin-wave velocity and ρ serves as Lagrangian multiplier. Interestingly, the bosonic spinons z locally determine the spin axis of the physical electrons, which leads to fractionalization of the spin and the charge degrees of freedom. Consequently, the effective charge carriers also couple to the internal gauge field. It was known for a long time that the existence of spin-density-wave order, i.e., $\langle z\rangle \neq 0$, is responsible for a Fermi surface reconstruction.[26,27] In a very simple approximation we can take this fact into account by introducing operators g_+ and g_- for the electronic pockets sitting at the antinodal points $(\pi,0)$ and f_{+1}, f_{+2}, f_{-1}, and f_{-2} for the hole degrees of freedom sitting at the nodal points $(\pi/2,\pi/2)$

$$\mathcal{L}_g = \bar{g}_+\left[\partial_\tau - iA_\tau - \frac{(\nabla - i\mathbf{A})^2}{2m_1} - \mu\right]g_+ + \bar{g}_-\left[\partial_\tau + iA_\tau - \frac{(\nabla + i\mathbf{A})^2}{2m_1} - \mu\right]g_- \tag{50}$$

and

$$\mathcal{L}_f = \sum_{q,a}\bar{f}_{qa}\left[\partial_\tau - iqA_\tau - \frac{(\nabla - iq\mathbf{A})^2}{2m_2} - \mu\right]f_{qa}, \tag{51}$$

where $q=\pm$ and $a=1,2$. As explained before, all the fermionic degrees are coupled to the internal $U(1)$ gauge field but carry different charges under the transformation.

We concentrate on the metallic SDW and ACL states with small Fermi pockets, i.e., for magnetic fields H bigger than the critical field strength H_{c2} above which superconductivity is destroyed. In the above model we again introduce disorder and the magnetic field on the level of the single-particle propagator with two scattering times τ_g and τ_f for electron and hole pockets, respectively. We realize that there are two cyclotron frequencies ω_{cg} and ω_{cf}. Additionally, there are two scattering times. The following discussion is again valid in the limit $\omega_{cg}\tau_g, \omega_{cf}\tau_f \lesssim 1$. The specific heat of the system is composed of four different contributions

$$c_V = \mathcal{N}_z c_V^z + \mathcal{N} c_V^g + \mathcal{N} c_V^f + c_V^A. \tag{52}$$

The first term is due to the bosonic spinons and was calculated in Ref. 28. The two following terms were calculated together with the magneto-oscillations in Eq. (29). Turning to the gauge field propagator we realize that the gauge field propagator is still given by Eq. (8), however Eq. (9) modifies to

$$\hat{\Pi}^{00} = \nu_g \frac{D_g q^2}{D_g q^2 - i\omega} + \nu_f \frac{D_f q^2}{D_f q^2 - i\omega},$$

$$\hat{\Pi}^{yy} = i\omega\sigma_{yy}^g + q^2\chi_g + i\omega\sigma_{yy}^f + q^2\chi_f,$$

$$\hat{\Pi}^{xy} = i\omega\sigma_{xy}^g + i\omega\sigma_{xy}^f, \tag{53}$$

which implies we can write

$$\hat{D}^{-1} = \hat{D}_g^{-1} + \hat{D}_f^{-1}. \tag{54}$$

A further modification comes into the picture due to the presence of the bosonic spinons. In the SDW state they condense, i.e., $\langle z\rangle^2 \sim \Delta_{AF}$, implying that the Higgs effect contributes a mass term leading to $\hat{\Pi}^{yy} = i\omega\sigma_{yy}^g + q^2\chi_g + i\omega\sigma_{yy}^f + q^2\chi_f + \Delta_{AF}$.

For the moment we will neglect this term and defer the discussion of the SDW state to the end of the section. As we discussed earlier, the oscillatory contributions to the thermodynamic and transport quantities entering Eq. (53) are of the form $e^{-\pi/\omega_{cg}\tau_g}$ and $e^{-\pi/\omega_{cf}\tau_f}$. In the following we will consider $e^{-\pi/\omega_{cf}\tau_f} \ll e^{-\pi/\omega_{cg}\tau_g}$. With this we can express the oscillating term to leading order in $\frac{\mu}{\omega_c}$ according to

$$\hat{D}_{\text{osc}}^{-1} = -2\cos\frac{2\pi\mu}{\omega_{cg}} e^{-\pi/\omega_{cg}\tau_g}\hat{D}_0^{-1} + 24\frac{\mu^2}{\omega_{cg}^2}\cos\frac{2\pi\mu}{\omega_{cg}} e^{-\pi/\omega_{cg}\tau_g}\zeta(T,\omega_{cg})\begin{pmatrix}0 & 0\\ 0 & \chi_{g0}q^2\end{pmatrix} + \mathcal{O}\left(e^{-2\pi/\omega_{cg}\tau_g}, e^{-\pi/\omega_{cf}\tau_f}, \left(\frac{\mu}{\omega_c}\right)^0\right). \tag{55}$$

As in our preceding discussion in Sec. IV B 2, the first term drops out and eventually we can write the whole expression as

$$\text{Im tr}\,\hat{D}_0 \cdot \hat{D}_{\text{osc}}^{-1} = -24\frac{Cq^6\omega + \tilde{C}q^2\omega^3}{Aq^8 + \tilde{A}q^4\omega^2 + B\omega^4}\frac{\mu^2}{\omega_{cg}^2} \times \cos\frac{2\pi\mu}{\omega_{cg}} e^{-\pi/\omega_{cg}\tau_g}. \tag{56}$$

Again, we are only interested in isolating the logarithmic behavior, for which we only need to know the constants A, B, and C. We find, following the earlier calculation, that the logarithmic part of the specific heat is given by

$$c_V^{A\mathrm{osc}} = -\frac{C}{A}\frac{\mu^2}{\omega_{cg}^2}\cos\frac{2\pi\mu}{\omega_{cg}}e^{-\pi/\omega_{cg}\tau_g} \times \ln\left(\sqrt{\frac{A}{B}}\frac{\Lambda^2}{T}\right)T\frac{\partial^2}{\partial T^2}T^2\zeta(T,\omega_{cg}) \tag{57}$$

with

$$A = D_{g0}^2 D_{f0}^2[(\sigma_{xy}^0)^2 - \nu_0\chi_0]^2,$$

$$B = (\sigma_{yy}^0)^4,$$

$$C = D_{g0}^2 D_{f0}^2 \nu_0^2 \sigma_{yy}^0 \chi_{g0}, \tag{58}$$

where

$$\sigma_{xy}^0 = \sigma_{xy}^{g0} + \sigma_{xy}^{f0},$$

$$\sigma_{yy}^0 = \sigma_{yy}^{g0} + \sigma_{yy}^{f0},$$

$$\nu_0 = \nu_{g0} + \nu_{f0},$$

$$\chi_0 = \chi_{g0} + \chi_{f0}. \tag{59}$$

In the SDW phase, the presence of a finite Higgs term $\Delta_{AF} \neq 0$ in the gauge field propagator introduces a new energy scale into the problem and the magneto-oscillations will essentially be given by LK theory. Without going into the details we can show this along the lines of the derivation of Eq. (57). The logarithmically in temperature diverging prefactor is now cutoff by the Higgs mass Δ_{AF}. A compact formulation of Eq. (57) treating both regimes is given by

$$c_V^{A\mathrm{osc}} = -\frac{C}{A}\frac{\mu^2}{\omega_{cg}^2}\cos\frac{2\pi\mu}{\omega_{cg}}e^{-\pi/\omega_{cg}\tau_g} \times \ln\left(\sqrt{\frac{A}{B}}\frac{\Lambda^2}{\max\left[\frac{\Delta_{AF}}{\chi},T\right]}\right)T\frac{\partial^2}{\partial T^2}T^2\zeta(T,\omega_{cg}), \tag{60}$$

which in the SDW phase at very low temperatures ($T \ll \frac{\Delta_{AF}}{\chi}$) corresponds to a simple renormalization of Fermi-liquid theory.

VI. SUMMARY AND DISCUSSION

In this paper we analyzed magneto-oscillations in the specific heat and the magnetization of two-dimensional nonrelativistic fermions coupled to a $U(1)$ gauge field, realizing a so-called algebraic charge liquid. Our results apply to a variety of different effective descriptions of strongly interacting fermionic theories, including the description of the $\nu=1/2$ fractional quantum-Hall state[19,29,30] and different effective low-energy gauge theory descriptions of the cuprate superconductors.[10,31–33] The starting point for all our calculations is the calculation of the oscillatory part of the free energy of the gauge field to Gaussian accuracy, see Eq. (41), which only depends upon input quantities of the Fermi liquid state, which obtains in the limit $\mathcal{N}\to\infty$. For the specific heat, we calculate a correction to the LK amplitude of the oscillations given in Eq. (43) and compare it to the standard LK result in Figs. 1 and 2. Most interestingly, we find nonlinear temperature dependence. We also calculate the magnetization oscillations due to the gauge field [Eq. (45)], which again shows very different temperature behavior of the amplitude compared to the Fermi-liquid result. We point out a couple of generic situations, in which our result holds and furthermore discuss its meaning in the context of the gauge theoretic description of the underdoped cuprates introduced by Galitski and Sachdev,[10] leading to Eq. (60) as the central result.

We note that Refs. 34 and 35 have investigated quantum oscillations in a non-Fermi-liquid state using the anti-de-Sitter/conformal field theory (AdS/CFT) correspondence, finding different deviations from the LK form.

ACKNOWLEDGMENTS

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APPENDIX A: THE GRAND POTENTIAL

The grand potential of the disordered Fermi gas in a magnetic field is readily calculated using the fermionic propagator

$$\Omega = -\beta^{-1}\frac{1}{2\pi l_B^2}\sum_{m,n}\ln[-G^{-1}(\omega_n,E_m)], \tag{A1}$$

where

$$G(\omega_n,E_m) = \frac{1}{i\omega_n + \mu - E_m + \frac{i}{2\tau}\mathrm{sgn}\ \omega_n} \tag{A2}$$

with $E_m=\omega_c(m+1/2)$. We use the following Poisson summation identity[7]

$$\sum_m f_n\left[\omega_c\left(m+\frac{1}{2}\right)\right] = \int_0^\infty \frac{dx}{\omega_c}f_n(x) - 2\sum_{l=1}^{\infty}\frac{(-1)^l}{2\pi l}\int_0^\infty dx f_n'(x)\sin\frac{2\pi l x}{\omega_c} \tag{A3}$$

with

$$f_n'(x) = \frac{-1}{i\omega_n + \mu - x + i\frac{\mathrm{sgn}\ \omega_n}{2\tau}} = -\frac{1}{2\pi\tau}\int_{-\infty}^{\infty} d\omega \frac{1}{i\omega_n - \omega}\frac{1}{(\omega+\mu-x)^2 + \frac{1}{4\tau^2}}. \tag{A4}$$

We can now perform the Matsubara sum yielding

$$\beta^{-1}\sum_n f_n'(x) = \int_{-\infty}^{\infty} d\omega n_f(\omega) G''(\omega). \tag{A5}$$

We furthermore use $\frac{1}{2\pi l_B^2} = \nu_0\omega_c$, which implies

$$\Omega = -\beta^{-1}\sum_n \nu_0 \int_0^\infty dE \ln\left(-i\omega_n + \mu - E - i\frac{\text{sgn}\,\omega_n}{2\tau}\right) + 2\nu_0\omega_c \sum_{l=1}^{\infty} \frac{(-1)^l}{2\pi^2 l}\int_0^\infty dE \sin\frac{2\pi l E}{\omega_c} \times \int d\omega n_f(\omega) G''(\omega,E). \tag{A6}$$

We will now concentrate on calculating the second term. We integrate the energy by parts to obtain

$$\int_0^\infty dE \sin\frac{2\pi lE}{\omega_c}\int d\omega n_f(\omega)G''(\omega,E) = \int d\omega n_f(\omega)\frac{\omega_c}{2\pi l}\cos\frac{2\pi lE}{\omega_c}[G''(\omega,0) - G''(\omega,\infty)] - \int d\omega n_f(\omega)\frac{\omega_c}{2\pi l}\int_0^\infty dE \cos\frac{2\pi lE}{\omega_c}\frac{d}{dE}G''(\omega,E). \tag{A7}$$

The first term on the right hand only produces one term. This can easily be checked, since for $E=0$ the expression is finite, whereas for $E\to\infty$ one can see, that the term vanishes, since the Fermi function restricts ω to be smaller than zero. Consequently, we obtain

$$\int_0^\infty dE \sin\frac{2\pi lE}{\omega_c}\int d\omega n_f(\omega)G''(\omega,E) = \int d\omega n_f(\omega)\frac{\omega_c}{2\pi l}G''(\omega,0) - \int d\omega n_f(\omega)\frac{\omega_c}{2\pi l}\int_0^\infty dE \cos\frac{2\pi lE}{\omega_c}\frac{d}{dE}G''(\omega,E). \tag{A8}$$

We can split the remaining task into two parts. For the first integral one obtains in the limit $\mu \gg \frac{1}{2\tau}, T$

$$\int d\omega n_f(\omega)\frac{\omega_c}{2\pi l}G''(\omega,0) = -\frac{\omega_c}{2l}. \tag{A9}$$

We now calculate the second integral

$$-\int d\omega n_f(\omega)\frac{\omega_c}{2\pi l}\int_0^\infty dE\cos\frac{2\pi lE}{\omega_c}\frac{d}{dE}G''(\omega,E) = -\left[n_f(\omega)\frac{\omega_c}{2\pi l}\int_0^\infty dE \cos\frac{2\pi lE}{\omega_c}G''(\omega,E)\right]_{\omega\to-\infty}^{\omega\to\infty} + \int d\omega\frac{dn_f(\omega)}{d\omega}\frac{\omega_c}{2\pi l}\int_0^\infty dE\cos\frac{2\pi lE}{\omega_c}G''(\omega,E). \tag{A10}$$

The first term again is easily analyzed. For $\omega\to\infty$ the Fermi distribution annihilates the expression. For $\omega\to-\infty$ we realize that the denominator of the Green's function diverges, since $E>0$, which implies that the denominator overall goes like $\frac{1}{(\omega+\mu)^2+\frac{1}{4\tau^2}}$ for $\omega\to-\infty$, thus going quadratically to zero. This implies

$$-\int d\omega n_f(\omega)\frac{\omega_c}{2\pi l}\int_0^\infty dE \cos\frac{2\pi lE}{\omega_c}\frac{d}{dE}G''(\omega,E) = \int d\omega\frac{dn_f(\omega)}{d\omega}\frac{\omega_c}{2\pi l}\int_0^\infty dE\cos\frac{2\pi lE}{\omega_c}G''(\omega,E). \tag{A11}$$

This expression can be treated by noting the derivative of the Fermi energy pins the ω integral to zero. This allows to perform the energy integration according to

$$\int_0^\infty dE\cos\frac{2\pi lE}{\omega_c}G''(\omega,E) = \int_{-\mu}^\infty dE \cos\frac{2\pi l(E+\mu)}{\omega_c}G''(\omega,E-\mu) \approx \int_{-\infty}^\infty dE\cos\frac{2\pi l(E+\mu)}{\omega_c}G''(\omega,E-\mu) = -\pi\left(\cos\frac{2\pi l\mu}{\omega_c}\cos\frac{2\pi l\omega}{\omega_c} - \sin\frac{2\pi l\mu}{\omega_c}\sin\frac{2\pi l\omega}{\omega_c}\right)e^{-l\pi/\omega_c\tau}. \tag{A12}$$

From there we can go on to solve the remaining integral. Since $\sin\frac{2\pi l\omega}{\omega_c}$ is an odd function of ω and $\frac{dn_f(\omega)}{d\omega}$ even, the integral over $\sin\frac{2\pi l\omega}{\omega_c}$ drops out leaving us with

$$-\int d\omega\frac{dn_f(\omega)}{d\omega}\frac{\omega_c}{2l}\cos\frac{2\pi l\mu}{\omega_c}\cos\frac{2\pi l\omega}{\omega_c}e^{-\pi l/\omega_c\tau} = \frac{\omega_c}{2l}\cos\frac{2\pi l\mu}{\omega_c}e^{-\pi/\omega_c\tau}\frac{2\pi^2 l\frac{T}{\omega_c}}{\sinh\frac{2\pi^2 lT}{\omega_c}}. \tag{A13}$$

We finally obtain (except for the diamagnetic contribution this can be compared to Ref. 21)

$$\Omega = \Omega_0 - \frac{\nu_0\omega_c^2}{2\pi^2}\sum_{l=1}^\infty\frac{(-1)^l}{l^2}\left(1 - \cos\frac{2\pi l\mu}{\omega_c}\frac{\lambda_l}{\sinh\lambda_l}e^{-l\pi/\omega_c\tau}\right) = \Omega_0 + \frac{\nu_0\omega_c^2}{24} + \frac{\nu_0\omega_c^2}{2\pi^2}\sum_{l=1}^\infty\frac{(-1)^l}{l^2}\cos\frac{2\pi l\mu}{\omega_c}\frac{\lambda_l}{\sinh\lambda_l}e^{-l\pi/\omega_c\tau} \tag{A14}$$

where $\lambda_l = \frac{2\pi^2 lT}{\omega_c}$. In the limit $\omega_c\tau \leq 1$ we can approximate the grand potential as

$$\Omega = \Omega_0 + \frac{\nu_0\omega_c^2}{24} - \frac{\nu_0\omega_c^2}{2\pi^2}\cos\frac{2\pi\mu}{\omega_c}\frac{\lambda_1}{\sinh\lambda_1}e^{-\pi/\omega_c\tau}. \quad (A15)$$

One can easily obtain the $\frac{T}{\omega_c}\to 0$ limit of this expression. This yields

$$\Omega = \Omega_0 + \frac{\nu_0\omega_c^2}{24} - \frac{\nu_0\omega_c^2}{2\pi^2}\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}$$
$$= \Omega_0 + \frac{(eB)^2}{48\pi m} - \frac{(eB)^2}{4\pi^3 m}\cos\frac{2\pi\mu}{\omega_c}e^{-\pi/\omega_c\tau}. \quad (A16)$$

APPENDIX B: THE FREE ENERGY OF THE OSCILLATORY PART OF THE PHOTON SYSTEM

In the following we sketch the isolation of the temperature-dependent part of the oscillatory part of the free energy. We start with an expression of the type

$$\mathcal{F}^A_{\rm osc} = \alpha\int_0^{\Lambda} q dq \int_{-\Lambda'}^{\Lambda'} d\omega n_b(\omega)\frac{\omega q^2}{Aq^4+\omega^2}, \quad (B1)$$

where $\Lambda \approx 2k_f$ and $\Lambda' \approx E_f$. We first perform the integration with respect to q

$$\mathcal{F}^A_{\rm osc} = \frac{\alpha}{4A}\int_{-\Lambda'}^{\Lambda'} d\omega n_b(\omega)\omega \ln\left(\frac{A\Lambda^4+\omega^2}{\omega^2}\right) \quad (B2)$$

and an integration by parts which leads to

$$\mathcal{F}^A_{\rm osc} = \frac{\alpha(\Lambda')^2}{8A}\ln\left[1 + A\left(\frac{\Lambda^2}{\Lambda'}\right)^2\right] + \frac{\alpha\Lambda^4}{8}\ln[A\Lambda^4 + (\Lambda')^2]$$
$$- \frac{\alpha}{8A}\int_{-\Lambda'}^{\Lambda'} d\omega n_b'(\omega)\omega^2 \ln\left(1 + \frac{A\Lambda^4}{\omega^2}\right)$$
$$- \frac{\alpha\Lambda^4}{8}\int_{-\Lambda'}^{\Lambda'} d\omega n_b'(\omega)\ln(\omega^2 + A\Lambda^4). \quad (B3)$$

The first two terms can be discarded since they have no temperature dependence. The last term will also not contribute a temperature-dependent part to leading order in $\frac{T}{\Lambda'}$ and $\frac{T}{\Lambda^2}$. Proceeding with the remaining parts of the integral we obtain

$$\mathcal{F}^A_{\rm osc} \approx -\frac{\alpha}{8A}T^2 \ln\frac{\sqrt{A}\Lambda^2}{T}\int_{-\infty}^{\infty} d\omega\frac{\omega^2}{1-\cosh\omega}$$
$$+ \frac{\alpha}{8A}T^2\int_{-\infty}^{\infty} d\omega\frac{\omega^2 \ln|\omega|}{1-\cosh\omega}$$
$$= \frac{\alpha}{6A}\pi^2 T^2 \ln\frac{\sqrt{A}\Lambda^2}{T} - 0.05880\frac{\alpha}{A}T^2. \quad (B4)$$

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